Unitary space of particle internal states

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A relativistic theory of particle internal properties is developed. Suppressing space-time information, internal wave functions and observables are constructed in a 3-dimensional complex space. The quantum numbers of a spinning point particle in this unitary space correspond with those of a low-mass hadron. Unitary-space physics is linked with space-time notions via the Penrose theory of twistors, where new flavors may be represented by many-twistor systems. It is shown here that a four-twistor particle fits into the unitary-space picture as a system of two points with equal masses and oppositely pointing unitary spins. Quantum states fall into the ISU(3) irreducible representations discovered by Sparling and the author. Full details of the computation involving SU(3) recoupling techniques are given.

I. INTRODUCTION

By the time physicists obtained a successful description of hadron systematics in terms of the SU(3) internal-symmetry group, the importance of this group and of its 3-dimensional complex linear space seemed to be obvious. Indeed, SU(3) has been referred to as *the* unitary group.[1] More recently the discovery of new families of hadrons and the concomitant efforts at widening the theoretical framework have appeared to undermine the privileged position of the 3-dimensional unitary scheme.

The purpose of the present work is, in a sense, to reinstate unitary symmetry. Thus the theory to be expounded here markedly deviates from the mainstream of current investigations in hadron physics. For one thing, our scenario does not even take place in space-time. It is situated in a 3-dimensional complex space with rotations and translations forming an inhomogeneous SU(3) group. This now is the unitary space.

Available experimental material on intrinsic particle properties points to the conclusion that we need not know much about space-time when trying to understand most of these properties. Rest mass or charge is unaltered whether the particle happens to be in a plane-wave state or in another space-time configuration. This is to be contrasted with computation methods of quantum field theory which use space-time state functions (or their Fourier transforms) carrying the full infinite burden of space-time information, and hiding essentials in the disguise of internal quantum numbers.

The unitary-space approach does the opposite. The enlargement of the SU(3) rotation group with translations means essentially that we elevate the linear space of internal quantum numbers to the rank of a *manifold*. A particle may have a trajectory and even spatial extension in the unitary space (Sec. II). Its quantum states are described by unitary-space functions.

A number of similar though different approaches to particle internal states have been proposed in the past. An inhomogeneous SU(3) symmetry scheme (with the inhomogenization differing from ours) has been investigated by Goebel,[2] Dullemond,[3] and Bose[4] with limited successes in applications. Quark internal wave functions in a 3-dimensional complex space were computed by Hoh.[5] While he reported no striking resemblance to the observed spectrum of hadrons, his "spherical" basis for the internal wave function shares many of the properties of the basis to be selected here. An essential difference between his and our approach is that we do not represent the state of a quark separately in the unitary space. Rather, we represent the state of the hadron as a whole.

The motivation for invoking unitary space lies in the Penrose theory of twistors,[6,7] some relevant details of which are discussed in Appendix A. A particle here is completely characterized by its twistor state function. There exists a procedure, well known among twistor theorists, for projecting out the more conventional space-time state function by integration over internal degrees of freedom. In addition to this, in the present paper we find a complementary procedure whereby the internal structure is exhibited but the space-time properties are suppressed. An explicit method for obtaining the internal-state function, based on the Hodges-Sparling contour integral,[8] for a two-twistor particle, is introduced in Sec. III. Such a particle has the merit that the features essential to the projection are present with it and a limited range of applications, being pointlike.[9] The method is extended to three-twistor systems in Sec. IV, where also an internal scalar product is defined.

Three-twistor particles, represented as spinning points in unitary space, possess several features[10] which make them good candidates for modeling low-mass hadrons. Will the unitary-space picture blow up as we go to higher energies?

Suppose we find an object in the real 3-space with an energy spectrum generated by the O(4) group and not merely by O(3) which gives the rotational isometries. (In fact, such a system does exist: a two-particle bound state being an example.[11]) There is no reason to think that the object with the O(4) symmetry gives us a hint that space is 4 dimensional. This is because of the semisimple nature of O(4) and of the local isomorphism $O(4)\simeq O(3)\otimes O(3)$. The result does not extend trivially to complex unitary space. The group SU(4) is simple. Now a four-twistor system has an ISU(4) state spectrum, and we show in Sec. V that there is an important sense in which the relation $ISU(4)\approx ISU(3)\otimes ISU(3)$ holds true. A four-twistor particle is pictured as two points with equal masses and oppositely equal spins in the 3-dimensional unitary space. And this is the sense in which we claim the unitary scheme reinstated. The result may be understood as a relativistic generalization of the spectrum generation for the Kepler problem.[11,12,13] It remains to be seen if a useful relativization of the Bohr formula can be derived in the present scheme.

The argument in Sec. V relies on the use of a naturally defined operator basis in the unitary space. This SU(3) operator basis is called the color triplet because it features certain previously postulated attributes of color.[14] It should be added, however, that in twistor theory new ways[15] open up in which phenomena (such as those involving particle statistics), related by some authors to color, can be explained.

The spectrum of free pointlike particle states in unitary space is provided by unitary irreducible representations (irreps) of the group ISU(3). These irreps have been manufactured using different techniques by Perjés and Sparling.[16] In Sec. VI we briefly survey one of these techniques which is Hecht's SU(3) coupling and recoupling formalism.[17] Section VII gives the full details of the derivation of ISU(3) generator matrix elements. Section VIII discusses the structure of the irreps and the discrete group of substitutions that connect them. The resulting systematics is compared with the observed spectrum of low-mass hadrons in Sec. IX.

Although our approach to unitary-space physics has a strong background and motivation arising from twistor theory, it is also self-contained without making a reference to twistors. Parts of this paper (Secs. III, IV, and Appendix A) which use twistors explicitly may be skipped when reading.

II. THE UNITARY SPACE

Unitary space is a 3-dimensional complex Kählerian manifold with metric[18]

$$dz_i^* dz^i = |dx|^2 + |dy|^2 + |dz|^2 , \quad i = 1, 2, 3 . \tag{2.1}$$

The unitary rotations

$$\vec{z}^R = \underline{U}\vec{z} , \quad \underline{U}^\dagger \underline{U} = \underline{1} \tag{2.2}$$

and the complex translations

$$\vec{z}' = \vec{z} + \vec{t} \tag{2.3}$$

constitute the isometries of this space. We require

$$\det \underline{U} = 1 \tag{2.4}$$

in order that the vector product

$$(\vec{u}\times\vec{v})_i \equiv \epsilon_{ijk} u^j v^k \tag{2.5}$$

be invariantly defined. The combined effect of isometries (2.2) and (2.3) can always be put in the form

$$\vec{z}' = \underline{U}(\vec{z} + \vec{t}) , \tag{2.6}$$

where $\underline{U} \in SU(3)$ and $\vec{t}$ is a complex 3-vector. This defines the 14-parameter inhomogeneous group ISU(3) with elements

$$g = (\underline{U}, \vec{t}) . \tag{2.7}$$

[In the standard theory of SU(3) representations,[19,20] the vector $\vec{t}$ belongs to the complex representation $(\lambda, \mu) = (1, 0)$. The inhomogenizations of a semisimple group G may be classified according to the representation(s) of G to which the inhomogeneous generators belong. Thus I recommend the notation $I^{(\Lambda)}G$ for inhomogeneous groups where Λ labels the representation(s) of the inhomogeneous part. One may conveniently drop the label Λ when it signifies a unique fundamental representation.]

One has the group composition law

$$(\underline{U}_2, \vec{t}_2)(\underline{U}_1, \vec{t}_1) = (\underline{U}_2\underline{U}_1, \vec{t}_1 + \underline{U}_1^\dagger \vec{t}_2) . \tag{2.8}$$

The infinitesimal operators of the SU(3) subgroup form a Hermitian and traceless matrix in unitary representations,

$$A_k^{\dagger i} = A_k^i , \quad A_i^i = 0 . \tag{2.9}$$

The adjoint of a *tensor operator* is defined

$$A_k^{\dagger i} \equiv (A_i^k)^\dagger . \tag{2.10}$$

The translation operator d^i belongs to a complex representation of SU(3), hence it is not Hermitian. The parameters of a Lie group can be chosen real.[11] Hence the Hermitian conjugates $d_i^\dagger$ must also be generators, and they indeed are.

We have the Lie algebra commutators for the ISU(3) group

$$[A^i_j, A^k_l] = \delta^i_l A^k_j - \delta^k_j A^i_l\,, \tag{2.11a}$$

$$[d^i, A^j_k] = \delta^i_k d^j - \tfrac{1}{3}\delta^j_k d^i, \quad [d^\dagger_i, A^k_j] = \tfrac{1}{3}\delta^k_j d^\dagger_i - \delta^k_i d^\dagger_j\,, \tag{2.11b}$$

$$[d^i, d^k] = 0\,, \quad [d^\dagger_i, d^\dagger_k] = 0\,, \tag{2.11c}$$

$$[d^i, d^\dagger_k] = 0\,. \tag{2.11d}$$

Given these geometric relations in the unitary space, we now consider physical objects here. We introduce a parametrization of physical systems by a time variable the nature of which will be discussed later in this section. All we require at the moment is Noether's theorem: The generators of the ISU(3) isometries provide conserved quantities for an isolated system. Translations are generated by the momentum d^i and its adjoint $d^\dagger_i$. The total angular momentum A^i_k generates unitary rotations. In fact we thus generalize the corresponding terminology for observables in the real 3-space which generate the subgroup E(3).

Under translations (2.3) the momenta are invariant and A^i_k pick up an "orbital part"

$$d^{i\,\prime} = d^i\,, \quad d^{\dagger\prime}_i = d^\dagger_i\,, \tag{2.12a}$$

$$A'^i_k = A^i_k + t^i d^\dagger_k + d^i t^*_k - \tfrac{1}{3}\delta^i_k (t^r d^\dagger_r + d^r t^*_r)\,. \tag{2.12b}$$

The behavior of static systems with $d^i = 0 = d^\dagger_i$ is rather trivial, and we exclude such systems from the present discussion. The ISU(3) invariant

$$m^2 \equiv 2 d^i d^\dagger_i \tag{2.13}$$

is then strictly positive. We refer to it as the *mass square*. The term mass square does not suggest itself from kinematics in unitary space, yet we are bound to adopt it in the twistor representation (cf. Appendix A).

In the real Euclidean 3-space the angular momentum can be transformed away by a suitable translation. This is not the case in unitary space. An intrinsic spin does exist in the unitary space, although a spin vector cannot be defined. Instead we proceed as follows. We introduce the SU(3) invariant B by

$$\Delta B = \Delta^i_k A^k_i\,, \tag{2.14}$$

where

$$\Delta^i_k = d^\dagger_k d^i \tag{2.15}$$

and $\Delta = \Delta^r_r = m^2/2$.

The quantity B is not an inhomogeneous Casimir operator as is seen from the commutation properties

$$[d^i, B] = \tfrac{2}{3} d^i\,, \qquad [d^\dagger_i, B] = -\tfrac{2}{3} d^\dagger_i\,. \tag{2.16}$$

Making use of the operator B we define the *spin tensor* S^i_k:

$$S^i_l = \epsilon^{ijk}\epsilon_{lmn}(A^m_j + \tfrac{1}{2}\delta^m_j B)\Delta^n_k\,. \tag{2.17}$$

The spin tensor has many esthetic properties. It is Hermitian and traceless:

$$S^{\dagger i}_k = S^i_k\,, \quad S^i_i = 0\,. \tag{2.18}$$

The commutators with the momentum

$$[d^i, S^j_k] = 0\,, \quad [d^\dagger_i, S^j_k] = 0 \tag{2.19}$$

show that the spin tensor is translationally invariant,

$$S'^i_k = S^i_k\,. \tag{2.20}$$

The momentum is a (non-Hermitian) eigenvector of S^i_k with vanishing eigenvalue

$$S^i_k d^k = 0 = d^\dagger_i S^i_k\,, \tag{2.21}$$

and we have

$$\det[S^i_k] = 0\,. \tag{2.22}$$

This holds nontrivially, setting $\det[S^i_k] \equiv \epsilon_{ijl} S^i_1 S^j_2 S^l_3$, in spite of the fact that the components of the spin tensor are operators, satisfying

$$[S^i_j, S^k_l] = (\Delta^k_j - \delta^k_j \Delta) S^i_l - (\Delta^i_l - \delta^i_l \Delta) S^k_j\,. \tag{2.23}$$

A theorem by Beltrametti and Blasi[21] implies that the number of independent Casimir operators of the group ISU(3) is two.[22] Of these, we are already in possession of the mass m. The spin operator S^i_k commutes with the momentum [Eqs. (2.19)]; hence the spin j given by

$$j(j+1) = \frac{1}{2\Delta^2} S^i_k S^k_i \tag{2.24}$$

must commute with all the generators. The spin is a Casimir operator of the ISU(3) group. In order to express the spin j in terms of the generators explicitly, we first eliminate the alternating symbols from definition (2.17) by using the identity

$$\epsilon^{ijk}\epsilon_{rst} = \delta^i_r\delta^j_s\delta^k_t - \delta^i_r\delta^k_s\delta^j_t + \delta^j_r\delta^k_s\delta^i_t - \delta^j_r\delta^i_s\delta^k_t + \delta^k_r\delta^i_s\delta^j_t - \delta^k_r\delta^j_s\delta^i_t\,. \tag{2.25}$$

We obtain the expression for the spin tensor

$$S^i_j = -\Delta A^i_j + \Delta^r_j A^i_r + A^r_j \Delta^i_r - (\tfrac{1}{2}B + 3)\Delta^i_j + (-\tfrac{1}{2}B + 1)\delta^i_j \Delta\,. \tag{2.26}$$

Plugging this form in Eq. (2.24) we get

$$j(j+1) = \tfrac{1}{4}B^2 - \tfrac{3}{2}B + \tfrac{1}{2}C_2 - (A^i_j A^j_k \Delta^k_i)\Delta^{-1}\,, \tag{2.27}$$

where

$$C_2 = A^i_k A^k_i \tag{2.28}$$

is an SU(3) [but not ISU(3)] scalar.

An elegant treatment of the center of mass in space-time has been given by Penrose.[6] We now apply his method to a particle in the unitary space. Upon displacing the coordinate origin in such a way that the orbital momentum vanishes, the origin is shifted to the center of mass. Then the angular momentum tensor $A_k'^i$ contains only a spin part which is orthogonal to the momentum:

$$A_k'^i d'^k = 0. \tag{2.29}$$

Use of the translation properties (2.12) gives

$$(A_k^i + d^i t_k^* + t^i d_k^\dagger)d^k - \tfrac{1}{3}(d^k t_k^* + t^k d_k^\dagger)d^i = 0. \tag{2.30}$$

This relationship splits into the scalar equation[23]

$$B + \tfrac{2}{3}(d^k t_k^* + t^k d_k^\dagger) + 2 = 0 \tag{2.31a}$$

and into

$$A_k^i d^k - Bd^i + t^i \Delta - (t^k d_k^\dagger)d^i = 0, \tag{2.31b}$$

which is analytic in t^i and is orthogonal to $d_i^\dagger$.

Let us denote the general solution of (2.31b) for t^i by Z^i. This is

$$Z^i(\tau) = -\frac{1}{\Delta}(A_k^i d^k - Bd^i) + i\tau d^i. \tag{2.32}$$

The points of the center of mass of a freely moving particle with conserved momentum lie on a line in the unitary space. The real parameter τ of the line may be taken as the proper time.

III. THE INTERNAL STATE: TWO-TWISTORS

We are now in possession of some basic physical notions of unitary space. So far, all this may seem very remote from ordinary physics. Space-time is known to be a real manifold offering little prospect for the unitary physicist. But I shall show presently that a unitary space can be spotted in the twistor structure of particles in the Minkowski space-time. This interpretation rests on the description of the internal state in twistor theory. First the requisite techniques are elucidated here on the simplest example of two-twistor particles.

In a two-twistor decomposition of a particle, let Z^α and W_α be constituents (cf. Appendix A). Then, by Eq. (A5), for any complex λ the operators

$$Z'^\alpha = Z^\alpha + \lambda I^{\alpha\beta} W_\beta, \quad W'_\alpha = W_\alpha - \lambda^* I_{\alpha\beta} Z^\beta \tag{3.1}$$

also describe constituents. The state function in the coordinate picture is obtained from $F(W_\alpha, Z^\alpha)$ by the contour integral[8]

$$\psi(x^{AA'})_{B\cdots KM'\cdots R'} = \frac{1}{(2\pi i)^2}\oint w_B \cdots w_K z_{M'} \cdots z_{R'} \times F(w_A, -ix^{AA'}w_A; ix^{AA'}z_{A'}, z_{A'})\Delta wz, \tag{3.2}$$

where[24]

$$\Delta wz = w_A dw^A \wedge z_{A'} dz^{A'}. \tag{3.3}$$

The twistors in the argument of F in Eq. (3.2),

$$(W_\alpha) = (w_A, -ix^{AA'}w_A), \quad (Z^\alpha) = (ix^{AA'}z_{A'}, z_{A'}), \tag{3.4}$$

are subject to the condition

$$W_\alpha Z^\alpha = 0. \tag{3.5}$$

The rest mass m and the spin j are simultaneous Casimir constants of the Poincaré and internal groups.[7] The twistor operator of the rest-mass square

$$\Box = -2|Z^\alpha I_{\alpha\beta} W^{\dagger\beta}|^2 \tag{3.6}$$

leads to the eigenvalue problem

$$(\Box + m^2)F = 0. \tag{3.7}$$

Solutions of Eq. (3.7) may be generated by a method due to Hodges and Sparling.[8] Let $G(W_\alpha, Z^\alpha)$ be an arbitrary but homogeneous two-twistor function. Then

$$F(W_\alpha, Z^\alpha) = \oint h(\lambda, \mu) G(W_\alpha + \lambda I_{\alpha\beta} Z^\beta, Z^\alpha + \mu I^{\alpha\beta} W_\beta) d\lambda \wedge d\mu \tag{3.8}$$

is a solution of (3.7), provided the function $h(\lambda, \mu)$ satisfies

$$2\frac{\partial^2 h}{\partial\lambda\partial\mu} = m^2 h. \tag{3.9}$$

My suggestion here is to look at Eq. (3.8) from a different angle. Let us regard (3.8) as a transformation of the internal data of the particle. The function $h(\lambda, \mu)$ is then associated with an internal state.

Let $G(W_\alpha, Z^\alpha)$ be a suitably chosen state function (of the lowest weight, e.g.) with mass m and spin j. Thus $G(W_\alpha, Z^\alpha)$ is already a solution of Eq. (3.7). The role of the function $G(W_\alpha, Z^\alpha)$ is to ensure the correct dependence on space-time coordinates. The integral transformation (3.8) produces another state function $F(W_\alpha, Z^\alpha)$ which may have internal quantum numbers different from those of G. On allowing for $h(\lambda, \mu)$ to vary, the function $F(W_\alpha, Z^\alpha)$ will sweep over internal degrees of freedom.

I do not propose that Eq. (3.8) represents the ultimate method for introducing an internal-state function; it merely serves to point out the *existence* of such a function. A further reassuring fact is that the internal observables commute with the kinematical twistor (A3),

$$[\text{any internal operator}, A^{\alpha\beta}] = 0. \tag{3.10}$$

(Note that the validity of this equation extends to all n-twistor systems.) Hence we infer that there must exist a procedure for obtaining the internal-state function by integrating over space-time degrees of freedom.

The use of Eq. (3.8) makes it possible to find out how internal operators are represented on the space of internal-state functions. We recast Eq. (3.8) in the form

$$F(W_\alpha, Z^\alpha) = \oint h(\lambda, \mu) e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}} G(W_\alpha, Z^\alpha)\, d\lambda \wedge d\mu \,. \tag{3.11}$$

Here we denote $\{W^\dagger Z\} = W^{\dagger\alpha} I_{\alpha\beta} Z^\beta$ and $\{WZ^\dagger\} = W_\alpha I^{\alpha\beta} Z^\dagger_\beta$. The equivalence of Eqs. (3.8) and (3.11) may be verified by comparing terms in power-series expansions in λ and μ.

Consider first the action of operator $d = \{W^\dagger Z\}$ on F:

$$\begin{aligned} dF(W_\alpha, Z^\alpha) &= \oint h(\lambda, \mu) \{W^\dagger Z\} e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}} G(W_\alpha, Z^\alpha)\, d\lambda \wedge d\mu \\ &= \oint h(\lambda, \mu) \frac{\partial}{\partial\lambda} \{ e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}} G(W_\alpha, Z^\alpha)\} d\lambda \wedge d\mu \,. \end{aligned} \tag{3.12}$$

By partial integration we obtain

$$dF(W_\alpha, Z^\alpha) = -\oint \frac{\partial h(\lambda, \mu)}{\partial\lambda} e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}} \times G(W_\alpha, Z^\alpha)\, d\lambda \wedge d\mu \,.$$

Hence the internal representation of operator d is

$$d = -\partial/\partial\lambda \,. \tag{3.13}$$

Similarly,

$$d^\dagger = -\partial/\partial\mu \tag{3.14}$$

[cf. Eq. (3.9)].

Next we take the homogeneity operator $p = W_\alpha W^{\dagger\alpha}$:

$$pF(W_\alpha, Z^\alpha) = \oint h(\lambda, \mu) \{[W_\alpha W^{\dagger\alpha}, e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}}] + e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}} p_0\} G\, d\lambda \wedge d\mu . \tag{3.15}$$

Here p_0 is the degree of homogeneity of $G(W_\alpha, Z^\alpha)$ in W_α. The commutator can be evaluated by use of Eqs. (A15):

$$[W_\alpha W^{\dagger\alpha}, e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}}] = (\mu\partial/\partial\mu - \lambda\partial/\partial\lambda) e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}} \,. \tag{3.16}$$

Partial integration in (3.15) yields

$$pF(W_\alpha, Z^\alpha) = \oint (\lambda\partial/\partial\lambda - \mu\partial/\partial\mu + p_0) h(\lambda, \mu) \times e^{\lambda\{W^\dagger Z\} + \mu\{WZ^\dagger\}} G(W_\alpha, Z^\alpha)\, d\lambda \wedge d\mu \,. \tag{3.17}$$

The function $F(W_\alpha, Z^\alpha)$ will be homogeneous of degree p_1 in W_α provided $h(\lambda, \mu)$ is an eigenfunction of the operator

$$p = \lambda\partial/\partial\lambda - \mu\partial/\partial\mu + p_0 \tag{3.18}$$

such that

$$ph(\lambda, \mu) = p_1 h(\lambda, \mu) \,. \tag{3.19}$$

The remaining internal operators $Z^\alpha Z^\dagger_\alpha$ and $Z^\alpha W_\alpha$ (a transition operator) may also be worked out along these lines if desired. We then obtain a representation of the two-twistor particle in which the space-time information is suppressed (excepting spin and mass) and the internal state is characterized by an analytic function $h(\lambda, \mu)$. The view has occasionally been expressed[7,25] that two-twistor particles are leptons. Supposing this will successfully be borne out, the present formalism may then provide a satisfactory framework for building lepton models. We will not attempt to do this here; rather we will carry out a detailed investigation of three-twistor internal space in what follows.

IV. RECOGNIZING UNITARY SPACE

The purpose of this section is to provide motivation for doing unitary physics. We find a realization of unitary space by twistor states. This would suggest to us an interpretation of unitary objects in more usual physical terms. Thus, later on we will be helped in selecting the appropriate labels of unitary quantum states.

We consider now a particle with three-twistor constituents (Appendix A). Internal transformations (A5) define a 15-parameter group. This has a 14-parameter subgroup transitive still over the allowable choices of the three twistors,[10] with group elements $g = (\underline{U}, \underline{\Lambda})$ where $\underline{U} \in SU(3)$. On identifying

$$t^a = \tfrac{1}{2}\epsilon^{abc}\Lambda_{bc} \tag{4.1}$$

we find that the group is isomorphic with ISU(3) of Sec. II. The vector t^a is an element of unitary space [cf. Eq. (2.3)]. The ISU(3) generators may be written in terms of twistor operators (A9) as

$$d^a = \tfrac{1}{2}\epsilon^{abc} d_{bc} \,, \quad d^\dagger_a = \tfrac{1}{2}\epsilon_{abc} d^{\dagger bc} \,, \tag{4.2a}$$

$$A^a_b = -B^a_b + \tfrac{1}{3}\delta^a_b B^r_r \,. \tag{4.2b}$$

These are precisely the algebraically independent internal twistor operators since the quantity B^r_r does not belong to the Lie algebra of ISU(3). However, B^r_r can be expressed in terms of the generators, using (A12b):

$$B^r_r = 3d^r A^s_r d_s \Delta^{-1} - 2\,. \qquad (4.3)$$

Definition (2.14) shows that $B = \frac{1}{3}B^r_r - 2$.

Operators (4.2) satisfy the Lie-algebra commutation relations (2.11). Comparing further with results of Sec. II we conclude that the internal observables of the three-twistor particle represent a structureless but possibly spinning point of unitary space.

A quantum state of a particle composed of three twistors is given by an analytic and homogeneous function of three-twistor variables $F(Z^\alpha_i)$ where $i = 1$, 2, and 3. It is not possible to generalize for three-twistor functions $F(Z^\alpha_i)$ the transform of Hodges and Sparling starting from (3.8) since there is now no way of arranging the twistor indices suitably. Following a suggestion of Penrose[26] we shall consider instead a generalization of the form (3.11) and set, formally,

$$F(Z^\alpha_a) = \oint \chi(\lambda_i, \mu^k) e^{\lambda_r d^r + \mu^r d^\dagger_r} G(Z^\alpha_a)\, d^3\lambda \wedge d^3\mu\,, \qquad (4.4)$$

where $\chi(\lambda_i, \mu^k)$ is analytic and

$$d^3\lambda \equiv \tfrac{1}{6}\epsilon^{ijk} d\lambda_i \wedge d\lambda_j \wedge d\lambda_k\,.$$

The procedure outlined in Eqs. (3.12)–(3.18) yields the representation of ISU(3) generators in unitary space

$$d^i = -\partial/\partial\lambda_i\,, \quad d^\dagger_i = -\partial/\partial\mu^i\,, \qquad (4.5a)$$

and (choosing function G with vanishing weights)

$$A^i_k = M^i_k(\lambda_j) - M^i_k(\mu^j)\,, \qquad (4.5b)$$

where M^i_k are the Bargmann operators[27,28]

$$\begin{aligned} 3M^3_3(\lambda) &= \lambda_1\partial/\partial\lambda_1 + \lambda_2\partial/\partial\lambda_2 - 2\lambda_3\partial/\partial\lambda_3\,,\\ M^2_2 - M^1_1 &= \lambda_1\partial/\partial\lambda_1 - \lambda_2\partial/\partial\lambda_2\,, \qquad (4.5c)\\ M^i_k &= \lambda_k\partial/\partial\lambda_i\,, \quad i \neq k\,. \end{aligned}$$

Representation (4.5b) and (4.5c) of the generators of SU(3) is, apart from a sign, the standard Bargmann-space representation [cf. Eq. (1.7) in Ref. 28]. The negative sign in (4.5b) results from our choice of $\chi(\lambda_i, \mu^k)$ as a representation function. The argument μ^k is taken from C^3 whereas $\lambda_i \in C^{3*}$ is a skewed product of two C^3 vectors.

Homogeneity of the twistor function $F(Z^\alpha_a)$ enforces that $\chi(\lambda_i, \mu^k)$ is an eigenfunction of operators A^1_1, A^2_2, and B. Since the components of the momentum d^i do not commute with A^i_k and B, they cannot be diagonal operators. Altogether 8 independent commuting Hermitian operators are needed[22] in order to label uniquely the functions χ. The five SU(3) labels are to be described in detail in Sec. VI and we have operator B and the two Casimir operators giving j and m.

The function $\chi(\lambda_i, \mu^k)$ provides a suitable description of a three-twistor internal quantum state. Although the form of the scalar product of three-twistor state functions is not known explicitly, the set of *internal* functions can be endowed with the scalar product

$$(\chi, \chi') = \int \chi(\lambda_i, \mu^k)^* \chi'(\lambda_i, \mu^k)\, d\lambda \wedge d\mu\,, \qquad (4.6)$$

where the integration extends over $C^3 \times C^{3*}$ and

$$d\lambda = \frac{d^3\lambda_i}{\pi^6}\exp(-\lambda_r\lambda^{*r})$$

and (4.7)

$$d\mu = \frac{d^3\mu^k}{\pi^6}\exp(-\mu^r\mu^*_r)\,.$$

The space of three-twistor internal states is thereby made a Bargmann space.

V. SYSTEMS WITH EXTENSION: FOUR-TWISTORS

The representation space of the standard quark picture[1,19] is a linear space which has little of the structural richness of a manifold. The proposition here is to place the quark model in the Kählerian manifold of unitary space where an object with extension and shape is a sensible notion. We now show that a system of *four-twistors* may be pictured as an object which has spatial extension in the manifold. Since we are concerned with internal properties only, we need not actually use twistors. The four-twistor particle will be described, in an abstract fashion, by its internal operators d_{ab}, $d^{\dagger ab}$, and B^a_b, satisfying (A11) and (A12) with $a, b = 1$, 2, 3, and 4.

The algebraic relations (A12) impose constraints on a four-twistor particle. Of these, $d_{a[b}d_{cd]} = 0$ give two Hermitian conditions on the momenta d_{ab} and $d_{[ab}B_{c]}{}^{[e}d^{\dagger fg]} = 0$ yield a further condition involving the U(4) generators B^a_b. It is not necessary to manipulate with individual components when solving the constraint equations. We introduce a 3+1 decomposition as follows:

$$\begin{aligned} &a_i \equiv d_{i4}\,, \quad d^i \equiv \tfrac{1}{2}\epsilon^{ijk}d_{jk}\,,\\ &b^i \equiv -B^i_4\,, \quad C \equiv -B^4_4\,, \quad i,j,k = 1,2,3\,, \qquad (5.1)\\ &C^\dagger = C\,. \end{aligned}$$

Operators $d^i, d^\dagger_i$, and the trace-free part of B^i_k generate the ISU(3) subgroup. Commutators (A11) become in this notation

$$[B^i_j, B^k_l] = \delta^k_j B^i_l - \delta^i_l B^k_j, \quad [d^i, d^j] = 0 = [d^i, d^\dagger_j],$$
$$[d^i, B^k_l] = \delta^k_l d^i - \delta^i_l d^k, \tag{5.2a}$$

and

$$[a_i, C] = -a_i, \tag{5.2b}$$
$$[b^i, C] = b^i, \tag{5.2c}$$
$$[b^i, b^\dagger_j] = \delta^i_j C + B^i_j, \tag{5.2d}$$
$$[d^i, b^j] = \epsilon^{ijk} a_k, \tag{5.2e}$$
$$[a_i, b^\dagger_j] = -\epsilon_{ijk} d^k, \tag{5.2f}$$
$$[a_i, B^k_l] = +\delta^k_i a_l, \tag{5.2g}$$
$$[b^i, B^k_l] = -\delta^i_l b^k, \tag{5.2h}$$

whereas

$$[a_i, a_j] = [a_i, a^{\dagger j}] = [a_i, d^j] = [a_i, d^\dagger_j] = [d^i, b^\dagger_j]$$
$$= [a_i, b^j] = [b^i, b^j] = [d^\dagger_i, C] = 0.$$

Equations (A12a) may be written as the single non-Hermitian condition

$$a_i d^i = 0. \tag{5.3}$$

The remaining constraint equations (A12b) are

$$d^i B^l_i d^\dagger_l = 0, \tag{5.4a}$$
$$d^i (b^l d^\dagger_l) + \epsilon^{ijk} a_j (B^l_k d^\dagger_l) = 0, \tag{5.4b}$$
$$d^i b^j a^{\dagger k} \epsilon_{jkl} + d^i C d^\dagger_l + \epsilon^{ijk} a_j B^m_k a^{\dagger n} \epsilon_{lmn} - \epsilon^{ijk} a_j b^\dagger_k d^\dagger_l = 0. \tag{5.4c}$$

Here (5.4a) is a Hermitian constraint, well known from the twistor theory of ISU(3) [cf. Eq. (4.3)]. Equations (5.4b) and (5.4c) follow from the commutators (5.2) and from (5.4a). The proof relies on the color algebra of Appendix B which, eventually, is also defined by (5.2) and (5.4a).

First I derive (5.4b), using the decomposition (4.2b) of B^i_k into irreducible parts. The commutator (5.2h) yields for the vector $f^\dagger_k$ defined in (B1)

$$[b^i, f^\dagger_k] = (b^r d^\dagger_r)\delta^i_k. \tag{5.5}$$

Multiplying Eq. (5.2e) with $f^\dagger_i$ from the right and using Table I and (5.5), I obtain $d^j(b^r d^\dagger_r) = \epsilon^{ijk} a_k f^\dagger_i$ which is Eq. (5.4b). A similar but somewhat more tedious computation employing the commutator

$$[b^i, f^k] = d^i b^k + \epsilon^{irs} a_r B^k_s \tag{5.6}$$

leads to constraint equation (5.4c), and this completes the proof.

It may be worthwhile to point out, for possible future applications, the existence of an algebraic constraint equation, formally analogous to Eq. (5.4a) but with the vector d^i replaced by

$$\varphi_i = \epsilon_{ijk} a^{\dagger j} b^k - \tfrac{1}{2} d^\dagger_i (C+1). \tag{5.7}$$

We multiply Eq. (5.4c) with $b^\dagger_i$ and b^l to obtain

$$\varphi^{\dagger k} B^i_k \varphi_i = 0. \tag{5.8}$$

Observe now that the Lie algebra given in (5.2) contains an ISU(3) subalgebra. In the previous section, the ISU(3) generators have been interpreted as observables of a point particle (with spin) in the unitary space. In order to find an interpretation of the full set of four-twistor internal operators, it will be of help to consider their properties under translations involving only three of the twistor constituents. We set in Eqs. (A13) $\Lambda_{i4} = 0$, and $t^i = \frac{1}{2}\epsilon^{ijk}\Lambda_{jk}$. Then we have, in addition to the ISU(3) relations (2.12),

$$a'_i = a_i, \quad C' = C, \tag{5.9a}$$
$$b'^i = b^i + \epsilon^{ijk} t^*_j a_k. \tag{5.9b}$$

A comparison of (5.9) with the $\vec{s}$ equation (B15) suggests that the vector b^i may be viewed as an orbital momentum and that a_i relates to momentum. We shall work out this idea in the rest of the present section.

Consider two spinning point particles with (d^i_1, A^i_{1k}) and (d^i_2, A^i_{2k}), respectively, in unitary space. Let the observables of different particles commute, i.e.,

$$[\text{any operator}_1, \text{ any operator}_2] = 0, \tag{5.10}$$

and let the masses of the particles be equal:

$$\Delta_1 = \Delta_2 \equiv \Delta_{12}. \tag{5.11}$$

The angular momentum and the momentum of the particles' center of mass,

$$A^i_k \equiv A^i_{1k} + A^i_{2k}, \quad d^i \equiv d^i_1 + d^i_2, \tag{5.12}$$

span an ISU(3) Lie algebra. We define the unitary operator

$$e^{2i\delta} \equiv \frac{\Delta_{12} + d^r_2 d^\dagger_{1r}}{\Delta_{12} + d^r_1 d^\dagger_{2r}} \tag{5.13}$$

(where this time $i = \sqrt{-1}$).

Under the exchange of particle 1 with particle 2, symbolically written as P,

TABLE I. Orthogonality properties of the color triplet, $\{v^i_{\underline{m}}\} = \{d^i, f^i, s^{\dagger i}\}$, with $\Lambda = \frac{1}{2}C_2 - j(j+1) - \frac{3}{4}B(B+2)$.

	$v^i_{\underline{m}}$ \\ $V^{\dagger \underline{n}}_i$	$d^\dagger_i$	$f^\dagger_i$	s_i
	d^i	Δ	0	0
$v^i_{\underline{m}} v^{\dagger \underline{n}}_i =$	f^i	0	$\Delta\Lambda$	0
	s^i	0	0	$\Delta^2\Lambda$

$$Pe^{i\delta}=e^{-i\delta}. \tag{5.14}$$

The operator δ is called the relative phase. Some interesting commutators involving the relative phase are

$$\begin{aligned}&[e^{i\delta},d^k]=0\,,\quad [e^{i\delta},A^k_l]=0\,,\\ &[e^{i\delta},s_1^{\dagger k}]=-e^{2i\delta}x^k\,,\quad [e^{i\delta},s_2^{\dagger k}]=-x^k\,,\\ &[e^{-i\delta},s_1^{\dagger k}]=x^k\,,\quad [e^{-i\delta},s_2^{\dagger k}]=e^{-2i\delta}x^k\,.\end{aligned} \tag{5.15}$$

Here s_{1k} and s_{2k} are the orbital momenta (B2) of particles 1 and 2, respectively, and x^k is defined

$$x^k\equiv\tfrac{1}{2}e^{i\delta}\frac{\Delta_{12}\epsilon^{klm}d^\dagger_{1l}d^\dagger_{2m}}{\Delta_{12}+d^r_2d^\dagger_{1r}} \tag{5.16}$$

such that

$$[x^k,d^i_1]=0=[x^k,d^i_2]. \tag{5.17}$$

Under permutation

$$Px^k=-x^k. \tag{5.18}$$

We now introduce a concept of relative momentum, vector a_i, in the complex unitary space, formally by

$$a_i\equiv e^{-i\delta}d^\dagger_{1i}-e^{i\delta}d^\dagger_{2i}. \tag{5.19}$$

This satisfies $a_id^i=0$ as required in (5.3). Exchange of the particles gives

$$Pa_i=-a_i \tag{5.20}$$

just as for relative momenta in real space. [In fact, if the momenta were Hermitian, we had in (5.13) $e^{i\delta}=1$.]

Next the relative orbital momentum b^i is defined, again formally,

$$b^i\equiv\frac{1}{\Delta_{12}}(s_1^{\dagger i}e^{-i\delta}-e^{i\delta}s_2^{\dagger i}). \tag{5.21}$$

The particular choice of factor ordering is quite important here, although

$$[b^k,e^{i\delta}]=0. \tag{5.22}$$

We find

$$Pb^i=-b^i. \tag{5.23}$$

The components of b^i commute among each other and

$$[b^i,b^\dagger_k]=-A^i_k+\delta^i_k(B_1+B_2)-2\Delta_{12}{}^{-1}(S^i_{1k}+S^i_{2k}). \tag{5.24}$$

Now the difference $B-C$ may be fully eliminated from commutators (5.2) by use of the decomposition (4.2b). All that remains of operators B and C is then contained in the quantity

$$c\equiv B+C+2. \tag{5.25}$$

The commutator (5.2d) takes the form

$$[b^i,b^\dagger_k]=-A^i_k+\delta^i_kc\,, \tag{5.26}$$

and from Eqs. (2.16), (5.2b), and (5.2c)

$$[d^i,c]=\tfrac{2}{3}d^i\,,\quad [a^{\dagger i},c]=\tfrac{2}{3}a^{\dagger i}\,,\quad [b^i,c]=\tfrac{2}{3}b^i\,. \tag{5.27}$$

Comparison with (5.24) shows that c may be identified as[29]

$$c=B_1+B_2\,, \tag{5.28}$$

and that the two spins must be oppositely equal, $S^i_{1k}=-S^i_{2k}$. Since both S^i_{1k} and S^i_{2k} are translationally invariant tensor operators of the SU(3) group defined in (5.12), this latter condition is a legitimate one. The total spin S^i_k is not necessarily zero, however.

In summary, we find that the total angular momentum A^i_k, the total momentum d^i, the relative momentum a_i, and the relative orbital momentum b^i defined in (5.12), (5.19), and (5.21), respectively, satisfy the four-twistor Lie algebra of (5.2). We thus achieved a *realization* of a four-twistor system in terms of two point particles with equal mass and with oppositely equal helicities, in the unitary space.

The group ISU(4) may be looked upon as a "complexified" IO(4), which, in turn, is an inhomogenized variant of O(4) [a 6-dimensional inhomogenization though, not the group E(4)]. Now O(4) has a semisimple Lie algebra, being locally O(3)⊗O(3). That is to say, the group O(3) gives the rigid rotations of a body in the real 3-space, whereas O(4) describes the separate rotations of *two* bodies in the 3-space. The Lie algebra of SU(4) is simple. But here our result shows that the relation ISU(4)≈ISU(3)⊗ISU(3) can be made sensible if interpreted in terms of the associated enveloping algebras.

VI. HECHT'S SU(3) FORMALISM

In this section we review some of the techniques developed by Hecht[17] for dealing with SU(3) representations. Hecht's work has not been arbitrarily selected from competing approaches; rather, it appears to be the only scheme available in the literature which is sufficiently general for $I^{(1,0)}$SU(3) calculations. Designed originally for nuclear shell-model computations, his notation had to be altered, mainly to bring it closer to current usage in particle physics.

The infinitesimal operators A^i_k of the group SU(3) satisfy the Lie-algebra commutators (2.11a). Unitary irreducible representations of SU(3) are labeled by a pair of non-negative integers[30] (λ,μ). A set of commuting infinitesimal operators is chosen:

$$\mathcal{Y} = A_1^1 + A_2^2, \quad \text{hypercharge} \tag{6.1a}$$

$$\mathcal{G}_z = \tfrac{1}{2}(A_1^1 - A_2^2), \quad \text{isospin projection}. \tag{6.1b}$$

The hypercharge and the isospin $\mathcal{G}$ have the eigenvalues in unitary representations, respectively,

$$Y = -\frac{2\lambda + \mu}{3} + p + q, \quad p = 0, 1, \ldots, \lambda \tag{6.2a}$$

$$I = \tfrac{1}{2}(\mu + p - q), \quad q = 0, 1, \ldots, \mu. \tag{6.2b}$$

A state $|a\rangle$ of the representation space will be labeled

$$|a\rangle = |(\lambda, \mu) Y I I_z\rangle. \tag{6.3}$$

An irreducible tensor operator $T_{YII_z}^{(\lambda,\mu)}$ is defined by its commutation properties

$$[\mathcal{Y}, T_{YII_z}^{(\lambda,\mu)}] = Y T_{YII_z}^{(\lambda,\mu)}, \quad [\mathcal{G}_z, T_{YII_z}^{(\lambda,\mu)}] = I_z T_{YII_z}^{(\lambda,\mu)}, \tag{6.4a}$$

$$[A_i^k, T_{YII_z}^{(\lambda,\mu)}] = \sum_{I'} \langle(\lambda,\mu) Y' I' I_z' | (\lambda,\mu) Y I I_z\rangle T_{Y' I' I_z'}^{(\lambda,\mu)}, \quad i \neq k. \tag{6.4b}$$

In the matrix elements of the infinitesimal generators A_i^k, the representation labels (λ, μ) will conveniently be dropped. Hecht has obtained these matrix elements in a neatly compact form

$$A_1^3 |YII_z\rangle = f(YII_z) |(Y+1)(I+\tfrac{1}{2})(I_z+\tfrac{1}{2})\rangle + f(Y, -(I+1) I_z) |(Y+1)(I-\tfrac{1}{2})(I_z+\tfrac{1}{2})\rangle, \tag{6.5a}$$

$$A_2^3 |YII_z\rangle = f(YI, -I_z) |(Y+1)(I+\tfrac{1}{2})(I_z-\tfrac{1}{2})\rangle - f(Y, -(I+1), -I_z) |(Y+1)(I-\tfrac{1}{2})(I_z-\tfrac{1}{2})\rangle, \tag{6.5b}$$

$$A_3^1 |YII_z\rangle = f((Y-1), -(I+\tfrac{3}{2})(I_z-\tfrac{1}{2})) |(Y-1)(I+\tfrac{1}{2})(I_z-\tfrac{1}{2})\rangle + f((Y-1)(I-\tfrac{1}{2})(I_z-\tfrac{1}{2})) |(Y-1)(I-\tfrac{1}{2})(I_z-\tfrac{1}{2})\rangle, \tag{6.5c}$$

$$A_3^2 |YII_z\rangle = -f((Y-1), -(I+\tfrac{3}{2}), -(I_z+\tfrac{1}{2})) |(Y-1)(I+\tfrac{1}{2})(I_z+\tfrac{1}{2})\rangle + f((Y-1)(I-\tfrac{1}{2}), -(I_z+\tfrac{1}{2})) |(Y-1)(I-\tfrac{1}{2})(I_z+\tfrac{1}{2})\rangle, \tag{6.5d}$$

where

$$f(YII_z) = \left[\frac{(I + I_z + 1)(p+1)(\lambda - p)(\mu + 2 + p)}{(2I+1)(2I+2)}\right]^{1/2}. \tag{6.6}$$

The isospin-raising operator A_1^2 possesses the matrix elements

$$\langle YII_z + 1 | A_1{}^2 | YII_z\rangle = [(I - I_z)(I + I_z + 1)]^{1/2}.$$

Note that $A_2^1 = A_1^{\dagger 2}$.

We shall now extend the short notation (6.3) of SU(3) labels for use with tensor operators. The Wigner-Eckart theorem for the matrix elements of the tensor operator $T(b)$ assumes the form

$$\langle a | T(b) | c\rangle = \langle c; b | a\rangle_\rho \langle a \| T \| c\rangle. \tag{6.7}$$

Here $\langle a \| T \| c\rangle$ is the reduced matrix element independent of Y, I, and I_z. The SU(3) Clebsch-Gordan coefficient $\langle c; b | a\rangle_\rho$ factorizes as

$$\langle c; b | a\rangle_\rho = \langle c; b \| a\rangle_\rho \langle I_c I_{zc}, I_b I_{zb} | I_a I_{za}\rangle, \tag{6.8}$$

where $\langle c; b \| a\rangle_\rho$ is an isoscalar factor, and $\langle I_c I_{zc}, I_b I_{zb} | I_a I_{za}\rangle$ an SU(2) Clebsch-Gordan coefficient. Greek subscripts label the multiplicity.

Clebsch-Gordan coefficients satisfy the orthogonality relations[19]

$$\sum_{\substack{Y_a I_a I_{za} \\ Y_b I_b I_{zb}}} \langle a; b | c\rangle_\rho \langle a; b | d\rangle_\sigma = \delta_{cd} \delta_{\rho\sigma}, \tag{6.9a}$$

$$\sum_{c\rho} \langle a; b | c\rangle_\rho \langle a'; b' | c\rangle_\rho = \delta_{\Pi_a \Pi_{a'}} \delta_{\Pi_b \Pi_{b'}}. \tag{6.9b}$$

Here and onward the projection indices Y, I, and I_z of state $|a\rangle$ are collectively denoted by Π_a and primes distinguish among states belonging to the same representation

$$|a\rangle = |(\lambda, \mu) \Pi_a\rangle, \quad |a'\rangle = |(\lambda, \mu) \Pi_{a'}\rangle. \tag{6.10}$$

The *conjugate* of state $|a\rangle$ is defined

$$|a^*\rangle = |(\mu\lambda), -Y, I, -I_z\rangle \equiv |(\mu\lambda), -\Pi_a\rangle, \tag{6.11}$$

and the Hermitian adjoints of tensor operator components $T(a)$ form a tensor operator belonging to the conjugate representation

$$T(a^*) = \xi_a T^\dagger(a) \tag{6.12a}$$

where ξ_a stands for the phase factor

$$\xi_a = (-1)^{(\lambda - \mu)/3 + I_z + Y/2}. \tag{6.12b}$$

The Clebsch-Gordan coefficients exhibit the symmetry property

$$\langle a; b | d\rangle_\rho = c\, \xi_b \left(\frac{\dim(d)}{\dim(a)}\right)^{1/2} \langle d; b^* | a\rangle_\rho \tag{6.13}$$

with

$$c = (-1)^{(\mu_a + \mu_b - \mu_d - \lambda_a - \lambda_b + \lambda_d)/3}. \tag{6.14}$$

The dimension of the irrep $|a\rangle$ is given

$$\dim(a) = \tfrac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)\,. \tag{6.15}$$

$6-(\lambda,\mu)$ coefficients are defined by

$$\mathfrak{U}\{a,b,c,d;e,f;\rho\} = \sum_{\Pi} u(I_aI_bI_cI_d;I_eI_f)\langle a;b\|e\rangle_{\rho_{ab}}\langle e;d\|c\rangle_{\rho_{ed}}\langle b;d\|f\rangle_{\rho_{bd}}\langle a;f\|c\rangle_{\rho_{af}}\,, \tag{6.16}$$

where $u(I_aI_bI_cI_d;I_eI_f)$ is a $6-j$ coefficient[31] of SU(2). The $6-(\lambda,\mu)$ coefficients satisfy the relation

$$\Sigma\langle a;f\|c\rangle_{\rho_{af}}\mathfrak{U}\{a,b,c,d;e,f;\rho\} = \sum_{\Pi_b,\Pi_d,\Pi_e}\langle b;d\|f\rangle_{\rho_{bd}}\langle a;b\|e\rangle_{\rho_{ab}}\langle e;d\|c\rangle_{\rho_{ed}}u(I_aI_bI_cI_d;I_eI_f)\,, \tag{6.17}$$

which provides a convenient means of computing the value of $\mathfrak{U}\{a,b,c,d;e,f;\rho\}$.

VII. MATRICES OF ISU(3) GENERATORS

Translation operators d^i (and $d_i^\dagger$) transform according to the (anti) triplet unitary irreducible representation (irrep) of the group SU(3). The components of the corresponding irreducible tensor operators may be identified as

$$\begin{aligned} &d^{(10)}_{1/3\,1/2\,1/2} = d^1\,, \quad d^{*(01)}_{-1/3\,1/2\,-1/2} = -d_1^\dagger\,,\\ &d^{(10)}_{1/3\,1/2\,-1/2} = d^2\,, \quad d^{*(01)}_{-1/3\,1/2\,1/2} = -d_2^\dagger\,,\\ &d^{(10)}_{-2/3\,00} = d^3\,, \quad d^{*(01)}_{2/3\,00} = d_3^\dagger\,. \end{aligned} \tag{7.1}$$

The components of tensor operators belonging to the contragredient representation have been given by Hecht[17]:

$$\begin{aligned} &T^{(11)}_{-1\,1/2\,1/2} = -\frac{1}{\sqrt{2}}T^3_2\,, \quad T^{(11)}_{-1\,1/2\,-1/2} = \frac{1}{\sqrt{2}}T^3_1\,,\\ &T^{(11)}_{000} = -\tfrac{1}{2}\sqrt{3}\,T^3_3\,, \quad T^{(11)}_{011} = -\frac{1}{\sqrt{2}}T^1_2\,,\\ &T^{(11)}_{010} = \tfrac{1}{2}(T^1_1 - T^2_2)\,, \quad T^{(11)}_{01-1} = \frac{1}{\sqrt{2}}T^2_1\,,\\ &T^{(11)}_{1\,1/2\,1/2} = -\frac{1}{\sqrt{2}}T^1_3\,, \quad T^{(11)}_{1\,1/2\,-1/2} = -\frac{1}{\sqrt{2}}T^2_3\,. \end{aligned} \tag{7.2}$$

We may employ the SU(3) recoupling techniques described in the previous section for obtaining the matrix elements of the ISU(3) generators d^i, $d_i^\dagger$, and A_k^i. It follows from Eqs. (7.1), (7.2), and from the Wigner-Eckart theorem (6.7) that the task then is equivalent to the determination of the corresponding reduced matrix elements. Furthermore, the matrices of the SU(3) generators are completely determined by the SU(3) representation labels (λ,μ). We have[17]

$$\langle\lambda,\mu\|A\|\lambda,\mu\rangle = -\frac{1}{\sqrt{3}}(g_{\lambda\mu})^{1/2}\,, \tag{7.3}$$

where

$$g_{\lambda\mu} \equiv \tfrac{3}{2}C_2 = \lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu\,. \tag{7.4}$$

Some of the reduced matrix elements of the translation operators vanish trivially. This follows from the decomposition of the product $(1,0)\otimes(\lambda,\mu)$ into irreducible terms. Application of Speiser's graphical method[31] (Fig. 1) yields that the remaining matrix elements contain the states situated at the head of an "up" or "down" or "sideways" weight vector of the (1, 0) triplet. Correspondingly, we employ the notation

$$\begin{aligned} U^2 &= \frac{\dim(\lambda+1,\mu)}{\Delta\dim(\lambda,\mu)}|\langle B-\tfrac{2}{3},(\lambda+1,\mu)\|d\|B,(\lambda,\mu)\rangle|^2\,,\\ D^2 &= \frac{\dim(\lambda,\mu-1)}{\Delta\dim(\lambda,\mu)}|\langle B-\tfrac{2}{3},(\lambda,\mu-1)\|d\|B,(\lambda,\mu)\rangle|^2\,,\\ S^2 &= \frac{\dim(\lambda-1,\mu+1)}{\Delta\dim(\lambda,\mu)}\\ &\quad\times|\langle B-\tfrac{2}{3},(\lambda-1,\mu+1)\|d\|B,(\lambda,\mu)\rangle|^2\,. \end{aligned} \tag{7.5}$$

The quantities U^2, D^2, and S^2 may be calculated from the diagonal matrix elements of operator equations (2.13), (2.14), and (2.27):

$$\langle a|d_i^\dagger d^i|a\rangle = \Delta\,, \tag{7.6a}$$

$$\langle a|A_k^i d^k d_i^\dagger|a\rangle = B\Delta\,, \tag{7.6b}$$

$$\langle a|A_j^i A_k^j d^k d_i^\dagger|a\rangle = [-j(j+1)+\tfrac{1}{4}B^2 - \tfrac{3}{2}B+\tfrac{1}{2}C_2]\Delta\,. \tag{7.6c}$$

The evaluation of Eq. (7.6a) proceeds, applying (6.7), as follows:

$$\begin{aligned} \langle a|d_i^\dagger d^i|a\rangle &= \sum_b\langle a|d_i^\dagger|b\rangle\langle b|d^i|a\rangle\\ &= \sum_{i,b}|\langle b|d^i|a\rangle|^2\\ &= \sum_{i,b}\langle a;(1,0)^i|b\rangle^2|\langle b\|d\|a\rangle|^2\,. \end{aligned} \tag{7.7}$$

Next we take the trace and use the orthogonality property (6.9b) of the Clebsch-Gordan coefficients:

$$\begin{aligned} \mathrm{Tr}\langle a|d_i^\dagger d^i|a\rangle &= \sum_{\lambda_a,\mu_b}\dim(b)|\langle b\|d\|a\rangle|^2\\ &= \dim(a)(U^2+D^2+S^2)\,. \end{aligned} \tag{7.8}$$

Hence Eq. (7.6a) yields

$$U^2+D^2+S^2=1\,. \tag{7.9}$$

Equations (7.6b) and (7.6c) contain operator expressions of the form

$$T^i_k d^k d^\dagger_i\,. \tag{7.10}$$

In Eq. (7.6b) we have $T^i_k=A^i_k$ and in (7.6c) we have $T^i_k=A^i_jA^j_k$. The quantity (7.10) involves an invariant coupling of irreducible tensor operators.[11] To make this manifest, we first need the values of the Clebsch-Gordan coefficients

$$\langle(11)Y_1I_1I_{z1};\,(10)Y_2I_2I_{z2}\|(10)Y_3I_3I_{z3}\rangle\,.$$

From the symmetry property (6.13) it follows that

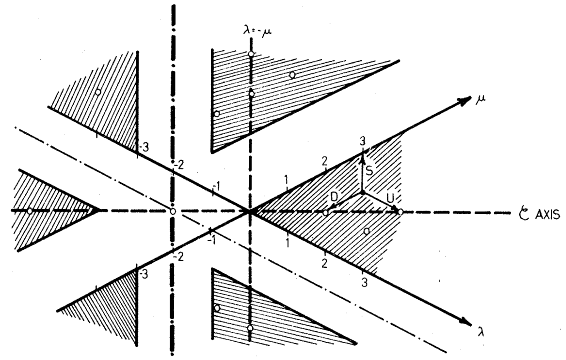

FIG. 1. The six sets of equivalent SU(3) representations provided by the substitution group. Substitutions are represented by reflections of the (λ,μ) plane. The decomposition of the direct product $(1,0)\otimes(\lambda,\mu)$ is obtained by Speiser's graphical method.

$$\langle(11)Y_1I_1;\,(10)Y_2I_2\|(10)Y_3I_3\rangle=(-1)^{I_1+I_2-I_3}\langle(10)Y_2I_2;\,(11)Y_1I_1\|(10)Y_3I_3\rangle\,. \tag{7.11}$$

The coupling of representation (1, 0), a triangular representation, involves no multiplicity problem. Values of the isoscalar factors appearing on the right-hand side of Eq. (7.11) are given in Ref. 17, Table 4, $\rho=1$. Unfortunately, the signs of the entries on p. 31, column $\rho=1$ in that reference are incorrect, when $\mu=0$ and they have to be reversed. The sign anomaly in the case of triangular representations follows from the sign convention (according to which, among the isoscalar factors with the largest value of I_2 and of I_{z2}, the one with the smallest value of Y_3 and Y_1 and with the largest value of I_{z3} is positive).

Using this information we obtain

$$T^i_k d^k d^\dagger_i=4\sum_{\text{all }\Pi\text{'s}} T^{(11)}_{\Pi_1} d^{(10)}_{\Pi_2} d^{*(01)}_{\Pi_3}\langle(11)\Pi_1;\,(10)\Pi_2|(10)\Pi\rangle\langle(10)\Pi;\,(01),-\Pi_3|(00)\rangle\,. \tag{7.12}$$

From symmetry relation (6.13),

$$\langle(10)\Pi;\,(01),-\Pi_3|(00)\rangle=\frac{1}{\sqrt{3}}\,\xi_3\delta_{\Pi\Pi_3}\,. \tag{7.13}$$

We are now in position to evaluate the quantity

$$\Phi=\mathrm{Tr}\langle a|T^i_k d^\dagger_i d^k|a\rangle\,. \tag{7.14}$$

Inserting complete sets of states, we get

$$\Phi=\frac{4}{\sqrt{3}}\sum\langle a|T^{(11)}_{\Pi_1}|a'\rangle\langle a'|d^{*(01)}_{\Pi_2}|b\rangle\langle b|d^{(10)}_{\Pi_3}|a\rangle\langle(11);\,(10)\Pi_2|(10)\Pi_3\rangle\xi_3\,. \tag{7.15}$$

The phase factor ξ_3 may be eliminated by the relationship [Eq. (6.12a)]

$$\begin{aligned}&\langle a|T^{(11)}_{-\Pi_1}|a'\rangle=\langle a'|T^{(11)}_{\Pi_1}|a\rangle^*\,\xi_{(11)}\,,\\&\langle a'|d^{*(01)}_{-\Pi_2}|b\rangle=\langle b|d^{(10)}_{\Pi_2}|a'\rangle^*\,\xi_{(10)}\,.\end{aligned} \tag{7.16}$$

We find then

$$\Phi=-\frac{4}{\sqrt{3}}\sum_{b,\text{all }\Pi\text{'s}}\langle a;\,(11)\Pi_1|a'\rangle\langle a';\,(10)\Pi_2|b\rangle\langle a;\,(10)\Pi_3|b\rangle\langle(11)\Pi_1;\,(10)\Pi_2|(10)\Pi_3\rangle|\langle b\|d\|a\rangle|^2\langle a\|T\|a\rangle^*$$

$$=-\frac{4}{\sqrt{3}}\sum_b \dim(b)\mathfrak{U}\{a(11)b(10);\,a(10)\rho\}|\langle b\|d\|a\rangle|^2\langle a\|T\|a\rangle^*\,, \tag{7.17}$$

and the value of the multiplicity index ρ is determined by the precise choice of operator T. The summation extends over the nontrivial terms $b=(\lambda,\mu-1)$, $(\lambda+1,\mu)$, and $(\lambda-1,\mu+1)$.

Equation (6.17) may now be used for the calculation of the $6-(\lambda\mu)$ coefficients $\mathfrak{U}\{(\lambda,\mu)(11)(\lambda',\mu')(10);\,(\lambda,\mu)(10)\rho\}$:

$$\langle(\lambda,\mu)Y,\tfrac{1}{2}\mu;(10)\tfrac{2}{3}0\|(\lambda',\mu')Y_LI_L\rangle\mathfrak{U}=\sum_{\substack{Y_2Y_3Y_{12}\\I_2I_{12}}}u(\tfrac{1}{2}\mu,I_2,\tfrac{1}{2}\mu,I_2;I_{12}0)$$
$$\times\langle(11)Y_2I_2;(10)Y_3I_2\|(10)-\tfrac{2}{3}0\rangle\langle(\lambda\mu)Y_1\tfrac{1}{2}\mu;(11)Y_2I_2\|(\lambda\mu)Y_{12}I_{12}\rangle$$
$$\times\langle(\lambda\mu)Y_{12}I_{12};(10)Y_3I_2\|(\lambda'\mu')Y_LI_L\rangle\,, \tag{7.18}$$

where the subscript L labels quantum numbers of a state with the lowest weights (i.e., with $p=0,\ q=0$). Let the "triangular delta" $\{I_1I_2I_3\}$ be defined by[32]

$$\{I_1I_2I_3\}\equiv\begin{cases}1 \text{ if } I_1,\ I_2, \text{ and } I_3 \text{ can be triangle sides,}\\ 0 \text{ if } I_1,\ I_2, \text{ and } I_3 \text{ cannot be triangle sides.}\end{cases} \tag{7.19}$$

Then the value of the $6-j$ symbol required in Eq. (6.17) is[33]

$$u(I_1I_2I_3I_4;I0)=(-)^{I-I_3-I_4}\left[\frac{2I+1}{(2I_1+1)(2I_2+1)}\right]^{1/2}\delta_{I_1I_3}\delta_{I_2I_4}\{I_1I_2I\}\,. \tag{7.20}$$

In Eq. (7.18) some "quark" [i.e., state (10)] coupling isoscalar factors occur. The values of general quark coupling isoscalar factors have been calculated by Asherova and Smirnov.[34] For convenience, we list these values in Table II. Substitution in Eq. (7.18) yields the nonvanishing $6-(\lambda\mu)$ coefficients

$$\mathfrak{U}(n)=-\tfrac{1}{2}\langle(\lambda,\mu)(Y_L+1)(I_L-\tfrac{1}{2});(11)00\|(\lambda,\mu)(Y_L+1)(I_L-\tfrac{1}{2})\rangle_\rho$$
$$-\frac{\sqrt{3}}{2}\left[\frac{(\mu+1)(\lambda+\mu+1)}{2\mu}\right]^{1/2}\langle(\lambda,\mu)(Y_L+1),\tfrac{1}{2}(\mu-1);(11)-1\tfrac{1}{2}\|(\lambda,\mu)Y_L,I_L\rangle_\rho\,,$$
$$\mathfrak{U}(p)=-\tfrac{1}{2}\langle(\lambda,\mu)Y_LI_L;(11)00\|(\lambda,\mu)Y_LI_L\rangle_\rho\,, \tag{7.21}$$
$$\mathfrak{U}(\lambda)=-\tfrac{1}{2}\langle(\lambda,\mu)Y_L+1,I_L+\tfrac{1}{2};(11)00\|(\lambda,\mu)(Y_L+1),(I_L+\tfrac{1}{2})\rangle_\rho$$
$$-\frac{\sqrt{3}}{2}\left[\frac{(\mu+1)\lambda}{2(\mu+2)}\right]^{1/2}\langle(\lambda,\mu)(Y_L+1),\tfrac{1}{2}(\mu+1);(11)-1,\tfrac{1}{2}\|(\lambda,\mu)Y_LI_L\rangle_\rho\,,$$

where we denote

$$\mathfrak{U}\{(\lambda\mu)(11)(\lambda'\mu')(10);(\lambda\mu)(10)\rho\}=\begin{cases}\mathfrak{U}_\rho(n) & \text{if } (\lambda',\mu')=(\lambda,\mu-1)\\ \mathfrak{U}_\rho(p) & \text{if } (\lambda',\mu')=(\lambda+1,\mu)\\ \mathfrak{U}_\rho(\lambda) & \text{if } (\lambda',\mu')=(\lambda-1,\mu+1)\,.\end{cases} \tag{7.22}$$

The multiplicity index ρ enumerates the possible choices of operator T_k^i in expression (7.10). For $\rho=1$ we set $T_k^i=A_k^i$. The isoscalar factors in (7.21) are directly given as

$$\langle a;(11)\Pi\|a'\rangle_1=\frac{\langle a'|A_\Pi^{(11)}|a\rangle}{\langle a\|A\|a\rangle}\,, \tag{7.23}$$

where the matrix elements $\langle a'|A_\Pi^{(11)}|a\rangle$ are listed in (6.5). We need not actually bother with factoring out the reduced matrix element $\langle a\|A\|a\rangle$ since it would reappear in expression (7.17). Isoscalar factors (7.23)

TABLE II. The quark coupling isoscalar factors $\langle(\lambda,\mu)Y_1I_1;(1,0)Y_2I_2\|(\lambda',\mu')YI\rangle$, adapted from Asherova and Smirnov, Ref. 34, where $Y=-\frac{1}{3}(2\lambda'+\mu')+p+q$ and $I=\frac{1}{2}\mu'+\frac{1}{2}p-\frac{1}{2}q$.

Y_2 I_2 / I_1	$(\lambda',\mu')=(\lambda+1,\mu)$	$(\lambda',\mu')=(\lambda-1,\mu+1)$	$(\lambda',\mu')=(\lambda,\mu-1)$
$\frac{1}{3}$ $\frac{1}{2}$ / $I+\frac{1}{2}$	$\left[\frac{q(\lambda-p+1)(\mu-q+1)}{(\lambda+1)(\lambda+\mu+2)(\mu+p-q+1)}\right]^{1/2}$	$-\left[\frac{pq(\lambda+\mu-q+2)}{(\lambda+1)(\mu+1)(\mu+p-q+1)}\right]^{1/2}$	$\left[\frac{(\lambda+\mu-q+2)(\mu-q+1)(\mu+p+1)}{(\lambda+\mu+2)(\mu+1)(\mu+p-q+1)}\right]^{1/2}$
$\frac{1}{3}$ $\frac{1}{2}$ / $I-\frac{1}{2}$	$\left[\frac{p(\lambda+\mu-q+2)(\mu+p+1)}{(\lambda+1)(\lambda+\mu+2)(\mu+p-q+1)}\right]^{1/2}$	$\left[\frac{(\lambda-p+1)(\mu-q+1)(\mu+p+1)}{(\lambda+1)(\mu+1)(\mu+p-q+1)}\right]^{1/2}$	$\left[\frac{pq(\lambda-p+1)}{(\lambda+\mu+2)(\mu+1)(\mu+p-q+1)}\right]^{1/2}$
$-\frac{2}{3}$ 0 / I	$\left[\frac{(\lambda-p+1)(\lambda+\mu-q+2)}{(\lambda+1)(\lambda+\mu+2)}\right]^{1/2}$	$-\left[\frac{(1+p)(\mu-q+1)}{(\lambda+1)(\mu+1)}\right]^{1/2}$	$\left[\frac{(1+q)(\mu+p+1)}{(\lambda+\mu+2)(\mu+1)}\right]^{1/2}$

have been evaluated in Ref. 17.

In the alternative situation $\rho=2$ we choose $T^i_k=A^i_rA^r_k$. A safe way of calculating the corresponding isoscalar factors in (7.21) is to remember that $T^{(11)}_{000}=-(\sqrt{3}/2)T^3_3$ and $T^{(11)}_{-1\,1/2\,1/2}=-(1/\sqrt{2})T^3_2$ if T^i_k is a traceless operator. So we take $T^i_k=A^i_rA^r_k-\frac{1}{3}\delta^i_kA^r_sA^s_r$. In order to carry out a procedure such as that for $\rho=1$ [cf. Eq. (7.23)], we must compute the matrix elements $\langle(\lambda\mu)|A^i_rA^r_k|(\lambda\mu)\rangle$. This is easily done by successive application of Eqs. (6.5). Let me show an example:

$$\begin{aligned}\langle(\lambda\mu)YII_z|A^3_1A^1_3|(\lambda\mu)YII_z\rangle &=\langle(\lambda\mu)|A^3_1\{f(YII_z)|Y+1,I+\tfrac{1}{2},I_z+1\rangle+f(Y_1,-(I+1)I_z)|(Y+1)(I-\tfrac{1}{2})(I_z+1)\rangle\}\\ &=\langle(\lambda\mu)|\{f(YII_z)f(YII_z)|(\lambda\mu)YII_z\rangle+f(Y,-(I+1),I_z)f(Y,-(I+1),I_z)\}|(\lambda\mu)YII_z\rangle\\ &=f^2(YII_z)+f^2(Y,-(I+1),I_z).\end{aligned} \qquad (7.24)$$

The calculation of the $6-(\lambda\mu)$ coefficients is completed by adding the diagonal terms given in (7.4). The results for both values of ρ are

$$\begin{aligned}&\frac{4}{\sqrt{3}}\langle\lambda\mu\|A\|\lambda\mu\rangle\mathfrak{u}_1(n)=\tfrac{1}{3}(\lambda+2\mu+6),\\ &\frac{4}{\sqrt{3}}\langle\lambda\mu\|A\|\lambda\mu\rangle\mathfrak{u}_1(p)=-\tfrac{1}{3}(2\lambda+\mu),\\ &\frac{4}{\sqrt{3}}\langle\lambda\mu\|A\|\lambda\mu\rangle\mathfrak{u}_1(\lambda)=\tfrac{1}{3}(\lambda-\mu+3),\\ &\frac{4}{\sqrt{3}}\langle\lambda\mu\|AA\|\lambda\mu\rangle\mathfrak{u}_2(n)=\tfrac{1}{3}(\lambda+\mu+1)\mu-2-\tfrac{1}{6}C_2,\\ &\frac{4}{\sqrt{3}}\langle\lambda\mu\|AA\|\lambda\mu\rangle\mathfrak{u}_2(p)=\tfrac{1}{3}(\lambda+\mu+1)\lambda+\tfrac{2}{3}(2\lambda+\mu)-\tfrac{1}{6}C_2,\\ &\frac{4}{\sqrt{3}}\langle\lambda\mu\|AA\|\lambda\mu\rangle\mathfrak{u}_2(\lambda)=-\tfrac{1}{3}\lambda(\mu+2)-2-\tfrac{1}{6}C_2.\end{aligned} \qquad (7.25)$$

Equations (7.6) take the form

$$\begin{aligned}&U^2+D^2+S^2=1,\\ &-(\mu+1)D^2+(\lambda+1)U^2=2B+1+\frac{\lambda-\mu}{3},\\ &\tfrac{1}{3}\lambda(\lambda+\mu+1)+[\tfrac{1}{3}(\lambda+\mu+1)(\mu-\lambda-3)+\lambda+\mu-1]D^2+\tfrac{2}{3}(2\lambda+\mu)U^2+[\lambda-2-\tfrac{1}{3}\lambda(\lambda+2\mu+6)]S^2=B^2+3B+\tfrac{1}{3}C_2-j(j+1).\end{aligned} \qquad (7.26)$$

This system of linear equations is readily solved for the reduced matrix elements of the translation operators:

$$\begin{aligned}U^2&=\frac{(\frac{1}{2}B-x-\frac{3}{2})^2-(j+\frac{1}{2})^2}{(\lambda+1)(\lambda+\mu+2)},\\ D^2&=\frac{(\frac{1}{2}B+y+\frac{1}{2})^2-(j+\frac{1}{2})^2}{(\mu+1)(\lambda+\mu+2)},\\ S^2&=\frac{-(\frac{1}{2}B+z-\frac{1}{2})^2+(j+\frac{1}{2})^2}{(\lambda+1)(\mu+1)},\end{aligned} \qquad (7.27)$$

where

$$x=\frac{2\lambda+\mu}{3},\quad y=\frac{\lambda+2\mu}{3},\quad z=\frac{\lambda-\mu}{3}. \qquad (7.28)$$

All the matrix elements of the generators of ISU(3) are now available in Eqs. (6.5) and (7.27). Next the structure of ISU(3) representations can be studied. This task has actually been carried out. For completeness, in the following section we review the properties of the ISU(3) representations discovered by Perjés and Sparling.[16]

VIII. STRUCTURE OF THE REPRESENTATIONS

The discrete unitary irreps of the group ISU(3) may be pictured[16] in the space of the labels B and (λ,μ). It is a corollary to the Wigner-Eckart theorem (6.7) that each allowable point of the $\{B,(\lambda,\mu)\}$ space represents an SU(3) multiplet. An irrep here is the innermost region bounded by zeros of the reduced matrix elements of the translation operators d^i and $d^\dagger_i$. These matrix elements are U^2, D^2, S^2, and

$$\tilde{U}^2 = \frac{(\frac{1}{2}B - x - \frac{1}{2})^2 - (j + \frac{1}{2})^2}{(\mu + 1)(\lambda + \mu + 2)},$$

$$\tilde{D}^2 = \frac{(\frac{1}{2}B + y + \frac{3}{2})^2 - (j + \frac{1}{2})^2}{(\lambda + 1)(\lambda + \mu + 2)}, \tag{8.1}$$

$$\tilde{S}^2 = \frac{-(\frac{1}{2}B + z + \frac{1}{2})^2 + (j + \frac{1}{2})^2}{(\lambda + 1)(\mu + 1)},$$

obtained by transposition of the reduced matrices. It follows from definition (7.5) that these quantities are non-negative in unitary representations.

The representations are identified by the Casimir labels j (integer or half-integer) and m (positive) but their structure does not depend essentially on m. They form an infinite wedge, the edge of which passes nearest to the origin $\{0, (0, 0)\}$. Their four boundary planes are

$$j + \tfrac{1}{2}B - x = 0\,, \tag{8.2a}$$

$$j - \tfrac{1}{2}B - y = 0\,, \tag{8.2b}$$

$$j + \tfrac{1}{2}B + z = 0\,, \tag{8.2c}$$

$$j - \tfrac{1}{2}B - z = 0\,. \tag{8.2d}$$

Planes c and d are parallel and they actually coincide when $j = 0$ such that the wedge slims into this plane (Figs. 2 and 3). When j is integer (half-integer), the allowed values of B are integer (half-integer) multiples of $\frac{2}{3}$.

The labeling of the representations selected is unique. It is not, however, these labels but the matrices of the infinitesimal generators that determine the structure of the representations. The labels are functions of diagonal operators, which are nonlinear in many cases. This system can be solved, with the eigenvalues of the diagonal operators kept fixed, for the labels. Usually there is more than one solution. This gives rise to a group of substitutions.[35]

Substitutions of the SU(3) group are known to form the dihedral group[36] D_3. The representation labels (λ, μ) chosen for SU(3) give the highest weight occurring in the representation [for both λ and μ non-negative, cf. Eqs. (6.2)]. This is the reason the substitution group turns out to be isomorphic[37] with the Weyl reflection group. The condition that λ and μ are non-negative no longer holds. The substitution group of SU(3) is generated by two of its elements such as

$$\mathcal{C}:\quad \begin{pmatrix} \lambda \\ \mu \\ Y \\ I \\ I_z \end{pmatrix} \to \begin{pmatrix} \mu \\ \lambda \\ -Y \\ +I \\ -I_z \end{pmatrix} \tag{8.3a}$$

and

$$\mathcal{P}:\quad \begin{pmatrix} \lambda \\ \mu \\ Y \\ I \\ I_z \end{pmatrix} \to \begin{pmatrix} \lambda + \mu + 1 \\ -\mu - 2 \\ Y \\ I \\ I_z \end{pmatrix}\,. \tag{8.4a}$$

The resulting sixtuplets of equivalent representations fall into six disjoint segments on the (λ, μ) plane (Fig. 1). Substitutions are represented here by reflections of the pattern.

With the SU(3) group enlarged to ISU(3), the substitution symmetries survive by adding the rules

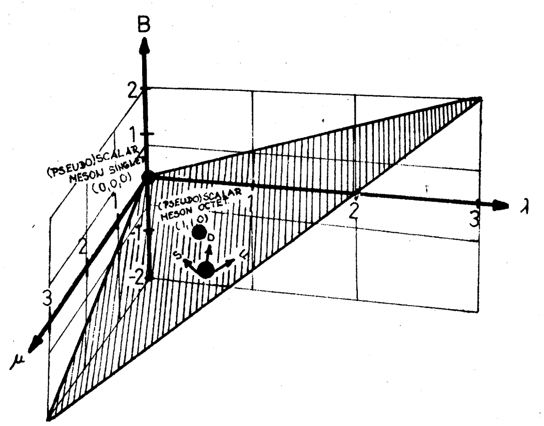

FIG. 2. Portion of the space of $j = 0$ representations of the ISU(3) group. Points $\{B, (\lambda, \mu)\}$ belonging to the unitary representation lie on a quadrant of a plane. The action of the laddering operators in this space is indicated by arrows.

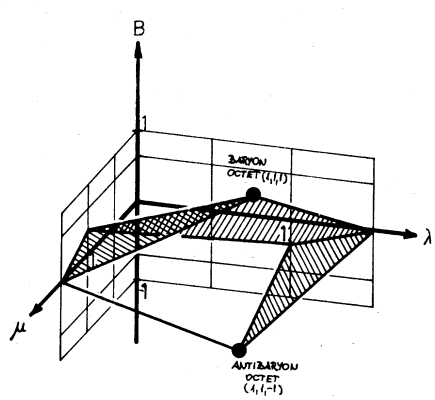

FIG. 3. Points of the $j = \frac{1}{2}$ unitary representations of the ISU (3) group fall into the interior of a wedge and into its four boundary planes. As was shown by Perjés and Sparling (Ref. 16), the low-mass hadron multiplets tend to occur at those zero-triality points of the ISU (3) unitary irreps which lie nearest to the origin.

$$\mathfrak{C}: \begin{pmatrix} j, m \\ B \end{pmatrix} \to \begin{pmatrix} j, m \\ -B \end{pmatrix}, \qquad (8.3b)$$

$$\mathcal{P}: \begin{pmatrix} j, m \\ B \end{pmatrix} \to \begin{pmatrix} j, m \\ B \end{pmatrix}. \qquad (8.4b)$$

Invariance, then, follows at once from (7.27) and (8.1). An additional and independent substitution symmetry for ISU(3) is under

$$j \to -j - 1. \qquad (8.5)$$

Hence, there are altogether 12 sets of equivalent ISU(3) representations.

IX. MODELS AND PARTICLES

It is tempting to make a comparison, at this stage of development of our scheme, with the observed systematics of hadrons. The link which connects the theory of unitary space with relativistic physics is the twistor representation. Any attempt at a classification of hadron states into irreps of the ISU(3) group will give at best approximate results, since particle interactions are ignored here. There is no reason to presume that interactions (other than gravitational) are insensitive to the intrinsic twistor structure of the particles involved. On the contrary, it is to be expected that the ISU(3) symmetry of the kinematical twistor (A3) will prove a broken symmetry in interactions. More impressive is the extent to which hadron states can be made to fit already.

Let us first stay at lower energies where new degrees of freedom such as charm do not show up yet. Here a hadron is intrinsically characterized by the eight quantum numbers called the spin, mass, baryon number, the two SU(3) multiplet labels, the hypercharge, and the two isospin labels. Then it is, at least, a strange coincidence that the simplest, pointlike objects in the unitary space have precisely this number of labels.[38]

Apparently the parallel goes much farther. One may identify, a suggestion by Perjés and Sparling,[16] the hadron SU(3) quantum numbers with (λ, μ), Y, I, and I_z, respectively [cf. Eq. (6.2)]. The identification of spin and mass with the Casimir eigenvalues j and m, respectively, is inherent to the twistor representation[7] (Appendix A). There remains the baryon number on one side and the label B on the other.

Conventionally, the baryon number has the eigenvalues 0 and ±1 for an elementary particle.[39] Is this not a peculiar observable? Most particle properties derive from group theory as eigenvalues of Hermitian operators. It is hard to imagine an operator fitting into a particle symmetry group which would have the eigenvalues 0 and ±1 in *all* irreps. This is one of the difficulties I feel to be present in the conventional formalism. The other concerns the quark occupation number operators. In the by-now-standard picture, each kind of quark bears a uniquely specified baryon number. This then tells the number of quarks (and antiquarks). That is to say, group theory, together with the subordination of baryon numbers, gives the number of quarks—again a peculiar situation. We have the two Casimir operators of the group SU(3) of which the quarks are elementary representations, and we have the two numbers of (anti) quarks. And these two pairs of quantum numbers are generally different. The behavior of quarks, objects defined by SU(3), depends on something lying outside group theory.

Here we present a more consistent picture. Our quantum number B has a discrete range which may extend over the whole real line $\mathbb{R}$. The stability of stable particles is not especially due to baryon conservation but primarily to vanishing of the matrix elements of transition operators d^i and $d_i^\dagger$ which change baryon number. In fact, all stable particles occur at such points of the ISU(3) irreps. The stable baryon and antibaryon octets with the quantum numbers $\{B, (\lambda, \mu)\} = \{\pm 1, (1, 1)\}$ are only found in the $j = \frac{1}{2}$ irrep (ignoring, for the moment, negative spin values). The baryon decuplet $\{1, (3, 0)\}$ and its anti-decuplet $\{-1, (0, 3)\}$ occur only in the $j = \frac{3}{2}$ irrep. Meson octets with $\{0, (1, 1)\}$ occur only in the $j = 0$ and 1 irreps and thus they may be identified with the octets of pseudoscalar and vector mesons, respectively. There is room in the $j = 0$ irrep for the pseudoscalar-meson singlet but a vector singlet is prohibited by the inequality $\lambda + \mu \geq 2j$ which is valid for all ISU(3) irreps. There are two possible ways of explaining the absence of this meson and generally of the unstable Regge recurrences: Either they lie in nonunitary points[16] or they belong (or mix) to higher SU(3) representations.[40] In any case, an extensive revision of our current thinking on this is implied. I expect that the issue will be settled by model calculations with interacting state functions in the unitary space.

A phenomenological approach to symmetry breaking avoiding the details of state functions has been proposed by Perjés and Sparling.[16] This attempts to be a generalization of the Gell-Mann-Okubo mass formula[1] using the ISU(3) group. The physical mass squares are given by the matrix elements of the operator T_3^3 belonging to a Hermitian octet. The behavior of the operator under the full inhomogeneous group is accounted for by constructing it from the generators of the group. The range of the octet operators that may be obtained in this way is then restricted by *CPT* invariance. Only octet operators with positive charge parity are allowable. (The statement of Ref. 16 that there are only two independent Hermitian octet operators of this kind is false because the baryon number B has negative charge parity such that the charge parity of a tensor operator can be suitably changed by multiplying with an odd power of B. However, we may explain the observed regularities in the mass parameters by assuming that all mass-splitting operators excepting Δ_3^3 give a negligible contribution, and that our original mass formula is a reasonable approximation.)

The particular identification of state labels with hadron quantum numbers described here already implies that the $\mathcal{C}$ reflection defined in (8.3) is charge conjugation. The physical significance of other elements of the substitution group is less clearcut. One may argue that the existence of substitutions follows from the nonlinear choice of representation labels and, as such, is not tied to physics. This had been my own opinion prior to the discovery by Penrose[41] that Chew-Frautschi plots with opposite signature show a symmetry with respect to the substitution (8.5). [It should be noted at this juncture that a lepton classification scheme involving the SU(2) substitutions (8.5) has been developed by Sparling.[25]] Returning to the dihedral subgroup of ISU(3) substitutions, with $\mathcal{C}$ allocated to charge conjugation, the remaining elements (cf. Fig. 1) await a physical interpretation. It is a remarkable fact[42] that the space-time reflections $\mathcal{P}$, $\mathcal{J}$, and $\mathcal{C}$ are elements of a dihedral group when acting on an appropriately chosen class of massive particle states. So, at least in some cases, an interpretation of ISU(3) substitutions appears to be at hand.

The familiarity with the structure of ISU(3) representations offers us an insight into the overall properties of hadron state functions in the unitary space. From the nondiagonal nature of the momentum operator d^i we infer that these functions are of spherical rather than plane-wave type. Therefore, a representation of interactions by some binding central tensor potential in the unitary space may be envisaged.

There are at least two different ways in which additional degrees of freedom enter this picture. First, there is the problem of representing several particles simultaneously. Each of them has a center of mass (operator Z^i) in the unitary space. Suppose some of the centers of mass are arranged appropriately for an interaction to take place (classically expressed, the particles meet in the unitary space). Will the particles interact? In the space-time, of course, they still may be galaxies away from each other. We may possibly regard space-time information as being the "intrinsic charge" from the unitary point of view. This is dual to the conventional picture in terms of space-time wave functions where the unitary information goes into the charges (Fig. 4).

Many related questions still remain to be solved in this field. Three-twistor particles have been seen to define points in the unitary space. New hadrons associated with new intrinsic degrees of freedom can be built from more than three twistors. In Sec. V we have shown that the unitary-space picture still extends to four-twistor systems, but the picture may remain valid quite generally.

ACKNOWLEDGMENTS

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APPENDIX A: THE TWISTOR REPRESENTATION

It is an important result in the theory of twistors that a relativistic particle can be decomposed into the sum of n twistors.[6] The number n of constituent twistors may be as few as just one for a massless particle and two for a massive particle. Let these constituents $Z_1^\alpha, Z_2^\alpha, \ldots, Z_n^\alpha$ be labeled by Roman indices $a, b, \ldots$ such that we write Z_a^α for an n-twistor constituent and $Z_\alpha^{\dagger a}$ for its conjugate. The Greek twistor indices range through the values 0, 1, 2, and 3.

The "partial momenta" $P^{a\alpha}$ of the constituents are defined by

$$P^{a\alpha} \equiv I^{\alpha\beta} Z_\beta^{\dagger a}, \tag{A1}$$

where

$$[I^{\alpha\beta}] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [I_{\alpha\beta}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{A2}$$

are "infinity twistors" breaking conformal symmetry as required by the properties of a massive particle.[8] The kinematical data of the particle form a symmetric twistor

$$A^{\alpha\beta} = Z_a^\alpha P^{a\beta} + Z_a^\beta P^{a\alpha} \tag{A3}$$

wherefrom the mass is obtained as

$$m^2 = 2(Z_a^\alpha P_{b\alpha}^\dagger)(Z_\beta^{\dagger a} P^{b\beta}) \tag{A4}$$

(with summation over the dummy labels).

When the mass of the particle is nonzero, the constituent twistors, together with the partial

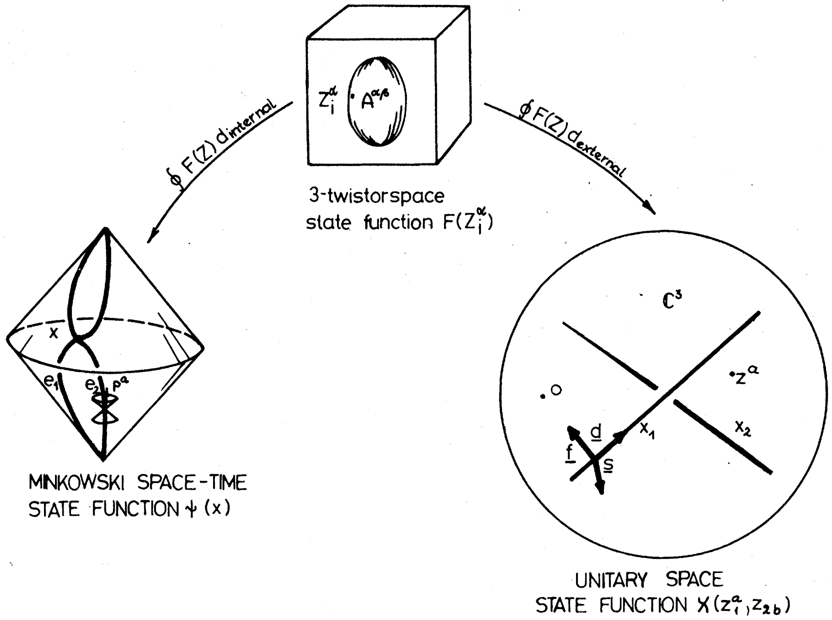

FIG. 4. Dual descriptions of a quantum state in space-time and in unitary space. The complete information is carried by a function $F(Z^\alpha_i)$ of points Z^α_i in the 3-twistor space. Contour integration over internal degrees of freedom gives the standard quantum field theory pictures in Minkowski space-time. Alternatively, by projecting out space-time information we obtain the unitary-space description of internal quantum numbers. Two particles meeting in Minkowski space-time do not necessarily interact because they may not have the appropriate charges. Space-time information is dually expressed by charges in the unitary picture. Two particles meeting in unitary space may still be galaxies away in space-time. A triplet of vector operators is associated with a particle in the unitary space.

momenta $P^{a\alpha}$, form a nondegenerate system. That is to say, they do not lie in a single plane of the four-dimensional twistor space. Then any $\binom{1}{0}$ twistor may be written as a linear combination of Z^α_a and $P^{a\alpha}$.

For a given physical particle, there is a set of possible choices of constituent twistors. It is not difficult to obtain the relation between two arbitrary selections of constituents, although the details of this have not been spelled out in previous works. Each of the new supporting twistors Z'^α_a of a massive particle is written in terms of the old n-twistors and partial momenta,

$$Z'^\alpha_a = U^b_a(Z^\alpha_b + \Lambda_{bc}P^{c\alpha}), \tag{A5}$$

where U^b_a and Λ_{ab} are complex numbers. Using $I^{\alpha\gamma}I_{\beta\gamma}=0$ in Eq. (A1), we obtain the new partial momenta

$$P'^{a\alpha} = U^{*a}_b P^{b\alpha}. \tag{A6}$$

Let Z^α_a be now an *arbitrary* n-twistor. Then the n-twistor Z'^α_a given by Eq. (A5) will represent the same kinematical data (A1) as does Z^α_a, provided

$$(U^b_a U^{*a}_c - \delta^b_c)Z^{(\alpha}_b P^{\dagger\beta)c} + U^b_a U^{*a}_c \Lambda_{be} P^{\dagger e(\alpha}P^{\dagger\beta)c} = 0. \tag{A7}$$

Since Z^α_a is arbitrary, the coefficients in the parentheses must vanish separately:

$$U^b_d U^{*d}_c - \delta^b_c = 0 \tag{A8a}$$

and consequently,

$$\Lambda_{(be)} = 0. \tag{A8b}$$

The meaning of this result is that an arbitrary internal transformation of the twistor constituents is of the form (A5) with $\underline{U}=[U^b_a]$ an $n\times n$ unitary matrix and $\Lambda=[\Lambda_{bc}]$ an $n\times n$ complex skew matrix.

Transformations (A5) define a group. This internal twistor group has the infinitesimal operators[22]

$$B^a_b = Z^\alpha_b Z^{\dagger a}_\alpha, \qquad d_{ab} = Z^\alpha_a I_{\alpha\beta} Z^\beta_b, \quad d^{\dagger ab} = Z^{\dagger a}_\alpha I^{\alpha\beta} Z^{\dagger b}_\beta, \tag{A9}$$

where

$$[Z^\alpha_a, Z^{\dagger b}_\beta] = \delta^b_a \delta^\alpha_\beta, \quad [Z^\alpha_a, Z^\beta_b] = 0. \tag{A10}$$

Operators d_{ab} and $d^{\dagger ab}$ are skew in their indices and each of them commutes with the other. The matrix B^a_b is Hermitian: $(B^b_a)^\dagger = B^a_b$. The nonvanishing commutators of the internal operators are

$$[B^a_b, B^c_d] = \delta^c_b B^a_d - \delta^a_d B^c_b\,, \tag{A11a}$$

$$[d_{ab}, B^d_c] = \delta^d_b d_{ac} - \delta^d_a d_{bc}\,, \tag{A11b}$$

$$[d^{\dagger ab}, B^c_d] = \delta^a_d d^{\dagger bc} - \delta^b_d d^{\dagger ac}\,. \tag{A11c}$$

From the four-dimensional nature of the twistor space it follows[22] that the internal operators are not all independent. They satisfy the nonlinear algebraic constraints[22]

$$d_{[ab} d_{c]d} = 0 \tag{A12a}$$

and

$$d_{[de} B_{f]}{}^{[a} d^{\dagger bc]} = 0\,. \tag{A12b}$$

When the number of twistor constituents is less than three, these relations are trivially satisfied and may be ignored. For a three-twistor system, only Eq. (A12b) gives a restriction. But generally both of Eqs. (A12) decrease the number of independent internal operators.

An internal translation [cf. Eq. (A5)]

$$\begin{aligned} Z'^{\alpha}_a &= Z^{\alpha}_a + \Lambda_{ab} I^{\alpha\beta} Z^{\dagger b}_{\beta}\,, \\ Z^{\dagger\prime a}_{\alpha} &= Z^{\dagger a}_{\alpha} + \Lambda^{*ab} I_{\alpha\beta} Z^{\beta}_b \end{aligned} \tag{A13}$$

affects the internal operators as follows:

$$\begin{aligned} d'_{ab} &= d_{ab}\,, \quad d^{\dagger\prime ab} = d^{\dagger ab}\,, \\ B'^a_b &= B^a_b + \Lambda_{bc} d^{\dagger ac} + \Lambda^{*ac} d_{bc}\,. \end{aligned} \tag{A14}$$

A quantum state is described by a homogeneous twistor function $F(Z^{\alpha}_a)$ with respective homogeneity degrees p_a in the twistor variables Z^{α}_a. In addition, the twistor function is to be an eigenfunction of physical operators. Further generality is achieved[9] by allowing state functions of the form $F(W^a_{\alpha}, Z^{\alpha}_b)$. This means effectively that for some twistor operators (those denoted by the kernel letter W) the representations of the twistor and the conjugate twistor are interchanged. Generally, a physical operator is constructed from W^a_{α}, Z^{α}_b, and

$$Z^{\dagger a}_{\alpha} = -\partial/\partial Z^{\alpha}_a\,, \quad W^{\dagger\alpha}_a = \partial/\partial W^a_{\alpha}\,. \tag{A15}$$

Examples of these operators are the generators of the Poincaré group of the form (A3) and the internal operators (A11). Note that the distinction between W^a_{α} and Z^{α}_a variables in the state function is merely for the purpose of exhibiting the representation properties.

The wave function conveniently describing the particle in a coordinate picture is obtained from $F(W^a_{\alpha}, Z^{\alpha}_b)$ by integrating over the light cone of space-time point $x^{AA'}$. In this process, the details of the intrinsic twistor structure are "averaged out." The method is explicitly shown on an economically manageable example, the two-twistor system, in Sec. III.

APPENDIX B: THE COLOR BASIS

Unitary space, a 3-dimensional complex manifold, has the isometry generators d^i, $d^{\dagger}_i$, and A^i_k with commutation properties (2.11). We define the non-Hermitian vector operator

$$f^i \equiv A^i_k d^k - B d^i\,, \tag{B1}$$

where $B \equiv (d^r d^{\dagger}_r)^{-1} d^i d^{\dagger}_k A^k_i$ [cf. Eq. (2.14)]. Operator f^i is left orthogonal to $d^{\dagger}_i$. This is to say that the factor f^i appears to the left of an antitriplet operator $(d^{\dagger}_i)$ in the equation $f^i d^{\dagger}_i = 0$. Care should be taken of factor ordering here because $[f^i, d^{\dagger}_k] = \delta^i_k \Delta - \Delta^i_k$, where $\Delta^i_k \equiv d^i d^{\dagger}_k$ and $\Delta \equiv \Delta^r_r$.

Furthermore, we introduce the antitriplet operator

$$s_i \equiv \epsilon_{ijk} d^j f^k\,, \tag{B2}$$

such that our vectors commute as follows

$$[f^i, d^k] = 0\,, \quad [d^i, s_k] = 0\,, \quad [s_i, s_k] = 0\,, \tag{B3}$$

but

$$[f^i, f^k] = \epsilon^{ikl} s_l\,, \quad [f^i, s_k] = d^i s_k\,. \tag{B4}$$

Operator s_i is orthogonal (both left and right) to d^i and f^i:

$$d^i s_i = 0 = f^i s_i\,. \tag{B5}$$

We refer to the set of vector operators (a triplet of triplets)

$$\{v^i_m\} \equiv \{d^i, f^i, s^{\dagger i}\} \tag{B6}$$

as the *color triplet*. The indices $m, n, \ldots$ label the colors d, f, and s as they take the values 1, 2, and 3, respectively. A notion of color has been introduced previously in strong-interaction physics with somewhat different properties.[14] This notion is saved in the present theory as an SU(3) degree of freedom independent of the flavors labeled by the indices $i, j, k, \ldots$.

Relations (B3)–(B5) are "analytic" in the operators, i.e., they do not involve adjoints. However, definition (B6) does contain the adjoint vector $s^{\dagger i}$. Consider now the adjoint triplet

$$\{v^{\dagger m}_i\} \equiv \{d^{\dagger}_i, f^{\dagger}_i, s_i\}\,. \tag{B7}$$

The vectors of the color triplet are left orthogonal to those of the adjoint triplet. The notation (B6) and (B7) has the advantage that tensors may thus be expressed compactly in terms of color components. An example: $T_{mn}{}^{p} = T_{ij}{}^{k} v^i_m v^j_n v^{\dagger p}_k$.

Some simple algebra provides the useful relations

$$f^i \Delta = \epsilon^{ijk} s_j d^{\dagger}_k\,, \tag{B8}$$

$$d^i \Delta \Lambda = -\epsilon^{ijk} s_j f^{\dagger}_k\,. \tag{B9}$$

Equations (B2), (B8), and (B9) express each of

TABLE III. The color commutators $[S_i^k = \epsilon_{ijl}\epsilon^{krs}(A_r^j + \delta_r^j B)\Delta_s^l]$.

$[d^i, d^k] = 0$	$[d^i, f^k] = 0$	$[d^i, s_k] = 0$	analytic commutators
	$[f^i, f^k] = \epsilon^{ikl} s_l$	$[f^i, s_k] = d^i s_k$	
		$[s_i, s_k] = 0$	
$[d^i, d_k^\dagger] = 0$	$[d^i, f_k^\dagger] = \delta_k^i \Delta - \Delta_k^i$	$[d^i, s^{\dagger k}] = \Delta \epsilon^{ikl} d_l^\dagger$	the nonanalytic ones
	$[f^i, f_k^\dagger] = -S_k^i + (\frac{3}{2}B + 2)(\Delta_k^i - \delta_k^i \Delta)$		
		$[f^i, s^k] = -2d^i s^{\dagger k} - \epsilon^{ikl}\Delta[d_l^\dagger(B+2) + f_l^\dagger] + \Delta\epsilon^{klm} d_l^\dagger A_m^i$	
		$[s_i, s^{\dagger k}] = \Delta(\Delta A_i^k + 2S_i^k - \delta_i^k \Delta B)$	

the color vectors in terms of a skewed product. The quantity Λ is given by

$$\Lambda = \tfrac{1}{2} C_2 - j(j+1) - \tfrac{3}{4} B(B+2) \tag{B10}$$

such that

$$f^i f_i^\dagger = \Delta\Lambda, \quad s^{\dagger i} s_i = \Delta^2 \Lambda, \tag{B11}$$

and

$$[d^i, \Lambda] = f^i. \tag{B12}$$

The color commutators with the SU(3) invariant B are

$$\begin{aligned} &[d^i, B] = \tfrac{2}{3} d^i, \\ &[f^i, B] = \tfrac{2}{3} f^i, \\ &[s^{\dagger i}, B] = -\tfrac{4}{3} s^{\dagger i}. \end{aligned} \tag{B13}$$

Tables I and III give the complete list of the orthogonality properties and commutators, respectively, of color vectors.

The color triplet is an almost complete set of vector operators. We establish this by considering the expression $s_k s^{\dagger i}$. Inserting here definition (B2) of s_k and that of s^i and using (2.25), we obtain the completeness relation

$$\delta_k^i \Delta^2 \Lambda = d^i \Delta\Lambda d_k^\dagger + f^i \Delta f_k^\dagger + s_k s^{\dagger i}. \tag{B14}$$

Hence, at those points of a representation space where $\Lambda = 0$, the color triplet and antitriplet collapse as a basis. Note that the expression (B14) for $s_k s^{\dagger i}$ has the further virtue that by its use both the commutator $[s_k, s^{\dagger i}]$ and the anticommutator $\{s_k, s^{\dagger i}\}$ are determined.

A clearcut interpretation of color vectors is obtained in terms of unitary kinematics. Consider a particle with momentum d^i and center-of-mass operator $Z^i(\tau)$. Equation (3.32) then shows that the vector f^i is proportional to the orthogonal distance of the center of mass from the origin (Fig. 4). From definition (B2), s_i is the orbital momentum vector. Thus f^i and s_i vanish for a particle moving radially in unitary space.

This interpretation is confirmed, incidentally, by the translation properties of the color vectors. The behavior of the ISU(3) infinitesimal operators under a shift of the origin $z'^i = z^i + t^i$ is given in Eqs. (2.12). Hence from (B1) and (B2) we obtain

$$\begin{aligned} &d'^i = d^i, \\ &f'^i = f^i + (d^r t^i - d^i t^r) d_r^\dagger, \\ &s_i' = s_i + \Delta\epsilon_{ijk} d^j t^k. \end{aligned} \tag{B15}$$

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