Relativistic paramagnetism: Quantum statistics

A. E. Delsante
School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia

N. E. Frankel
School of Physics, University of Melbourne, Melbourne, Australia
and Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, United Kingdom
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The statistical mechanics of a noninteracting, ultradegenerate system of relativistic, spin-1/2, magnetic moments in an arbitrarily strong magnetic field is studied.

I. INTRODUCTION

A nonrelativistic, quantum-mechanical treatment of Pauli paramagnetism was first given by Pauli[1] in 1927. A relativistic treatment using classical statistical mechanics was not given until much later, in a paper by Frankel *et al.*[2] in 1967. In that paper the quantum-mechanical treatment was suggested, and we present that treatment here.

We will discuss the quantum statistical mechanics of a gas of relativistic, ultradegenerate ($T=0°$K), noninteracting spin-$\frac{1}{2}$ magnetic moments in a magnetic field. Expressions for the Fermi energy and total internal energy will be given exactly where possible or in some appropriate asymptotic limit otherwise.

Apart from extending the nonrelativistic statistical mechanics, an additional motivation for this work springs from recent interest in the structure of pulsars.[3] It is now generally agreed that pulsars are rapidly rotating neutron stars supporting strong magnetic fields of the order of 10^{13} G at the surface[4]; the size of the field in the interior is unknown and could be even larger.

In turn, neutron stars[5] themselves are thought to consist mainly of dense neutrons, whose gravitational collapse is prevented by their degeneracy pressure, with a thin crust of ordinary matter and a core whose composition is doubtful. In view of the recent interest in the possible existence of black holes, the problem of the maximum mass of a neutron star has attained new importance, since a knowledge of this maximum mass would enable astronomers to distinguish between black holes and neutron stars. This problem was first considered in the work of Oppenheimer and Volkoff,[6] who used the equation of state for relativistic, noninteracting neutrons to calculate the maximum mass of a simple model neutron star. Since then much work has gone into obtaining more accurate equations of state for dense matter, but the problem of the maximum mass is still not completely settled.[7] Furthermore, the effect of a very strong magnetic field on the equation of state has not been treated.[8]

The magnetic energy of a neutron in a field of the order of 10^{13} G is approximately 10^{-7} of its rest-mass energy and, therefore, makes a completely negligible contribution to the equation of state of a neutron star. What is more, the field is only known to exist at the surface of the star. However, these known physical systems stimulate the interesting question as to what would be the equation of state of a relativistic neutron star where the magnetic field was much more intense and, furthermore, where this intense field permeated throughout the star. We will obtain expressions for the total energy which would enable an equation of state to be obtained and used in future (numerical) calculations of the mass-radius relationship of such a highly idealized model magnetic neutron star. We have reported some of the preliminary results in this paper elsewhere.[9]

II. THE SINGLE-PARTICLE ENERGY SPECTRUM

The Dirac equation for the system is

$$(i\hbar c\partial\!\!\!/ + \tfrac{1}{2}\mu\sigma_{\mu\nu}F^{\mu\nu} - mc^2)\Phi = 0\,,$$

which has been solved[2] to yield the following positive-energy levels:

$$E(\vec{p},s) = [p^2c^2 + \mu^2B^2 + m^2c^4 + 2\mu Bs(p^2c^2\sin^2\theta + m^2c^4)^{1/2}]^{1/2}\,, \qquad (1)$$

where m is the mass of the particle, $s=\pm1$ is the spin quantum number, θ is the angle between the momentum $\vec{p}$ and the magnetic field B which we take to be uniform and in the z direction, and μ is the magnetic moment of the particle.

We note that Eq. (1) can also be obtained from the energy levels of a charged fermion (e.g., electron) with an anomalous magnetic moment in a magnetic field via the following procedure: The energy levels of such a particle are[10]

$$E=(p_z^2c^2+\{mc^2[1+(B/B_c)(2n+s+1)]^{1/2}+s\bar{\mu}B\}^2)^{1/2}, \tag{2}$$

where $B_c=m^2c^3/e\hbar$, $n=0,1,2,\ldots$, and $\bar{\mu}$ is the anomalous part of the particle's magnetic moment. If we let the charge of the particle go to zero, then the magnetic moment becomes entirely anomalous. There are no Landau levels for a neutral particle, so to recover (1) from (2) we note that the limit must be taken such that $(B/B_c)(2n+s+1)\to p_\perp^2/m^2c^2$. We then obtain from (2)

$$E=[p^2c^2+m^2c^4+\mu^2B^2+2\mu Bs(p_\perp^2c^2+m^2c^4)^{1/2}]^{1/2},$$

which directly gives (1) since $p_\perp^2=p^2\sin^2\theta$ and now $\bar{\mu}=\mu$.

The spectrum given in (1) has some very interesting properties. Let $E_0=mc^2$ and $a=|\mu B|$ and consider the spectrum for $s=+1$, which corresponds to particles with spin up (since μ for a *neutron* is negative):

$$E(\vec{p},1)=[p^2c^2+a^2+E_0^2-2a(p^2c^2\sin^2\theta+E_0^2)^{1/2}]^{1/2}. \tag{3}$$

Equation (3) is displayed in Fig. 1 for various values of a and θ as a function of p, along with the corresponding equation for spin down. We see that for $a<E_0$ the spectrum is monotonically increasing (this is, of course, always true for spin down). However, for $a>E_0$ and $\theta\neq\pi/2$ there is a smooth minimum which becomes a cusp at $\theta=\pi/2$. In fact, at $\theta=\pi/2$ the energy of the particle is zero for $pc=(a^2-E_0^2)^{1/2}$. Hence at zero temperature we expect the particles to fill the levels starting at the lowest energy—that is, zero. Thus the intense magnetic field depresses the single-particle energies for spin-up particles, with maximum depression occurring for particles moving at right angles to the field. The consequences of this peculiar behavior for the Fermi energy will be quite marked, as we shall see.

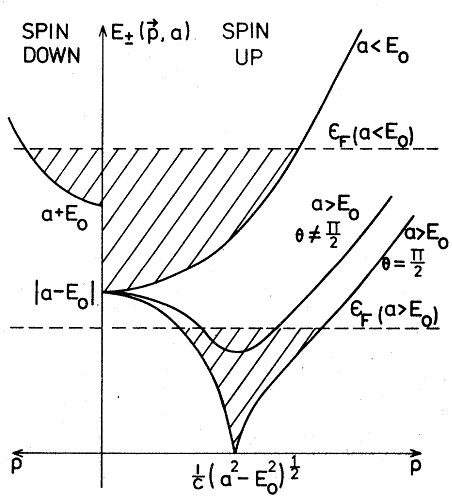

FIG. 1. Single-particle energy and typical Fermi energies for spin-up and spin-down particles as a function of momentum p.

The fact that for some momenta the energy difference between positive- and negative-energy states can be considerably less than $2E_0$, and in a particular case, zero, implies that pair production is possible. The possibility of pair production for charged fermions with an anomalous magnetic moment has been discussed by O'Connell[11] and Chiu *et al.*[12] The latter have shown that when pair production does occur, it does not do so at the expense of the magnetic field energy, but requires some thermal energy (however small). This is so for our spectrum (1), with the exception that the depression of the spin-up energy states occurs for all $a>E_0$, which is not the case for (2). We will have more to say about the spectrum and its properties in Sec. VII.

III. THE FERMI ENERGY

The Fermi energy ϵ_F is obtained from the number equation

$$N=\sum_{\vec{p},s}n_{\vec{p},s}=\sum_{\vec{p}}n_{\vec{p},1}+\sum_{\vec{p}}n_{\vec{p},-1}=N_++N_-,$$

where $N_\pm$ is the number of spins up or down and $n_{\vec{p},s}$ is taken to be the $T=0$°K Fermi-Dirac distribution function. Hence, in the thermodynamic limit where the sums over $\vec{p}$ may be taken as integrals we have

$$N_\pm=\frac{2\pi V}{(2\pi\hbar)^3}\int\int p^2dp\sin\theta\,d\theta, \tag{4}$$

where the appropriate limits on p and θ are obtained from the constraint condition

$$[p^2c^2+E_0^2+a^2\mp(p^2c^2\sin^2\theta+E_0^2)^{1/2}]^{1/2}\leq\epsilon_F. \tag{5}$$

Solving (5) yields the following integrals:

$$N_+=\frac{4\pi V}{(2\pi\hbar c)^3}\left(\int_{p_1}^{p_2}p^2dp\int_{\arcsin\alpha}^{\pi/2}\sin\theta\,d\theta+\int_0^{p_1}p^2dp\int_0^{\pi/2}\sin\theta\,d\theta\right), \tag{6}$$

$$N_-=\frac{4\pi V}{(2\pi\hbar c)^3}\left(\int_{\bar{p}_1}^{\bar{p}_2}p^2dp\int_0^{\arcsin\alpha}\sin\theta\,d\theta+\int_0^{\bar{p}_1}p^2dp\int_0^{\pi/2}\sin\theta\,d\theta\right), \tag{7}$$

where

$$p_1 = [\epsilon_F^2 - (a - E_0)^2]^{1/2}, \quad p_2 = [(\epsilon_F + a)^2 - E_0^2]^{1/2},$$

$$\bar{p}_1 = [(\epsilon_F - a)^2 - E_0^2]^{1/2}, \quad \bar{p}_2 = [\epsilon_F^2 - (a + E_0)^2]^{1/2},$$

and

$$\alpha = \frac{1}{p}\left[\left(\frac{p^2 + a^2 + E_0^2 - \epsilon_F^2}{2a}\right)^2 - E_0^2\right]^{1/2}.$$

Note that pc has been replaced by p everywhere.

Integration gives

$$N_+ = \frac{2\pi V}{(2\pi\hbar c)^3}\left\{\tfrac{1}{3}[\epsilon_F^2 - (a - E_0)^2]^{1/2} \times [2\epsilon_F^2 + (a - E_0)(a + 2E_0)] + a\epsilon_F^2\left[\frac{\pi}{2} + \arcsin\left(\frac{a - E_0}{\epsilon_F}\right)\right]\right\}, \tag{8}$$

$$N_- = \frac{2\pi V}{(2\pi\hbar c)^3}\left\{\tfrac{1}{3}[\epsilon_F^2 - (a + E_0)^2]^{1/2} \times [2\epsilon_F^2 + (a + E_0)(a - 2E_0)] - a\epsilon_F^2\left[\frac{\pi}{2} - \arcsin\left(\frac{a + E_0}{\epsilon_F}\right)\right]\right\}. \tag{9}$$

The equation for ϵ_F is of course obtained by adding N_+ and N_-. Note that for N_- to exist we must have $\epsilon_F \geq a + E_0$, and for N_+ to exist as given by (8) we must have $\epsilon_F \geq |a - E_0|$; in fact, these conditions were assumed in obtaining the limits in the integrals (6) and (7). If we now put $\epsilon_F = a + E_0$ in (9), we see that $N_- = 0$ and therefore $N_+ = N$. Hence there is some value of a, say a_0, for which $N_- = 0$ and $\epsilon_F = a_0 + E_0$. The equation for a_0 is clearly obtained by putting $a = a_0$, $\epsilon_F = a_0 + E_0$, and $N_+ = N$ into (8), for which we then obtain

$$N = \frac{2\pi V}{(2\pi\hbar c)^3}\left\{\tfrac{2}{3}(a_0 E_0)^{1/2}(3a_0^2 + 5a_0 E_0) + a_0(a_0 + E_0)^2\left[\frac{\pi}{2} + \arcsin\left(\frac{a_0 - E_0}{a_0 + E_0}\right)\right]\right\}. \tag{10}$$

Thus as a increases from zero (where $N_+ = N_- = N/2$), N_- decreases until a reaches a_0, where $N_- = 0$ and all the spins are up. Then as a increases beyond a_0, the appropriate equation for ϵ_F must be (8) with $N_+ = N$, and where now $|a - E_0| \leq \epsilon_F \leq a + E_0$. So far this is strictly analogous to the behavior of the corresponding nonrelativistic equations.[13]

However, (8) must eventually run into trouble, since given that N is constant, we see that as a increases, ϵ_F must decrease to maintain the right-hand side of (8) constant. However, ϵ_F must be larger than $|a - E_0|$ for (8) to be valid, and so there must be some value of a, say a_1, for which ϵ_F reaches its minimum value $|a_1 - E_0|$.

The equation for a_1 is obtained simply by putting $N_+ = N$, $a = a_1$, and $\epsilon_F = |a_1 - E_0|$ in (8) [note that, in fact, we must have $a_1 > E_0$, otherwise (8) would give $N = 0$]:

$$\frac{N}{V} \equiv \rho = \frac{2\pi^2 a_1}{(2\pi\hbar c)^3}(a_1 - E_0)^2. \tag{11}$$

Beyond a_1, ϵ_F must be less than $a - E_0$, and (8) cannot be used. We must return to (5) and resolve it under the condition $\epsilon_F \leq a - E_0$, $a > E_0$. When this is done we obtain the integral

$$N = \frac{4\pi V}{(2\pi\hbar c)^3}\int_{\bar{p}_1}^{p_2} p^2 dp \int_{\arcsin\alpha}^{\pi/2} \sin\theta\, d\theta. \tag{12}$$

The derivation of (12) will be discussed fully in connection with the total energy in Sec. IV. Integration gives

$$\epsilon_F = \left[\frac{(2\pi\hbar c)^3 \rho}{2\pi^2 a}\right]^{1/2}. \tag{13}$$

Collecting the results, we have the following:

(a) For $0 \leq a \leq a_0$, where a_0 is given by (10), ϵ_F is given by (8) and (9).

(b) For $a_0 \leq a \leq a_1$, where a_1 is given by (11), ϵ_F is given by (8) with $N_+ = N$.

(c) For $a \geq a_1$, ϵ_F is given by (13).

Note that for $a \geq a_1$, ϵ_F does not depend on E_0, and that as $B \to \infty$, $\epsilon_F \to 0$. The reason for the latter behavior can be seen from Fig. 1. As noted before, for $a < E_0$ the energy spectrum is monotonically increasing for both spin up and spin down, and so $\epsilon_F \geq |a - E_0|$. Once a is greater than E_0, however, the spectrum for spin up has a local minimum and ϵ_F can decrease below $a - E_0$, trapping particles in this minimum. Eventually, as a approaches infinity, particles will be trapped in the cusp at $p = (a^2 - E_0^2)^{1/2}/c$, where they will be confined to an increasingly narrow range of p values near $(a^2 - E_0^2)^{1/2}/c$ and θ values near $\pi/2$.

Equations (8), (9), and (13) show that ϵ_F and its first derivative are monotonically decreasing functions of a, with ϵ_F at $a = 0$ being given by $[E_0^2 + (\hbar c)^2(3\pi^2\rho)^{2/3}]^{1/2}$. It is interesting to note that there will be a discontinuity in some higher derivative of ϵ_F at $a = a_1$.

We need to check that (8) reduces to the nonrelativistic, weak-field result, namely

$$N_+ = \frac{4\pi V}{3(2\pi\hbar c)^3}[2E_0(\epsilon_F + a)]^{3/2}. \tag{14}$$

To do this, we note that in (8), ϵ_F includes the rest-mass energy, whereas ϵ_F in (14) does not. Thus we replace ϵ_F by $\epsilon_F + E_0$ in (8), assume $a \ll E_0$, $\epsilon_F \ll E_0$ and expand the arcsin function appropriately. This then recovers (14), and similarly for N_-.

IV. THE TOTAL ENERGY

The total internal energy of the system is given by

$$U=\sum_{\vec{p},s} E(\vec{p},s) n_{\vec{p},s}\,.$$

Taking the $\vec{p}$ sums to integals and writing $U=[2\pi V/(2\pi\hbar c)^3]\,(I_+ + I_-)$, we obtain the following.

(i) For $0 \le a \le a_0$:

$$\begin{aligned}I_+ = &\frac{1}{3\epsilon_F}[\epsilon_F^2-(a-E_0)^2]^{1/2}[\tfrac{1}{3}\epsilon_F^2(17a^2+6aE_0-3E_0^2)+\tfrac{4}{3}a^2(a+2E_0)(a-E_0)+2\epsilon_F^4]\\ &+\tfrac{4}{3}a\epsilon_F(a^2+\epsilon_F^2)\left[\arcsin\left(\frac{a-E_0}{\epsilon_F}\right)+\frac{\pi}{2}\right]-\tfrac{1}{3}(a^2+E_0^2)^2\ln\left|\frac{\epsilon_F+[\epsilon_F^2-(a-E_0)^2]^{1/2}}{a-E_0}\right|\\ &-\frac{2}{3a}\int_{p_1}^{p_2} p\,dp[(p^2+E_0^2)^{1/2}+a]\{A^2E(\delta_1,r)-[(p^2+E_0^2)^{1/2}-a]^2F(\delta_1,r)\}\\ &-\frac{2}{3a}\int_0^{p_1} p\,dp[(p^2+E_0^2)^{1/2}+a]\{A^2E(\delta_2,r)-[(p^2+E_0^2)^{1/2}-a]^2F(\delta_2,r)\},\end{aligned}\tag{15}$$

where

$$\delta_1=\arcsin\left\{\frac{1}{2\epsilon_F^2}[(p^2+E_0^2)^{1/2}+a]^2\left[1-\left(\frac{E_0^2+\alpha^2p^2}{p^2+E_0^2}\right)^{1/2}\right]\right\}^{1/2},$$

$$\delta_2=\arcsin\left\{\frac{[(p^2+E_0^2)^{1/2}+a]^2[1-E_0/(p^2+E_0^2)^{1/2}]}{2[p^2+(a-E_0)^2]}\right\}^{1/2},$$

$$r=\frac{2\sqrt{a}(p^2+E_0^2)^{1/4}}{a+(p^2+E_0^2)^{1/2}},\quad A^2=p^2+a^2+E_0^2\,.$$

p_1, p_2, and α have been defined in Sec. III and F and E are elliptic integrals of the first and second kind, respectively. Also,

$$\begin{aligned}I_- = &\tfrac{1}{6}\epsilon_F\,[\epsilon_F^2-(a+E_0)^2]^{1/2}[2\epsilon_F^2+(a+E_0)(3a-E_0)]\\ &+\tfrac{2}{3}a\epsilon_F^3\left[-\frac{\pi}{2}+\arcsin\left(\frac{a+E_0}{\epsilon_F}\right)\right]-\tfrac{1}{6}(a+E_0)^4\ln\left|\frac{\epsilon_F+[\epsilon_F^2-(a+E_0)^2]^{1/2}}{a+E_0}\right|\\ &+\frac{2}{3a}\int_0^{\bar{p}_1} p\,dp[(p^2+E_0^2)^{1/2}+a]\left[A^2E\left(\tfrac{1}{2}\arcsin\frac{p}{(p^2+E_0^2)^{1/2}},\ r\right)\right.\\ &\qquad\left.-[(p^2+E_0^2)^{1/2}-a]^2F\left(\tfrac{1}{2}\arcsin\frac{p}{(p^2+E_0^2)^{1/2}},\ r\right)\right]\\ &-\frac{2}{3a}\int_{\bar{p}_1}^{\bar{p}_2} p\,dp[(p^2+E_0^2)^{1/2}+a]\{A^2E(\gamma,r)-[(p^2+E_0^2)^{1/2}-a]^2F(\gamma,r)\},\end{aligned}\tag{16}$$

where

$$\gamma=\tfrac{1}{2}\arcsin\left(\frac{p(1-\alpha^2)^{1/2}}{(p^2+E_0^2)^{1/2}}\right).$$

This is as far as we can take the integrals for the total energy.

(ii) For $a_0 \le a \le a_1$, U is given by (15) with I_+ replaced by I.

(iii) Let $a \ge a_1$. For this region a rather remarkable result appears, so the integral will be treated in some detail. As was pointed out in Sec. III, for $a \ge a_1$ we must solve (5) for $\epsilon_F \le a-E_0$, $a > E_0$.

Let $C^2=a^2+E_0^2-\epsilon_F^2$, so that the equation to be solved is

$$C^2+p^2 \le 2a(p^2\sin^2\theta+E_0^2)^{1/2}\,,$$

or

$$f(p^2)=p^4+p^2(2C^2-4a^2\sin^2\theta)+C^4-4a^2E_0^2 \le 0\,,\tag{17}$$

where we have written pc as p.

Now $f(p^2)=0$ at

$$p_1{}^2(\theta)=2a^2\sin^2\theta-C^2+[(2a^2\sin^2\theta-C^2)^2-C^4+4a^2E_0]^{1/2} \tag{18}$$

and

$$p_2{}^2(\theta)=2a^2\sin^2\theta-C^2-[(2a^2\sin^2\theta-C^2)^2-C^4+4a^2E_0]^{1/2}\,.$$

Clearly, from a graphical consideration of $f(p^2)$, we require that $p_1{}^2(\theta)$ and $p_2{}^2(\theta)$ be non-negative, which in turn means we require

$$C^2\leqslant 2a^2\sin^2\theta\,. \tag{19}$$

Also, we require $p_1{}^2(\theta)$ and $p_2{}^2(\theta)$ to be real, which from (18) means that the condition

$$g(\sin^2\theta)=a^2\sin^4\theta-C^2\sin^2\theta+E_0{}^2\geqslant 0 \tag{20}$$

must be satisfied. Now $g(\sin^2\theta)=0$ at

$$\sin^2\theta=\frac{C^2\pm[C^4-4a^2E_0{}^2]^{1/2}}{2a^2}, \tag{21}$$

and so to satisfy (19) we must take the positive sign in (21). Hence from a graphical consideration of (20) and (21) we deduce that the allowed values of θ are between

$$\arcsin\left[\frac{C^2+(C^4-4a^2E_0{}^2)^{1/2}}{2a^2}\right]^{1/2}$$

and $\pi/2$. We take the possible range of θ to be 0 to $\pi/2$ and double the integral. The region of integration is displayed in Fig. 2, and the integral for the energy is as follows:

$$U=\frac{4\pi V}{(2\pi\hbar c)^3}I,$$

where

$$I=\int_{p_1(\pi/2)}^{p_2(\pi/2)}p^2dp\int_{\arcsin\alpha}^{\pi/2}\sin\theta\,d\theta[p^2+a^2+E_0{}^2-2a(p^2\sin^2\theta+E_0{}^2)^{1/2}]^{1/2}, \tag{22}$$

where $p_1(\pi/2)$ and $p_2(\pi/2)$ are given by (18) and are equal to p_2 and $\bar{p}_1$, defined in Sec. III.

We now make the following changes of variable: Let $p=[(a+x\epsilon_F)^2-E_0{}^2]^{1/2}$, so that $-1\leqslant x\leqslant 1$, and let $y=\cos\theta$. Then the θ integral in (22) becomes

$$\int_0^{(1-\alpha^2)^{1/2}}dy((a+x\epsilon_F)^2+a^2-2a\{(a+x\epsilon_F)^2-y^2[(a+x\epsilon_F)^2-E_0{}^2]\}^{1/2})^{1/2}\,.$$

Now let

$$Z^2=(a+x\epsilon_F)^2-y^2[(a+x\epsilon_F)^2-E_0{}^2]\,.$$

Then (22) becomes

$$I=\int_{-1}^{1}dx\,\epsilon_F(a+x\epsilon_F)\int_{a+x\epsilon_F-\epsilon_F{}^2(1-x^2)/2a^2}^{a+x\epsilon_F}\frac{ZdZ[(a+x\epsilon_F)^2+a^2-2aZ]^{1/2}}{[(a+x\epsilon_F)^2-Z^2]^{1/2}}\,. \tag{23}$$

Clearly, this double integral is independent of E_0, since ϵ_F is also independent of E_0 for $a\geqslant a_1$. Thus for $a\geqslant a_1$ the total internal energy of the system is independent of the mass of its constituent particles—a very strange result indeed. Of course, this procedure can also establish independently that ϵ_F is independent of E_0, as shown by Eq. (13).

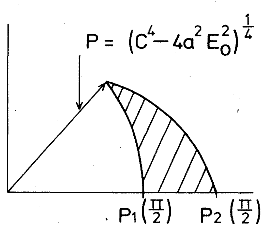

FIG. 2. Region of integration in the (p,θ) plane for the total energy when $a\geq a_1$.

In fact, we have not evaluated the integrals in (23) exactly. However, given that the total energy is independent of E_0, it must be equal to the zero-mass result, for which we have readily evaluated the integrals exactly.

For the zero-mass case, we see from (10) and (11) that $a_0=a_1$, and the total energy is found to be as follows:

$$U=\frac{4\pi V}{(2\pi\hbar c)^3}I,$$

where

(i) for $0 \leq a \leq a_0$,

$$I=\tfrac{1}{2}\epsilon_F(\epsilon_F^2-a^2)^{1/2}(\epsilon_F^2+a^2/6)+\tfrac{2}{3}a\epsilon_F^3\arcsin\left(\frac{a}{\epsilon_F}\right)+\frac{a^4}{12}\ln\left[\frac{\epsilon_F}{a}+\left(\frac{\epsilon_F^2}{a^2}-1\right)^{1/2}\right], \tag{24}$$

(ii) for $a \geq a_0$,

$$U=\tfrac{3}{5}N\epsilon_F, \tag{25}$$

where ϵ_F is still given by (11).

To reiterate, (25) is a completely general result; it is always valid for $a \geq a_1$.

Hence, as $a\to\infty$, the total energy tends to zero, as does the magnetization, since at zero temperature it is given by $M=-\partial U/\partial B$.[14] In view of the shape of the single-particle energy spectrum as shown in Fig. 1, it is not surprising that the Fermi energy, and hence the total energy, since $U<N\epsilon_F$, tends to zero. What is remarkable is that the Fermi energy and total energy (and hence other thermodynamic quantities) are completely independent of the mass of the particles beyond a well-defined value of the magnetic field.

V. EXPANSIONS OF THE TOTAL ENERGY

Although (15) and (16) can be used to evaluate numerically the total energy as a function of the magnetic field in the region $0 \leq a \leq a_1$ for various values of the density, it is useful to have an explicit analytical expression for the energy in some suitable limit. This is especially true if the equation of state is to be used in calculating the mass-radius relationship of a model neutron star consisting of relativistic neutrons in a strong magnetic field. Now for neutrons, $a=E_0$ implies a magnetic field strength of approximately 10^{20} G; hence, since neutron-star magnetic fields are of the order of 10^{13} G, the most useful limit to take in the total energy is $a \ll E_0$, retaining the density as an arbitrary parameter as much as possible. What is more, a/E_0 forms a natural expansion parameter in its own right.

(i) *The region* $0 \leq a \leq a_0$. Here the integrals for the energy are essentially the same as those for the Fermi energy given in (6) and (7):

$$U=\frac{4\pi V}{(2\pi\hbar c)^3}(J_+ + J_-),$$

where

$$J_+=\int_{p_1}^{p_2}p^2dp\int_{\arcsin\alpha}^{\pi/2}\sin\theta\,d\theta\,E(\vec{p},1)+\int_0^{p_1}p^2dp\int_0^{\pi/2}\sin\theta\,d\theta\,E(\vec{p},1) \tag{26}$$

and

$$J_-=\int_{\bar{p}_1}^{\bar{p}_2}p^2dp\int_0^{\arcsin\alpha}\sin\theta\,d\theta\,E(\vec{p},-1)+\int_0^{\bar{p}_1}p^2dp\int_0^{\pi/2}\sin\theta\,d\theta\,E(\vec{p},-1). \tag{27}$$

Since $a \ll E_0$, we may expand as follows:

$$\begin{aligned}E(\vec{p},1)&=(p^2+E_0^2)^{1/2}-\frac{a(p^2\sin^2\theta+E_0^2)^{1/2}}{(p^2+E_0^2)^{1/2}}+\frac{a^2p^2\cos^2\theta}{2(p^2+E_0^2)^{3/2}}+\cdots,\\ E(\vec{p},-1)&=(p^2+E_0^2)^{1/2}+\frac{a(p^2\sin^2\theta+E_0^2)^{1/2}}{(p^2+E_0^2)^{1/2}}+\frac{a^2p^2\cos^2\theta}{2(p^2+E_0^2)^{3/2}}+\cdots.\end{aligned} \tag{28}$$

When the expansions (28) are inserted into J_+ and J_-, the resulting double integrals can be evaluated exactly. The results are as follows:

$$\begin{aligned}J_+ + J_- =\;&\tfrac{1}{4}p_1(p_1^2+E_0^2)^{3/2}-\tfrac{1}{8}E_0^2p_1(p_1^2+E_0^2)^{1/2}-\tfrac{1}{8}E_0^4\ln\left[\frac{p_1+(p_1^2+E_0^2)^{1/2}}{E_0}\right]\\ &+\tfrac{1}{4}\bar{p}_2(\bar{p}_2^2+E_0^2)^{3/2}-\tfrac{1}{8}E_0^2\bar{p}_2(\bar{p}_2^2+E_0^2)^{1/2}-\tfrac{1}{8}E_0^4\ln\left[\frac{\bar{p}_2+(\bar{p}_2^2+E_0^2)^{1/2}}{E_0}\right]\\ &-\bar{p}_2(\bar{p}_2+E_0^2)^{1/2}(\tfrac{2}{15}\epsilon_F^2-\tfrac{3}{10}aE_0-\tfrac{2}{9}a^2)-p_1(p_1^2+E_0^2)^{1/2}(\tfrac{2}{15}\epsilon_F^2+\tfrac{3}{10}aE_0-\tfrac{2}{9}a^2)\\ &+\tfrac{1}{12}a^2\Bigg\{p_1(p_1^2+E_0^2)^{1/2}+\frac{2E_0^2p_1}{(p_1^2+E_0^2)^{1/2}}-3E_0^2\ln\left[\frac{p_1+(p_1^2+E_0^2)^{1/2}}{E_0}\right]\\ &\qquad+\bar{p}_2(\bar{p}_2^2+E_0^2)^{1/2}+\frac{2E_0^2\bar{p}_2}{(\bar{p}_2^2+E_0^2)^{1/2}}-3E_0^2\ln\left[\frac{\bar{p}_2+(\bar{p}_2^2+E_0^2)^{1/2}}{E_0}\right]\Bigg\}\\ &-\frac{p_1}{30(p_1^2+E_0^2)^{1/2}}(2\epsilon_F^4-2a^2\epsilon_F^2+aE_0\epsilon_F^2+3a^3E_0-a^2E_0^2)\end{aligned}$$

$$-\frac{\bar{p}_2}{30(\bar{p}_2^{\,2}+E_0^{\,2})^{1/2}}(2\epsilon_F^{\,4}-2a^2\epsilon_F^{\,2}-aE_0\epsilon_F^{\,2}-3a^3E_0-a^2E_0^{\,2})$$
$$+\frac{(\epsilon_F+a)}{15a}\left\{\left[E\left(\theta',\frac{1}{k}\right)+E\left(\phi,\frac{1}{k}\right)-E\left(\frac{\pi}{2},\frac{1}{k}\right)\right](3\epsilon_F^{\,4}+\tfrac{11}{3}a^2\epsilon_F^{\,2}-\tfrac{1}{3}a^4)\right.$$
$$\left.-\left[F\left(\theta',\frac{1}{k}\right)+F\left(\phi,\frac{1}{k}\right)-F\left(\frac{\pi}{2},\frac{1}{k}\right)\right](3\epsilon_F^{\,4}-6a\epsilon_F^{\,3}+\tfrac{8}{3}a^2\epsilon_F^{\,2}-\tfrac{28}{3}a^3\epsilon_F-\tfrac{1}{3}a^4)\right\}+\cdots, \qquad (29)$$

where

$$\theta'=\arcsin\left(\frac{\epsilon_F+a-E_0}{2\epsilon_F}\right)^{1/2}, \quad \phi=\arcsin\left(\frac{\epsilon_F+a+E_0}{2\epsilon_F}\right)^{1/2}, \quad k=\frac{\epsilon_F+a}{(4a\epsilon_F)^{1/2}},$$

ϵ_F is given by (8) and (9), and $F(\pi/2,1/k)$, $F(\theta',1/k)$ and $E(\pi/2,1/k)$, $E(\theta',1/k)$ are complete and incomplete elliptic integrals of the first and second kind, respectively.

The above expression can be expanded by noting that the modulus of all the elliptic integrals is $1/k=(4a\epsilon_F)^{1/2}/(\epsilon_F+a)$. This is a small parameter, since in the region $0\leqslant a\leqslant a_0$, $\epsilon_F\geqslant a+E_0$ and we have assumed $a/E_0\ll 1$ in (28). Using the small-modulus expansions of elliptic integrals,[15] we obtain

$$J_+ + J_- = \tfrac{1}{4}p_1(p_1^{\,2}+E_0^{\,2})^{3/2}-\tfrac{1}{8}p_1E_0^{\,2}(p_1^{\,2}+E_0^{\,2})^{1/2}-\tfrac{1}{8}E_0^{\,4}\ln\left[\frac{p_1+(p_1^{\,2}+E_0^{\,2})^{1/2}}{E_0}\right]$$
$$+\tfrac{1}{4}\bar{p}_2(\bar{p}_2^{\,2}+E_0^{\,2})^{3/2}-\tfrac{1}{8}E_0^{\,2}\bar{p}_2(\bar{p}_2^{\,2}+E_0^{\,2})^{1/2}-\tfrac{1}{8}E_0^{\,4}\ln\left[\frac{\bar{p}_2+(\bar{p}_2^{\,2}+E_0^{\,2})^{1/2}}{E_0}\right]$$
$$+a\left\{\tfrac{1}{3}E_0^{\,3}\ln\left[\frac{p_1+(p_1^{\,2}+E_0^{\,2})^{1/2}}{\bar{p}_2+(\bar{p}_2^{\,2}+E_0^{\,2})^{1/2}}\right]+\tfrac{1}{6}E_0\bar{p}_2(\bar{p}_2^{\,2}+E_0^{\,2})^{1/2}-\tfrac{1}{6}E_0p_1(p_1^{\,2}+E_0^{\,2})^{1/2}\right.$$
$$+\epsilon_F^{\,3}\left[\tfrac{2}{3}\theta'-\tfrac{3}{5}\sin\theta'\cos\theta'+\tfrac{4}{15}\sin^3\theta'\cos\theta'-\frac{p_1}{30\epsilon_F}\left(\frac{13E_0}{\epsilon_F}+7\right)-\frac{\pi}{3}\right.$$
$$\left.\left.+\tfrac{2}{3}\phi-\tfrac{3}{5}\sin\phi\cos\phi+\tfrac{4}{14}\sin^3\phi\cos\phi+\frac{\bar{p}_2}{30\epsilon_F}\left(\frac{13E_0}{\epsilon_F}-7\right)\right]\right\}+\cdots. \qquad (30)$$

This is still complicated expression as the parameters p_1, $\bar{p}_2$, θ', and ϕ are also functions of a. However, this complexity is necessary if we wish to retain the maximum flexibility in the choice of the range of the density. This can be seen by noting, for example, that for very small a, ϵ_F is given approximately by $(E_0^{\,2}+\sigma^2)^{1/2}$, where $\sigma=\hbar c(3\pi^2\rho)^{1/3}$; hence expressions such as $p_1=(\epsilon_F^{\,2}-E_0^{\,2}+2aE_0-a^2)^{1/2}$ cannot be expanded further unless we specify the relative sizes of a and σ.

We note that putting $a=0$ in (30) quickly recovers the well-known result for the total internal energy of a relativistic noninteracting Fermi gas.[13]

(ii) *The region* $a_0\leqslant a\ll E_0$. In this region $J_-=0$ and the exact result for J_+ [within the approximation (28)] is

$$J_+=\tfrac{1}{4}p_1(p_1^{\,2}+E_0^{\,2})^{3/2}-\tfrac{1}{8}E_0^{\,2}p_1(p_1^{\,2}+E_0^{\,2})^{1/2}-\tfrac{1}{8}E_0^{\,4}\ln\left[\frac{p_1+(p_1^{\,2}+E_0^{\,2})^{1/2}}{E_0}\right]$$
$$-p_1(p_1^{\,2}+E_0^{\,2})^{1/2}(\tfrac{2}{15}\epsilon_F^{\,2}+\tfrac{3}{10}aE_0-\tfrac{2}{9}a^2)+\frac{a^2}{12}\left\{p_1(p_1^{\,2}+E_0^{\,2})^{1/2}+\frac{2E_0^{\,2}p_1}{(p_1^{\,2}+E_0^{\,2})^{1/2}}-3E_0^{\,2}\ln\left[\frac{p_1+(p_1^{\,2}+E_0^{\,2})^{1/2}}{E_0}\right]\right\}$$
$$-\frac{p_1}{30(p_1^{\,2}+E_0^{\,2})^{1/2}}(2\epsilon_F^{\,4}-2a^2\epsilon_F^{\,2}+aE_0\epsilon_F^{\,2}+3a^3E_0-a^2E_0^{\,2})$$
$$+\frac{(\epsilon_F+a)}{15a}[(3\epsilon_F^{\,4}+\tfrac{11}{3}a^2\epsilon_F^{\,2}-\tfrac{1}{3}a^4)E(\theta',1/k)-(3\epsilon_F^{\,4}-6a\epsilon_F^{\,3}+\tfrac{8}{3}a^2\epsilon_F^{\,2}-\tfrac{28}{3}a^3\epsilon_F-\tfrac{1}{3}a^4)F(\theta',1/k)]+\cdots. \qquad (31)$$

In the region $a_0\leqslant a\ll E_0$, we can simplify (31) much more than was possible with (29). This is because the conditions $a_0\leqslant a$ and $a\ll E_0$ require $a_0\ll E_0$, which upon examination of (10) is seen to be equivalent to a low-density condition: For $a_0\ll E_0$, (10) yields

$$a_0=\frac{(\hbar c)^2}{E_0}\left(\frac{3\pi^2\rho}{4}\right)^{2/3}\left[1+O\left(\frac{\hbar c\rho^{1/3}}{E_0}\right)+\cdots\right]. \qquad (32)$$

Thus in this region we have the condition on the density

$$(\hbar c)\left[\frac{3\pi^2\rho}{4}\right]^{1/3} \ll E_0 .$$

This can be used to obtain an approximation for ϵ_F. To do this, we notice that for zero density, $\epsilon_F = E_0 - a$ in the region $a_0 \leq a \leq a_1$, where $a_0 = 0$, $a_1 = E_0$ for zero density. Thus for very low density we write

$$\epsilon_F \simeq E_0 - a + \delta(\rho,a) . \tag{33}$$

We note at this point that since $\epsilon_F = E_0 + a_0$ at $a = a_0$, we must have $\delta(\rho,a_0) = 2a_0$. We now solve (8) for $\delta(\rho,a)$ assuming that δ/E_0 and a/E_0 are both small. The result is

$$\delta(\rho,a) \simeq \bar{\delta}(\rho)(1 + a/3E_0 - \bar{\delta}/2E_0 + \cdots) , \tag{34}$$

where

$$\bar{\delta}(\rho) = \frac{(\hbar c)^2}{2E_0}(6\pi^2\rho)^{2/3} .$$

Inserting (33) into (31) and after much expansion and algebra we obtain

$$\frac{U}{N} = E_0\left(1 - \frac{a}{E_0} + \frac{3\bar{\delta}}{5E_0} + \text{higher-order terms}\right). \tag{35}$$

This expression is valid for $a \geq a_0$, $a \ll E_0$.

(iii) *A different expansion of the energy*. Another way of obtaining an expansion of (26) and (27) is to make a Taylor expansion of J_+ and J_- about $a = 0$. This is done as follows: In order to use the expression

$$J_\pm(a) \simeq J_\pm(0) + aJ_\pm'(0) + \frac{a^2}{2}J_\pm''(0) + \cdots$$

we first need to interchange the order of integrations in (26) and (27). (The reason for this will be given below.) When this is done the total energy is given by

$$U = \frac{4\pi V}{(2\pi\hbar c)^3}(K_+ + K_-) ,$$

where

$$K_+(a) = \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{p_1(\theta)} p^2 dp\, E(\vec{p},1) \tag{36}$$

and

$$K_-(a) = \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{p_2(\theta)} p^2 dp\, E(\vec{p},-1) , \tag{37}$$

where $p_1(\theta)$ and $p_2(\theta)$ are defined by (18).

To differentiate K_+ and K_- we use the standard formula: If

$$f(x) = \int_{C_1(x)}^{C_2(x)} g(x,y)dy ,$$

then

$$f'(x) = \int_{C_1(x)}^{C_2(x)} \frac{\partial g(x,y)}{\partial x} dy + g(x,C_2(x))\frac{\partial C_2(x)}{\partial x} - g(x,C_1(x))\frac{\partial C_1(x)}{\partial x} . \tag{38}$$

[If (38) is applied to the alternative expressions (26) and (27), then repeated differentiation of the terminal $\arcsin\alpha$ produces spurious singularities at $a = 0$, which are difficult to deal with—the forms (36) and (37) avoid this problem.]

Thus applying (38) to (36) and (37) we find

$$K_+(a) + K_-(a) \simeq K_+(0) + K_-(0) - \tfrac{1}{2}a^2\left\{\epsilon_F(\epsilon_F^2 - E_0^2)^{1/2} + E_0^2 \ln\left[\frac{\epsilon_F + (\epsilon_F^2 - E_0^2)^{1/2}}{E_0}\right]\right\} + \cdots , \tag{39}$$

where

$$K_+(0) + K_-(0) = \tfrac{1}{2}\epsilon_F^3(\epsilon_F^2 - E_0^2)^{1/2} - \tfrac{1}{4}E_0^2\epsilon_F(\epsilon_F^2 - E_0^2)^{1/2} - \tfrac{1}{4}E_0^4 \ln\left[\frac{\epsilon_F + (\epsilon_F^2 - E_0^2)^{1/2}}{E_0}\right] \tag{40}$$

and is a well-known expression.[13] Note that in (39) and (40), ϵ_F is evaluated at $a = 0$, where it is given by

$$\epsilon_F = [E_0^2 + (\hbar c)^2(3\pi^2\rho)^{2/3}]^{1/2} .$$

The nonrelativistic limit of (39) can easily be taken by assuming $\hbar c(3\pi^2\rho)^{1/3} \ll E_0$ and expanding ϵ_F accordingly. The result is the well-known nonrelativistic one[13]

$$\frac{U}{N} = E_0\left[1 + \frac{3(\epsilon_F^2 - E_0^2)}{10E_0^2} + \cdots - \frac{3a^2}{2(\epsilon_F^2 - E_0^2)}\left(1 + \frac{\epsilon_F^2 - E_0^2}{6E_0^2} + \cdots\right) + \cdots\right]. \tag{41}$$

Note that (39) is valid only in the region $0 \leq a \leq a_0$ since we have Taylor expanded about $a = 0$.

(iv) *Discussion of results*. For the region $a_0 \leq a \ll E_0$ the expansion of the energy is straightforward and is given by (35). In the region $0 \leq a \ll E_0$, however, we have two different expansions given by (30) [or more fully by (29)] or (39).

These very different looking expressions can be understood if we make the following observations. Firstly, in the low-density limit (39) yields (41), and the condition for its validity is then

$$a \ll \frac{(\hbar c)^2 (3\pi^2 \rho)^{2/3}}{\sqrt{5}\, E_0}\,.$$

However, we know from (32) that

$$a_0 = \frac{(\hbar c)^2}{E_0}\left(\frac{3\pi^2\rho}{4}\right)^{2/3},$$

so for low density (39) is in fact further restricted to the region $0 \leq a \ll a_0$. On the other hand, (30) was obtained with the sole restriction $a \ll E_0$, and since $a_0 \ll E_0$ for low density, we see that (30) therefore has a considerably wider range of validity. In fact, the entire range $0 \leq a \leq a_0$ can be covered for suitably low density, but with an attendant increase in complexity.

Secondly, we expect that (30) should reduce to (39) in the limit of very small a (or equivalently, very high density). While to check this would require a very tedious calculation since it would require a careful expansion of (29) to order a^2, a quick study of (30) reveals that for very small a (or very high density) the terms linear in a do in fact exactly cancel in this limit.

One remaining region readily amenable to expansion is the region $a \ll E_0$ and $a \lesssim a_0$. A Taylor expansion of (30) in powers of $(a - a_0)$ could be obtained, and at $a = a_0$ it would obviously join to (35). We do not give it here.

VI. SOME RESULTS IN ONE AND TWO DIMENSIONS

For completeness and to illustrate the effect of phase space on the behavior of ϵ_F, similar calculations have been done in one and two dimensions.

(i) *One dimension.* The single-particle energy levels are now, with $\theta \equiv 0$ in (1),

$$E(\vec{p}, s) = [p^2c^2 + (E_0 + sa)^2]^{1/2}, \quad s = \pm 1\,.$$

Let $n_\pm = \pi\hbar c N_\pm / V$. Then

(i) for $0 \leq a \leq a_0$, where $a_0 = n^2/4E_0$,

$$n_+ = n/2 + 2aE_0/n\,,$$

$$n_- = n/2 - 2aE_0/n\,,$$

$$\epsilon_F = \frac{1}{2n}(n^2 + 4a^2)^{1/2}(n^2 + 4E_0^2)^{1/2},$$

and

$$U = \frac{V}{2\pi\hbar c}\left[n\epsilon_F + (a - E_0)^2 \ln\left|\frac{\epsilon_F + n_+}{a - E_0}\right| + (a + E_0)^2 \ln\left|\frac{\epsilon_F + n_-}{a + E_0}\right|\right];$$

(ii) for $a \geq a_0$, $n_+ = n$, and

$$\epsilon_F = [n^2 + (a - E_0)^2]^{1/2},$$

$$U = \frac{V}{2\pi\hbar c}\left[n\epsilon_F + (a - E_0)^2 \ln\left|\frac{\epsilon_F + n}{a - E_0}\right|\right].$$

Note that in one dimension only the two regions $a \leq a_0$ and $a \geq a_0$ occur.

(ii) *Two dimensions.* The reasoning here follows exactly that of the three-dimensional case. Let $n_\pm = (2\pi\hbar c)^2 N_\pm / V$. Then solving the constraint condition (5) for $\epsilon_F \geq a + E_0$ gives the following integrals:

$$n_+ = \pi(\epsilon_F^2 - E_0^2) + a\int_0^{2\pi} d\theta (a^2 \sin^4\theta + D\sin^2\theta + E_0^2)^{1/2}, \tag{42}$$

$$n_- = \pi(\epsilon_F^2 - E_0^2) - a\int_0^{2\pi} d\theta (a^2 \sin^4\theta + D\sin^2\theta + E_0^2)^{1/2}, \tag{43}$$

where $D = \epsilon_F^2 - a^2 - E_0^2$. Hence

$$\epsilon_F = (n/2\pi + E_0^2)^{1/2}, \tag{44}$$

a constant. We have not been able to evaluate the integrals in (42) and (43) exactly. As in the three-dimensional case, putting $\epsilon_F = a + E_0$ in (43) and integrating gives $n_- = 0$. The value of a for which this occurs is clearly

$$a_0 = (n/2\pi + E_0^2)^{1/2} - E_0\,.$$

Thus (42), (43), and (44) are valid for $0 \leq a \leq a_0$. For $a_0 \leq a \leq a_1$, ϵ_F is given by (42) with $n_+ = n$, and as before a_1 is found by putting $a = a_1$, $\epsilon_F = a_1 - E_0$, and $n_+ = n$ in (42) and integrating, with the result

$$n = (4a_1^2 - 8a_1E_0)[\pi/2 - \arcsin(E_0/a_1)^{1/2}] + 4a_1^2(E_0/a_1)^{1/2}(1 - E_0/a_1)^{1/2}\,.$$

Finally, for $a \geq a_1$ we use the same reasoning which led to (12) (see Sec. IV) to obtain the two-dimensional version

$$n = \int_{\arcsin\beta}^{\pi - \arcsin\beta} d\theta \int_{p_-(\theta)}^{p_+(\theta)} p\, dp + \int_{\pi + \arcsin\beta}^{2\pi - \arcsin\beta} d\theta \int_{p_-(\theta)}^{p_+(\theta)} p\, dp\,,$$

where

$$\beta^2 = [-D + (D^2 - 4a^2E_0^2)^{1/2}]/2a^2\,,$$

and

$$p_\pm(\theta) = [2a^2 \sin^2\theta + D \pm 2a(a^2 \sin^4\theta + D\sin^2\theta + E_0^2)^{1/2}]^{1/2}$$

This reduces to

$$n = 8a\int_{\arcsin\beta}^{\pi/2} d\theta (a^2 \sin^4\theta + D\sin^2\theta + E_0^2)^{1/2}\,.$$

The results for the total energy are rather complicated and are not given here. However, it is interesting to note that for $E_0=0$, $U=2N\epsilon_F/3$ for all values of a—the same result as in three dimensions for $a \geq a_1$.

We have inquired as to whether or not both ϵ_F and U are independent of the rest mass m for $a > a_1$, here as in the three-dimensional case. We have not succeeded in answering this question because of the complexity of the integrals involved.

In conclusion, we can say that ϵ_F increases with increasing a (for sufficiently large values) in one dimension, in two dimensions it is constant in one region, and in three dimensions it is monotonically decreasing.

VII. DISCUSSION AND CONCLUSION

We first summarize the results given in Sec. V for the energy. In the region $0 \leq a \leq a_0$ we have two different expansions. For a very small field (or equivalently, very high density) we have the Taylor expansion given by (39). In the low-density, weak-field limit this reduces to the nonrelativistic result given by (41). In the high-density limit $[\hbar c(3\pi^2\rho)^{1/3} \gg E_0]$, (39) becomes

$$\frac{U}{N}=\tfrac{3}{4}\sigma\left\{1+\frac{E_0^2}{\sigma^2}+\cdots -\frac{a^2}{\sigma^2}\left[1+\frac{E_0^2}{\sigma^2}\ln\left(\frac{2\sigma}{E_0}\right)+\cdots\right]+\cdots\right\},$$

where $\sigma=\hbar c(3\pi^2\rho)^{1/3}$.

A more general (but much more complicated) result is obtained by expanding the energy integrals for $a \ll E_0$. The results are given in (29) or (30). These expressions can cover the entire range $0 \leq a \leq a_0$ provided $a_0 \ll E_0$, whereas (39) only covers the range $0 \leq a \ll a_0$ when $a_0 \ll E_0$.

In the region $a_0 \leq a \leq a_1$ the energy for $a \ll E_0$ is given by (35). This is a low-density expansion. Finally, in the region $a \geq a_1$ we have the exact result $U=2N\epsilon_F/3$, which is valid for any density.

From these various expressions for the energy, the equation of state is most easily obtained by using the equation

$$P=-\frac{U}{V}+\frac{N\epsilon_F}{V},$$

where P is the pressure, which is derived elsewhere.[16] In particular, we note that for $a \geq a_1$ we obtain $P=U/2V$. This is an interesting result in that it is an example of a noninteracting system which violates the condition $P \leq U/3V$, which was once thought to be a consequence of relativity. Zeldovich[17] has obtained another counter example, using particles interacting via a vector meson field. In our case the violation occurs because of the neutron's interaction with an external field.

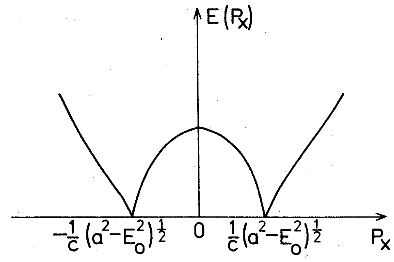

FIG. 3. Energy for a spin-up particle in the cusp at $\theta=\pi/2$ moving solely in the x direction as a function of its momentum p_x for $a > E_0$.

In addition, using $P=U/2V$ and (13) and (25) shows that the pressure vanishes as $B\rightarrow\infty$. To see why this is so in terms of momentum transfer at a wall,[18] consider the case of a very large field. The Fermi energy is then close to zero, and all the particles are confined to the cusp in the energy spectrum at $p=(a^2-E_0^2)^{1/2}/c$ (see Fig. 1). Furthermore, nearly all the particles are moving at right angles to the field, in the x and y directions. Consider now a wall perpendicular to the x direction, and consider only particles moving along the x axis. Since $\theta=\pi/2$, their energy is given by

$$E(p_x)=\left|(p_x^2c^2+E_0^2)^{1/2}-a\right| .$$

This is shown in Fig. 3. The velocity of a particle with momentum p_x is given by

$$v_x=\frac{\partial E(p_x)}{\partial p_x}=\pm\frac{p_xc^2}{(p_x^2c^2+E_0^2)^{1/2}},$$

where the positive sign if taken when p_x is to the right of the cusp. Since the field is so large, all particles essentially either have momentum $(a^2-E_0^2)^{1/2}/c$ with velocities $\pm c(a^2-E_0^2)^{1/2}/a$, or momentum $-(a^2-E_0^2)^{1/2}/c$ with velocities $\pm c(a^2-E_0^2)^{1/2}/a$. Thus, of the particles hitting the wall [say those with velocity $c(a^2-E_0^2)^{1/2}/a$], half have momentum $(a^2-E_0^2)^{1/2}/c$ and half have momentum $-(a^2-E_0^2)^{1/2}/c$. Hence the momentum transfers at the wall cancel.

This argument will of course only be exact when the field is actually infinite—the Fermi energy is then exactly zero, and all the particles are confined to the tips of the cusps in Fig. 3 (where the density of states is now infinite). Thus the momentum of any particle can only be $\pm\infty$ and its velocity $\pm c$. However, note that a particle can have positive momentum and negative group velocity (and vice versa), as well as the more conventional positive momentum and velocity, and, in fact, this unusual behavior is the very reason, as we have just seen, for the momentum transfers canceling exactly and hence the pressure vanishing.

However, the vanishing of the pressure is strictly a zero-temperature result. In real systems, T is never exactly zero, and to make the zero-temperature approximation, we just require that ϵ_F/kT to be very large. In this case, however, no matter how small kT is, ϵ_F will eventually be much smaller for a sufficiently large magnetic field. The gas will then become classical, with the pressure given by the classical ideal-gas result $P=\rho kT$.

While we have ignored the role of interactions between the particles, needless to say in this limit of ultrastrong magnetic field where $\epsilon_F \to 0$, particle interactions will play an increasingly important role, especially in the super-dense case.

The high-field behavior we have discussed is dependent on the particular form of the energy spectrum given in (1). We note first that (1) should follow from the addition of a quantized spin and the application of a Lorentz boost to the energy of a neutral particle at rest, since the Dirac equation itself should contain nothing more than the spin (introduced by adding the term $\frac{1}{2}\mu\sigma_{\mu\nu}F^{\mu\nu}$) and Lorentz covariance. As a simple example, in one dimension the energy of a particle with spin at rest is $|E_0 \pm a|$ (although strictly speaking a magnetic field does not exist in one dimension, and we have a two-state system which is identically equivalent to an electric dipole) which must transform via a Lorentz boost to $[(E_0 \pm a)^2 + p^2c^2]^{1/2}$. In principle, then it should be possible (although more difficult, since the magnetic moment of a neutral particle is entirely anomalous) to perform the same transformations in two and three dimensions and thus obtain (1).

The energy spectrum given in (1) may well be drastically changed for high magnetic fields by nonnegligible quantum-electrodynamical and other analogous corrections to the mass and magnetic moment of the neutron. These types of corrections have been considered for the electron in calculations by various authors,[19-21] although as pointed out in Ref. 20 more work is still needed here. In our case analogous calculations would need to be done for a neutral particle whose magnetic moment is entirely anomalous. It may well be that for extremely strong magnetic fields the spectrum is dramatically altered. Of course if our spectrum is changed, among other things, the argument concerning pair production given in Sec. II may well need to be revised, as is the case for the electron spectrum in an intense magnetic field, pointed out in Refs. 19 and 20.

Nevertheless, it is still of considerable interest to study the statistical mechanics of the energy spectrum given by (1), for all magnetic field strengths. It is remarkable that such a simple system can give such unusual behavior in its own right, and perhaps some of this behavior persists even when radiative corrections are taken into account. In any case, the very weak to moderate-field results, $0 \leq a \ll E_0$, will still be correct for all densities (within the noninteracting-gas approximation), and it is clear from (28) that these results will not always be equivalent to the nonrelativistic ones.

We have seen that this relatively simple system of noninteracting, relativistic, ultradegenerate magnetic moments in an arbitrarily strong magnetic field displays a rich variety of physical behavior. We hope that these results and the questions they give rise to will make some contribution to the very difficult and complex structure of relativistic statistical mechanics. It is also hoped that the results may have some application to the study of the relativistic magnetic stars which may occur in astrophysics.

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[14]Note that the total magnetization defined by $M = -\partial U/\partial B$, where U is the total internal energy, is quite different from that defined by $M = \mu(N_+ - N_-)$, since it is clear from the results for N_+ and N_- given in Sec. III that the magnetization calculated from the second definition will saturate at $a = a_0$ and remain constant thereafter, whereas the magnetization calculated from the first definition tends to zero as the field tends to infinity, as is shown here. The only case where the two definitions give identical results is when the spectrum is linear in the field, which is certainly not the case for our spectrum (1), except in the very weak-field limit. Both definitions are valid ones for the magnetization, but they describe two different aspects of the system's response to a magnetic field, and in fact experiments can be set up to measure each of these different magnetizations. The same comments also apply for other dimensions—for example in one dimension one can show from the results for N_+ and the energy (given in part i of Sec. VI) that both magnetizations saturate in the high-field limit, but with opposite signs: the first definition gives $M \to -\mu N$, and the second gives $M \to \mu N$. Detailed results for each of these magnetizations can be obtained directly from the expressions given in this paper. One normally calls a system paramagnetic if $\partial M/\partial B \equiv \psi > 0$, and diamagnetic if $\psi < 0$. For the nonrelativistic systems where the spectrum is linear in B this is unambiguous (e.g., Ref. 13). However, when, as in our system, the spectrum is not a linear function of B, ψ as defined above may be positive or negative as we have seen. Of course, the slope of an M vs. B curve when $M \equiv \mu(N_+ - N_-)$ is always positive. We have chosen the use of the word paramagnetism in the title solely for historical (Pauli, Ref. 1) purposes.

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