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Nonlocal charges in two dimensions

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We exhibit some correspondences between known properties of local charges and the properties of nonlocal charges in the O(n) σ model in two dimensions. By determining the magnitudes of the matrix elements of the nonlocal charges between one-particle states and using their multiplicative action on asymptotic multiparticle states, one sees that production is forbidden for essentially the same reason that an infinite number of local conservation laws forbid production. The way that continuation to $n = 2$ may correspond to the massive Thirring model at $g = -\pi/2$ is briefly discussed.

Nonlocal symmetries are of interest for several reasons. One is that local symmetries that leave the vacuum invariant must be exact,[1] whereas nonlocal symmetries need not be.[2] Another reason has to do with the existence in various $(1+1)$-dimensional models of infinitely many conservation laws, which one hopes may have analogs in $(3+1)$ dimensions, without being so restrictive as to prevent particle production. But the naive extension of the infinite number of local conservation laws to $(3+1)$ dimensions would imply a trivial S matrix,[3] hence one wonders whether the nonlocal conservation laws may be less restrictive.

At the level of charges, it is well known that local conserved charges must commute with the S matrix S. This is essentially because a local conserved charge acts additively[4,5] on asymptotic multiparticle states, and hence its behavior is determined by how it affects one-particle states, on which S acts as the identity. Since a nonlocal conserved charge Q need not commute[6] with S, one can always write $Q=M+N$, where M and N are Hermitian if Q is, such that

$$S(M+N)S^{-1}=M-N\,. \tag{1}$$

A particularly simple special case is $M=0$, so that one has a conserved charge that anticommutes with the S matrix. Whereas a charge that commutes with S gives information about S-matrix elements only when one knows how the charge acts on the states, the mere existence of a charge that anticommutes with S implies that there must exist at least one pair of antipodal eigenstates of S, i.e., $|a\rangle$ and $|b\rangle$ such that

$$S|a\rangle=\exp(i\delta_a)|a\rangle,\quad S|b\rangle=\exp(i\delta_b)|b\rangle\,, \tag{2}$$

and

$$\delta_a-\delta_b=\pi\,.$$

For example, the simplest nonlocal charges in the O(n) σ model in two dimensions have M's that are integrals of the spatial components j_1 of conserved currents,[7] and hence the M's vanish on those states of which all particle momenta are zero. Since kinematic factors require some eigenvalues of S to become unity at such a threshold, one suspects that there would be other eigenvalues of S with a threshold phase of π. From the known S matrix for this model[8] one readily verifies that such is indeed the case.

Recently Zamolodchikov has emphasized that the nonlocal charges of Ref. 7 satisfy a multiplicative law on asymptotic multiparticle states.[9] The one-parameter family of charges has the form

$$R(w)=P\exp\left[w(1-w^2)^{-1}\int_{(t,-\infty)}^{(t,\infty)}(\epsilon_{\mu\nu}J^\nu-wJ_\mu)dx^\mu\right], \tag{3}$$

where J_μ are matrices with $J_\mu^{ab}=\phi^a\partial_\mu\phi^b-\phi^b\partial_\mu\phi^a$, P denotes path ordering in matrix multiplication, and under suitable boundary conditions $R(w)$ is independent of t. If the upper limit of integration in (3) is left at x_μ, the $R(w,x)$ thus defined is an x-dependent rotation that generates one solution from another solution to the classical field equations. One can also use a spinor representation for J_μ and generate one solution from another by conjugation. For the classical theory this makes no difference. But as one takes x_1 to infinity in $R(w,x)$ to obtain the conserved charges $R(w)$, the representation used does make a difference, and we will return to this question later. Coming back to the action of $R(w)$ on asymptotic states, we write down the multiplicative action postulated by Zamolodchikov:

$$R^{ab}(w)|\theta_1c_1,\ldots,\theta_kc_k,\ \text{out}\rangle=\sum_{a_i=1}^{n}R^{aa_1}(w)|\theta_1c_1\rangle_0R^{a_1a_2}(w)|\theta_2c_2\rangle_0\cdots R^{a_{k-1}b}(w)|\theta_kc_k\rangle_0\,, \tag{4}$$

where $\theta_i = \sinh^{-1}(p_i^1/m)$, c_i is the internal-symmetry index, and $\theta_1 < \theta_2 < \cdots < \theta_k$. It is shown in Ref. 9 that Eq. (4) leads to the factorization relations for S-matrix elements that were previously derived by other methods. Although the multiplicative action was postulated for plane-wave states, presumably one has in mind wave packets with nonoverlapping rapidities centered at θ_i, such that for large time the spatial overlaps in wave functions decrease faster than any inverse power of t. Then the integral in (3) can be split into k intervals each covering only the essential support of one wave packet, and the path-ordered matrix multiplication leads to the structure (4). Because of the short-distance singularities of products of fields, the above argument has only a heuristic value; however, for the lowest-order nonlocal charge in a power-series expansion in w, Lüscher has analyzed[7] the short-distance singularities and established its multiplicative (or non-Abelian additive, in the terminology of Ref. 7 for the charges in question) action on multiparticle states. In view of this, and of the compatibility of Zamolodchikov's assumption with factorization, we will accept the multiplicative action, and use it in the form of Eq. (4) for simplicity.

It follows immediately that, as in the case of local charges, the action of these nonlocal charges is again determined by their one-particle matrix elements. They fail to commute with the S matrix, but only in a very simple and readily understandable manner: Because, for a given rapidity ordering the ordering in the spatial position of wave packets is reversed from $t=-\infty$ to $t=\infty$, the multiplicative action on the in-states reads[9]

$$R^{ab}(w)\,|\theta_1 c_1, \ldots, \theta_k c_k, \text{ in}\rangle = \sum_{a_i} R^{a_{k-1}b}(w)\,|\theta_1 c_1\rangle_i\, R^{a_{k-2}a_{k-1}}(w)\,|\theta_2 c_2\rangle_i \cdots R^{aa_1}(w)\,|\theta_k c_k\rangle_i\,. \tag{5}$$

Now in the case of a local charge, because its action is determined by its one-particle matrix elements, which are limited in complexity, there is a certain "uniqueness" property. By this we mean that under fairly general conditions a vector charge that transforms like the 4-momentum is in fact proportional to the momentum operator,[4] and a triplet of charges that transform like isospin operators are indeed proportional to the isospin generators of the theory.[10] This is the basic reason for the absence of production when higher local tensor charges exist, because they correspond to higher powers of momentum, the additive conservation of which leads to the persistence of individual momenta, apart from permutations. Since with Eq. (4) the nonlocal charges are also determined by their one-particle matrix elements, are their actions also essentially unique? We shall see that at least the magnitudes of the one-particle matrix elements can be determined for real w.

One makes use of the O(n) transformation property of the charges, but, instead of exploiting their Poincaré transformation properties, it seems easier for the purpose at hand to exploit their conservation in two-particle scattering as Zamolodchikov did[9] to first arrive at the relations

$$\begin{aligned} f_2(\theta) &= \lambda f_1(\theta)(i\lambda z + i\theta)^{-1}\,, \\ f_3(\theta) &= -\lambda f_1(\theta)(\pi + i\lambda z + i\theta)^{-1}\,, \end{aligned} \tag{6}$$

where $\lambda = 2\pi(n-2)^{-1}$, and the f_i are defined by

$$\begin{aligned} R^{ab}\,|\theta c\rangle &= f_1(\theta)\delta^{ab}\,|\theta c\rangle + f_2(\theta)\delta^{ac}\,|\theta b\rangle \\ &\quad + f_3(\theta)\delta^{bc}\,|\theta a\rangle\,. \end{aligned} \tag{7}$$

One now invokes the unitarity of $R(w)$ for real w to conclude

$$\begin{aligned} &\mathrm{Re}(f_1^* f_2) = 0\,, \\ &2\,\mathrm{Re}(f_3^* f_1 + f_3^* f_2) + n\,|f_3|^2 = 0\,, \end{aligned} \tag{8a}$$

and

$$|f_1|^2 + |f_2|^2 = 1\,. \tag{8b}$$

Because $R^\dagger$ acts in the reverse order to R on multiparticle states, the one-particle unitarity Eq. (8) also ensures multiparticle unitarity. One finds that one of the conditions in (8a) is identically satisfied, and that the solution is

$$\begin{aligned} &\mathrm{Im} z = 0\,, \\ &|f_1| = |\lambda z + \theta|\,[\lambda^2 + (\lambda z + \theta)^2]^{-1/2}\,. \end{aligned} \tag{9}$$

Thus the three f_i's are determined up to a common phase, with the real parameter z corresponding to w, but expressing more simply the covariance property under a boost, since θ appears only in the combination $(\lambda z + \theta)$.

It is now easy to see how the existence of these symmetries prevents particle production. One may consider

$$F = \langle \theta_1 c_1, \theta_2 c_2, \ldots, \theta_n c_n; \text{ out}\,|(Q^\dagger)^{ab} Q^{a'b'}|\,\theta c, -\theta d; \text{ in}\rangle\,,$$

which equals $\sum_\alpha S_\alpha(\theta_i,\theta)\,u_\alpha(\lambda z,\theta)$ when the charges act to the right, and $\sum_\beta S_\beta(\theta_i,\theta)\,v_\beta(z,\theta)$ when the charges act to the left. The u_α are analytic functions of $y=\lambda z$, with poles at $y=\mp(\theta\pm i\pi)$, $y=\mp\theta$, and branch points at $y=\mp(\theta\pm i\lambda)$, whereas the v_β have poles at $y=-(\theta_i\pm i\pi)$, $y=-\theta_i$, and branch points at $y=-(\theta_i\pm i\lambda)$. For $n\geq 3$, $|\theta_i|<|\theta|$, and the two sides can be equal only if the sum F is in fact free from singularities. The f_i approaches either a constant or zero as $|y|$ approaches infinity; and, since an entire function bounded at infinity is a constant, $(d/dy)F=0$. But the explicit forms of the f_i show that all states $|\theta c', -\theta d;\text{ in}\rangle$ can be reached by letting $(d/dy)(Q^\dagger)^{ab}Q^{a'b'}$ act on $|\theta c, -\theta d;\text{ in}\rangle$, and varying a, b, a', b' and c,d; hence, production from two-particle states is forbidden.

That the nonlocal charges forbid the $2\to 4$ particle process was first shown by Weisz[11] from an explicit examination of the many coupled equations, and the absence of $2\to n$ particle processes at threshold was shown by Lüscher and extended above the threshold by a continuity argument.[7] The above demonstration has the virtue of bringing out the parallels with the case of local charges: Instead of all powers of momentum being conserved one has functions of momentum being conserved; instead of additive conservation one has multiplicative conservation, both sharing the essential ingredient that the one-particle matrix elements completely determine the charge. It would seem that any set of nonlocal charges in four-dimensional theories, to be useful in a context with nontrivial scattering, must not share these features.

We next consider the massive Thirring model, or the solitons in the sine-Gordon theory. The sine-Gordon field is known to be related to the angle between the tangent vectors in lightlike directions (see Pohlmeyer, Ref. 7) of the O(3) σ model; noting, however, the existence of an O(2) symmetry in the soliton sector, Zamolodchikov postulated the existence of nonlocal O(2) charges, with multiplicative action on asymptotic states containing the Thirring particles.[9] It was shown that this assumption again leads to the known factorization relations:

$$F_1(\theta,w)/F_2(\theta,w)=\sinh[G(\theta+a)]/\sinh(Gb)\,,$$
$$F_0(\theta,w)/F_2(\theta,w)=\cosh[G(\theta+a)]/\cosh(Gb)\,, \tag{10}$$

where $F_0=2f_1+f_2+f_3$, $F_1=f_2-f_3$, $F_2=f_2+f_3$, $b=-i\pi/2$, and G is related to the Thirring coupling constant g by $G=1+2g\pi^{-1}$. Applying again the unitarity of the charges for real w, one finds $\mathrm{Im}a=-\pi/2$. The magnitude of f_1 being determined from Eq. (8b), the f_i are once more fixed, for a given w [or given $\mathrm{Re}(a)$], up to a common phase. These f_1's are of course different from those of the $O(n)$ σ model, the S matrix of which does not contain a parameter corresponding to the coupling constant g. Nevertheless, one may ask whether the continuation of the $O(n)$ σ-model S matrix to $n=2$ corresponds to that of the massive Thirring model[12] for a particular value of g. The answer is that it does for $g=-\pi/2$, which is the critical value at which the mass term acquires dimension two. [The $G\to 0$ limit of Eq. (10) also agrees with the $\lambda\to\infty$, $z\to 0$ limit of Eq. (6) with λz fixed.] If the massive Thirring model exists at this critical value of the coupling constant, it seems reasonable that the S matrix is given by this confluence of the limits of two different models, with the fermion-fermion (or antifermion-antifermion) scattering amplitude $u(\theta)$ being simply

$$u(\theta)=\frac{\Gamma(i\theta/2\pi)\Gamma(2^{-1}-(i\theta/2\pi))}{\Gamma(-i\theta/2\pi)\Gamma(2^{-1}+(i\theta/2\pi))} \tag{11}$$

and the fermion-antifermion transmission amplitude $t(\theta)$ and reflection amplitude $r(\theta)$ being

$$t(\theta)=(i\pi-\theta)^{-1}\theta u(\theta)\,,\quad r(\theta)=\theta^{-1}i\pi t(\theta)\,. \tag{12}$$

In (11) and (12) θ refers to the rapidity difference.

Finally, we return to the question of the matrix representation. While Ref. 9 adopts the vector representation for the matrix J in Eq. (3), Ref. 7 adopts a "spinor" representation, and in general these may lead to different consequences. For O(3) there are nine charges in the first procedure and only four in the second, for a given w. It is not immediately obvious that the two sets necessarily have different implications, since both forbid production. However, for the O(2) problem we have just considered, the spinor representation yields only two charges, Q_0 and Q_1, with $Q_0|f,\theta\rangle=h_0(\theta)|f,\theta\rangle$, $Q_0|\bar f,\theta\rangle=h_0(\theta)|\bar f,\theta\rangle$, $Q_1|f,\theta\rangle=-ih_1(\theta)|f,\theta\rangle$, and $Q_1|\bar f,\theta\rangle=ih_1(\theta)|\bar f,\theta\rangle$, ($f$ and $\bar f$ refer to fermion and antifermion, respectively). Assuming multiplicative action, one finds that if $h_0(\theta)\neq h_1(\theta)\times$ constant, there can be no reflection, contradicting (12), whereas if h_0 is proportional to h_1 no constraint is imposed on the two-particle S matrix. In contrast, the vector representation yields four charges which impose the factorization constraints. Thus, the two sets of charges do have different consequences.

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