Integral equation method for effecting Kinnersley-Chitre transformations. II

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(Received 29 May 1979)

Our previously presented integral equation formulation of the Kinnersley-Chitre transformation theory is generalized to the case of electrovac-to-electrovac transformations. The solution of the integral equation for a case in which the kernel has a finite number of simple poles is obtained. In particular, we show that when the transformation corresponding to one simple pole is applied to Minkowski space, one obtains the Ehlers transform of the extreme charged Kerr-NUT (Newman-Unti-Tambourino) space. We also find the general solution corresponding to a confluence of two simple poles.

I. INTRODUCTION

In an earlier paper[1] (designated as I) we demonstrated how vacuum-to-vacuum Kinnersley-Chitre (KC) transformations[2] can be effected by solving a linear integral equation of the Cauchy type. Preliminary studies[3] have indicated that the contact which has thus been made with the well-developed field of linear integral equations, and with complex function theory in particular, is apt to be fruitful in the further elucidation of the solution-generating theory. Since it seemed plausible that our approach would admit an extension to electrovac fields, we sought and found just such a generalization.

It is gratifying to be able to report that with a suitable reinterpretation *exactly the same linear integral equation can be used to effect electrovac-to-electrovac KC transformations*. Where in our formulation of the vacuum problem we used 2×2 matrices, we now employ 3×3 matrices. In particular, it is possible to associate with any given stationary axially symmetric electrovac spacetime a certain complex 3×3 matrix potential $F(s)$, which depends not only upon the nonignorable spacetime coordinates but also upon an additional complex variable s. It is assumed that $F(s)$ is analytic in an open neighborhood of $s=0$, and that

$$F(0)=\begin{pmatrix} 0 & i & 0\\ -i & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

in the representation we shall employ.

If $F_0(s)$ is the potential associated with a particular seed metric, then

$$F(s)\equiv[I+sf(s)]F_0(s) \tag{1.1}$$

is the potential associated with another stationary axially symmetric electrovac spacetime, providing that the 3×3 matrix function $f(s)$ is analytic in an open neighborhood of $s=0$, and that it satisfies the linear integral equation

$$f(t)+\frac{1}{2\pi i}\int_C ds\,\frac{[f(s)+s^{-1}I]K(s)}{s-t}=0\,, \tag{1.2}$$

where the kernel $K(s)$ is given by the similarity transformation

$$K(s)\equiv F_0(s)\{\exp[\gamma(s)\mathfrak{C}(s)]-I\}[F_0(s)]^{-1}\,. \tag{1.3}$$

Here

$$\mathfrak{C}(s)\equiv\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & -is/2\end{pmatrix},$$

while $\gamma(s)$ is an arbitrary spacetime-independent 3×3 Hermitian matrix[4] function of s, analytic in an annulus about $s=0$ inside the region of analyticity of $F_0(s)$ and $f(s)$. The contour C is any closed positively oriented contour surrounding $s=0$ and within this annulus, while the point t in Eq. (1.2) lies in the interior of the region enclosed by C.

In the next section we shall define the F-potential associated with an electrovac spacetime, and we shall describe a calculational procedure which may be employed in its determination.

II. DIFFERENTIAL EQUATION FOR THE 3×3 F-POTENTIAL[5]

We shall denote the spacelike Killing vector by $\underline{K}_S$ and the timelike Killing vector by $\underline{K}_T$, assigning the symbols K_S and K_T to the respective covectors (1-forms).[6] We assume that

$$\rho^2\equiv-(\underline{K}_S\wedge\underline{K}_T)\cdot(\underline{K}_S\wedge\underline{K}_T)\neq 0\,, \tag{2.1}$$

except on a set of measure zero (e.g., on the axis). It is convenient to introduce a two-dimensional duality operator *. For any 0-, 1-, or 2-form α such that the Grassman product

$\alpha\ulcorner(K_S K_T)$ vanishes, we define

$$*\alpha \equiv \rho^{-1}(\alpha K_S K_T)\ulcorner(e^1e^2e^3e^4), \qquad (2.2)$$

where the 4-form $e^1e^2e^3e^4$ is the invariant volume element. In particular, we shall employ this two-dimensional duality operator in order to define another field z such that

$$*dz = -d\rho \text{ or } *d\rho = +dz. \qquad (2.3)$$

A complex $\mathcal{E}$-potential exists whenever one has a closed self-dual 2-form W, whose Lie derivative with respect to some vector field $\underline{K}$ vanishes. From these attributes of W one can infer immediately that $K\ulcorner W$ is an exact differential, i.e., there exists a complex potential $\mathcal{E}$ such that $d\mathcal{E} = K\ulcorner W$.

In the case of stationary axially symmetric electrovac spacetimes we introduce three natural closed self-dual 2-forms (W_S, W_T, W_M). The simplest is that associated with the Maxwell 2-form $\mathfrak{F}$. It is

$$W_M \equiv 2P\mathfrak{F}, \qquad (2.4)$$

where $P = \frac{1}{2}(1 - iD)$ projects onto the subspace of eigenvalue $+i$ of the duality operator D. The sourcefree Maxwell equations give us $dW_M = 0$. Furthermore, we assume that the Lie derivatives of $\mathfrak{F}$ with respect to $\underline{K}_S$ and $\underline{K}_T$ vanish. It follows that we may introduce complex potentials φ_S and φ_T such that

$$d\varphi_S = K_S\ulcorner W_M, \quad d\varphi_T = K_T\ulcorner W_M.$$

Two additional closed self-dual 2-forms are provided by

$$W_S \equiv -4P(\tfrac{1}{2}dK_S + \varphi_S^*\mathfrak{F}) \qquad (2.5)$$

and

$$W_T \equiv -4P(\tfrac{1}{2}dK_T + \varphi_T^*\mathfrak{F}). \qquad (2.6)$$

The KC transformations apply only to those axially symmetric stationary electrovac spacetimes which satisfy the condition that the scalar product of $\mathfrak{F}$ with the 2-form $K_S K_T$ vanishes. Assuming this condition now, one may show that the Lie derivative of each of the potentials (φ_S, φ_T) with respect to each of the Killing vectors $(\underline{K}_S, \underline{K}_T)$ vanishes, and from this one may conclude that the Lie derivative of each of the 2-forms (W_S, W_T) with respect to each of the Killing vectors $(\underline{K}_S, \underline{K}_T)$ vanishes.

What is less obvious is that if $K_S\ulcorner W$ and $K_T\ulcorner W$ are exact differentials, W being any of the closed self-dual 2-forms, then $K_M\ulcorner W$ is also an exact differential, where

$$\underline{K}_M \equiv 2i(\varphi_S^*\underline{K}_T - \varphi_T^*\underline{K}_S). \qquad (2.7)$$

In other words, the Lie derivative of each of the 2-forms (W_S, W_T, W_M) with respect to each of the vector fields $(\underline{K}_S, \underline{K}_T, \underline{K}_M)$ vanishes. This conclusion is based upon the observation that

$$\begin{aligned} d(K_M\ulcorner W) &= 2i[d\varphi_S^*(K_T\ulcorner W) - d\varphi_T^*(K_S\ulcorner W)] \\ &= 2i[(K_S\ulcorner W_M^*)(K_T\ulcorner W) - (K_T\ulcorner W_M^*)(K_S\ulcorner W)] \\ &= -2iK_S\ulcorner[W_M^*(K_T\ulcorner W) + (K_T\ulcorner W_M^*)W] \\ &= -2i(K_S K_T)\ulcorner(W_M^* W) = 0, \end{aligned}$$

where we use the fact that the 4-form $W_M^* W$ necessarily vanishes. In conclusion, we may introduce a complex 3×3 matrix potential $F^{(1)}$ such that

$$dF^{(1)} \equiv \begin{pmatrix} K_S \\ K_T \\ K_M \end{pmatrix} \ulcorner (W_S, W_T, W_M), \qquad (2.8)$$

where K_M is the covector of $\underline{K}_M$.

We suspect that the significance of the vector field $\underline{K}_M$ is not yet fully appreciated. Some day, perhaps, the solution-generating theory will be extended to solutions without symmetries, and non-Killing fields such as $\underline{K}_M$ may play a role in that hypothetical future theory. However, the present paper is not concerned with such speculations.

For purposes of comparison with Kinnersley and Chitre it should be noted that our 3×3 matrix $F^{(1)}$ can be expressed in terms of their 2×2 matrix H, their 2×1 matrix φ, their 1×2 matrix $L^{(1,1)}$, and their 1×1 matrix $K^{(1,1)}$ as

$$F^{(1)} = \begin{pmatrix} H & \varphi \\ 2iL^{(1,1)} & 2iK^{(1,1)} \end{pmatrix}. \qquad (2.9)$$

As was amply discussed in Ref. 2, H and φ satisfy the relation

$$\varphi\varphi^\dagger = h + iz\epsilon - \tfrac{1}{2}(H + H^\dagger), \qquad (2.10)$$

where

$$h \equiv -\begin{pmatrix} \underline{K}_S \\ \underline{K}_T \end{pmatrix} \cdot (\underline{K}_S, \underline{K}_T)$$

and

$$\epsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From the definitions (2.7) and (2.8) it follows immediately that

$$\begin{pmatrix} -4\epsilon\varphi\varphi^\dagger\epsilon & -2i\epsilon\varphi \\ -2i\varphi^\dagger\epsilon & 1 \end{pmatrix} dF^{(1)} = 0. \qquad (2.11)$$

Substituting Eq. (2.10) into Eq. (2.11), we have

$$\left\{ \begin{pmatrix} -4\epsilon(h+iz\epsilon)\epsilon & 0 \\ 0 & 1 \end{pmatrix} - 2i[\mathfrak{E}F^{(1)}\Omega + \Omega F^{(1)\dagger}\mathfrak{E}] \right\} dF^{(1)} = 0\,,$$

where

$$\mathfrak{E} \equiv \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad \Omega \equiv \begin{pmatrix} i\epsilon & 0 \\ 0 & 1 \end{pmatrix}.$$

However, because H and φ satisfy the "self-duality relations"

$$h\epsilon dH = -i\rho * dH\,, \quad h\epsilon d\varphi = -i\rho * d\varphi\,, \tag{2.12}$$

this equation simplifies to

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - 2i[\mathfrak{E}F^{(1)}\Omega + \Omega F^{(1)\dagger}\mathfrak{E}] \right\} dF^{(1)} = -4i\mathfrak{E}(z+\rho^*)dF^{(1)}\,. \tag{2.13}$$

The key step in the development of the F-potential for electrovac fields consists of multiplying Eq. (2.13) by an arbitrary complex parameter t and deducing that

$$\mathcal{K}(t)\Gamma(t) = it\mathfrak{E}dF^{(1)}\,, \tag{2.14}$$

where

$$\Gamma(t) \equiv t[1 - 2t(z+\rho^*)]^{-1}dF^{(1)}\,, \tag{2.15}$$

$$\mathcal{K}(t) \equiv i\mathfrak{E}(t) - it[\mathfrak{E}F^{(1)}\Omega + \Omega F^{(1)\dagger}\mathfrak{E}]\,, \tag{2.16}$$

and

$$\mathfrak{E}(t) \equiv \begin{pmatrix} \epsilon & 0 \\ 0 & -\frac{1}{2}it \end{pmatrix}.$$

Noting that

$$dF^{(1)\dagger}\mathfrak{E}\Gamma(t) = 0\,, \tag{2.17}$$

and evaluating the exterior derivative of Eq. (2.14), we see that $\Gamma(t)$ satisfies the relation

$$d\Gamma(t) = \Gamma(t)\Omega\Gamma(t)\,. \tag{2.18}$$

This is, however, the integrability condition for the existence of a potential $F(t)$ satisfying the defining equation

$$dF(t) = \Gamma(t)\Omega F(t)\,. \tag{2.19}$$

For a given choice of $\Gamma(t)$ this equation does not, of course, define the potential $F(t)$ uniquely. We shall select $F(t)$ so that it also satisfies the conditions

$$F(0) = \Omega^{-1}\,, \tag{2.20}$$

$$\dot{F}(0) \equiv \lim_{t\to 0}\frac{\partial F}{\partial t} = F^{(1)}\,, \tag{2.21}$$

and

$$F(t)^\dagger\mathcal{K}(t)F(t) = i\mathfrak{E}(t)\,. \tag{2.22}$$

That these subsidiary conditions can be imposed upon $F(t)$ can be shown as follows.

First, it should be observed that if $F_1(t)$ and $F_2(t)$ are both solutions of Eq. (2.19) for the same choice of $\Gamma(t)$, then

$$d[F_1(t)^{-1}F_2(t)] = 0\,;$$

that is, $F_2(t) = F_1(t)u(t)$, where $u(t)$ is a spacetime-independent matrix function of t.

Since $\Gamma(0) = 0$ and $\dot{\Gamma}(0) = dF^{(1)}$, Eq. (2.19) implies that $dF(0) = 0$ and $d\dot{F}(0) = dF^{(1)}\Omega F(0)$. We follow the conventions of Kinnersley and Chitre when we select $F(t)$ so that Eqs. (2.20) and (2.21) are satisfied.

Finally, if Eq. (2.19) is multiplied on the left by $F(t)^\dagger\mathcal{K}(t)$, and then Eq. (2.14) is employed, one obtains

$$F(t)^\dagger\mathcal{K}(t)dF(t) = itF(t)^\dagger\mathfrak{E}dF^{(1)}\Omega F(t)\,.$$

Adding to this equation its own Hermitian conjugate, one readily concludes that

$$d[F(t)^\dagger\mathcal{K}(t)F(t)] = 0\,;$$

that is, $F(t)^\dagger\mathcal{K}(t)F(t)$ is a spacetime-independent Hermitian matrix function of t. Taking advantage of the remaining arbitrariness of $F(t)$, we can choose it so that Eq. (2.22) is satisfied, thereby obtaining complete agreement with the conventions of Kinnersley and Chitre.

In conclusion, by solving Eq. (2.19) subject to the subsidiary conditions (2.20)–(2.22) we can associate with any given electrovac spacetime a 3×3 F-potential. These are the starting equations from which one can deduce the integral equation (1.2) which permits one to effect KC transformations.

In I we presented an interim derivation of the vacuum-to-vacuum KC transformation equations and of the vacuum specialization of our integral equation. That approach relied very heavily upon the infinite hierarchy of potentials machinery which Kinnersley and Chitre had developed. Since, however, the final integral equation made no reference to the infinite hierarchy, we felt that eventually a more direct derivation would be found.

It must be admitted that we first found the present electrovac generalization of our integral equation by using exactly those techniques which were described in I. Recently, however, we discovered that the use of the KC infinite hierarchy of potentials can be obviated by applying the known solution of the classic Hilbert (Riemann) problem, discussed in Ref. 3, to the analysis of Eqs. (2.19)–(2.22). This results in the rapid identification of the KC transformation group and the elegant derivation of our integral equation. It may even allow us to prove certain outstanding conjectures. However, these sophisticated aspects of the theory will be addressed elsewhere. The present paper is concerned instead with the practical application of the solution-generating techniques, which we hope will be of interest to a wide audience.

III. SOME PREVIOUSLY OBTAINED F-POTENTIALS

To carry out a KC transformation upon a given spacetime, one must know $F(s)$ for that seed metric. In time we shall probably have quite an extensive catalog of F-potentials. Up to the present time, however, almost all the F-potentials which have been constructed pertain to vacuum metrics. In the case of Minkowski space Kinnersley and Chitre[2] found that

$$F^{\mathrm{MS}}(s)=\begin{pmatrix} -\dfrac{\lambda-1+2sz}{2\lambda s} & i\dfrac{\lambda+1-2sz}{2\lambda} & 0 \\ -\dfrac{i}{\lambda} & \dfrac{s}{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $\lambda(s):=[(1-2sz)^2+4s^2\rho^2]^{1/2}$, while in the case of a static vacuum metric they found that

$$F(s)=\begin{pmatrix} e^{-\psi(0)} & 0 & 0 \\ 0 & e^{\psi(0)} & 0 \\ 0 & 0 & 1 \end{pmatrix} F^{\mathrm{MS}}(s) \begin{pmatrix} e^{-\psi(s)} & 0 & 0 \\ 0 & e^{\psi(s)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

where $\psi(s)$ is an s-dependent generalization of Weyl's potential function ψ, found by solving the equation

$$d\psi(s)=[\lambda(s)]^{-1}[(1-2sz)-2s\rho*]d\psi, \tag{3.3}$$

subject to the condition $\psi(0)=\psi$.

In I we presented among other things the F-potential of Kerr-NUT (Newman-Unti-Tambourino) space. This result was obtained by solving our integral equation using Schwarzschild space as the seed metric, and selecting

$$\gamma(s)=\beta(s)\begin{pmatrix} 1 & -is^{-1} & 0 \\ is^{-1} & s^{-2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

where $\beta(s)$ is regular everywhere except possibly at the origin. We suspect that this method will ultimately replace the evaluation of $F(s)$ by solving Eq. (2.19).

As far as we know, the only electrovac spacetime for which $F(s)$ has previously been calculated is the Reissner-Nordström solution, which was the subject of study of T. Jones.[7] In the next section we shall describe some new electrovac results which we have obtained by solving the integral equation (1.2).

IV. n SIMPLE POLES

In I we considered vacuum-to-vacuum transformations generated by

$$\gamma(s)=\beta(s)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\ 1\ 0),$$

where

$$\beta(s)\equiv\sum_{i=1}^{n}\frac{\alpha_i s}{s-u_i}, \tag{4.1}$$

α_i and u_i being nonzero real constants. We now consider the electrovac-to-electrovac transformations generated by

$$\gamma(s)\equiv\beta(s)C(s)C(s)^\dagger, \tag{4.2}$$

where $C(s)$ is a 3×1 matrix which satisfies the condition

$$C(s)^\dagger\mathfrak{C}(s)C(s)=0. \tag{4.3}$$

This is a natural generalization of the vacuum-to-vacuum transformations which we considered previously. We shall see that the same method of solving the integral equation (1.2) may be used in connection with the transformation generated by (4.2).

The kernel (1.3) assumes the form

$$K(s)=\beta(s)\Gamma(s)\tilde{\Gamma}(s), \tag{4.4}$$

where

$$\Gamma(s)\equiv F_0(s)C(s) \tag{4.5}$$

and

$$\tilde{\Gamma}(s)\equiv C(s)^\dagger\mathfrak{C}(s)F_0(s)^{-1}. \tag{4.6}$$

We will consider the case when $\Gamma(s)\tilde{\Gamma}(s)$ is analytic in some open neighborhood of $s=0$ containing the points $s=u_i$ $(1\leqslant i\leqslant n)$. Choosing the contour C so that it encloses all the poles u_i, we obtain from Eq. (1.2) the result

$$f(t)+\sum_{i=1}^{n}\alpha_i\left(\frac{X(t)-X(u_i)}{t-u_i}\right)=0, \tag{4.7}$$

where

$$X(t)\equiv[I+tf(t)]\Gamma(t)\tilde{\Gamma}(t). \tag{4.8}$$

Condition (4.3) implies that

$$\tilde{\Gamma}(t)\Gamma(t)=0. \tag{4.9}$$

Therefore, Eqs. (4.7) and (4.8) can be solved quite easily. We find that

$$f(t)=\left\{\sum_{k=1}^{n}\frac{\alpha_k[X(u_k)-\Gamma(t)\tilde{\Gamma}(t)]}{t-u_k}\right\}\times[I-\beta(t)\Gamma(t)\tilde{\Gamma}(t)], \qquad (4.10)$$

where the t-independent matrices $X(u_k)$ $(1\leq k\leq n)$ remain to be determined.

To determine $X(u_k)$ first substitute from Eq. (4.10) into Eq. (4.8) and take the limit as $X(t)$ as $t\to u_i$. With the aid of Eq. (4.9) we obtain

$$X(u_i)=\sum_{j=1}^{n}[I+u_jf(u_j)]\Gamma(u_j)Z_{ji}\tilde{\Gamma}(u_i)+\Gamma(u_i)\tilde{\Gamma}(u_i), \qquad (4.11)$$

where

$$Z_{ji}\equiv\alpha_j u_i\tilde{\Gamma}(u_j)\left(\frac{\Gamma(u_i)-\Gamma(u_j)}{u_i-u_j}\right), \qquad (4.12)$$

and the diagonal elements are obtained by taking the limit as $u_j\to u_i$. Equation (4.11) is equivalent to

$$\sum_{j=1}^{n}[I+u_jf(u_j)]\Gamma(u_j)(\delta_{ji}-Z_{ji})=\Gamma(u_i).$$

Therefore, we conclude that

$$X(u_k)=\sum_{i=1}^{n}\Gamma(u_i)[(I-Z)^{-1}]_{ik}\tilde{\Gamma}(u_k). \qquad (4.13)$$

When the $X(u_k)$ of Eq. (4.13) are substituted back into Eq. (4.10) one obtains the 3×3 matrix $f(t)$. Finally, Eq. (1.1) gives the potential $F(t)$ corresponding to the transformed electrovac spacetime. In the next section we shall show that in the case $n=1$ one obtains a family of metrics which includes the extreme charged Kerr-NUT solution. The latter arises when $F_0(s)$ is the potential associated with Minkowski space, i.e., the one given in Eq. (3.1).

V. ONE SIMPLE POLE

When $n=1$ Eq. (4.10) reduces to

$$f(t)=\alpha\frac{[X(u)-\Gamma(t)\tilde{\Gamma}(t)]}{t-u}\left[I-\alpha t\frac{\Gamma(t)\tilde{\Gamma}(t)}{t-u}\right], \qquad (5.1)$$

where

$$X(u)=\frac{\Gamma(u)\tilde{\Gamma}(u)}{1-\alpha u\tilde{\Gamma}(u)\dot{\Gamma}(u)}. \qquad (5.2)$$

In particular,

$$f(0)=-\frac{\alpha}{u}\left[\frac{\Gamma(u)\tilde{\Gamma}(u)}{1-\alpha u\tilde{\Gamma}(u)\dot{\Gamma}(u)}-\Gamma(0)\tilde{\Gamma}(0)\right]. \qquad (5.3)$$

The changes in the complex Ernst-potentials are therefore given by

$$\Delta\mathcal{E}=if_T{}^S(0)=-\frac{i\alpha}{u}\left[\frac{\Gamma_T(u)\tilde{\Gamma}^S(u)}{1-\alpha u\tilde{\Gamma}(u)\dot{\Gamma}(u)}-\Gamma_T(0)\tilde{\Gamma}^S(0)\right] \qquad (5.4)$$

and

$$\Delta\varphi=f_T{}^M(0)=-\frac{\alpha}{u}\left[\frac{\Gamma_T(u)\tilde{\Gamma}^M(u)}{1-\alpha u\tilde{\Gamma}(u)\dot{\Gamma}(u)}-\Gamma_T(0)\tilde{\Gamma}^M(0)\right]. \qquad (5.5)$$

We shall now specialize to a static vacuum seed metric, i.e., we assume that $F_0(s)$ is given by Eq. (3.2). In this case

$$\Gamma_S(s)=\frac{e^{-\psi(0)}}{2\lambda(s)}\left\{i[\lambda(s)+1-2sz]e^{\psi(s)}C^T(s)-[\lambda(s)-1+2sz]e^{-\psi(s)}\frac{C^S(s)}{s}\right\},$$
$$\Gamma_T(s)=\frac{se^{\psi(0)}}{\lambda(s)}\left(e^{\psi(s)}C^T(s)-ie^{-\psi(s)}\frac{C^S(s)}{s}\right), \qquad (5.6)$$
$$\Gamma_M(s)=C^M(s),$$

and

$$\tilde{\Gamma}^S(s)=se^{\psi(0)}\left(e^{\psi(s)}C^T(s)^*-ie^{-\psi(s)}\frac{C^S(s)^*}{s}\right),$$
$$\tilde{\Gamma}^T(s)=\tfrac{1}{2}e^{-\psi(0)}\left\{-i[\lambda(s)+1-2sz]e^{\psi(s)}C^T(s)^*+[\lambda(s)-1+2sz]e^{-\psi(s)}\frac{C^S(s)^*}{s}\right\}, \qquad (5.7)$$
$$\tilde{\Gamma}^M(s)=-\tfrac{1}{2}isC^M(s)^*.$$

Of course, the components of $C(s)$ are not independent, for by Eq. (4.3) we have

$$\tfrac{1}{2}i|C^M(s)|^2=C^T(s)\frac{C^S(s)^*}{s}-C^T(s)^*\frac{C^S(s)}{s}. \qquad (5.8)$$

From Eqs. (5.6) and (5.7) it follows that

$$\Gamma_T(0)\tilde{\Gamma}^S(0) = -|C^S(0)|^2$$

and

$$\Gamma_T(0)\tilde{\Gamma}^M(0) = 0\,.$$

Setting $\alpha = 1$, we obtain from Eqs. (5.4) and (5.5) the following expressions for the changes in the potentials $\mathcal{E}$ and φ:

$$\Delta\mathcal{E} = -iue^{2\psi(0)}\frac{\left(e^{\psi(u)}C^T(u) - ie^{-\psi(u)}\frac{C^S(u)}{u}\right)\left(e^{\psi(u)}C^T(u)^* - ie^{-\psi(u)}\frac{C^S(u)^*}{u}\right)}{\lambda(u)[1 - u\tilde{\Gamma}(u)\dot{\Gamma}(u)]} - \frac{i}{u}|C^S(0)|^2\,, \tag{5.9}$$

$$\Delta\varphi = \tfrac{1}{2}iue^{\psi(0)}\frac{\left(e^{\psi(u)}C^T(u) - ie^{-\psi(u)}\frac{C^S(u)}{u}\right)C^M(u)^*}{\lambda(u)[1 - u\tilde{\Gamma}(u)\dot{\Gamma}(u)]}\,. \tag{5.10}$$

It remains only to evaluate the denominator $\lambda(u)[1 - u\tilde{\Gamma}(u)\dot{\Gamma}(u)]$. We note that

$$\tilde{\Gamma}(u)\dot{\Gamma}(u) = C(u)^\dagger\,\mathfrak{E}(u)[\dot{C}(u) + F_0(u)^{-1}\dot{F}_0(u)C(u)]\,,$$

where

$$F_0(u)^{-1}\dot{F}_0(u) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\psi}(u) + \begin{bmatrix} e^{\psi(u)} & 0 & 0 \\ 0 & e^{-\psi(u)} & 0 \\ 0 & 0 & 1 \end{bmatrix} F^{\mathrm{MS}}(u)^{-1}\dot{F}^{\mathrm{MS}}(u) \begin{bmatrix} e^{-\psi(u)} & 0 & 0 \\ 0 & e^{\psi(u)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$F^{\mathrm{MS}}(u)^{-1}\dot{F}^{\mathrm{MS}}(u) = \frac{\lambda(u) - 1}{2u\lambda(u)}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + [\lambda(u)]^{-2}\begin{bmatrix} \frac{1 - 2uz}{2u} & i\frac{\lambda(u) + 1 - 2uz}{2} & 0 \\ i\frac{\lambda(u) - 1 + 2uz}{2u^2} & \frac{1 - 2uz}{2u} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} 1 - u\tilde{\Gamma}(u)\dot{\Gamma}(u) = 1 &- uC(u)^\dagger\,\mathfrak{E}(u)\dot{C}(u) - \frac{u}{2}\left(\frac{C^S(u)^*}{u}C^T(u) + C^T(u)^*\frac{C^S(u)}{u}\right)\left(2u\dot{\psi}(u) + \frac{\lambda(u) - 1}{\lambda(u)}\right) \\ &- \frac{iu}{2}[\lambda(u)]^{-2}\left(\left|\frac{C^S(u)}{u}\right|^2 e^{-2\psi(u)}[\lambda(u) - 1 + 2uz] - |C^T(u)|^2 e^{2\psi(u)}[\lambda(u) + 1 - 2uz]\right. \\ &\left. + \tfrac{1}{2}|C^M(u)|^2(1 - 2uz)\right)\,. \end{aligned}$$

Following Hoenselaers, Kinnersley, and Xanthopoulos[8] we introduce polar coordinates (r, θ) such that

$$\lambda(u) = 2ur\,, \quad 1 - 2uz = -2ur\cos\theta\,.$$

One then obtains

$$\begin{aligned} \lambda(u)[1 - u\tilde{\Gamma}(u)\mathring{\Gamma}(u)] = 2ur[1 - uC(u)^\dagger\,\mathfrak{E}(u)\dot{C}(u)] &- \frac{u}{2}\left(\frac{C^S(u)^*}{u}C^T(u) + C^T(u)^*\frac{C^S(u)}{u}\right)[(2u)^2 r\dot{\psi}(u) + 2ur - 1] \\ &- \frac{iu}{2}\left(\left|\frac{C^S(u)}{u}\right|^2 e^{-2\psi(u)}(1 + \cos\theta) - |C^T(u)|^2 e^{2\psi(u)}(1 - \cos\theta) - \tfrac{1}{2}|C^M(u)|^2\cos\theta\right)\,. \end{aligned} \tag{5.11}$$

Notice that the complex potentials depend only upon the values of $C(s)$ and $\dot{C}(s)$ at $s = u$, and not upon the detailed functional dependence of $C(s)$ on s. While there is a constant imaginary contribution to $\Delta\mathcal{E}$ which depends upon $C^S(0)$, such a contribution is of no physical significance. We shall for simplicity assume in the following that $C^S(0) = 0$.

When the seed metric is Minkowski space, both $\Delta\mathcal{E}$ and $\Delta\varphi$ are inversely proportional to an expression of the form $a + br + c\cos\theta$, where a, b, and c are constants, among which there is a relation derivable from Eq. (5.8). A detailed comparison with the known complex potentials of the extreme charged Kerr solution reveals that we have obtained a solution which differs from that one only by an Ehlers transformation. Thus, this is a natural electrovac generalization of the result found by Hoenselaers, Kinnersley, and Xanthopoulos[8] in the vacuum case.

Aside from the extreme charged Kerr solution, the electrovac spacetimes which correspond to the Ernst-potentials given in Eqs. (5.9)–(5.11) are probably all new, although the vacuum counterparts were found earlier.[8] All these solutions, of course, emerge from the $n=1$ specialization of our more general solution embodied in Eqs. (4.10) and (4.12)–(4.14).

VI. POLES OF HIGHER ORDER

It is unnecessary to consider separately the KC transformations induced by $\gamma(s)$ of the form (4.2) with $\beta(s)$ containing poles of order higher than the first, for the results of such transformations can be constructed from our solution (4.10) by permitting appropriate confluences of the simple poles at $s=u_i$ $(1 \leq i \leq n)$.

As a simple illustration we have considered the case $n=2$ with

$$u_1 = u+\epsilon\,,\quad \alpha_1 = \tfrac{1}{2}(\alpha+\beta\epsilon^{-1})\,,$$
$$u_2 = u-\epsilon\,,\quad \alpha_2 = \tfrac{1}{2}(\alpha-\beta\epsilon^{-1})\,, \tag{6.1}$$

where ϵ is a small parameter which we ultimately let tend toward zero. The solution $f(s)$ of the integral equation (1.2) corresponding to

$$\beta(s) = \frac{\alpha s}{s-u} + \frac{\beta s}{(s-u)^2} \tag{6.2}$$

can be obtained directly from our solution (4.8), and is given by

$$f(t) = \left(1 - \alpha q_1 - \beta\dot{q}_1 - \frac{\beta^2}{4} q_4\right)^{-1} \left(\frac{M_1 - \alpha N(t)}{t-u} + \frac{M_2 - \beta N(t)}{(t-u)^2}\right)\left[I - \left(\frac{\alpha}{t-u} + \frac{\beta}{(t-u)^2}\right) t N(t)\right], \tag{6.3}$$

where $N(t) \equiv \Gamma(t)\tilde{\Gamma}(t)$, and the t-independent fields are given by

$$M_1 \equiv \alpha N(u) + \beta\dot{N}(u) - \tfrac{1}{2}\beta^2[q_3 N(u) + \dot{q}_1\dot{N}(u) + q_2(\dot{\Gamma}(u)\tilde{\Gamma}(u) - \Gamma(u)\dot{\tilde{\Gamma}}(u)) - 2q_1\dot{\Gamma}(u)\dot{\tilde{\Gamma}}(u)]\,,$$
$$M_2 \equiv \beta N(u) - \tfrac{1}{2}\beta^2[(\dot{q}_1 - q_2)N(u) - 2q_1\dot{\Gamma}(u)\tilde{\Gamma}(u)]\,,$$
$$q_1 \equiv up\,,\quad q_2 \equiv u\dot{\tilde{\Gamma}}(u)\dot{\Gamma}(u) - p\,,$$
$$q_3 \equiv \tfrac{1}{2}u[-\ddot{p} + \tfrac{1}{3}\tilde{\Gamma}(u)\dddot{\Gamma}(u) + \ddot{\tilde{\Gamma}}(u)\dot{\Gamma}(u)] - \dot{p} - \dot{\tilde{\Gamma}}(u)\dot{\Gamma}(u)\,,$$
$$q_4 \equiv u^2[p\ddot{p} - \dot{p}^2 - p(\tfrac{1}{3}\tilde{\Gamma}(u)\dddot{\Gamma}(u) + \ddot{\tilde{\Gamma}}(u)\dot{\Gamma}(u)) + (\dot{\tilde{\Gamma}}(u)\dot{\Gamma}(u))^2]\,,$$

and

$$p \equiv \tilde{\Gamma}(u)\dot{\Gamma}(u)\,.$$

As a check upon the result (6.3) we evaluated explicitly the change in the complex $\mathcal{E}$-potential for the case when the seed metric is Minkowski space and

$$C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

In this case we obtained the result

$$\Delta\mathcal{E} = \frac{-2i\beta_0 r^3 + 4i\beta_1 r^2\cos\theta - 4\beta_1{}^2\sin^2\theta\cos\theta}{r^4 + i\beta_0 r^3(1-\cos\theta) - 2i\beta_1 r^2(1+\cos\theta - 2\cos^2\theta) - \beta_1{}^2\sin^2\theta(1-\cos\theta)^2}\,, \tag{6.4}$$

where, again following Ref. 8, we have introduced coordinates (r, θ) such that

$$\lambda(u) = 2ur\,,\quad 1 - 2uz = -2ur\cos\theta\,,\quad \rho = r\sin\theta\,,$$

and new parameters

$$\beta_0 \equiv \tfrac{1}{4}\alpha\,,\quad \beta_1 \equiv \beta/16u^2\,.$$

The result (6.4) agrees with that given in Ref. 8 for the case of a combined "rank 0" and "rank 1" transformation of Minkowski space, thus lending credibility to the more general result (6.3).

VII. CONCLUDING REMARKS

The breakthrough made by Kinnersley and Chitre which was subsequently developed further by Hoenselaers, Kinnersley, and Xanthopoulos has significantly altered the character of the stationary axially symmetric field problem. Previously there existed no constructive technique for generating new solutions of arbitrary complexity which did not suffer from the blemish of producing solutions with undesirable asymptotic properties. Now, using KC transformations, one can produce asymptotically flat solutions characterized by arbitrarily prescribed multipole moments. Admittedly the procedures which have been developed thus far are tedious when compared to simple Ehlers or Harrison transformations, but they do work, and in principle the calculations could be done on an electronic data processor. As a result we no longer feel that the primary objective should remain the working out of specific vacuum and electrovac solutions. Rather, one should take advantage of the insights which the development of the KC transformation theory has provided in order to discover a really clever way of describing the general solution. It is our hope that the introduction of complex variable techniques via our integral equation formulation of the KC transformation theory will facilitate the discovery of such a description.

ACKNOWLEDGMENT

This work was supported in part by National Science Foundation Grants No. PHY75-08750 and No. PHY 79-08627.

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[1]I. Hauser and F. J. Ernst, Phys. Rev. 20, 362 (1979) (henceforth designated as I).

[2]W. Kinnersley and D. M. Chitre, J. Math. Phys. 18, 1538 (1977); 19, 1926 (1978); 19, 2037 (1978).

[3]In particular, we have found that the analysis and the solving of the integral equation is facilitated by considering the equivalent *homogeneous Hilbert problem in several unknown functions*, whose theory is given, for example, in Chap. 18 of *Singular Integral Equations* by N. I. Muskhelishvili, edited by J. Radok (Noordhoff, Groningen, The Netherlands, 1953). A paper on the applications of the homogeneous Hilbert problem for our particular class of kernels is being prepared by the authors.

[4]$[\gamma(s)]^\dagger$ means the Hermitian conjugate of $\gamma(s^*)$, and $[\gamma(s)]^*$ means the complex conjugate of $\gamma(s^*)$, where s^* denotes the complex conjugate of s. With this understanding, the statement that $\gamma(s)$ is *Hermitian* means $[\gamma(s)]^\dagger=\gamma(s)$. Likewise, we say that $\mathfrak{G}(s)$ is *anti-Hermitian*, which means $[\mathfrak{G}(s)]^\dagger=-\mathfrak{G}(s)$.

[5]Our notations and conventions concerning differential forms, Grassmann products, duality operations, etc., were described in an appendix to the paper by I. Hauser and F. J. Ernst, J. Math. Phys. 19, 1316 (1978).

[6]If instead one considers spacetimes with two commuting spacelike Killing vectors, then the transformation theory applies, but one must alter a few signs in the equations contained in this section.

[7]T. Jones, J. Math. Phys. (to be published).

[8]C. Hoenselaers, W. Kinnersley, and B. Xanthopoulos, Phys. Rev. Lett. 42, 481 (1979); J. Math. Phys. (to be published).