Must ultrabaric matter be superluminal?

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We examine the question of whether or not special relativity requires that the pressure must be less than the energy density of matter. To do this, we study a model of matter consisting of a classical one-dimensional lattice of point particles interacting via a potential satisfying the three-dimensional Klein-Gordon equation. Despite the fact that for this model the pressure p can exceed the energy density ρc^2, giving rise to an adiabatic sound speed $c_s = (dp/d\rho)^{1/2} > c$, and in the low-frequency limit to a group velocity $d\omega/dk > c$ and phase velocity $\omega/k > c$, for this type of lattice model, the formally calculated speed c_s is not a signal speed and we find that the true signal propagation speed $v_{\text{signal}} < c$. Thus special relativity alone does not guarantee that $p < \rho c^2$. We briefly discuss other constraints on $p(\rho)$, none of which seem sufficiently rigorous to rule out the possibility that $p > \rho c^2$ at high densities. The significance of the present result for the upper mass limit of neutron stars and the existence of black holes is also considered.

I. INTRODUCTION

The equation of state $p(\rho)$ of matter at high densities ($\rho \gg 10^{14}$ g/cm^3) is totally unknown. Nonetheless, it is often asserted that very general physical principles place rigorous restrictions on the allowed $p(\rho)$.[1,2] In this paper we examine the following question: What restriction does special relativity place on the allowed equation of state of matter at high densities? In particular, does the condition that no signal can propagate faster than the speed of light necessarily imply that $p < \rho c^2$?

It appears that in much of the literature which addresses this problem, it is tacitly assumed that, given an equation of state $p(\rho)$, the adiabatic sound speed $c_s = (dp/d\rho)^{1/2}$ necessarily represents the propagation speed v_s for a signal in the medium. Since it is v_s which is definitely constrained by special relativity, we must also examine several related questions. Given a medium with particles interacting via a potential $V(r)$, what is the connection between $V(r)$ and $p(\rho)$? If the medium is dispersive, what is the associated dispersion relation $\omega(k)$? Most crucially, what is the connection between the group velocity $v_g = d\omega/dk$, the phase velocity $v_p = \omega/k$, the adiabatic sound speed $c_s = (dp/d\rho)^{1/2}$, and the actual propagation speed of a signal v_s?

We are motivated to study these questions because of their possible applicability to the dense interiors of neutron stars, where the interaction potential between hadrons is unknown. Taking the point of view that the repulsive hard core exhibited at lower densities becomes arbitrarily rigid at higher densities, one might have that $p(\rho) \to \infty$ for finite ρ. However, one could immediately rule out such a possibility if $v_s \to \infty$ as well, assuming that special relativity and causality are still true at high densities. In the limit of indefinitely rigid interactions, matter could become incompressible. We are therefore also led to consider what equation of state could lead to the incompressible condition ρ = constant and to check if, as seems obvious *prima facie*, such a situation also requires that $v_s \to \infty$. The importance of the answers to all these questions to our understanding of the upper mass limit of neutron stars and the existence of black holes will be examined in Sec. V.

II. SOME EARLIER RESTRICTIONS ON $p(\rho)$

For a gas consisting of free, noninteracting particles in three dimensions, one can rigorously prove on the basis of special-relativistic kinematics alone that $p \leq \frac{1}{3}\rho c^2$. [In N dimensions, one has $p \leq (1/N)\rho c^2$.] Such a result also applies to a pure radiation field. However, if repulsive interactions between particles are included, this limit must be reconsidered. In 1961, Zeldovitch[3] constructed an explicit classical model of matter consisting of particles interacting via neutral-vector-meson exchange which has $p \to \rho c^2$ and $dp/d\rho \to c^2$ as $\rho \to \infty$. On the basis of this model, Zeldovitch concludes that $c_s = (dp/d\rho)^{1/2} < c$ is a rigorous upper limit to any allowed stiffness of matter. This conclusion, however, depends on equating the signal propagation speed v_s with c_s, an assumption which is not necessarily valid. Similarly, it is known that the group velocity v_g need not equal the signal velocity. For example, as first discussed by Sommerfeld and Brillouin,[4] the group velocity of *electromagnetic* waves in a dispersive medium can exceed c for frequencies near resonance. That v_s for electromagnetic waves must be less than c is obvious. This is made absolutely clear when one considers not the macroscopic index of refraction but the underlying microscopic events in which the wave is propagating at speed c between scatterings by the individual charges in the medium. One then finds directly that $v_s < c$. Because of such confusion, we wish to carefully distinguish between equations of state which are "ultrabaric," i.e., which have $p > \rho c^2$, which may or may not be physically allowed, and those which are superluminal, i.e., with $v_s > c$, which are excluded by causality.

In a series of intriguing papers, Bludman and Ruderman[5,7] and Ruderman[6] examined several models of matter in which the adiabatic sound speed c_s exceeds the speed of light. Two types of models were studied in detail: lattices of particles interacting via neutral-vector-meson exchange and a classical field model.

In the particle models, individual particles interact with each other by a short-range repulsive Yukawa interaction. Despite the fact that the particles interact through ordinary retarded neutral-vector fields, there exists a regime of high density in which $p > \rho c^2$ and in which the sound speed $c_s > c$. They conclude that such a model is noncausal by identifying the signal propagation speed with the sound speed. They conclude that "such apparently noncausal behavior (occurs) whenever the calculated self-energy of a particle exceeds its renormalized rest energy." They also conclude that Lorentz invariance alone imposes no restriction on c_s or on the ratio of the pressure to the energy density. Nonetheless, most physicists would tend to disallow such models if they did indeed violate causality (which we take here to mean that $v_s > c$). In the second paper,[6] Ruderman takes the point of view that the origin of the noncausality lies in the choice of boundary conditions imposed on the problem, much as the motion of a point electron in classical electrodynamics can show pre-acceleration (or runaway solutions). In the last paper,[7] they conclude for a quantum-mechanical model that real matter, if it is stable at very high densities, must not show "noncausal sound propagation."

The other class of models examined by Bludman and Ruderman[5] are Lorentz-invariant nonlinear field theories which, in the limit of low densities, reduce to a noninteracting Klein-Gordon field. For such models, the dispersion relation leads to a group velocity in excess of the speed of light for some frequency ranges. However, by applying the classical analysis of Brillouin and Sommerfeld to these models, Fox, Kuper, and Lipson[8] were able to show that the acoustic branch of the model is causal. However, they could say nothing about the optical branch. (Though they find that a necessary condition for causality violation is that the infinite frequency limit of the phase velocity shall exceed the speed of light, their result has no bearing on the particle models whose dispersion relation is not derived from a wave equation.)

In the following discussion, we shall only address the problem of classical particles interacting via a field. Bludman and Ruderman have previously considered a one-dimensional lattice of particles with one-dimensional interactions and a three-dimensional lattice with three-dimensional interactions. We have examined a one-dimensional lattice of particles with three-dimensional interactions, which exhibits all of the physical features of the Bludman-Ruderman models, and yet is sufficiently tractable to provide a clearer view of the origin of the apparent noncausal behavior of ultrabaric matter. We present the results of our analysis in the following section.

III. AN ULTRABARIC EQUATION OF STATE

To clarify the problems posed by the Bludman-Ruderman model, we consider a one-dimensional lattice of point particles interacting via a potential A_μ satisfying the three-dimensional Klein-Gordon equation,

$$\left[\Box^2 - \left(\frac{m^* c}{\hbar}\right)^2\right] A_\mu = -\frac{4\pi}{c} J_\mu ,$$

where $J_\mu = \rho(\vec{x},t) U_\mu$, $A_\mu = \{\phi, \vec{A}\}$, $\rho(\vec{x},t)$ is the charge density (not electric charge), m^* is the mass of the neutral vector meson mediating the repulsive interaction, and U_μ is the four-velocity.

The force experienced by a particle of charge g in the lattice parallel to its motion is

$$\vec{F}(\vec{x},t) = -g\left[\vec{\nabla}\phi(\vec{x},t) + \frac{1}{c}\frac{\partial \vec{A}(\vec{x},t)}{\partial t}\right]. \tag{1}$$

The point charges are assumed to be regularly spaced at equilibrium positions $l\vec{a}$ (l is an integer). The deviation from equilibrium of the lth particle is denoted by $\vec{x}_l(t)$ and assumed to have the standard normal-mode form

$$\vec{x}_l(t) = \vec{\epsilon}\, e^{i(\vec{k}\cdot l\vec{a} - \omega_0 t)}, \tag{2}$$

with $\vec{\epsilon}$ parallel to $\vec{a}$ and where $|\vec{\epsilon}| \ll |\vec{a}|$. Then

$$\rho(\vec{x}', t') = g \sum_{l \neq n} \delta(\vec{x}' - l\vec{a} - \vec{x}_l(t')) \tag{3}$$

and

$$\vec{J}(\vec{x}', t') = g \sum_{l \neq n} \dot{\vec{x}}'(t')\,\delta(\vec{x}' - l\vec{a} - \vec{x}_l(t')), \tag{4}$$

where we have placed the observation point on the

nth particle and have excluded its contribution to the charge and current densities.

The potentials are computed with the aid of the Green's function for the three-dimensional Klein-Gordon equation:

$$G(\vec{x}-\vec{x}',t-t')=\frac{1}{|\vec{x}-\vec{x}'|}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\times\exp\left[i\left(\frac{\omega^2}{c^2}-\mu^2\right)^{1/2}|\vec{x}-\vec{x}'|\right],$$

where $\mu\equiv m^*c/\hbar$ and where $G(r,\tau)=0$ for $r>c\tau$ ($\tau\equiv t-t'$, $r\equiv|\vec{x}-\vec{x}'|$). With this, ϕ and $\vec{A}$ are

$$\phi(\vec{x},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3x'dt'\rho(\vec{x}',t')G(\vec{x}-\vec{x}',t-t'), \tag{5}$$

$$\vec{A}(\vec{x},t)=\frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3x'dt'\vec{J}(\vec{x}',t')G(\vec{x}-\vec{x}',t-t'). \tag{6}$$

Placing the observation point at the position of the nth particle and using Eqs. (3) and (4) yields

$$\phi[\vec{x}_n(t)]=g\sum_{l\neq n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega\,dt'}{2\pi}\frac{e^{-i\omega(t-t')}\exp[-(\mu^2-\omega^2/c^2)^{1/2}|(n-l)\vec{a}+\vec{x}_n(t)-\vec{x}_l(t')|]}{|(n-l)\vec{a}+\vec{x}_n(t)-\vec{x}_l(t')|}$$

and

$$\vec{A}[\vec{x}_n(t)]=\frac{g}{c}\sum_{l\neq n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega\,dt'}{2\pi}\frac{\dot{\vec{x}}_l(t')e^{-i\omega(t-t')}\exp[-(\mu^2-\omega^2/c^2)^{1/2}|(n-l)\vec{a}+\vec{x}_n(t)-\vec{x}_l(t')|]}{|(n-l)\vec{a}+\vec{x}_n(t)-\vec{x}_l(t')|}.$$

Since we wish to obtain the force on the nth particle to first order in the deviations from equilibrium, and since that force as given by (1) is

$$\vec{F}[\vec{x}_n(t)]=-g\left\{\vec{\nabla}_{\vec{x}_n(t)}\phi[\vec{x}_n(t)]+\frac{1}{c}\frac{\partial\vec{A}[\vec{x}_n(t)]}{\partial t}\right\},$$

it is clear that $\phi[\vec{x}_n(t)]$ must be computed to second order in the $\vec{x}_l$'s while $\vec{A}[\vec{x}_n(t)]$ need only be computed to first order in the $\vec{x}_l$'s.

Expanding the quantity $|(n-l)\vec{a}+\vec{x}_n(t)-\vec{x}_l(t')|$ and using the fact that $\delta(\omega)=(1/2\pi)\int_{-\infty}^{\infty}dt'e^{i\omega t'}$ results in, to the required order in the $\vec{x}_l$'s,

$$\begin{aligned}\phi[x_n(t)]=\frac{g}{a}\sum_{l\neq n}\frac{1}{|n-l|}\Bigg\{&x_n^2(t)e^{-\mu|n-l|a}\left[\tfrac{1}{2}\mu^2+\frac{\mu}{|n-l|a}+\frac{1}{(n-l)^2a^2}\right]\\&-x_n(t)x_l(t)\exp\left[-\left(\mu^2-\frac{\omega_0^2}{c^2}\right)^{1/2}|n-l|a\right]\\&\times\left[\mu^2-\frac{\omega_0^2}{c^2}+\frac{2(\mu^2-\omega_0^2/c^2)^{1/2}}{|n-l|a}+\frac{2}{(n-l)^2a^2}\right]\Bigg\}\end{aligned}$$

and

$$A[x_n(t)]=-\frac{i\omega_0 g}{ac}\sum_{n\neq l}\frac{1}{|n-l|}x_l(t)\exp\left[-\left(\mu^2-\frac{\omega_0^2}{c^2}\right)^{1/2}|n-l|a\right],$$

where only terms containing $x_n(t)$ have been retained for $\phi[x_n(t)]$ and where the vector notation has been dropped since the motion is one dimensional.

Now using Eqs. (1) and (2) results in the dispersion relation

$$\begin{aligned}-m\omega_0^2x_n(t)=-\frac{g^2}{a}\sum_{l\neq n}\frac{1}{|n-l|}\Bigg\{&x_n(t)e^{-\mu|n-l|a}\left[\mu^2+\frac{2\mu}{|n-l|a}+\frac{2}{(n-l)^2a^2}\right]\\&-x_l(t)\exp\left[-\left(\mu^2-\frac{\omega_0^2}{c^2}\right)^{1/2}|n-l|a\right]\left[\mu^2+\frac{2(\mu^2-\omega_0^2/c^2)^{1/2}}{|n-l|a}+\frac{2}{(n-l)^2a^2}\right]\Bigg\}.\end{aligned}$$

Consider this relation for $\omega_0<\mu c$. If $\mu a\gg1$ we can sum only over nearest neighbors with negligible error. Then if we note that Eq. (2) implies

$$x_{n+1}(t)+x_{n-1}(t)=2x_n(t)\cos ka,$$

the dispersion relation becomes

$$\left[m+\frac{\mu a}{c^2}\,\frac{g^2e^{-\mu a}}{a}\,\cos ka\right]\omega_0^{\,2}\cong\frac{4g^2\mu^2e^{-\mu a}}{a}\,\sin^2(\tfrac{1}{2}ka)\,. \tag{7}$$

In the long-wavelength limit ($ka \ll 1$) we have

$$\omega_0^{\,2}\cong\frac{k^2c^2(\mu a)^2(g^2e^{-\mu a}/a)}{mc^2+(\mu a)(g^2e^{-\mu a}/a)}\,. \tag{8}$$

The phase and group velocities in this limit are

$$v_p=\frac{\omega_0}{k}=v_g=\frac{d\omega_0}{dk}=c\left[\frac{(\mu a)^2(g^2e^{-\mu a}/a)}{mc^2+(\mu a)(g^2e^{-\mu a}/a)}\right]^{1/2}.$$

If $(\mu a)(g^2e^{-\mu a}/a)\gg mc^2$, then

$$v_p=v_g=(\mu a)^{1/2}c\gg c\,.$$

The computational method used here was that employed by Bludman and Ruderman.[5] To verify the validity of this technique the calculation was repeated using the explicitly retarded, closed form of the Klein-Gordon equation Green's function[9]

$$G(\vec{x}-\vec{x}',t-t')=\frac{\delta(t-t'-|\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}-\frac{\mu J_1\{\mu c[(t-t')^2-|\vec{x}-\vec{x}'|^2/c^2]^{1/2}\}}{[(t-t')^2-|\vec{x}-\vec{x}'|^2/c^2]^{1/2}}\,u_{-1}\!\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right).$$

The results obtained by this method were identical to those obtained above.[11]

We may derive the one-dimensional "pressure" for the chain model by considering the interactions when all the particles are at rest. In this case retarded effects do not enter into the calculations.

The interaction energy between particles i and j in the chain is

$$\phi_{ij}=\frac{g^2e^{-\mu|r_i-r_j|}}{|r_i-r_j|}\,.$$

If n is the number density (number per unit length) and ϵ the energy density, then the energy per particle ϵ/n is

$$\epsilon/n=mc^2+\tfrac{1}{2}g^2\sum_l{}'\,\frac{e^{-\mu|l|a}}{|l|a}$$

$$=mc^2+g^2\sum_{l=1}^{\infty}\frac{e^{-\mu l a}}{la}\,.$$

Taking the sum yields

$$\epsilon/n=mc^2-\frac{g^2}{a}\ln(1-e^{-\mu a})\,. \tag{9}$$

But $n=1/a$ so that

$$\epsilon=\frac{1}{a}\left[mc^2-\frac{g^2}{a}\ln(1-e^{-\mu a})\right]. \tag{10}$$

Now the first law of thermodynamics $dE=TdS-PdV$ implies that

$$P=-\frac{\partial E}{\partial V}$$

for a system at zero temperature. If there are N particles in the system then

$$E=(\epsilon/n)N\,,\quad V=Na\,,$$

where E is the total energy of the system and V is the total volume (length) of the system. Then

$$P=-\frac{1}{N}\frac{\partial}{\partial a}\,[N(\epsilon/n)]=-\frac{\partial(\epsilon/n)}{\partial a}\,,$$

that is,

$$P=\frac{g^2}{a^2}\left[\frac{(\mu a)e^{-\mu a}}{1-e^{-\mu a}}-\ln(1-e^{-\mu a})\right]. \tag{11}$$

In the nearest-neighbor approximation $\mu a\gg 1$. Thus the energy density becomes

$$\epsilon\cong\frac{1}{a}\left[mc^2+\frac{g^2e^{-\mu a}}{a}\right]$$

and the pressure becomes

$$P\cong\frac{g^2}{a^2}\,(\mu a)e^{-\mu a}\,. \tag{12}$$

Now (10) implies that

$$\frac{g^2e^{-\mu a}}{a^2}\cong\epsilon-\frac{mc^2}{a}$$

so that

$$P\cong\mu a\left(\epsilon-\frac{mc^2}{a}\right). \tag{13}$$

For $\epsilon\gg mc^2/a$

$$P\cong(\mu a)\epsilon\,, \tag{14}$$

with $c_s^{\,2}=dP/d\rho=(\mu a)c^2$. Since $\mu a\gg 1$, $c_s^{\,2}\gg c^2$.

With $p>\epsilon$, it might seem that such a medium does not conserve energy when expanded from a dense state to a low-density state. That the system is conservative, however, may be seen by integrating directly the expression (11) for p from a state with $p=p_0$ with energy E_0 to $p=0$ and E

$=Nmc^2$. One will find that $\int p\,dv = E_0 - Nmc^2$. That this must be so, of course, follows from the definition of p by the first law of thermodynamics with $T=0$ and with no dissipation.

IV. ORIGIN OF THE APPARENT DISCREPANCY WITH SPECIAL RELATIVITY

In order to try to clarify the origin of the apparent noncausality exhibited by our model, let us consider a Newtonian analog (which neglects retardation effects), i.e., the familiar chain of mass points coupled by springs in Newtonian mechanics. The force on the nth mass is

$$m\ddot{x}_n(t) = -\gamma[2x_n(t) - x_{n-1}(t) - x_{n+1}(t)],$$

where γ is the spring constant. Again taking $x_n(t)$ of the form

$$x_n(t) = \epsilon e^{ikna}e^{-i\omega_0 t},$$

the dispersion relation becomes

$$m\omega_0^2 = 4\gamma\sin^2(\tfrac{1}{2}ka). \tag{15}$$

In the limit of long wavelengths ($ka \ll 1$) this becomes

$$\omega_0^2 \cong \frac{\gamma k^2 a^2}{m} = \left(\frac{\gamma a^2}{mc^2}\right)k^2c^2, \tag{16}$$

with group and phase velocities

$$v_p = \frac{\omega_0}{k} = v_g = \frac{d\omega_0}{dk} = \left(\frac{\gamma a^2}{mc^2}\right)^{1/2} c.$$

If we adjust γa^2 to be larger than mc^2, the group and phase velocities will exceed c. At this point the similarity of Eqs. (7) (for $ka \ll 1$) and (15) should be noted. The similarity is more than a coincidence. It will be shown that (7) is the result of an implicitly Newtonian calculation even though retarded Green's functions were used in deriving it.

In order to see how such a situation could arise we now examine the propagation of the force. Equation (7) is the dispersion relation for the chain *with nearest-neighbor interactions*. The force on the nth particle can only come from its interaction with the fields produced by the $(n-1)$th and $(n+1)$th particles through Eq. (1). If one considers the propagation of a disturbance (in the increasing n direction) through a chain initially at rest, it is clear that the nth particle will not experience a force until the perturbed field generated by the movement of the $(n-1)$th particle has reached it. It is therefore clear that a disturbance cannot propagate through a chain at rest faster than the interparticle force can be transmitted: The force propagation speed will set the upper limit for the propagation speed. Thus we consider first the classical analog of the one-dimensional Klein-Gordon field:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_A{}^2}\frac{\partial^2}{\partial t^2} - \mu^2\right)z(x,t) = 0.$$

This is the wave equation for the transverse displacement $z(x,t)$ of a stretched string embedded in an elastic medium which provides a restoring force proportional to z. In this case, z is the analog of either the scalar potential ϕ or the vector potential A.

When one end (at $x=0$) of the string is driven with a force $F(t) = F_0 e^{-i\omega_0 t}$ the resulting displacements are proportional to[10]

$$z(x,t) \propto e^{-i\omega_0 t}\exp\left[i\left(\frac{\omega_0{}^2}{c_A{}^2} - \mu^2\right)^{1/2}x\right], \quad \text{for } \omega_0 > \mu c_A$$

$$z(x,t) \propto e^{-i\omega_0 t}\exp\left[-\left(\mu^2 - \frac{\omega_0{}^2}{c_A{}^2}\right)^{1/2}x\right], \quad \text{for } \omega_0 < \mu c_A.$$

For $\omega_0 > \mu c_A$ there is wave motion with phase velocity

$$v_p(\omega_0) = \frac{c_A}{(1 - \mu^2 c_A{}^2/\omega_0{}^2)^{1/2}}.$$

However, for $\omega_0 < \mu c_A$ there is no wave motion. The motion of the string is damped in space and is everywhere in phase with the driving force. That is, no retardation of any kind is evident, and it appears that the force is instantaneously transmitted to every point on the string. This is a general feature of the one- and three-dimensional Klein-Gordon equations for a steady sinusoidal force of frequency $\omega_0 < \mu c_A$. *Following Brillouin and Sommerfeld,*[4] *we take a signal to be a wave train which vanishes prior to a starting time* t_0. Thus we consider the response of the string to a signal of the same frequency:

$$F(t) = \begin{cases} 0, & t < t_0 \\ F_0 e^{-i\omega_0 t}, & t > t_0 \end{cases} \text{ at } x = 0.$$

Thus

$$F(x',t') = F_0 e^{-i\omega_0 t'}u_{-1}[t' - t_0]\,\delta(x')$$

and

$$z(x,t) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dt' F(x',t')G(x-x',t-t'),$$

where the one-dimensional Klein-Gordon equation Green's function is

$$G(x-x',t-t')=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega(t-t')}\exp\left[-\left(\mu^2-\frac{\omega^2}{c_A{}^2}\right)^{1/2}|x-x'|\right].$$

Thus

$$z(x,t)\propto\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx'dt'd\omega}{2\pi}\,e^{-i\omega(t-t')}\exp\left[-\left(\mu^2-\frac{\omega^2}{c_A{}^2}\right)^{1/2}|x-x'|\right]e^{-i\omega_0t'}u_{-1}(t'-t_0)\delta(x')$$

$$=\int_{t_0}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega dt'}{2\pi}\,e^{-i\omega t}\exp\left[-\left(\mu^2-\frac{\omega^2}{c_A{}^2}\right)^{1/2}x\right]e^{i(\omega-\omega_0)t'}.$$

We now assume $t_0>0$ and introduce a convergence factor in order to perform the integration over t'. When the resulting contour integral is evaluated, it is found that

$$z(x,t)=0 \quad \text{for } t<t_0+x/c_A$$

and

$$z(x,t)\propto e^{-i\omega_0t}\exp\left[-\left(\mu^2-\frac{\omega_0{}^2}{c_A{}^2}\right)^{1/2}x\right]$$

$$\text{for } t>t_0+x/c_A\,.$$

Thus the complete solution is

$$z(x,t)\propto e^{-i\omega_0t}\exp\left[-\left(\mu^2-\frac{\omega_0{}^2}{c_A{}^2}\right)^{1/2}x\right]\times u_{-1}(t-t_0-x/c_A)\,.$$

Now we see that the displacement field propagates down the string at speed c_A. Ahead of this wavefront there is no motion. Behind the wavefront the motion of the string is in phase with the force.

This behavior is the time varying analog of the well-known case in electromagnetism of the field of a constantly moving charge. If the charge has been in constant, uniform motion for a long period of time, then the electric field, even at great distances, will point directly at the present position of the charge and not at its retarded position. The field appears to have instantaneous knowledge of the position of the charge. However, if the charge is suddenly accelerated, the changed field will propagate outward at speed c.

Before proceeding to an example for the three-dimensional Klein-Gordon equation it will be instructive to calculate the dispersion relation for the Klein-Gordon field itself:

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\mu^2\right)\phi(x,t)=0\,.$$

With $\phi(x,t)\;\alpha\;e^{i(kx-\omega t)}$ we obtain

$$-k^2+\frac{\omega^2}{c^2}-\mu^2=0\,.$$

This yields a group velocity

$$v_g(\omega)=\frac{d\omega}{dk}=\frac{ck}{(k^2+\mu^2)^{1/2}}=c\left(1-\frac{\mu^2c^2}{\omega^2}\right)^{1/2}.$$

For $\omega>\mu c$ the group velocity is less than c but for $\omega<\mu c$ it becomes imaginary and is a clear indication of anomalous behavior. The dispersion relation here is similar to that for propagation in a waveguide and $\omega=\mu c$ corresponds to the cutoff frequency below which the waves are evanescent (damped).[12]

To clarify the situation further we present an example for the three-dimensional Klein-Gordon equation. It is a somewhat unphysical example but is mathematically convenient and illustrates the point. Consider a point charge with $\omega_0<\mu c$:

$$\rho(\vec{x}',t')=ge^{i\omega_0t'}\delta(\vec{x}')\,.$$

Then, using the Green's function as before,

$$\phi(\vec{x},t)=\frac{g}{|\vec{x}|}\,e^{i\omega_0t}\exp\left[-\left(\mu^2-\frac{\omega_0{}^2}{c^2}\right)^{1/2}|\vec{x}|\right].$$

Now consider the response to a point charge initially at rest which begins to oscillate at time t_0:

$$\rho(\vec{x}',t')=ge^{i\omega_0t'}\delta(\vec{x}')u_{-1}(t'-t_0)\,,\quad t_0>0\,.$$

In this case

$$\phi(\vec{x},t)=\frac{g}{|\vec{x}|}\,e^{i\omega_0t}\exp\left[-\left(\mu^2-\frac{\omega_0{}^2}{c^2}\right)^{1/2}|\vec{x}|\right]\times u_{-1}\left(t-t_0-\frac{|\vec{x}|}{c}\right).$$

Thus, the force is actually transmitted at speed c. Clearly, since a signal cannot be propagated through the lattice any faster than the interparticle forces can be transmitted, the upper limit of the signal propagation speed through the lattice is c.

If we solve a normal-mode problem with a frequency $\omega<\mu c$ with normal-mode displacements proportional to $\exp(-i\omega_0t)$, we will inevitably obtain a classical Newtonian dispersion relation where the group velocity can exceed c because the normal-mode displacement form does not have the character of a signal and will result in an *apparent* infinite interaction speed for the force. Then the normal-mode problem is not an initial-value problem. In the analysis of Sec. III a dispersion relation that was valid for low frequencies only was obtained ($\omega \ll \mu_0 c$). A signal as defined by Brillouin and Sommerfeld will contain all components of frequency because there is a well defined time at which the signal starts. Thus, the behavior of the system for high frequencies is necessary in order to characterize the propagation of a transient disturbance, i.e., a signal.

The examples of a point charge previously considered illustrate the problem nicely. For the first case of a steady harmonic oscillation, the resulting field appears to be transmitted instantaneously (i.e., with no phase delay). However, when the same charge is started from rest, the disturbance in the field propagates at c and its causal nature is evident.

The normal-mode displacement form [Eq. (2)] is analogous to the former case of the point charge that has been harmonically oscillating from $t=-\infty$ and gives rise to a field which *apparently* propagates instantaneously. Nonetheless, *signals* will propagate with a velocity $v_s \leq c$.

One may still wonder how the adiabatic sound speed can exceed the speed of light, and yet signals propagate at speed less than c. One reason may be that the $p(\rho)$ relation arises from a static calculation, ignoring the dynamics of the medium. One would get the same equation of state whether one used Newtonian theory or special relativity. However, when we consider the dynamics of the system we must take new considerations into account, namely the retardation of the interaction forced upon us by special relativity. The notion that c_s is a signal propagation speed is a carryover from Newtonian hydrodynamics. In that case, one has assumed infinite interaction speed but finite temperature, so that the characteristic speed of a sound wave is tied to the thermal velocity, which itself is a function of the static thermodynamic properties of the material. Therefore, the static and dynamic calculations give the same result in Newtonian hydrodynamics. In our model, we have finite interaction speed and zero temperature. Thus in our model the adiabatic sound speed is not a dynamically meaningful speed, but only a measure of the local stiffness.

Another point is that a lattice does not have an infinite range of allowed frequencies of vibration, while it does have an infinite range of wave vectors. The continuum limit of the lattice model does have an infinite range of allowed frequencies of vibration. If we consider a signal to be a waveform which vanishes prior to some time t_0, then its Fourier spectrum will contain components at all frequencies. This suggests that the group velocity v_g computed from a lattice dispersion relation is not capable of accurately giving the velocity of propagation of the waveform because the dispersion relation lacks the necessary high-frequency information which defines the signal boundary. A continuum dispersion relation does not have this problem. So the apparent disagreement between one's intuitive association of c_s with v_s may be due to the misapplication of continuum notions to a lattice.

V. IMPLICATIONS FOR THE NEUTRON-STAR UPPER MASS LIMIT

In the preceding section, we have shown, at least classically, that one can have $p=\mu a \rho c^2 \equiv \alpha\rho c^2$ with $\alpha > 1$. For the particular model under discussion above, we have $dp/d\rho = \alpha c^2 = \text{constant}$. Now given the equation of state of matter and a theory of gravity, one can compute the mass of an object in hydrostatic equilibrium between the pressure support and the gravitational attraction of the matter. For objects of a solar mass or greater $M_\odot \simeq 2\times 10^{33}$ g and mean density greater than nuclear density $\rho \equiv \rho_N = 2\times 10^{14}$ g/cm^3, Newtonian gravitational theory must be replaced by a more correct theory of gravity to determine the structure of the resulting configurations since GM/Rc^2 is large.

As is well known, for a given equation of state the hydrostatic equilibrium equation can be integrated (with the central density as a parameter) to yield a sequence of objects of varying mass.[13] At present, there appear to be only two physically motivated viable theories of gravity: general relativity and the bimetric theory of Nathan Rosen.[14] (By viable, we mean internally consistent and in agreement with all current observational tests.) In both theories, such a procedure leads to a mass-central density relation containing a maximum. To the left of this peak, all stars are stable to small perturbations; to the right, unstable. The value of the maximum mass depends crucially on the equation of state and the theory of gravitation.[15]

In general relativity, we have previously shown[16] that, for an equation of state of the form $p=\alpha\rho c^2$ above some density ρ_M (then matched to a softer equation of state with $p \ll \rho c^2$), for infinite central density

$$M=\left(\frac{c^2}{G}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\alpha}{\alpha^2+6\alpha+1}\right)^{3/2}\frac{1}{{\rho_M}^{1/2}} .$$

This function has a maximum for $\alpha=1$. For finite central densities, we found the same result: If the adiabatic sound speed is a constant, the maximum upper mass limit occurs if $\alpha=1$. Assuming such a stiff equation of state to be applicable above nuclear density [we actually use $p=c^2(\rho-\rho_M)+p_M$

in order to match onto the equation of state at lower densities], we found $M_{max} \lesssim 4.8\ M_\odot$ if $\rho_M = \rho_N$.

In the bimetric theory, however, the situation is quite different.[17] Despite the fact that for a given equation of state there always exists some upper mass limit, if one chooses $p > \rho c^2$, M_{max} is larger than it is for $p < \rho c^2$. An approximate scaling argument (also applicable for Newtonian gravitation) shows that $M_{max} \propto \alpha^{3/2}$. Thus in the bimetric theory, by increasing $dp/d\rho$ without limit, the maximum mass of neutron stars can become arbitrarily large. Therefore, at least in the bimetric theory, if matter at high densities is very rigid, collapse to a singularity can be naturally avoided.

In connection with the present discussion, which allows the existence of ultrabaric matter, we wish to add a remark about the extreme limit of the present discussion. For incompressible matter $\rho =$ constant. It is of interest in general relativity because it allows the hydrostatic equilibrium equation to be integrated exactly.[18] Furthermore, it places an absolute upper limit on the allowed mass for equilibrium configurations.[19] It is easy to show that, for an object composed of matter which is incompressible above a density ρ_0, the mass of the object must be

$$M \leqq \left(\frac{8c^2}{9G}\right)^{3/2}\left(\frac{3}{32\pi}\right)^{1/2}\frac{1}{\rho_0^{1/2}} \cong 8.07\left(\frac{\rho_N}{\rho_0}\right)^{1/2} M_\odot .$$

The incompressible model has never been derived as the general limit of an explicit equation of state. (It is not, for example, the limit of $p = \alpha\rho^n$ with either α, n, or both $\to \infty$.) And since $\rho =$ constant implies that $dp/d\rho$ is infinite, the mass corresponding to this case is not generally considered to be a physical possibility. In view of the result of the previous section that $dp/d\rho > c^2$ does not necessarily violate causality, it is perhaps possible that a model of matter can be constructed to yield the equation of state in question with $dp/d\rho > c^2$ but with a subluminal signal propagation speed. In that event, the physically allowable upper mass limit for a neutron star in general relativity would be ~8 $M_\odot$. In the bimetric theory, however, the upper mass limit would be infinite.

We would therefore like to present an equation of state that will yield incompressibility in a suitable limit. Consider the equation of state

$$p(\rho) = \frac{\epsilon\rho_0^{1/2}c^2}{(\rho_0 - \rho)^{1/2}} \tag{17}$$

for $\rho \leq \rho_0$. If we assume that $\epsilon \ll \rho_0$, then this equation of state will yield an enormous variation of pressure for a small change in density. For example if $\rho = \rho_0$, p is infinite, while if $\rho = \rho_0 - \epsilon$, then

$$p(\rho_0 - \epsilon) = \frac{\epsilon\rho_0^{1/2}c^2}{\epsilon^{1/2}} = \left(\frac{\epsilon}{\rho_0}\right)^{1/2}\rho_0 c^2 \ll \rho_0 c^2 .$$

To numerically compute a model neutron star, one chooses a $\rho(0)$ very slightly less than ρ_0 and integrates the hydrostatic equilibrium equation radially outward until $p = (\epsilon/\rho_0)^{1/2}\rho_0 c^2$.

Now if we take the limit $\epsilon \to 0$, we are integrating the hydrostatic equilibrium equation from $p = \infty$ at $r = 0$ to $p \cong 0$ at the star surface. Our numerical calculations for $\epsilon = 0.01\rho_0$ gave $M = 7.8\ M_\odot$ at a radius of 26.7 km, close to the exact values for $\rho = \rho_N$. That this equation of state really does result in an incompressible star can be seen by examining the general relativistic hydrostatic equilibrium equation[13]

$$\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2}\,\frac{[1 + p(r)/\rho(r)c^2][1 + 4\pi r^3 p(r)/M(r)c^2]}{1 - 2GM(r)/rc^2} .$$

If we set $dp/dr = (d\rho/dr)(dp/d\rho)$ this becomes

$$\frac{d\rho}{dr} = -\frac{GM(r)\rho(r)[1 + p(r)/\rho(r)c^2][1 + 4\pi r^3 p(r)/M(r)c^2]}{r^2(dp/d\rho)[1 - 2GM(r)/rc^2]} . \tag{18}$$

If ρ is approximately constant, then $M(r) \cong 4\pi r^3\rho_0/3$ and using (17), Eq. (18) becomes

$$\frac{d\rho}{dr} \cong -\frac{8\pi G\rho_0^{3/2} r}{3c^2}\,\frac{1}{1 - 8\pi G\rho_0 r^2/3c^2}\,\frac{[(\rho_0 - \rho) + (\epsilon/\rho_0^{1/2})(\rho_0 - \rho)^{1/2}][(\rho_0 - \rho)^{1/2} + 3\epsilon/\rho_0^{1/2}]}{\epsilon} .$$

$|d\rho/dr|$ will be at a maximum when $r \cong R$ and $\rho \cong \rho_0 - \epsilon$. Then

$$\frac{d\rho}{dr} \cong -\frac{2GM}{Rc^2}\left(\frac{1}{1 - 2GM/Rc^2}\right)\frac{\rho_0}{R}\left[1 + \left(\frac{\epsilon}{\rho_0}\right)^{1/2}\right]\frac{\epsilon^{1/2}}{\rho_0^{1/2}}\left[1 + 3\left(\frac{\epsilon}{\rho_0}\right)^{1/2}\right]$$

$$\cong -\left(\frac{\rho_0}{R}\right)\left(\frac{\epsilon}{\rho_0}\right)^{1/2} .$$

As $\epsilon \to 0$ the density gradient vanishes and the exact incompressible case is recovered.

VI. CONCLUSIONS

The question of whether or not the pressure can exceed the energy density in physically realizable matter, of course, must be asked of parts of physics other than special relativity alone. However, none of the discussions we have been able to find in the literature seem to be of a sufficiently general nature to rigorously guarantee that the pressure of matter must be less than its energy density. For example, Bludman and Ruderman showed[7] only that "a quantum version of the same model, if it is stable against spontaneous pair production, can be neither ultrabaric nor superluminal (i.e., $p < \rho c^2$ and $dp/d\rho < c^2$) if, at high density, the correlation energy increases faster than the number of particles." Assuming both special relativity and conservation of energy, and even considering what appears to be only the kinetic pressure of a one-dimensional lattice, Geroch and Hegyi[20] were unable to restrict pressure for matter at high densities to $p < \rho c^2$.

With the restriction removed that the energy density exceed the pressure, neutron stars may exist with indefinitely large mass, avoiding the necessity of the ultimate evolution of sufficiently massive stars to physical singularities.

In conclusion we have shown that a particular classical model of matter with $p > \rho c^2$ at high densities has a signal propagation speed $v_s < c$. Therefore, ultrabaric matter need not be noncausal and is not excluded by special relativity.

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