Gauge field configurations in curved spacetimes. I

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We study classical solutions of SU(2) Yang-Mills field equations, with or without coupled scalar fields, in curved spacetimes. We consider, essentially, only static, spherically symmetric background metrics, for both Lorentz and Euclidean signatures. Section I presents the motivation for searching such solutions. In Sec. II, the equations of motion for a class of *Ansätze* and several static solutions are given, namely a class of scalar fields compatible with a point monopole and particular solutions, for gauge fields alone, in background Schwarzschild and de Sitter metrics, respectively. Some interesting properties are discussed. In Sec. III a finite-action, Prasad-Sommerfield-type solution is constructed for the O(4,1) de Sitter metric. In Sec. IV it is shown how one single, simple de Sitter solution can lead to various well-known flat-space solutions, and to new ones, through a systematic exploitation of conformal equivalence. Self-duality constraints are formulated explicitly in Sec. V for a static spherically symmetric metric. Certain results for the Robinson-Bertotti metric are given in Sec. VI.

I. INTRODUCTION

We study, in the following sections, classical solutions of Yang-Mills (YM) field equations, with and without coupled scalars, in curved spacetimes. The motivations for searching such solutions are as follows.

The basic importance of the topological properties of the classical solutions of gauge field equations is now generally recognized.[1] The gravitational field, irrespective of the strength of its coupling constant, can qualitatively modify the spacetime topology in quite nontrivial fashions. This has, naturally, important consequences, concerning both the possible forms and the physical interpretation of the gauge field solutions in such spacetimes. These should be explored systematically. Not only should one consider solutions of the closed system (including both the gravitational and YM fields), but also the wider class of solutions for gauge fields for given background metrics. These can as a first approximation incorporate important nonperturbative features.

In the path-integral formalism, so well suited to gauge fields, when gravitational effects and perturbations of the metric are to be included, one can look for classical solutions in certain previleged classes of background metrics in searching for possible extrema, about which fluctuations are then to be studied.

There is another fascinating aspect. Certain models of hadronic structures postulate *strong* gravitational forces inside extended hadrons.[2-5] These are treated, in certain respects, as "microuniverses" with, usually, a "strong microcosmological constant." Though no detailed, realistic model is available, this is again an intriguing possibility worth exploring. One may, as a first step, assume that certain nonperturbative, nonlinear features can be simulated, to a reasonable extent, by a background metric, inducing, for example, a baglike structure. Then if the model involves gauge fields, such as gluonic ones, as a next step one can look for classical solutions in such a metric.

Thus, from different points of view, we are led to the study of classical solutions of YM field equations for given metrics.

Though in certain cases (self-dual solutions for Euclidean signature and one particular point-monopole solution being examples) we obtain solutions of the combined system of the YM and the gravitational fields, we will not impose this as a necessary condition. We will solve only the matter field equations for given background metrics.

In this paper we will consider only static spherically symmetric metrics and only SU(2) gauge fields and scalar isotriplets.

Certain considerations can be generalized to other metrics and gauge groups. We intend to pursue these topics elsewhere. But, within these restrictions, we obtain here, for these relatively simple but important cases, a variety of solutions with interesting properties.

Let us now summarize our notations and conventions. The Hermitian SU(2) gauge field components are

$$A_\mu(x) = A_\mu^a(x)\,\frac{\tau^a}{2}\,, \quad \begin{matrix}\mu = 0, 1, 2, 3\\ a = 1, 2, 3\end{matrix} \tag{1.1}$$

and the field tensor is, with a normalized coupling constant,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i[A_\mu(x), A_\nu(x)]\,. \quad (1.2)$$

(Changing the coupling constant from 1 to g_y, the solutions change simply by a factor g_y^{-1} and the action by one of g_y^{-2}.) As scalar field we consider always an isotriplet

$$\Phi(x) = \Phi^a(x)\,\frac{\tau^a}{2}\,, \quad (1.3)$$

the covariant derivatives being defined by

$$D_\mu \Phi(x) = \partial_\mu \Phi(x) + i[A_\mu(x), \Phi(x)]\,. \quad (1.4)$$

The Lagrangian density is, with a scalar potential $V(\Phi)$ and a metric $g_{\mu\nu}$ of determinant g,

$$\mathcal{L} = |g|^{1/2}\{2\,\mathrm{Tr}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(D_\mu\Phi)(D^\mu\Phi)] - V(\Phi)\} \quad (1.5)$$

with signature $(-+++)$. The equations of motion, for the matter fields only, which we will try to solve are thus

$$\begin{aligned} &|g|^{-1/2}D_\mu(|g|^{1/2}F^{\mu\nu}) - i[\Phi, D^\nu\Phi] = 0\,, \\ &|g|^{-1/2}D_\mu(|g|^{1/2}D^\mu\Phi) - i\,\frac{\delta V}{\delta\Phi} = 0\,. \end{aligned} \quad (1.6)$$

We have written the contribution of the potential symbolically as $\delta V/\delta\Phi$ and here

$$D_\mu(|g|^{1/2}F^{\mu\nu}) = \partial_\mu(|g|^{1/2}F^{\mu\nu}) + |g|^{1/2} i[A_\mu, F^{\mu\nu}]\,. \quad (1.7)$$

II. EQUATIONS OF MOTION AND STATIC SOLUTIONS

Let the background metrics be given by

$$ds^2 = (\mp)N\,dt^2 + N^{-1}dr^2 + r^2 d\Omega \quad (d\Omega \equiv d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \quad (2.1)$$

where

$$N = 1 - \frac{2M}{r} + \frac{(\mp Q^2) + P^2}{r^2} - \frac{\Lambda}{3}r^2\,, \quad (2.2)$$

the upper and lower signs corresponding to Lorentz and Euclidean signatures, respectively.

Consider now the Lagrangian (1.5) and the ansatz

$$\begin{aligned} &A_0 = 0\,, \\ &\vec{A} = [K(r,t) - 1]\,i[\hat{\Phi}, \vec{\nabla}\hat{\Phi}]\,, \end{aligned} \quad (2.3)$$

and

$$\Phi = \frac{H(r,t)}{r}\,\hat{\Phi}\,,$$

where

$$\hat{\Phi} \equiv \frac{\vec{\tau}}{2}\cdot\frac{\vec{x}}{r} = \frac{\vec{\tau}}{2}\cdot\hat{r} \quad (2.4)$$

[$\hat{\Phi}$ should not be identified with the scalar field though the ansatz (2.3) gives a simple proportionality.] The equations of motion (1.6) now reduce to

$$(\mp)N^{-1}\ddot{K} + NK'' + N'K' = \frac{1}{r^2}[K(K^2-1) + KH^2]\left(\frac{\partial K}{\partial t} \equiv K_t \equiv \dot{K},\ \frac{\partial K}{\partial r} \equiv K_r \equiv K'\right), \quad (2.5)$$

$$(\mp)N^{-1}\ddot{H} + NH'' + rN'\left(\frac{H}{r}\right)' = \frac{2}{r^2}HK^2 + \frac{\delta V}{\delta\Phi}\,r\,. \quad (2.6)$$

Though we will study classical solutions to start with, we have included a simple time dependence in the ansatz for eventual study of more general cases.

The variable transformation

$$r^* = \int dr/N \quad (2.7)$$

leads to the equations

$$\frac{r^2}{N}(K_{r^*r^*} \mp K_{tt}) = K(K^2-1) + KH^2\,, \quad (2.8)$$

$$\frac{r^2}{N}(H_{r^*r^*} \mp H_{tt}) = 2K^2H + rN'H + r^3\,\frac{\delta V}{\delta\Phi}\,. \quad (2.9)$$

Let us now present certain static solutions. In this section we will continue to use Eqs. (2.5) and (2.6) directly. Equations (2.8) and (2.9) will be useful later on.

(i) Point monopole and a class of scalar field. It is well known that for

$$\frac{\delta V}{\delta\Phi} = 0\,,\quad K = 0\,,\quad H/r = c_1 \quad (2.10)$$

one obtains the point-monopole solution compatible with the Reissner-Nordström metric.[6-8]

For (2.10) one has a solution for the total (gravity, YM) system. But for a given background metric (2.1) we note briefly the following more general possibility, namely

$$\frac{\delta V}{\delta\Phi} = 0\,,\quad K = 0\,,$$

and

$$\frac{H}{r} = c_1 + c_2\int\frac{dr}{Nr^2}\,, \quad (2.11)$$

where c_1 and c_2 are constants. As an example we note that for

$$N = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}$$

and

$$\alpha \equiv M^2 - P^2 - Q^2 , \tag{2.12}$$

$$\int \frac{dr}{Nr^2} = \begin{cases} -\frac{1}{\sqrt{\alpha}} \tanh^{-1}\left(\frac{r-M}{\sqrt{\alpha}}\right) & \text{for } \alpha > 0 \\ -\frac{1}{(|\alpha|)^{1/2}} \tan^{-1}\left(\frac{r-M}{(|\alpha|)^{1/2}}\right) & \text{for } \alpha < 0 \\ -\frac{1}{r-M} & \text{for } \alpha = 0 . \end{cases} \tag{2.13}$$

For completeness we add that for flat space one has, in addition, the well-known finite-energy Prasad-Sommerfield solution.[9] We will come back to it from a different point of view in Sec. III.

(ii) Schwarzschild solutions. Now

$$N = \left(1 - \frac{2M}{r}\right),$$

and for

$$A_0 = 0 , \quad H = 0$$

and

$$\vec{A} = [K(r) - 1] i [\hat{\Phi}, \vec{\nabla}\hat{\Phi}] \tag{2.14}$$

we have found the solutions

$$K = \left(\frac{r - 2M\beta}{r + 6M(\beta - 1)}\right) , \tag{2.15}$$

where

$$\beta = \tfrac{1}{2}(3 \pm \sqrt{3}) .$$

(A change of sign for K corresponds to a simple global gauge transformation by $U = e^{i\pi\hat{\Phi}}$.) For

$$\beta = \tfrac{1}{2}(3 + \sqrt{3}) \tag{2.16}$$

one obtains a finite energy static solution in the external region ($2M < r < \infty$), the static energy (mass) being given by,[8] integrating ($-T_0^0$),

$$E = 4\pi \int_{2M}^{\infty} dr \left[NK'^2 + \frac{1}{2r^2} (K^2 - 1)^2 \right] , \tag{2.17}$$

the integrand being now nonsingular at $r = 2M$. For both solutions K is regular in the interior region $r < 2M$. But K^2 does not $\to 1$ as $r \to 0$ and (2.17) thus also has a divergence at the essential Schwarzchild singularity. The integral can be evaluated exactly and one gets

$$E = \frac{4\pi}{M} c(\beta) , \tag{2.18}$$

where for (2.16) $c(\beta)$ is a finite but lengthy expression in β.

Approximately, one has

$$E = \frac{4\pi}{M} (0.24) . \tag{2.19}$$

Note that though $K \to 1$ and $A_\mu \to 0$ as $M \to 0$, the integral behaves as M^{-1}. For a coupling constant g_y the energy is of course

$$\frac{4\pi}{Mg_y^2} c(\beta) . \tag{2.20}$$

Considered as a monopole-type solution in the exterior region it is seen that (for 2.16) K starts from a negative value at $r = 2M$ and vanishes for

$$r = 2M\beta , \tag{2.21}$$

where the field is exactly that of a point monopole. Then K changes sign and $(K - 1)$ (and hence $\vec{A}$) $\to 0$ as $r \to \infty$. As the Schwarzchild parameter M increases the energy E decreases and the range (2.21) becomes greater.

It is, however, quite interesting to consider the solution on the Euclidean section (Appendix B).

One then has a periodic time with a period

$$T = 8\pi M \tag{2.22}$$

and the full domain of the Kruskal coordinates correspond to $2M < r < \infty$. Thus, one has a static solution with a *finite action* (Euclidean):

$$S = 8\pi ME = 32\pi^2 c(\beta) . \tag{2.23}$$

Let us now compare this with the self-dual and anti-self-dual solutions which are, respectively [Appendix B]

$$A_0 = \pm \frac{N'}{2} \hat{\Phi} , \qquad \vec{A} = (N^{1/2} - 1) i [\hat{\Phi}, \vec{\nabla}\hat{\Phi}] , \tag{2.24}$$

with

$$N = 1 - \frac{2M}{r} .$$

These have, respectively, the Pontryagin number

$$P_y = \pm 1$$

and action

$$S = 8\pi^2 . \tag{2.25}$$

The (non-self-dual) solution (2.15) [with (2.16)] has

$$F_{0i} = 0 \quad (i = 1, 2, 3)$$

and evidently $P_y = 0$. Its action is a little less than $8\pi^2$.

Thus, it has certain features one would expect of an instanton-anti-instanton bound state, the "binding" being strong enough to lower the action even below $8\pi^2$. It is as if the A_0's of (2.24) [and the P_y's of (2.25)] are simply added so as to cancel, whereas $N^{1/2}$ in (2.24) is altered due to a binding interaction. As a comparison one may note that as

$$r \to \infty$$

$$N^{1/2} \to 1 - \frac{M}{r}$$

and

$$K \to 1 - \frac{M}{r}(8\beta - 6). \tag{2.26}$$

The situation can be compared to that found by Kerner[10] for excited levels of flat-space Prasad-Sommerfield solutions. These higher-energy solutions behave as if a monopole-antimonopole pair is being added at each step to a monopole. But the energy (at least in the numerical method used there) is a little more than the sum of those of the components.

In flat Euclidean space one has only self-dual or anti-self-dual finite-action solutions, and to prove rigorously whether finite-action non-self-dual solutions exist or not seems to be a difficult problem. Here the continuation via the Kruskal coordinates (Appendix B) leads to a (finite-action, non-self-dual) solution resembling a bound state.

(iii) Complex de Sitter solutions. In the preceding case one found a simple interesting homographic solution. It is not difficult to verify (to adopt a more unified viewpoint) that a simple ansatz

$$K = \frac{\alpha r^n + 1}{\beta r^n + 1} \quad (\alpha, \beta \text{ constants}) \tag{2.27}$$

works for the Schwarzchild case, i.e.. for

$$N = 1 - \frac{2M}{r} \quad \text{with } n = -1 \tag{2.28}$$

in (2.27) and also for the de Sitter case, i.e., for

$$N = 1 \mp \frac{\Lambda}{3} r^2 \quad \text{with } n = 2. \tag{2.29}$$

For the latter case, however, we get a *complex* solution, namely

$$K = \frac{ar^2 \pm (3/\Lambda)}{br^2 \pm (3/\Lambda)}, \tag{2.30}$$

where

$$a = \pm i\sqrt{3}, \quad b = -\tfrac{3}{7}(2 \pm i\sqrt{3}) \tag{2.31}$$

for each sign in (2.29).

We give this complex solution for comparison with the Schwarzchild case. A parallel calculation of the energy and the Euclidean action for the interior region $[r < (3/\Lambda)^{1/2}]$ for the O(4, 1) case give finite but, of course, complex values. In Secs. III and IV we will exploit the conformal equivalence of the de Sitter metric with the flat one to construct various solutions. Here also similar techniques can naturally be employed. But since that leads to flat-space solutions with actions not only complex but also divergent, we merely mention one particularly simple result.

Starting from (2.30) and (2.31) and using suitable variable transformations one can show that for the Minkowski and the flat Euclidean cases, respectively, where

$$ds^2 = (\mp dt^2 + dr^2 + r^2 d\Omega), \tag{2.32}$$

one has the solutions

$$K = \frac{ar^2 \pm t^2}{br^2 \pm t^2}. \tag{2.33}$$

Here a and b are given by (2.31).

III. A FINITE-ACTION PRASAD-SOMMERFIELD-TYPE SOLUTION FOR THE DE SITTER METRIC

Mecklenburg and O'Brien[11] have given a divergent time-dependent generalization of the Prasad-Sommerfield (PS) solution[9] for the Minkowski metric. The flat-space equations to be solved are, for the ansatz (2.3),

$$\begin{aligned} r^2(K_{rr} - K_{tt}) &= K(K^2 - 1) + KH^2, \\ r^2(H_{rr} - H_{tt}) &= 2HK^2. \end{aligned} \tag{3.1}$$

They search for a variable $u(r, t)$ such that, assuming

$$\begin{aligned} &K(r, t) = K(u), \quad H(r, t) = H(u), \\ &r^2(K_{rr} - K_{tt}) = u^2 K_{uu}, \\ &r^2(H_{rr} - H_{tt}) = u^2 H_{uu}. \end{aligned} \tag{3.2}$$

Their solution is, in the final form,

$$u = \frac{r}{r^2 - t^2} \tag{3.3}$$

and

$$K = \frac{Cu}{\sinh(Cu)}, \quad H = Cu\coth(Cu) - 1 \tag{3.4}$$

where C is the scale parameter. Their u is singular on the light cone and the action is divergent even on the Euclidean section, though as

$$t \to -it, \quad (r^2 - t^2) \to (r^2 + t^2).$$

We will show that a similar technique leads to a finite-action solution for the O(4, 1) de Sitter case. This can be done by systematically exploiting its conformal equivalence with flat space.

In fact, from this point of view, the flat-space transformation (3.3) can be completed as follows. Let

$$u=\frac{r}{r^2-t^2}\,,\quad v=\frac{t}{r^2-t^2} \tag{3.5}$$

when (for this transformation which sends the light cone to infinity)

$$\begin{aligned} ds^2&=(-dt^2+dr^2+r^2d\Omega)\\ &=(u^2-v^2)^{-1}(-dv^2+du^2+u^2d\Omega)\,. \end{aligned} \tag{3.6}$$

Thus

$$r^2(K_{rr}-K_{tt})=u^2(K_{uu}-K_{vv}) \tag{3.7}$$

and so on. Supposing now that K and H do not depend on v, one gets the previous result.

Let us now consider analogous transformations for the de Sitter line element

$$\begin{aligned} ds^2=&-\left(1-\frac{\Lambda}{3}r^2\right)dt^2\\ &+\left(1-\frac{\Lambda}{3}r^2\right)^{-1}dr^2+r^2d\Omega\,. \end{aligned} \tag{3.8}$$

We consider the region within the horizon [at $r=(3/\Lambda)^{1/2}$], i.e., we will study the possibility of the existence of a certain type of finite-action gauge and scalar field configuration in such a simple, closed (macro or micro) universe. Let

$$\left(\frac{\Lambda}{3}\right)^{1/2}r=\tanh\rho\quad\text{and}\left(\frac{1}{3}\right)^{1/2}t=\tau \tag{3.9}$$

[in fact, $\rho=(\Lambda/3)^{1/2}r^*=(\Lambda/3)^{1/2}\int dr/N$] when

$$ds^2=\frac{3}{\Lambda}(\cosh\rho)^{-2}[-d\tau^2+d\rho^2+\sinh^2\rho d\Omega]\,. \tag{3.10}$$

We will now introduce a scalar potential

$$V(\Phi)=\frac{\Lambda}{3}(2\,\mathrm{Tr}\Phi^2) \tag{3.11}$$

which is exactly what is necessary to ensure conformal invariance.[12] The equations of motion (2.8) and (2.9) now reduce to

$$\begin{aligned} &(\sinh\rho)^2(K_{\rho\rho}-K_{tt})=K(K^2-1)+KH^2\,,\\ &(\sinh\rho)^2(H_{\rho\rho}-H_{tt})=2HK^2\,. \end{aligned} \tag{3.12}$$

The action

$$\begin{aligned} S=-4\pi\int\!\!\int dr\,dt\Big\{&NK'^2-N^{-1}\dot{K}^2+\tfrac{1}{2}Nr^2\left[\left(\frac{H}{r}\right)'\right]^2\\ &-\tfrac{1}{2}N^{-1}\dot{H}^2+\frac{1}{2r^2}(K^2-1)^2+\frac{1}{r^2}K^2H^2+\frac{\Lambda}{3}H^2\Big\}\quad[0\le r\le(\Lambda/3)^{1/2},\,-\infty<t<+\infty] \end{aligned} \tag{3.13}$$

is now transformed to, with $0\le\rho<\infty,\,-\infty<\tau<\infty$

$$S=-4\pi\int\!\!\int d\rho\,d\tau\left\{(K_\rho{}^2-K_\tau{}^2)+\tfrac{1}{2}(H_\rho{}^2-H_\tau{}^2)+\frac{1}{(\sinh\rho)^2}[\tfrac{1}{2}(K^2-1)^2+K^2H^2]-\tfrac{1}{2}(H/\sinh\rho\cosh\rho)_{,\rho}\right\}. \tag{3.14}$$

The last term, a derivative, does not contribute to the equations of motion. It may be verified that the rest yields directly (3.12).

We now introduce a second transformation. The intermediate step (3.9), (3.12) will be useful later on. That is why we are proceeding in two stages. Let

$$\xi_\pm=\tanh(\tfrac{1}{2}\tau_\pm)\,, \tag{3.15}$$

where $\xi_\pm=\xi\pm\eta$, $\tau_\pm=\tau\pm\rho$. The metric is now

$$ds^2=\frac{3}{\Lambda}\,\frac{4}{(1+\eta^2-\xi^2)^2}(-d\xi^2+d\eta^2+\eta^2d\Omega)\,. \tag{3.16}$$

Equations (3.12) now become

$$\begin{aligned} &\eta^2(K_{\eta\eta}-K_{\xi\xi})=K(K^2-1)+KH^2\,,\\ &\eta^2(H_{\eta\eta}-H_{\xi\xi})=2HK^2\,. \end{aligned} \tag{3.17}$$

Hence we have a PS-type solution

$$K=\frac{\lambda\eta}{\sinh\lambda\eta}\,,\quad H=(\lambda\eta\coth\lambda\eta-1)\,, \tag{3.18}$$

where λ is a scale parameter. We can consider it as a time-dependent solution in (r,t). But the action is evaluated easily in terms of (η,ξ). The action now is transformed to, with

$$0\le\eta\le1\,,\quad -1\le\xi\le1\,, \tag{3.19}$$

$$\begin{aligned} S=-4\pi\int\!\!\int d\eta\,d\xi\Big\{&(K_\eta{}^2-K_\xi{}^2)+\tfrac{1}{2}(H_\eta{}^2-H_\xi{}^2)\\ &+\frac{1}{\eta^2}[(K^2-1)^2+K^2H^2]\Big\}\,. \end{aligned} \tag{3.20}$$

[The action has no singularity for our solution, even at the limit corresponding to $r=(3/\Lambda)^{1/2}$.] In transforming we have dropped in (3.14) the term

$$\frac{\partial}{\partial\rho}(H/\sinh\rho\cosh\rho)\,.$$

Not only does it not contribute to the equations of motion, but it can also easily be seen to vanish both for

$$\eta\to0\ (\rho\to0)\ \text{and}\ \eta\to1\ (\rho\to\infty)\,.$$

One can now integrate (3.20) and one gets, the ξ integration giving simply a factor of 2,

$$S=4\pi\left[1+\frac{2\lambda^3\cosh\lambda}{(\sinh\lambda)^3}-\frac{3\lambda^2}{(\sinh\lambda)^2}-\lambda^2\right]. \tag{3.21}$$

The integrand is formally the same as for the flat-space PS solution. The domains of the variables, being different, lead to (3.21).

We have presented the result for Lorentz signature. But as a particular prescription for going over to Euclidean signature, one may note, from (3.15), that $\xi\to -i\xi$ corresponds simply to $t\to -it$. The Euclidean action is positive.

IV. MERONS AND RELATED FLAT-SPACE SOLUTIONS VIA THE DE SITTER METRIC

We continue the theme of Sec. III by exploiting, now in the opposite sense, the conformal equivalence properties.

The role of the conformal transformations in the construction of meron solutions has, of course, been duly recognized and utilized from the very beginning by its authors.[13] Our aim is to show how, starting from one very simple de Sitter solution (which is immediately evident from the equations of motion for certain coordinates), a number of important flat-space solutions can be derived formally. This is done quite systematically by introducing a general form of transformation, leading to a line element explicitly conformally equivalent to a flat one. One obtains a corresponding transformed solution which, due to the invariance properties of gauge fields, is formally also a flat-space solution. Varying the parameters of the transformation now yields different well-known cases and also leads to a new solution.

We proceed as follows. We start again from (3.8), maintain (3.9), (3.10), but generalize (3.15) to

$$\xi_\pm=\frac{ae^{\tau_\pm}-b}{ce^{\tau_\pm}+d}. \tag{4.1}$$

For $a=b=c=d=1$ we get back (3.15). Indeed, particular cases of transformations we will use are familiar in the study of conformal infinity. But the generalized form is useful for our purpose.

Now,

$$ds^2=\frac{3}{\Lambda}\left(\frac{\tanh\rho}{\eta}\right)^2(-d\xi^2+d\eta^2+\eta^2d\Omega), \tag{4.2}$$

where (considering, for example the case $b\neq 0$, and normalizing to $b=1$)

$$\left(\frac{\tanh\rho}{\eta}\right)=(ad+c)[cd(\eta^2-\xi^2)+(ad-c)\xi+a]^{-1}. \tag{4.3}$$

We consider now gauge fields only ($\Phi=0$) and start again with the ansatz

$$A_0=0,\quad \vec{A}=(K-1)i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]. \tag{4.4}$$

Equation (3.12)

$$(\sinh\rho)^2(K_{\rho\rho}-K_{\tau\tau})=K(K^2-1) \tag{4.5}$$

is transformed as before [under (4.1)] to

$$\eta^2(K_{\eta\eta}-K_{\xi\xi})=K(K^2-1). \tag{4.6}$$

Now (4.5) has evidently the solution

$$K=\pm\cosh\rho. \tag{4.7}$$

The transformed solution (for $b=1$) is

$$K=\pm\frac{a+(ad-c)\xi+cd(\eta^2-\xi^2)}{\{[(a-c\xi)^2-c^2\eta^2][(1+d\xi)^2-d^2\eta^2]\}^{1/2}}. \tag{4.8}$$

We are thus led (ignoring the conformal factor) formally to the flat-space solution which is, with a change of notation, for

$$ds^2=(-dt^2+dr^2+r^2d\Omega),$$

$$K=\pm\frac{a+(ad-c)t+cd(r^2-t^2)}{\{[(a-ct)^2-c^2r^2][1+dt)^2-d^2r^2]\}^{1/2}}. \tag{4.8$'$}$$

They can of course be verified directly. The parameters a, c, d can now be varied to extract various interesting cases.

Case (i)—meron or antimeron. For $a=d=0$, $c=\pm1$, and $t\to -it$ (Euclidean case),

$$K=\pm\frac{t}{[r^2+t^2]^{1/2}}=\pm\cos\left(\tan^{-1}\frac{r}{t}\right). \tag{4.9}$$

These are, respectively, the meron and antimeron solutions *in the gauge* $A_0=0$ or the "neutral gauge."[14,15] The action is well known to be logarithmically divergent.

Case (ii)—finite-action Minkowski solution. For imaginary c and d and *real* a such that

$$\frac{a}{c}=\frac{1}{d}=-i\lambda, \tag{4.10}$$

we have a real K given by

$$K=\pm\frac{\lambda^2+t^2-r^2}{\{[r^2+(\lambda-it)^2][r^2+(\lambda+it)^2]\}^{1/2}}. \tag{4.11}$$

These can be shown to be gauge equivalent to finite-action Minkowski solution found in Ref. 13.

Case (iii)—meron and antimeron. The related well-known meron-antimeron solution is, of course, obtained from (4.11) on

$t\to -it$ (Euclidean case).

Now

$$K=\pm\frac{\lambda^2-(r^2+t^2)}{\{[(\lambda-t)^2+r^2][(\lambda+t)^2+r^2]\}^{1/2}} \tag{4.12}$$

$$=\pm\cos\left[\tan^{-1}\left(\frac{r}{t+\lambda}\right)-\tan^{-1}\left(\frac{r}{t-\lambda}\right)\right]. \tag{4.13}$$

We are again in the neutral ($A_0=0$) gauge. The action is again logarithmically divergent. [Starting with $b=0$ in (4.1) one can again arrive at this solution in a similar fashion. We will not enumerate explicitly all the alternative possibilities.]

Case (iv)—solution singular on a spherical surface. For

$$\frac{a}{c}=\frac{1}{d}=\lambda\,,\quad \text{all real}$$

and

$$t\to -it \quad \text{(Euclidean case)}$$

one finally obtains for $(\lambda\neq 0)$,

$$K=\pm\frac{\lambda^2+t^2+r^2}{\{[(\lambda+r)^2+t^2][(\lambda-r)^2+t^2]\}^{1/2}}\,. \tag{4.14}$$

This is singular for

$$t=0\,,\quad r=|\lambda|>0\,,$$

which represents a two-sphere in the space coordinates.

We have attempted elsewhere[15] to initiate the explicit construction of a hierarchy of divergent solutions. Here we have obtained another remarkable example via the de Sitter solutions (4.7). The Euclidean baglike aspect is attractive, but the action is strongly divergent.

For completeness we note the following points. Suitably combining the two transformations that lead from (4.7) to (4.9) and (4.13) one obtains the transformation that converts a meron solution to a meron-antimeron pair (by bringing in the antimeron from infinity). The result can be summarized as follows.

Starting with

$$ds^2=(dt^2+dr^2+r^2d\Omega)$$

and setting

$$r=\frac{2\lambda\chi}{\chi^2+(\zeta-\lambda)^2}\,,\quad t=\frac{\lambda^2-(\chi^2+\zeta^2)}{\chi^2+(\zeta-\lambda)^2}\,, \tag{4.15}$$

one gets

$$ds^2=\frac{4\lambda^2}{[\chi^2+(\zeta-\lambda)^2]^2}(d\zeta^2+d\chi^2+\chi^2d\Omega) \tag{4.16}$$

and

$$K=\cos\left(\tan^{-1}\frac{r}{t}\right)$$
$$=\cos\left[\tan^{-1}\left(\frac{\chi}{\zeta+\lambda}\right)-\tan^{-1}\left(\frac{\chi}{\zeta-\lambda}\right)\right]. \tag{4.17}$$

A second application of such a transformation $(\chi,\zeta\to\tilde{\chi},\tilde{\zeta}$, say) gives an antimeron only by sending the meron off to infinity. Thus, the cycle closes.

We see that in the context of conformal equivalence classes of metrics the meron and meron-antimeron solutions become effectively the same, since the transformations involved can change the structure and locations of singularities. Similarly, in the context of invariance under general variable transformations[5] the multimeron solutions should be considered to be effectively the same as a single meron one.

Finally, we note that had we started, instead of (3.8), from the O(3, 2) line element

$$ds^2=-\left(1+\frac{\Lambda}{3}r^2\right)dt^2+\left(1+\frac{\Lambda}{3}r^2\right)^{-1}dr^2+r^2d\Omega \tag{4.18}$$

and proceeded analogously, it would not have been necessary to pass via imaginary parameters to obtain [as in (4.9)–(4.13)] the meron-antimeron solutions. Instead of repeating the analogous steps, we give in the next subsection a more general class of de Sitter solutions which will be related to the finite-action Minkowski solutions constructed by using the "O(4) × O(2)" formalism.[16,17]

A class of nonstatic de Sitter solutions. Starting from (4.18) and setting

$$\left(\frac{\Lambda}{3}\right)^{1/2}r=\tan\rho\,,\quad \left(\frac{\Lambda}{3}\right)^{1/2}t=\tau \tag{4.19}$$

we get

$$ds^2=\frac{3}{\Lambda}\,\frac{1}{\cos^2\rho}[-d\tau^2+d\rho^2+(\sin\rho)^2d\Omega]\,, \tag{4.20}$$

and for the ansatz (4.4)

$$(\sin\rho)^2(K_{\rho\rho}-K_{\tau\tau})=K(K^2-1)\,. \tag{4.21}$$

Consider the solution

$$K=\cos\rho \tag{4.22}$$

which leads, for example, to the meron solution (4.9) through

$$\xi=e^{\tau}\cos\rho\,,\quad \eta=e^{\tau}\sin\rho \tag{4.23}$$

and

$$\xi\to -i\xi\,.$$

The gauge field

$$A_0=0\,,\qquad \vec{A}=(\cos\rho-1)i[\hat{\Phi},\vec{\nabla}\hat{\Phi}] \tag{4.24}$$

after a gauge transformation by

$$U=e^{i\rho\hat{\Phi}} \tag{4.25}$$

takes the form

$$A_0=0\,,\qquad \vec{A}=\sin\rho\cos\rho\vec{\nabla}\hat{\Phi}-\sin^2\rho i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]+\hat{\rho}\hat{\Phi}\,. \tag{4.26}$$

This form of the solution can be simply generalized to give a τ-dependent ansatz, namely

$$A_0 = 0,$$
$$\vec{A} = f(\tau)\{\cos\rho\,\sin\rho\,\vec{\nabla}\hat{\Phi} - \sin^2\rho\, i\,[\hat{\Phi}, \vec{\nabla}\hat{\Phi}] + \hat{\rho}\hat{\Phi}\}\,. \tag{4.27}$$

The equations of motion reduce to

$$\frac{d^2 f}{d\tau^2} = 2f(f-1)(f-2)\,. \tag{4.28}$$

This equation can be completely solved in terms of elliptic functions.[16,17]

As compared to the O(4)×O(2) formalism,[16,17] though the ansatz (4.27) is, in a certain sense, slightly less simple, as a compensating feature, here we do not need the introduction of extra coordinates and subsequent projections.

Solutions similarly related to more general Minkowski solutions[18,19] can also be obtained in this fashion. But the formalism becomes less simple and we will not discuss them here. Our purpose was to show how simply and systematically many well-known results can be obtained via the de Sitter space, while new results, such as (3.21) and (4.14), also emerge. We remind the reader that the famous BPST instanton solution is again obtained directly from the spin connections of (3.16) after $\xi \to -i\xi$ and using the formulas given in Appendix B (see, for example, Wilczek[13]).

Instead of the de Sitter metric one can also start from the more general Robertson-Walker line element

$$ds^2 = -d\tau^2 + Q^2(\tau)\left(\frac{d\rho^2}{1-\lambda\rho^2} + \rho^2 d\Omega\right) \quad (\lambda = 0, \pm 1)\,. \tag{4.29}$$

As far as the object of this section is concerned, the same solutions are obtained after suitable coordinate transformations.

The YM fields obtained in this metric, starting from known Minkowski-space solutions[20] or using the spin connections,[21] have been discussed by certain authors.

V. SELF-DUALITY CONSTRAINTS FOR CURVED SPACETIMES

Elsewhere[22] we have studied the construction of self-dual (and anti-self-dual) SU(2) YM fields through spin connections for a class of metrics. For such Euclidean fields the energy-momentum tensor vanishes and hence the metric is not perturbed. On the other hand, the use of spin connections leads to only one type of solution for a given metric. Can one construct other self-dual solutions?

The general method for construction of self-dual solutions, using methods of algebraic geometry,[23] leads, in flat space, to formulas such that the extraction of explicit expressions for A_μ is a difficult task and one has quite complicated results except for the previously known $(5P_y+4)$ parameter solutions. For curved backgrounds, the adequate generalization of such methods, even for SU(2) gauge fields, will presumably present further complications. Hence it is worthwhile to approach the problem from different points of view.

Here, as a first step, we formulate successively Witten[24] and then more general Yang[25] type self-duality constraint equations for curved spaces. (The anti-self-dual case can, of course, be treated analogously.)

Again we restrict ourselves to the metrics (2.1) with Euclidean signature, namely

$$ds^2 = N dt^2 + N^{-1} dr^2 + r^2 d\Omega\,. \tag{5.1}$$

The relevant domains of r and t are to be determined for each case separately. We will formulate the equations directly in terms of N.

We start with the restricted ansatz

$$A_0 = d(r,t)\hat{\Phi}\,,$$
$$\vec{A} = a(r,t)\vec{\nabla}\hat{\Phi} + [b(r,t)-1]i[\hat{\Phi}, \vec{\nabla}\hat{\Phi}] + c(r,t)\vec{r}\hat{\Phi}\,. \tag{5.2}$$

(Setting further $c=0$ one gets the ansatz of Charap and Duff.[26]) For *diagonal* metrics, the self-duality constraints can be shown to reduce to

$$\frac{F_{0i}}{(g_{00}g_{ii})^{1/2}} = \frac{F_{jk}}{(g_{jj}g_{kk})^{1/2}} \quad (i,j,k: 1,2,3 \text{ cyclic})\,. \tag{5.3}$$

From (5.1), (5.2), and (5.3) one gets finally

$$(\dot{a} - db) = -N(b' + ca)\,,$$
$$(\dot{b} + da) = N(a' - cb)\,,$$
$$(\dot{c} - d') = -\frac{1}{r^2}(a^2 + b^2 - 1)\,. \tag{5.4}$$

From the first two equations one obtains

$$c = -\partial_r\left(\tan^{-1}\frac{b}{a}\right) - N^{-1}\partial_t[\ln(a^2+b^2)^{1/2}]\,,$$
$$d = -\partial_t\left(\tan^{-1}\frac{b}{a}\right) + N\partial_r[\ln(a^2+b^2)] \tag{5.5}$$

(assuming for the moment that these expressions are well defined). Substituting in the third equation of (5.4) we get

$$\partial_t\{N^{-1}\partial_t[\ln(a^2+b^2)^{1/2}]\} + \partial_r\{N\partial_r[\ln(a^2+b^2)^{1/2}]\} = \frac{1}{r^2}(a^2+b^2-1)\,. \tag{5.6}$$

Using $\partial_t N = 0$ and with $r^* = \int dr/N$, defining

$$a^2 + b^2 = e^\chi , \tag{5.7}$$

finally we have

$$(\partial_t{}^2 + \partial_{r^*}{}^2)\chi = \frac{2N}{r^2}(e^\chi - 1). \tag{5.8}$$

For $N = 1$ we get back Witten's case.[24]

Defining

$$\chi = \rho + 2\int \frac{dr}{Nr} , \tag{5.9}$$

(5.8) gives

$$(\partial_t{}^2 + \partial_{r^*}{}^2)\rho = 2K(r)e^\rho , \tag{5.10}$$

where

$$K(r) = \frac{N}{r^2}\exp\left(2\int \frac{dr}{Nr}\right) \tag{5.11}$$

$$= -\frac{d}{dr^*}\left(\frac{1}{r}\right)\exp\left(2\int \frac{dr^*}{r}\right) \equiv \tilde{K}(r^*). \tag{5.12}$$

The condition

$$K(r) = \text{constant} \tag{5.13}$$

leads to

$$N = 1 + \lambda r^2 . \tag{5.14}$$

Thus the flat and the conformally equivalent constant curvature de Sitter spaces are selected out. For the Schwarzchild case,

$$N = \left(1 - \frac{2M}{r}\right), \quad K(r) = N^3 . \tag{5.15}$$

More generally, setting

$$\zeta = e^{\rho/2} \tag{5.16}$$

one has

$$\frac{(\partial_t{}^2 + \partial_{r^*}{}^2)(\ln\zeta)}{\zeta^2} = \tilde{K}(r^*). \tag{5.17}$$

The problem has thus been reduced (in terms of r^*, t) to one of finding a surface whose Gaussian curvature is given by $-\tilde{K}$.[27]

The self-dual solution from spin connection[22]

$$A_0 = \tfrac{1}{2}N'\hat{\Phi} ,\qquad \vec{A} = (N^{1/2} - 1)i[\hat{\Phi}, \vec{\nabla}\hat{\Phi}] \tag{5.18}$$

(with $N > 0$ for the domain of interest) corresponds to

$$a = 0, \quad b = \sqrt{N}, \quad c = 0, \quad d = \tfrac{1}{2}N'$$

and hence to

$$e^\chi = N . \tag{5.19}$$

Equation (5.8) now reduces to

$$N'' = \frac{2}{r^2}(N - 1), \tag{5.20}$$

which is satisfied by

$$N = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right) .$$

For completeness we note that a very particular solution of (5.4) [for which (5.5) is not well defined] is

$$a = b = c = 0, \quad d = 1/r . \tag{5.21}$$

One thus gets the *point-dyon* solutions, which, for example, for the Schwarzschild case[26] have

$$P_y = 2 . \tag{5.22}$$

(For the corresponding anti-self-dual solution with $d = -1/r$, $P_y = -2$.)

We now pass on to a more general case. To exploit the symmetry of (5.1) we choose the coordinates

$$y = \tan\frac{\theta}{2}e^{i\varphi}, \quad z = \tfrac{1}{2}(r^* + it),\qquad \bar{y} = \tan\frac{\theta}{2}e^{-i\varphi}, \quad \bar{z} = \tfrac{1}{2}(r^* - it), \tag{5.23}$$

which should be compared to the choice of Yang.[25] We will restrict ourselves to real (t, r, θ, φ). Correspondingly, we have

$$A_y = \frac{1}{2y}(\sin\theta A_\theta - iA_\varphi),\qquad A_z = (NA_r - iA_0), \tag{5.24}$$

and similar expressions for $A_{\bar{y}}$ and $A_{\bar{z}}$. From (5.1) and (5.3) one now obtains

$$F_{yz} = F_{\bar{y}\bar{z}} = 0 ,\qquad (1 + y\bar{y})^2 F_{y\bar{y}} + \frac{r^2}{N}F_{z\bar{z}} = 0 . \tag{5.25}$$

The last equation can also be written as

$$g^{\mu\bar{\mu}}F_{\mu\bar{\mu}} = 0 \quad (\mu = y, z) \tag{5.26}$$

since

$$ds^2 = 4N\,dz\,d\bar{z} + \frac{4r^2}{(1 + y\bar{y})^2}dy\,d\bar{y} . \tag{5.27}$$

Since in our conventions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] ,$$

from the first two equations of (5.25) we obtain, corresponding to Yang's K gauge,

$$A_y = -iK^{-1}(\partial_y K), \quad A_z = -iK^{-1}(\partial_z K),\qquad A_{\bar{y}} = i(\partial_{\bar{y}}K)K^{-1}, \quad A_{\bar{z}} = i(\partial_{\bar{z}}K)K^{-1}, \tag{5.28}$$

with

$$K = (1 - v^2)^{-1/2}(1 + \vec{v}\cdot\vec{\tau}) . \tag{5.29}$$

These give

$$A_\mu = -i(1-v^2)^{-1}[\vec{v}_\mu - i(\vec{v}\times\vec{v}_\mu)]\cdot\vec{\tau} \quad \begin{pmatrix}\vec{v}_\mu \equiv \partial_\mu \vec{v} \\ \mu = y, z\end{pmatrix} \tag{5.30}$$

and $A_{\bar{\mu}}$ is obtained on conjugation. Substituting in the last equation of (5.25) we get

$$g^{\mu\nu}[\tfrac{1}{2}(1-v^2)\vec{v}_{\mu\mu} + (\vec{v}\cdot\vec{v}_\mu)\vec{v}_\mu + (\vec{v}\cdot\vec{v}_\mu)\vec{v}_\mu - (\vec{v}_\mu\cdot\vec{v}_\mu)\vec{v} + i(\vec{v}_\mu\times\vec{v}_\mu)]\cdot\vec{\tau} = 0 \quad (\mu = y, z). \tag{5.31}$$

It may be verified that (5.18) corresponds to

$$\vec{v} = \left(\frac{\sqrt{N}-1}{\sqrt{N}+1}\right)\hat{r} \tag{5.32}$$

when (5.31) reduces to (5.20).

More generally, setting

$$\vec{v} = (\tanh\tfrac{1}{4}x)\hat{r} \tag{5.32$'$}$$

one finds that (5.31) reduces to (5.8).

In the R gauge of Yang,[25] one has (replacing Yang's φ by λ since we reserve φ for the azimuthal angle)

$$\lambda A_\mu = -i\begin{pmatrix} -\frac{\lambda_\mu}{2} & 0 \\ \rho_\mu & \frac{\lambda_\mu}{2}\end{pmatrix}, \quad \lambda A_{\bar{\mu}} = -i\begin{pmatrix} -\frac{\lambda_{\bar{\mu}}}{2} & \bar{\rho}_{\bar{\mu}} \\ 0 & \frac{\lambda_{\bar{\mu}}}{2}\end{pmatrix} \quad (\lambda_\mu \equiv \partial_\mu \lambda). \tag{5.33}$$

Equation (5.26) now leads to

$$\begin{aligned} &g^{\mu\bar{\mu}}(\lambda\lambda_{\mu\bar{\mu}} - \lambda_\mu\lambda_{\bar{\mu}} + \rho_\mu\bar{\rho}_{\bar{\mu}}) = 0, \\ &g^{\mu\bar{\mu}}(\lambda\rho_{\mu\bar{\mu}} - 2\rho_\mu\lambda_{\bar{\mu}}) = 0, \quad (\mu = y, z). \\ &g^{\mu\bar{\mu}}(\lambda\bar{\rho}_{\bar{\mu}\mu} - 2\bar{\rho}_{\bar{\mu}}\lambda_\mu) = 0. \end{aligned} \tag{5.34}$$

A more suggestive alternative form is

$$\begin{aligned} &g^{\mu\bar{\mu}}\left[\frac{1}{\lambda^2}(\ln\lambda)_{\mu\bar{\mu}} + \left(\frac{\rho_\mu}{\lambda^2}\right)\left(\frac{\bar{\rho}_{\bar{\mu}}}{\lambda^2}\right)\right] = 0, \\ &g^{\mu\bar{\mu}}\left(\frac{\rho_\mu}{\lambda^2}\right)_{\bar{\mu}} = 0, \\ &g^{\mu\bar{\mu}}\left(\frac{\bar{\rho}_{\bar{\mu}}}{\lambda^2}\right)_\mu = 0. \end{aligned} \tag{5.34$'$}$$

The solution (5.18) can now be shown to correspond to

$$\lambda = \frac{\sqrt{N}(1+y\bar{y})}{(N+y\bar{y})}, \quad \rho = \frac{y(-1+N)}{(N+y\bar{y})}. \tag{5.35}$$

Comparing with (5.32) we see that this solution is simpler in the K gauge. Let us finally note that using (5.26) the action can be written as

$$S = 2\,\mathrm{Tr}[F_{y\bar{y}}F_{z\bar{z}} + F_{y\bar{z}}F_{\bar{y}z}]. \tag{5.36}$$

For these coordinates, the factor $|g|^{1/2}$ cancels the factor $(g^{y\bar{y}}g^{z\bar{z}})$ obtained after using (5.26). In the R gauge this gives (compare Ref. 28)

$$\begin{aligned} S = 4\Big\{ &-\left[(\ln\lambda)_{y\bar{y}} + \frac{\rho_y\bar{\rho}_{\bar{y}}}{\lambda^2}\right]\left[(\ln\lambda)_{z\bar{z}} + \frac{\rho_z\bar{\rho}_{\bar{z}}}{\lambda^2}\right] \\ &+ \left[(\ln\lambda)_{y\bar{z}} + \frac{\rho_y\bar{\rho}_{\bar{z}}}{\lambda^2}\right]\left[(\ln\lambda)_{\bar{y}z} + \frac{\bar{\rho}_{\bar{y}}\rho_z}{\lambda^2}\right]\Big\} \\ &+ 2\lambda^2\left[-\left(\frac{\bar{\rho}_{\bar{y}}}{\lambda^2}\right)_y\left(\frac{\rho_z}{\lambda^2}\right)_{\bar{z}} - \left(\frac{\rho_y}{\lambda^2}\right)_{\bar{y}}\left(\frac{\bar{\rho}_{\bar{z}}}{\lambda^2}\right)_z \right. \\ &\left. + \left(\frac{\rho_z}{\lambda^2}\right)_{\bar{y}}\left(\frac{\bar{\rho}_{\bar{z}}}{\lambda^2}\right)_y + \left(\frac{\rho_y}{\lambda^2}\right)_{\bar{z}}\left(\frac{\bar{\rho}_{\bar{y}}}{\lambda^2}\right)_z\right]. \end{aligned} \tag{5.37}$$

The negative signs can all be absorbed by using (5.34), i.e., such relations as

$$g^{y\bar{y}}\left[(\ln\lambda)_{y\bar{y}} + \frac{\rho_y\bar{\rho}_{\bar{y}}}{\lambda^2}\right] = -g^{z\bar{z}}\left[(\ln\lambda)_{z\bar{z}} + \frac{\rho_z\bar{\rho}_{\bar{z}}}{\lambda^2}\right].$$

The $g^{\mu\bar{\mu}}$ which canceled out in (5.36) now reappear.

We have formulated the constraint equations and have verified how known solutions appear in this context. We hope to study further elesewhere the contents of these equations. One may consider, for example, the possibility of adapting Bäcklund-type transformations[29] to curved spaces. Though these transformations do not seem to lead, in flat space, to new solutions with desirable properties, the situation needs not obligatorily be the same for curved spaces. There, apparent extra difficulties are sometimes compensated by agreeable new possibilities. The finite-action non-self-dual solution (2.15) is such an example in a different direction.

For spaces conformally equivalent to flat ones, one may, of course, formally introduce all the known flat-space solutions and examine their properties in the relevant spacetime regions.

VI. ROBINSON-BERTOTTI METRIC

We present here a few remarks concerning gauge fields in a background Robinson-Bertotti metric.[30,31] Starting with the form

$$ds^2 = Q^2\left(-\lambda^2 d\tau^2 + \frac{1}{\lambda^2}d\lambda^2 + d\Omega\right) \tag{6.1}$$

with the Maxwell field (for zero magnetic charge)

$$\mathcal{A}_0 = Q\lambda, \quad \vec{\mathcal{A}} = \vec{0}, \tag{6.2}$$

the transformation

$$\lambda \to Q/r, \quad \tau \to t/Q \tag{6.3}$$

gives

$$ds^2 = \frac{Q^2}{r^2}(-dt^2 + dr^2 + r^2 d\Omega) \tag{6.4}$$

and

$$\mathcal{A}_0 = \frac{Q}{r} .$$

In Ref. 31 some analogy of this metric to the field of a heavy charged nucleus has been noted. Here, starting from different forms of the RB metric, we will compare certain results with those obtained in the preceding sections. There will be both correspondences and typical differences.

Thus, for example, starting with the form

$$ds^2 = Q^2[-(x^2-1)dt^2 + (x^2-1)^{-1}dx^2 + d\Omega] \quad (6.5)$$

one can again introduce parameters as in Sec. IV. Defining

$$\rho = -\int \frac{dx}{x^2-1} \quad \text{or} \quad x = \tanh\rho \quad (6.6)$$

and

$$\zeta_+ = \frac{ae^{t_+} - b}{ce^{t_+} + d}, \quad \zeta_- = \frac{ae^{t_-} + b}{ce^{t_-} - d}$$
$$(t_\pm \equiv t \pm \rho, \quad \zeta_\pm \equiv \zeta \pm x), \quad (6.7)$$

one can again construct a class of metric explicitly conformal to the Minkowski one. [Note the difference between (6.7) and (4.1).] Instead of writing the general formulas, let us briefly note the following results. For $a=b=c=d=1$

$$\zeta_+ = \tanh\left(\frac{t_+}{2}\right), \quad \zeta_- = \coth\left(\frac{t_-}{2}\right). \quad (6.8)$$

The equations of motion for the simple ansatz

$$A_0 = 0, \quad \vec{A} = (K-1)i[\hat{\Phi}, \vec{\nabla}\hat{\Phi}] \quad (6.9)$$

now reduce to, in the coordinates ρ, t and for $K = K(\rho)$,

$$(\cosh\rho)^2 K_{\rho\rho} = -K(K^2-1). \quad (6.10)$$

Now one has the *imaginary* solutions

$$K = \pm i \sinh\rho . \quad (6.11)$$

It can be shown that, in spite of such differences, one can again deduce (through 6.7), in a fashion similar to that of Sec. IV, such flat-space solutions as meron and meron-antimeron.

Let us note another interesting property of the RB metric. From the spin connections of the Euclidean version of (6.4) (using the results indicated in Appendix B) one gets the SU(2) fields[32]

$$A_0 = \pm\hat{\Phi}\frac{1}{r}, \quad \vec{A} = -i[\hat{\Phi}, \vec{\nabla}\hat{\Phi}]. \quad (6.12)$$

Thus one has *point dyons* with

$$\vec{E} = \pm \frac{1}{r^2}\hat{\Phi}\hat{r}, \quad \vec{B} = \frac{1}{r^2}\hat{r}\hat{\Phi} \quad (6.13)$$

(where $E_i = F_{0i}$, $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$, $i,j,k = 1,2,3$). The properties will, however, be different from those of (5.21) though the solutions are formally the same.[33]

One gets

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 2Q^{-4} \quad (6.14)$$

for this case.

Let us now introduce the Kruskal-type coordinates (Appendix B) as follows. Starting from (6.5), we use the transformation

$$e^{ct} = \left(\frac{\eta+\xi}{\eta-\xi}\right)^{1/2}, \quad \left|\frac{x-1}{x+1}\right|^c = \tfrac{1}{4}(\eta^2 - \xi^2) \quad (6.15)$$

when

$$ds^2 = Q^2\left[\frac{1}{4c^2}(x^2-1)\left|\frac{x-1}{x+1}\right|^{-c}(-d\xi^2 + d\eta^2) + d\Omega\right]. \quad (6.16)$$

Let us now consider the region *within* the horizons $x = \pm 1$, such that

$$ds^2 = Q^2\left[-\frac{1}{4c^2}(1-x)^{1-c}(1+x)^{1+c}(-d\xi^2 + d\eta^2) + d\Omega\right]. \quad (6.17)$$

For $c=1$, this is regular at the horizon $x=1$. We choose this case and introduce the continuation

$$\xi \to -i\xi$$

when

$$t \to -it$$

with a period (for $c=1$),

$$T = 2\pi . \quad (6.18)$$

We note that for the corresponding Maxwell field to be real, along with

$$\mathcal{A}_0 \to i\mathcal{A}_0, \quad (6.19)$$

we need also, for example,

$$Q \to -iQ.$$

Thus finally

$$(ds^2)_{\text{cntd}} = Q^2[\tfrac{1}{4}(1+x)^2(d\xi^2 + d\eta^2) - d\Omega]. \quad (6.20)$$

Similarly, in the continued version of (6.5), the change of sign of Q^2 compensates the one due to the fact that (x^2-1) becomes negative for $|x|<1$. A negative sign, however, now appears before $d\Omega$. For the RB metric, with its high symmetry and particular topology $(R_2\times S_2)$, one cannot, in any case, associate a direct, familiar physical interpretation with these coordinates.

Formally, using (6.14) which remains invariant, such a continuation leads to a finite action $(8\pi^2|P_y|)$, where the indices [compare (B22)] are given (since the integrand has no singularity even at $x=-1$):

$$P_y=\pm\frac{4\pi}{32\pi^2}\,2\int_0^{2\pi}dt\left(\lim_{\delta\to 0}\int_{-1+\delta}^{1}dx\right)$$
$$=\pm 1 \qquad (6.21)$$

for the two cases (6.13), respectively. This should be compared to the values (±2) for the Schwarzschild case,[26] as indicated in (5.21).

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APPENDIX A

We give here, for completeness, the equatios of motion for a general static, spherically symmetric metric with A_0 nonzero. Let

$$ds^2=A(r)dt^2+B(r)dr^2+\gamma^2 d\Omega\,. \qquad (A1)$$

Let

$$A_0=d(r,t)\hat{\Phi}\quad\left(\hat{\Phi}\equiv\frac{\vec{\tau}\cdot\hat{r}}{2}\right),$$
$$\vec{A}=a(r,t)\vec{\nabla}\hat{\Phi}+(b(r,t)-1)i[\hat{\Phi},\vec{\nabla}\hat{\Phi}]+c(r,t)\hat{r}\hat{\Phi}\,, \qquad (A2)$$

and the scalar field

$$\Phi=f(r,t)\hat{\Phi}\,.$$

The equations of motion reduce to

$$A^{-1}\ddot{f}+B^{-1}f''+\left[\frac{2}{r}+\frac{1}{2}\left(\frac{A'}{A}-\frac{B'}{B}\right)\right]f'B^{-1}-\frac{2f}{r^2}(a^2+b^2)-\frac{\delta V}{\delta\Phi}=0\,, \qquad (A3)$$

$$(\dot{c}'-d'')+(\dot{c}-d')\left[\frac{2}{r}-\frac{1}{2}\left(\frac{A'}{A}+\frac{B'}{B}\right)\right]+\frac{2B}{r^2}[(a\dot{b}-\dot{a}b)+d(a^2+b^2)]=0\,, \qquad (A4)$$

$$A^{-1}(\ddot{c}-\dot{d}')-\frac{2}{r^2}[(ab'-ba')+c(a^2+b^2)]=0\,, \qquad (A5)$$

$$BA^{-1}(\ddot{a}-2\dot{b}d-b\dot{d}-ad^2)+\frac{1}{2}\left(\frac{A'}{A}-\frac{B'}{B}\right)(a'-cb)+(a''-2cb'-c'b-ac^2)-\frac{aB}{r^2}(a^2+b^2+r^2f^2-1)=0\,, \qquad (A6)$$

$$BA^{-1}(\ddot{b}+2\dot{a}d+a\dot{d}-bd^2)+\frac{1}{2}\left(\frac{A'}{A}-\frac{B'}{B}\right)(b'+ac)+(b''+2ca'+ac'-bc^2)-\frac{bB}{r^2}(a^2+b^2+r^2f^2-1)=0\,. \qquad (A7)$$

APPENDIX B

We collect here, for ready reference, some known, useful results concerning static, spherically symmetric metrics, their maximal extensions and Euclidean continuations, spin connections and associated SU(2) gauge fields.

Let us consider the line element

$$ds^2=-N\,dt^2+N^{-1}dr^2+r^2d\Omega\quad(d\Omega\equiv d\theta^2+\sin^2\theta\,d\varphi^2)\,, \qquad (B1)$$

where

$$N=\Lambda-\frac{2M}{r}+\frac{Q^2+P^2}{r^2}-\frac{\Lambda}{3}r^2\,, \qquad (B2)$$

M being the mass of the central body (or that associated with the essential singularity at $r=0$), Q the electric and P the magnetic charge, and Λ the cosmological constant. Let

$$r^*=\int\frac{dr}{N}\,, \qquad (B3)$$

defining the Kruskal-type coordinates as

$$\exp(2cr^*)=\tfrac{1}{4}(\eta^2-\xi^2)\,,$$
$$\exp(ct)=\left(\frac{\eta+\xi}{\eta-\xi}\right)^{1/2}\,, \qquad (B4)$$

and

$$ds^2=\frac{1}{4c^2}N\exp(-2cr^*)(-d\xi^2+d\eta^2)+r^2d\Omega\,. \qquad (B5)$$

The roots of N determine the spacetime regions and c may be chosen to remove an apparent singularity. See Ref. 30 for detailed discussions of various important cases.

Now it is seen that if the passage to the Euclidean section[34,35] is defined through

$$\xi \to -i\xi ,$$

the corresponding continuation $t \to -it$ makes the Euclidean t periodic with a period:

$$T = \frac{2\pi}{c} . \tag{B6}$$

The domain of r is chosen to be that for which (B5) remains positive definite. Thus for

$$N = \left(1 - \frac{2M}{r}\right) \text{ (Schwarzschild)}$$

one gets

$$T = 8\pi M, \quad 2M < r < \infty . \tag{B7}$$

For

$$N = \left(1 - \frac{\Lambda}{3} r^2\right) \text{ [de Sitter O(4, 1)]}$$

one gets

$$T = 2\pi\left(\frac{3}{\Lambda}\right)^{1/2}, \quad 0 \leq r < \left(\frac{3}{\Lambda}\right)^{1/2} . \tag{B8}$$

The more general case

$$N = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \tag{B9}$$

is discussed in detail in Ref. 22, where other references are quoted. For the charged case ($Q, P \neq 0$), for the associated electric field to be real one should also introduce (since the Maxwellian $\mathcal{a}_0 \to -i\mathcal{a}_0$)

$$Q \to -iQ \tag{B10}$$

and hence $Q^2 \to -Q^2$ in (B1). For more general considerations involving Kerr-type metrics see Ref. 36.

The periodicity in time permits the existence of finite-action, *static* Euclidean gauge fields. To compare with the examples furnished in this paper, let us briefly recapitulate the results concerning SU(2) gauge fields associated with the spin connections.[35,37]

Considering directly Euclidean signature, let

$$g_{\mu\nu} = L^a_\mu L^b_\nu \delta_{ab} \tag{B11}$$

and

$$\hat{\gamma}_\mu = L^a_\mu \gamma_a \tag{B12}$$

in terms of the tetrads L^a_μ and flat-space Euclidean Dirac matrices γ_a, so that

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2g_{\mu\nu} . \tag{B13}$$

The formula for the spin connection is[35]

$$\begin{aligned} B^{ab}_\mu = {} & \tfrac{1}{2} L^{a\nu}(L^b_{\nu,\mu} - L^b_{\mu,\nu}) - \tfrac{1}{2} L^{b\nu}(L^a_{\nu,\mu} - L^a_{\mu,\nu}) \\ & + \tfrac{1}{2} L^{a\nu} L^{b\sigma}(L_{c\nu,\sigma} - L_{c\sigma,\nu}) L^c_\mu . \end{aligned} \tag{B14}$$

These have the transformation properties of local SO(4) gauge fields. For *diagonal* metrics one may choose (no sum over μ)

$$L^a_\mu = (g_{\mu\mu})^{1/2} \delta_{a\mu}, \quad L^\mu_a = (g_{\mu\mu})^{-1/2} \delta_{a\mu} \tag{B15}$$

when

$$\hat{\gamma}_\mu = (g_{\mu\mu})^{1/2} \gamma_\mu, \quad \hat{\gamma}^\mu = (g_{\mu\mu})^{-1/2} \gamma_\mu \tag{B16}$$

and

$$B^{ab}_\mu = \{(g_{\mu\mu})^{1/2}\}_{,\nu} (g_{\nu\nu})^{-1/2} (\delta_{a\mu}\delta_{b\nu} - \delta_{b\mu}\delta_{a\nu}) . \tag{B17}$$

Defining

$$B_\mu = \tfrac{1}{4} \Sigma_{ab} B^{ab}_\mu , \tag{B18}$$

where

$$\Sigma_{ab} = \frac{1}{2i} [\gamma_a, \gamma_b]$$

are the SO(4) generators,

$$B_\mu = \frac{i}{4} (\hat{\gamma}^\nu \hat{\gamma}_{\mu,\nu} - \hat{\gamma}_{\mu,\nu} \hat{\gamma}^\nu) . \tag{B19}$$

A block diagonal form of B_μ permits an immediate separation into two SU(2) gauge fields.

Thus finally one obtains for (B1) the SU(2) fields[22]

$$\begin{aligned} B_0 &= \pm \frac{N'}{2} \hat{\Phi} , \\ \vec{B} &= (\sqrt{N} - 1) i [\hat{\Phi}, \vec{\nabla} \hat{\Phi}] . \end{aligned} \tag{B20}$$

These are respectively self-dual and anti-self-dual for

$$N'' = \frac{2}{r^2} (N - 1) \tag{B21}$$

which holds for

$$N = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2$$

[i.e., for $P^2 - Q^2 = 0$ in the continued version of (2.1)]. For (B7) and also for (B8) these fields correspond respectively to the Pontryagin indices[22]

$$P_y = \frac{1}{32\pi^2} \int_0^T dt \int r^2 dr \, \mathrm{Tr}(F^*_{\mu\nu} F^{\mu\nu}) = \pm 1 \tag{B22}$$

and action

$$S = 8\pi^2 . \tag{B23}$$

Several other interesting cases can be found in Refs. 22, 36, and 38.

As a final comment we note that the periodicity of time, permitting the existence of the above-mentioned solutions, also affects their significance. The well-known tunnel-effect interpretation of flat-space instantons[39] is not valid here. For the Schwarzschild case, for example, they have rather a thermodynamic interpretation.[26]

Though we do not study gravitational instantons in this paper, we would like to refer the reader to certain sources[40] where different possible interpretations associated to different types of asymptotic behaviors are discussed in that context.

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