Higher-order asymptotic-freedom corrections to photon-photon scattering

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We generalize Witten's calculation of the photon structure function F_2^γ to the next-to-the-leading order of asymptotic freedom. Except for the second moment of F_2^γ, the result is independent of the unknown matrix elements of quark and gluon operators between the photon states. The nonleading corrections turn out to be large.

I. INTRODUCTION

It is well known by now that asymptotic-freedom predictions[1,2] as calculated in the leading order of the effective coupling constant are consistent[3] with the scaling violations observed in deep-inelastic data. The theoretical calculations needed to obtain these leading-order predictions are rather straightforward and have been obtained already several years ago.[1,2] On the other hand, the calculations of the next-to-the-leading-order asymptotic-freedom effects are much more involved and have only been studied during the past two years.[4-9] These next-to-the-leading-order effects are now theoretically well understood although their detailed confrontation with the data remains to be done.

So far the calculations of higher-order corrections have concentrated on deep-inelastic scattering off hadronic targets. In spite of the fact that corrections in question are not small, they can be absorbed to a large extent in the redefinition of the parameter Λ, the sole free parameter of the theory.[6] The phenomenological study of higher-order corrections is complicated by the fact that any asymptotic-freedom expression for the hadronic deep-inelastic structure functions involves matrix elements of local operators between hadronic states which are uncalculable by present methods. These matrix elements must be treated as free parameters in fitting data. Since the magnitude of higher-order corrections varies only slowly with Q^2 some of the higher-order effects can be absorbed (in the range of Q^2 available experimentally) in the unknown matrix elements in question.[6] It is therefore of interest to look for processes which at least in the first few orders in the effective coupling constant are free of the unknown matrix elements of local operators.

One such process is the deep-inelastic scattering off photon targets. This process can be studied in e^+e^- collisions[10] as shown in Fig. 1 where one of the virtual photons is very far off shell (large Q^2) and the other one is close to the mass shell (small p^2).

In quantum chromodynamics (QCD) the process of Fig. 1 has been analyzed by Witten[11] using operator-product-expansion and renormalization-group methods. He has obtained definite predictions for the photon structure functions which in the leading order of asymptotic freedom are independent of the unknown hadronic matrix elements. The asymptotic-freedom result for the shape of the photon structure function F_2^γ, differs substantially from simple parton-model predictions.[12,13] Witten's result has been recently rederived by Llewellyn Smith[14] in the framework of perturbative QCD and by DeWitt *et al.*[15] and Brodsky *et al.*[16] in the framework of the Altarelli-Parisi approach.[17] For a recent review of the phenomenological implications of these results we refer the interested reader to Refs. 14 and 16.

If we write generally the moments of $F_2^\gamma(x, Q^2)$

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x, Q^2) = a_n \ln\frac{Q^2}{\Lambda^2} + \tilde{a}_n \ln\ln\frac{Q^2}{\Lambda^2} + b_n + O\left(\frac{1}{\ln(Q^2/\Lambda^2)}\right), \tag{1.1}$$

then what Witten has calculated are the coefficients a_n.

In this paper we shall extend Witten's calculation to higher orders and evaluate the constants $\tilde{a}_n$ and b_n. As observed by Witten[11] the constants b_n and

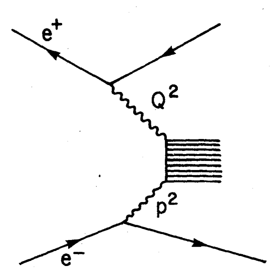

FIG. 1. Dominant contribution to the process $e^+e^- \to$ hadrons $+ e^+e^-$.

$\tilde{a}_n$ do not depend on the unknown matrix elements of local operators except for b_2. In other words for the first three terms in the expansion in Eq. (1.1), we obtain for $n>2$ definite asymptotic-freedom predictions in terms of a single free parameter Λ. For $n=2$ there is an additional free parameter in b_2 which involves the photon matrix element of the hadronic energy momentum tensor.

It is obvious from the above that the process under consideration is, from a theoretical point of view, an excellent place to study properties of higher-order corrections. Unfortunately, experimental tests of our results may prove to be difficult.

Our paper is organized as follows. In Sec. II and following Witten we derive a formal expression for the moments of F_2^γ valid to any order in the effective quark gluon coupling $\bar{g}^2$ and to first order in the electromagnetic coupling e^2. Using this expression we find in Sec. III the parameters a_n, b_n, and $\tilde{a}_n$ of Eq. (1.1) in terms of one-loop and two-loop anomalous dimensions, one-loop and two-loop contributions to the β function, and one-loop corrections to the Wilson coefficient functions. Section IV contains all information needed for the numerical evaluation of the parameters a_n, $\tilde{a}_n$, and b_n. Numerical results and their discussion are presented in Sec. V. For completeness we include formulas for the longitudinal photon structure function in Sec. VI. Section VII contains a brief summary of our paper.

II. BASIC FORMALISM

In the short-distance analysis the moments of the photon structure function $F_2^\gamma(x,Q^2)$ are given as follows[18]:

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x,Q^2) = \sum_i C_n^i\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right)\langle\gamma|O_i^n|\gamma\rangle, \tag{2.1}$$

where $Q^2=-q^2$, x is the Bjorken variable, g^2 is the renormalized strong coupling constant, μ^2 is the subtraction scale at which the theory is renormalized, and $\alpha=e^2/4\pi$ is the electromagnetic coupling constant. The sum on the right-hand side, of Eq. (2.1) runs over spin-n, twist-2 operators such as fermion nonsinglet operator $O_{\rm NS}$, singlet fermion and gluon operators O_ψ and O_G, and photon operator O_γ. The latter operator which is not present in the deep-inelastic scattering off hadronic targets, is the analog of the gluon operator O_G with the non-Abelian field strength tensor $G_{\alpha\beta}$ replaced by the electromagnetic tensor $F_{\alpha\beta}$. As noted by Witten,[11] O_γ must be included in the analysis of photon-photon scattering. The reason is that although the Wilson coefficients C_n^γ are $O(\alpha)$ the matrix elements $\langle\gamma|O_\gamma^n|\gamma\rangle$ are $O(1)$. Therefore, the photon contribution in Eq. (2.1) is of the same order in α as the contributions of quark and gluon operators. The latter have Wilson coefficients $O(1)$, but matrix elements in photon states $O(\alpha)$. We want to evaluate Eq. (2.1) to lowest order in α but to all orders in g.

In what follows it will be useful to work with matrix notation. The coefficient functions are described by the column vector

$$\vec{C}_n\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right)=\begin{bmatrix} C_n^\psi\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) \\ C_n^G\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) \\ C_n^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) \\ C_n^\gamma\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right)\end{bmatrix}. \tag{2.2}$$

The renormalization-group equation which governs the Q^2 dependence of $\vec{C}_n$ can then to lowest order in α be written as follows:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\vec{C}_n\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) = \gamma_n(g^2,\alpha)\vec{C}_n\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right), \tag{2.3}$$

where $\gamma_n(g^2,\alpha)$ is the anomalous-dimension matrix whose elements are equal to the elements of the *transposed* anomalous-dimension matrix as defined by Gross and Wilczek.[2] To lowest order in α this matrix has the form

$$\gamma_n(g^2,\alpha)=\begin{bmatrix}\hat{\gamma}_n(g^2) & 0 \\ \vec{K}_n(g^2,\alpha) & 0\end{bmatrix}, \tag{2.4}$$

with $\hat{\gamma}_n(g^2)$ being the standard hadronic anomalous-dimension matrix

$$\hat{\gamma}_n(g^2)=\begin{bmatrix}\gamma_{\psi\psi}^n(g^2) & \gamma_{G\psi}^n(g^2) & 0 \\ \gamma_{\psi G}^n(g^2) & \gamma_{GG}^n(g^2) & 0 \\ 0 & 0 & \gamma_{\rm NS}^n(g^2)\end{bmatrix}, \tag{2.5}$$

and $\vec{K}(g^2,\alpha)$ standing for the three-component row vector

$$\vec{K}_n(g^2,\alpha)=[K_\psi^n(g^2,\alpha), K_G^n(g^2,\alpha), K_{\rm NS}^n(g^2,\alpha)]. \tag{2.6}$$

The vector $\vec{K}_n$ represents the mixing between the photon operator and the remaining three operators.

In the notation of Eq. (2.5), its components are $\gamma^n_{\psi\gamma}$, $\gamma^n_{G\gamma}$, and $\gamma^n_{\mathrm{NS}\gamma}$. We prefer, however, to use separate notation for the mixing in question because it depends on both g and α. It is also the notation of Witten.[11]

The solution of (2.3) is given by[1,2]

$$\vec{C}_n\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) = \left[T\exp\left(\int_{\bar{g}(Q^2)}^{g} dg' \, \frac{\gamma_n(g'^2,\alpha)}{\beta(g')}\right)\right] \times \vec{C}_n(1,\bar{g}^2,\alpha)\,, \tag{2.7}$$

with $\bar{g}^2$ being the effective strong-interaction coupling constant which satisfies the following equation:

$$\frac{d\bar{g}^2}{dt} = \bar{g}\,\beta(\bar{g})\,; \quad \bar{g}(t=0) = g\,. \tag{2.8}$$

Here $t = \ln(Q^2/\mu^2)$. The T ordering in Eq. (2.7) is necessary because $[\gamma(g'^2), \gamma(g''^2)] \neq 0$ and is defined as follows:

$$T\exp\left[\int_{\bar{g}}^{g} dg' \, \frac{\gamma(g'^2)}{\beta(g')}\right] = 1 + \int_{\bar{g}}^{g} dg' \, \frac{\gamma(g'^2)}{\beta(g')} + \int_{\bar{g}}^{g} dg' \int_{\bar{g}}^{g'} dg'' \, \frac{\gamma(g'^2)}{\beta(g')}\,\frac{\gamma(g''^2)}{\beta(g'')} + \cdots\,. \tag{2.9}$$

Writing the T-ordered exponential as

$$T\exp\left[\int_{\bar{g}}^{g} dg' \, \frac{\gamma_n(g'^2)}{\beta(g')}\right] \equiv \begin{pmatrix} M_n & 0 \\ \vec{X}_n & 1 \end{pmatrix}, \tag{2.10}$$

where M_n is a 3 by 3 matrix and $\vec{X}_n$ is a three-component row vector, we find from (2.4) and (2.10)

$$M_n\left(\frac{Q^2}{\mu^2}, g^2\right) = T\exp\left[\int_{\bar{g}}^{g} dg' \, \frac{\hat{\gamma}_n(g'^2)}{\beta(g')}\right] \tag{2.11}$$

and

$$\vec{X}_n\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) = \int_{\bar{g}}^{g} dg' \, \frac{\vec{K}_n(g'^2,\alpha)}{\beta(g')} \, T\exp\left[\int_{\bar{g}}^{g'} dg'' \, \frac{\hat{\gamma}_n(g''^2)}{\beta(g'')}\right]. \tag{2.12}$$

On the other hand, Eqs. (2.1), (2.7), and (2.10) give

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x,Q^2) = \sum_i \langle\gamma|O_i^n|\gamma\rangle \left[M_n\left(\frac{Q^2}{\mu^2}, g^2\right)\vec{C}_n(1,\bar{g}^2,\alpha)\right]_i + \vec{X}_n\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right)\vec{C}_n(1,\bar{g}^2,\alpha) + C_n^\gamma(1,\bar{g}^2,\alpha) \tag{2.13}$$

where i now runs over ψ, G, and NS and

$$\vec{C}_n(1,\bar{g}^2,\alpha) \equiv \begin{bmatrix} C_n^\psi(1,\bar{g}^2,\alpha) \\ C_n^G(1,\bar{g}^2,\alpha) \\ C_n^{\mathrm{NS}}(1,\bar{g}^2,\alpha) \end{bmatrix}. \tag{2.14}$$

In addition, we have used the fact that

$$\langle\gamma|O_\gamma|\gamma\rangle = 1\,. \tag{2.15}$$

Equation (2.13) is valid to any order in the effective coupling constant $\bar{g}^2$ and to the first order in α. The only unknown quantities in Eq. (2.13) are the matrix elements $\langle\gamma|O_i^n|\gamma\rangle$. Fortunately, due to asymptotic freedom the sum $\sum_i$ in Eq. (2.13), except for $n=2$, goes to zero for large Q^2 as in the case of deep-inelastic scattering on hadronic targets. It then follows[11] that the parameters a_n and $\tilde{a}_n$ for all n and b_n for $n>2$ can be found by evaluating the last two terms of Eq. (2.13). We shall now evaluate these terms and consequently a_n, $\tilde{a}_n$, and b_n.

III. PHOTON STRUCTURE FUNCTION BEYOND THE LEADING ORDER

We begin with the evaluation of the T-ordered exponential which enters Eq. (2.12). We first expand the anomalous-dimension matrix $\hat{\gamma}_n(g^2)$ in powers of g:

$$\gamma^n_{ij}(g^2) = \gamma^{0,n}_{ij}\,\frac{g^2}{16\pi^2} + \gamma^{(1),n}_{ij}\,\frac{g^4}{(16\pi^2)^2} + \cdots\,, \quad i,j = \psi, G\,, \tag{3.1}$$

$$\gamma^n_{\mathrm{NS}}(g^2) = \gamma^{0,n}_{\mathrm{NS}}\,\frac{g^2}{16\pi^2} + \gamma^{(1),n}_{\mathrm{NS}}\,\frac{g^4}{(16\pi^2)^2} + \cdots\,. \tag{3.2}$$

Then writing in an obvious notation

$$\hat{\gamma}_n(g^2) = \hat{\gamma}_n^0(g^2) + \hat{\gamma}_n^{(1)}(g^2) + \cdots \tag{3.3}$$

and using Eq. (2.9), we obtain neglecting terms $O(\bar{g}^4)$

$$T\exp\left[\int_{\bar g}^{g} dg' \frac{\hat\gamma_n(g'^2)}{\beta(g')}\right]=\exp\left[\int_{\bar g}^{g} dg' \frac{\hat\gamma_n^0(g'^2)}{\beta(g')}\right] + \int_{\bar g}^{g} dg' \exp\left[\int_{g'}^{g} dg'' \frac{\hat\gamma_n^0(g''^2)}{\beta(g'')}\right]\frac{\hat\gamma_n^{(1)}(g'^2)}{\beta(g')}\exp\left[\int_{\bar g}^{g'} dg'' \frac{\hat\gamma_n^0(g''^2)}{\beta(g'')}\right]. \quad (3.4)$$

To proceed further we evaluate

$$\exp\left[\int_{g_2}^{g_1} dg' \frac{\hat\gamma_n^0(g'^2)}{\beta(g')}\right] \quad (3.5)$$

which appears three times in Eq. (3.4). This is easily done by writing

$$\hat\gamma_n^0(g^2)=\frac{g^2}{16\pi^2}\sum_i \lambda_i^n P_i^n , \quad (3.6)$$

where the λ_i^n are the eigenvalues of the matrix $\hat\gamma_n^0$ and P_i^n are the corresponding projection operators. Explicitly

$$P_\pm^n=\frac{1}{\lambda_\pm^n-\lambda_\mp^n}\begin{bmatrix}\gamma_{\psi\psi}^{0,n}-\lambda_\mp^n & \gamma_{G\psi}^{0,n} & 0\\ \gamma_{\psi G}^{0,n} & \gamma_{GG}^{0,n}-\lambda_\mp^n & 0\\ 0 & 0 & 0\end{bmatrix}, \quad (3.7)$$

$$P_{\rm NS}^n=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \quad (3.8)$$

$$\lambda_\pm^n=\tfrac{1}{2}\{\gamma_{\psi\psi}^{0,n}+\gamma_{GG}^{0,n} \pm[(\gamma_{\psi\psi}^{0,n}-\gamma_{GG}^{0,n})^2+4\gamma_{\psi G}^{0,n}\gamma_{G\psi}^{0,n}]^{1/2}\} \quad (3.9)$$

and

$$\lambda_{\rm NS}^n=\gamma_{\rm NS}^{0,n}. \quad (3.10)$$

Using the known properties of the projection operators

$$P_i^nP_j^n=\begin{cases}0, & i\neq j,\\ P_i^n, & i=j,\end{cases} \quad (3.11)$$

and expanding $\beta(g)$ as follows:

$$\beta(g)=-\frac{g^3}{16\pi^2}\beta_0-\frac{g^5}{(16\pi^2)^2}\beta_1, \quad (3.12)$$

we obtain to the desired order

$$\exp\left[\int_{g_2}^{g_1} dg' \frac{\hat\gamma_n^0(g'^2)}{\beta(g')}\right]=\sum_i P_i^n\left(\frac{g_2^{\,2}}{g_1^{\,2}}\right)^{\lambda_i^n/2\beta_0}\left(1+\frac{\lambda_i^n}{2\beta_0^{\,2}}\beta_1\frac{g_1^{\,2}-g_2^{\,2}}{16\pi^2}\right). \quad (3.13)$$

Inserting (3.13) into (3.4) we finally obtain

$$T\exp\left(\int_{\bar g}^{g} dg' \frac{\hat\gamma_n(g'^2)}{\beta(g')}\right)=\sum_i P_i^n\left(\frac{\bar g^2}{g^2}\right)^{\lambda_i^n/2\beta_0}\left(1+\frac{\lambda_i^n}{2\beta_0^{\,2}}\beta_1\frac{g^2-\bar g^2}{16\pi^2}\right) - \frac{g^2}{16\pi^2}\sum_{i,j}\frac{P_i^n\hat\gamma_n^{(1)}P_j^n}{2\beta_0+\lambda_i-\lambda_j}\left(\frac{\bar g^2}{g^2}\right)^{\lambda_j^n/2\beta_0}\left[1-\left(\frac{\bar g^2}{g^2}\right)^{1+(\lambda_i-\lambda_j)/2\beta_0}\right]. \quad (3.14)$$

We next expand the components of $\vec{\rm K}_n$ as follows:

$$K_j^n(g^2,\alpha)=-\frac{e^2}{16\pi^2}K_j^{0,n}-\frac{e^2g^2}{(16\pi^2)^2}K_j^{(1),n}; \quad j=\psi,\ {\rm NS} \quad (3.15)$$

and

$$K_G^n(g^2,\alpha)=-\frac{e^2g^2}{(16\pi^2)^2}K_G^{(1),n}. \quad (3.16)$$

Inserting Eqs. (3.14)–(3.16) into (2.12) we obtain, after dropping terms which vanish for $Q^2\to\infty$,

$$\vec{\rm X}_n\left(\frac{Q^2}{\mu^2},g^2,\alpha\right)=\frac{e^2}{2\beta_0}\frac{\vec{\rm K}_n^0}{\bar g^2}\sum_i P_i^n\frac{1}{1+\lambda_i^n/2\beta_0} - \frac{e^2}{16\pi^2}\frac{\vec{\rm K}_n^0}{2\beta_0}\left[2\beta_1\sum_i\frac{P_i^n}{(1+\lambda_i^n/2\beta_0)\lambda_i^n}+\sum_{i,j}\frac{P_i^n\hat\gamma_n^{(1)}P_j^n}{\lambda_j^n(1+\lambda_i^n/2\beta_0)}\right]+\frac{e^2}{16\pi^2}\vec{\rm K}_n^{(1)}\sum_i\frac{P_i^n}{\lambda_i^n}, \quad (3.17)$$

where we have defined two row vectors

$$\vec{K}_n^{(0)} = [K_\psi^{0,n}, 0, K_{\rm NS}^{0,n}] \qquad (3.18a)$$

and

$$\vec{K}_n^{(1)} = [K_\psi^{(1)n}, K_G^{(1)n}, K_{\rm NS}^{(1)n}]. \qquad (3.18b)$$

Equation (3.17) is valid for $n>2$. We next write

$$C_n^j(1,\bar{g}^2,\alpha) = \begin{cases} e^2\delta_\psi\left(1+\dfrac{\bar{g}^2}{16\pi^2}B_\psi^n\right), & j=\psi, \\ e^2\delta_\psi\dfrac{\bar{g}^2}{16\pi^2}B_G^n, & j=G, \\ e^2\delta_{\rm NS}\left(1+\dfrac{\bar{g}^2}{16\pi^2}B_{\rm NS}^n\right), & j={\rm NS} \\ \dfrac{e^4}{16\pi^2}\delta_\gamma B_\gamma^n, & j=\gamma, \end{cases} \qquad (3.19)$$

where δ_j depend on the quark charges and are given in Sec. IV.

Inserting (3.17) and (3.19) into Eq. (2.13) we finally obtain for $n>2$

$$\int_0^1 dx\,x^{n-2}F_2^\gamma(x,Q^2) = \alpha^2\left(\frac{16\pi^2}{\beta_0\bar{g}^2}a_n + b_n\right) = \alpha^2\left(a_n\ln\frac{Q^2}{\Lambda^2} + \tilde{a}_n\ln\ln\frac{Q^2}{\Lambda^2} + b_n\right). \qquad (3.20)$$

In order to obtain the last formula we have expanded $\bar{g}^2(Q^2)$, the solution of Eq. (2.8) with $\beta(g)$ given by (3.12), in powers of $\bar{g}_0^2(Q^2)$, the effective coupling constant calculated in the one-loop approximation, with the result

$$\bar{g}^2(Q^2) = \bar{g}_0^2(Q^2) - \frac{\beta_1}{\beta_0}\frac{\bar{g}_0^4(Q^2)}{16\pi^2}\ln\ln\frac{Q^2}{\Lambda^2} + O(\bar{g}_0^6), \qquad (3.21)$$

where

$$\bar{g}_0^2(Q^2) = \frac{16\pi^2}{\beta_0\ln(Q^2/\Lambda^2)}. \qquad (3.22)$$

The parameters a_n, $\tilde{a}_n$, and b_n are given as follows:

$$a_n = \frac{1}{2}\left[\frac{K_\psi^{0,n}\delta_\psi}{d}\left(1+\frac{\gamma_{GG}^{0,n}}{2\beta_0}\right) + \frac{K_{\rm NS}^{0,n}\delta_{\rm NS}}{1+\gamma_{\rm NS}^{0,n}/2\beta_0}\right], \qquad (3.23)$$

$$\tilde{a}_n = \frac{\beta_1}{\beta_0^2}a_n, \quad n>2 \qquad (3.24)$$

and

$$b_n = \frac{K_\psi^{(1),n}\delta_\psi\gamma_{GG}^{0,n}}{\lambda_+^n\lambda_-^n} - \frac{K_G^{(1),n}\delta_\psi\gamma_{\psi G}^{0,n}}{\lambda_+^n\lambda_-^n} + \frac{K_{\rm NS}^{(1),n}\delta_{\rm NS}}{\gamma_{\rm NS}^{0,n}} + \frac{1}{2\beta_0}(K_\psi^{0,n}\delta_\psi R_\psi^n + K_{\rm NS}^{0,n}\delta_{\rm NS}R_{\rm NS}^n) + B_\gamma^n\delta_\gamma, \quad n>2. \qquad (3.25)$$

Here, we have defined

$$d \equiv \left(1+\frac{\lambda_-^n}{2\beta_0}\right)\left(1+\frac{\lambda_+^n}{2\beta_0}\right), \qquad (3.26)$$

$$R_\psi^n = \frac{B_\psi^n}{d}\left(1+\frac{\gamma_{GG}^{0,n}}{2\beta_0}\right) - \frac{B_G^n}{d}\frac{\gamma_{G\psi}^{0,n}}{2\beta_0} + \frac{\Delta^n}{d\lambda_+^n\lambda_-^n} - \frac{\beta_1}{\beta_0}\frac{(\gamma_{GG}^{0,n})^2 + 2\beta_0\gamma_{GG}^{0,n} + \gamma_{\psi G}^{0,n}\gamma_{G\psi}^{0,n}}{d\lambda_+^n\lambda_-^n}, \qquad (3.27)$$

$$R_{\rm NS}^n = \frac{B_{\rm NS}^n}{1+\gamma_{\rm NS}^{0,n}/2\beta_0} - \frac{\gamma_{\rm NS}^{(1),n} + 2\beta_1}{\gamma_{\rm NS}^{0,n}(1+\gamma_{\rm NS}^{0,n}/2\beta_0)} \qquad (3.28)$$

and

$$\Delta^n = \gamma_{G\psi}^{(1),n}\gamma_{\psi G}^{0,n} - \gamma_{\psi\psi}^{(1),n}\gamma_{GG}^{0,n} + \frac{\gamma_{G\psi}^{0,n}\gamma_{G\psi}^{(1),n}\gamma_{GG}^{0,n} + \gamma_{GG}^{0,n}\gamma_{G\psi}^{(1),n}\gamma_{\psi G}^{0,n} - \gamma_{\psi G}^{0,n}\gamma_{GG}^{(1),n}\gamma_{G\psi}^{0,n} - \gamma_{GG}^{0,n}\gamma_{\psi\psi}^{(1),n}\gamma_{GG}^{0,n}}{2\beta_0}. \qquad (3.29)$$

Equations (3.20) and (3.23)–(3.25) are the main results of our paper. Equation (3.23) has been previously obtained by Witten.[11] On the other hand, Eqs. (3.24) and (3.25) are new.

The equations above are valid for $n>2$. For completeness we quote the result for $n=2$. Since this moment depends on the unknown photon matrix element of the hadronic energy-momentum tensor, we shall not use it in the numerical calculations.

For $n=2$ we have

$$\int_0^1 dx F_2^\gamma(x,Q^2) = \alpha^2\left(\frac{16\pi^2}{\beta_0\bar{g}^2}a_2 + a_2'\ln\bar{g}^2 + b_2'\right) \qquad (3.30)$$

$$= \alpha^2\left(a_2\ln\frac{Q^2}{\Lambda^2} + \tilde{a}_2\ln\ln\frac{Q^2}{\Lambda^2} + b_2\right), \qquad (3.31)$$

where a_2 is evaluated from (3.23) and

$$\tilde{a}_2 = \frac{\beta_1}{\beta_0{}^2} a_2 - a_2' , \tag{3.32}$$

with a_2' given as follows:

$$a_2' = -\frac{\delta_\psi \gamma^0_{GG}}{2\lambda_+ \beta_0}\left(K_\psi^{(1)} + K_G^{(1)} - \frac{\beta_1}{\beta_0} K_\psi^{(0)}\right). \tag{3.33}$$

The index $n=2$ has been dropped on the right-hand side of the Eq. (3.33).

We finally quote for comparison the asymptotic behavior of the moments of the photon structure function as obtained in the simple parton model (PM)

$$\int_0^1 dx\, x^{n-2} F_2^\gamma\big|_{\rm PM} = \alpha^2 P_n \ln \frac{Q^2}{\Lambda_{\rm PM}{}^2} , \tag{3.34}$$

where

$$P_n = 4\delta_\gamma \frac{n^2+n+2}{n(n+1)(n+2)} . \tag{3.35}$$

Notice that P_n can be obtained from a_n by putting there all anomalous dimensions but $K_\psi^{0,n}$ and $K_{\rm NS}^{0,n}$ equal to zero. We shall now give all the information needed for the numerical evaluation of a_n, $\tilde{a}_n$, and b_n.

IV. MAGIC NUMBERS OF ASYMPTOTIC FREEDOM

All the quantities necessary to evaluate a_n, $\tilde{a}_n$, and b_n have been already calculated in the literature. It has been recognized[5] in the past year that anomalous dimensions in two loops and the $\bar{g}^2$ corrections to $C_n^i(1,\bar{g}^2)$, i.e., B_i^n, are renormalization-prescription dependent. Any physical quantity cannot of course depend on renormalization-scheme, and the renormalization-prescription dependences of B_i^n and of two-loop anomalous dimensions cancel in the expressions for physical quantities. However, in order for the cancellation to occur both B_i^n and $\gamma_n^{(1)}$ have to be calculated in the same scheme. In what follows all the expressons listed in this section correspond to 't Hooft's minimal-subtraction scheme.[19] In fact, this is the only scheme at present in which all the quantities relevant for our calculation are known.[5,6] A nice property of this scheme is that all quantities below are gauge independent.[20]

The formulas of this section are for SU(3) color gauge theory with f flavors. Quarks may have arbitrary charges, although our results depend only on the average charge squared $\langle e^2\rangle$ and the average of the fourth power of the charge $\langle e^4\rangle$.

A. Anomalous dimensions in one loop

For the pure hadronic sector anomalous dimensions in one-loop approximations have been calculated in Refs. 1 and 2. They are[2]

$$\gamma_{\psi\psi}^{0,n} = \gamma_{\rm NS}^{0,n} = \frac{8}{3}\left(1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n}\frac{1}{j}\right), \tag{4.1}$$

$$\gamma_{\psi G}^{0,n} = -4f\,\frac{(n^2+n+2)}{n(n+1)(n+2)} , \tag{4.2}$$

$$\gamma_{G\psi}^{0,n} = -\frac{16}{3}\,\frac{(n^2+n+2)}{n(n^2-1)} , \tag{4.3}$$

$$\gamma_{GG}^{0,n} = 6\left(\frac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_{j=2}^{n}\frac{1}{j}\right) + \frac{4}{3} f. \tag{4.4}$$

The anomalous dimensions $K_\psi^{0,n}$ and $K_{\rm NS}^{0,n}$ are obtained from (4.2) by changing group-theory factors. They are

$$K_\psi^{0,n} = 8\,\frac{n^2+n+2}{n(n+1)(n+2)}\,3f\langle e^2\rangle \tag{4.5}$$

and

$$K_{\rm NS}^{0,n} = 8\,\frac{n^2+n+2}{n(n+1)(n+2)}\,3f(\langle e^4\rangle - \langle e^2\rangle^2) . \tag{4.6}$$

B. Two-loop anomalous dimensions

For the hadronic sector the two-loop anomalous dimensions have been calculated in Ref. 5. We give only their numerical values in Table I since the corresponding analytic expressions of Ref. 5 are rather complicated. The nondiagonal elements differ by a sign from those of Ref. 5 as we use the definitions of Gross and Wilczek.[2] The two-loop anomalous dimensions $K_\psi^{(1),n}$, $K_{\rm NS}^{(1),n}$, and $K_G^{(1),n}$ can be obtained from $\gamma_{\psi G}^{(1),n}$ and $\gamma_{GG}^{(1),n}$ by picking in the relevant formulas of Ref. 5 the terms proportional to $C_F T(R)$, removing $T(R)$, and inserting relevant charge factors as in Eqs. (4.5) and (4.6). As the result of this procedure we obtain

$$K_\psi^{(1),n} = \tfrac{4}{3} B_n^{fg} 3f\langle e^2\rangle , \tag{4.7}$$

$$K_{\rm NS}^{(1),n} = \tfrac{4}{3} B_n^{fg} 3f(\langle e^4\rangle - \langle e^2\rangle^2) , \tag{4.8}$$

$$K_G^{(1),n} = -\tfrac{4}{3} B_n^{gg} 3f\langle e^2\rangle , \tag{4.9}$$

where the values of B^{fg} and B^{gg} can be found in Ref. 5. Numerical values for $K_\psi^{(1),n}$, $K_{\rm NS}^{(1),n}$, and $K_G^{(1),n}$ are given in Table II.

C. One-loop corrections to $C_n(1,\bar{g}^2)$

These corrections have been calculated in Ref. 6 and recalculated in Ref. 5. We have

$$B_{\rm NS}^n = B_\psi^n = \frac{4}{3}\left(3\sum_{j=1}^{n}\frac{1}{j} - 4\sum_{j=1}^{n}\frac{1}{j^2} - \frac{2}{n(n+1)}\sum_{j=1}^{n}\frac{1}{j} + 4\sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j} + \frac{3}{n} + \frac{4}{n+1} + \frac{2}{n^2} - 9\right) + \frac{1}{2}\gamma_{\psi\psi}^{0,n}(\ln 4\pi - \gamma_E) , \tag{4.10}$$

TABLE I. Coefficients of $g^4/(16\pi^2)^2$ in the anomalous dimensions $\gamma_{NS}^{(1)}$, $\gamma_{\psi\psi}^{(1)}$, $\gamma_{\psi G}^{(1)}$, $\gamma_{G\psi}^{(1)}$, and $\gamma_{GG}^{(1)}$ for $f=3$ and $f=4$. This table has been calculated on the basis of the results of Ref. 5.

	$\gamma_{NS}^{(1)}$		$\gamma_{\psi\psi}^{(1)}$		$\gamma_{\psi G}^{(1)}$		$\gamma_{G\psi}^{(1)}$		$\gamma_{GG}^{(1)}$	
n	3	4	3	4	3	4	3	4	3	4
2	77.70	71.37	65.84	55.56	−45.25	−60.34	−65.84	−55.56	45.25	60.34
4	133.25	120.14	132.6	119.28	7.75	10.34	−28.64	−27.40	178.9	151.61
6	164.26	147.00	164.1	146.82	16.56	22.08	−18.46	−18.28	242.9	201.94
8	186.68	166.39	186.6	166.34	19.47	25.96	−13.94	−14.08	287.6	238.16
10	204.5	181.78	204.4	181.74	20.44	27.25	−11.40	−11.67	323.1	267.48
12	219.3	194.63	219.3	194.58	20.63	27.51	−9.78	−10.11	353.1	292.44
14	232.1	205.7	232.1	205.7	20.46	27.29	−8.65	−9.00	379.0	314.2
16	243.3	215.4	243.3	215.4	20.11	26.82	−7.81	−8.17	402.1	333.7
18	253.3	224.1	253.3	224.1	19.68	26.25	−7.16	−7.52	422.8	351.2
20	262.3	231.9	262.3	231.9	19.22	25.63	−6.64	−7.00	441.6	367.3

$$B_G^n = 2f\left[\frac{4}{n+1} - \frac{4}{n+2} + \frac{1}{n^2} - \frac{n^2+n+2}{n(n+1)(n+2)}\left(\sum_{j=1}^{n}\frac{1}{j}+1\right)\right] + \frac{1}{2}\gamma_{\psi G}^{0,n}(\ln 4\pi - \gamma_E)\,, \quad (4.11)$$

where γ_E is the Euler-Mascheroni constant $\gamma_E = 0.5772$. We shall comment on the terms $(\ln 4\pi - \gamma_E)$ at the end of this section.

B_γ^n is given in terms of B_G^n as follows:

$$B_\gamma^n = \frac{2B_G^n}{f}\,. \quad (4.12)$$

D. Parameters $\beta_0, \beta_1, \delta_\psi, \delta_{NS}, \delta_\gamma$

β function parameters β_0 and β_1 have been calculated in Refs. 1, 2 and Ref. 21, respectively, and are given as follows:

$$\beta_0 = 11 - \tfrac{2}{3}f \quad (4.13)$$

and

$$\beta_1 = 102 - \tfrac{38}{3}f\,. \quad (4.14)$$

For δ_ψ, δ_{NS}, and δ_γ we have

$$\delta_\gamma = 3f\langle e^4\rangle\,, \quad (4.15)$$

$$\delta_\psi = \langle e^2\rangle\,, \quad (4.16)$$

$$\delta_{NS} = 1\,. \quad (4.17)$$

This completes the list of parameters needed to evaluate a_n, b_n, and $\tilde{a}_n$.

E. Comments on $(\ln 4\pi - \gamma_E)$

The terms $(\ln 4\pi - \gamma_E)$ which occur in B_ψ^n, B_{NS}^n, B_G^n, and B_γ^n are artifacts of the dimensional regularization scheme and it should be possible to absorb them through a redefinition of the scale parameter Λ as discussed in Ref. 6. In fact, as can be shown by means of the formulas of the present section

$$b_n = \bar{b}_n - a_n(\ln 4\pi - \gamma_E)\,, \quad (4.18)$$

where $\bar{b}_n$ is free of the $(\ln 4\pi - \gamma_E)$ terms. Therefore, Eq. (3.20) can be written as

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x, Q^2) = \alpha^2\left(a_n \ln\frac{Q^2}{\bar{\Lambda}^2} + \tilde{a}_n \ln\ln\frac{Q^2}{\bar{\Lambda}^2} + \bar{b}_n\right), \quad (4.19)$$

TABLE II. Coefficients of $e^2g^2/(16\pi^2)^2$ in the anomalous dimensions K_ψ^n, K_{NS}^n, and K_G^n for $f=3$ and $f=4$.

	$K_{NS}^{(1)}$		$K_\psi^{(1)}$		$K_G^{(1)}$	
n	3	4	3	4	3	4
2	3.247	4.871	29.23	48.71	−29.23	−48.71
4	3.707	5.560	33.36	55.60	−24.83	−41.39
6	3.790	5.685	34.11	56.84	−23.11	−38.52
8	3.707	5.560	33.36	55.60	−22.40	−37.33
10	3.577	5.365	32.19	53.64	−22.05	−36.74
12	3.431	5.147	30.88	51.47	−21.84	−36.40
14	3.289	4.993	29.60	49.33	−21.71	−36.19
16	3.156	4.733	28.40	47.33	−21.63	−36.05
18	3.031	4.547	27.28	45.47	−21.57	−35.95
20	2.916	4.373	26.24	43.74	−21.53	−35.88

TABLE III. Numerical values of the parameters a_n, $\tilde{a}_n$, $\bar{b}_n$, and p_n for $f=3$ and $f=4$.

	a_n		$\tilde{a}_n$		$\bar{b}_n$		p_n	
n	3	4	3	4	3	4	3	4
2	0.660	1.245	0.353	0.529			0.889	1.679
4	0.276	0.504	0.218	0.373	−0.604	−1.028	0.489	0.924
6	0.175	0.317	0.138	0.235	−0.418	−0.716	0.349	0.660
8	0.127	0.230	0.100	0.170	−0.327	−0.561	0.274	0.518
10	0.0989	0.179	0.0781	0.132	−0.269	−0.463	0.226	0.427
12	0.0806	0.146	0.0637	0.108	−0.228	−0.394	0.193	0.364
14	0.0678	0.122	0.0536	0.0904	−0.198	−0.343	0.168	0.318
16	0.0584	0.105	0.0461	0.0777	−0.175	−0.303	0.149	0.282
18	0.0511	0.0919	0.0404	0.0680	−0.157	−0.271	0.134	0.253
20	0.0453	0.0815	0.0358	0.0603	−0.142	−0.245	0.122	0.230

where

$$\bar{\Lambda} = \Lambda \exp[\tfrac{1}{2}(\ln 4\pi - \gamma_E)]. \tag{4.20}$$

In other words, we can absorb all $(\ln 4\pi - \gamma_E)$ terms by redefining the parameter Λ. Numerical values for a_n, $\tilde{a}_n$, and $\bar{b}_n$ are given in Table III.

V. NUMERICAL RESULTS

In this section we shall evaluate the moments of the photon structure function as given by the formulas (3.20)–(3.29) and compare the results to the leading-order and parton-model predictions. We shall also invert moment equations and present approximate analytic expressions for the photon structure functions as given by the parton model, asymptotic freedom in the leading order, and asymptotic freedom with higher-order corrections. Finally, we shall make a comparison of next-to-the-leading-order effects calculated here with those present in deep-inelastic scattering off hadronic targets.

First, however, we make a few comments. Our formulas for the structure functions are only exact up to the terms of $O(\bar{g}^2)$ which we have not calculated. Generally the formula (3.20) can be written as

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x,Q^2) = \alpha^2 \left(\frac{16\pi^2}{\beta_0 \bar{g}^2} a_n + b_n + \sum_{i=1} r_n^{(i)} [\bar{g}^2]^i + \sum_{i=1} h_n^{(i)} [\bar{g}^2]^{d_n^{(i)}} \right). \tag{5.1}$$

The parameters a_n, b_n, and $r_n^{(i)}$ can be calculated in perturbation theory. The coefficients $h_n^{(i)}$ are, on the other hand, uncalculable by present methods as they require the values of photon matrix elements of gluon and quark operators. Since the $d_n^{(i)}$ are positive, the first two terms in Eq. (5.1) will dominate at sufficiently large values of Q^2. At small values of Q^2 of 0 (few GeV^2) it is conceivable that the remaining terms will not be negligible. The study of the latter terms is beyond the scope of this paper, and we shall only present the results for the first two terms.

Our second comment concerns the heavy-quark-mass effects which are not taken into account in our formulas. These mass effects occur in the Wilson coefficient functions, in the anomalous dimensions, and in the β function. These effects have been studied by Hill and Ross[22] in the leading order and we shall comment on this paper later. In the case of the next-to-the-leading-order corrections, the inclusion of mass effects is a formidable task since this would require the calculation of renormalization-group functions in a mass-sensitive renormalization scheme. In what follows we shall present the results for $f=3$ and $f=4$ with the standard charge assignment as in the Weinberg-Salam-Glashow-Iliopoulos-Maiani model. We do not present the numerical results for $f>4$, although they can be easily obtained from formulas (3.20)–(3.29). The reason is that the effect of the b quark even far above its production threshold is suppressed relative to the charm contribution by a factor of 16 due to its charge. On the other hand, the t-quark contribution is not expected to be of any significance below $Q^2 \approx 100\ \mathrm{GeV}^2$.

In Table III we have presented the numerical values for the coefficients a_n, $\tilde{a}_n$, $\bar{b}_n$, and p_n as functions of n. As noted by Witten[11] a_n decreases faster to zero than p_n for increasing n and, therefore, the photon structure function as given by the leading-order expression is suppressed at large values of x relative to the parton-model predictions. The parameters $\bar{b}_n$ are negative and with increasing n decrease slightly slower than a_n. Consequently, the importance of higher-order contributions increases with n. Their effect is to further suppress the structure function at large values of x relative to leading-order predictions.

In order to calculate the moments of the photon structure function, we must specify the parameter Λ. In deep-inelastic scattering off hadronic targets this parameter is found by fitting the theory to existing data. As discussed in Refs. 23 and 6 the scale parameter Λ cannot be determined meaningfully from experiment without calculating at least next-to-the-leading-order effects. In particular, the value of Λ, if determined on the basis of leading-order expression, can be different in deep-inelastic scattering off hadronic targets and in the photon-photon scattering discussed here. On the other hand, if next-to-the-leading-order effects are included in the phenomenological analysis, Λ can be determined in a theoretically meaningful way for both processes. Therefore, in our analysis we shall take the value of Λ which has been obtained in Refs. 6 and 24 by fitting the asymptotic-freedom formulas to the moments of F_3 as measured by the BEBC group.[24]

As pointed out in Ref. 6, even if the next-to-the-leading-order corrections are included in the phenomenological analysis, there is some freedom in defining the parameter Λ or equivalently the effective coupling constant. As discussed in Sec. IV one can redefine the parameter Λ by absorbing in it the $(\ln 4\pi - \gamma_E)$ terms. Generally one can absorb into Λ any constant term proportional to $\gamma^{0,n}$ in deep-inelastic scattering off hadronic targets and proportional to a_n in photon-photon scattering. Any such redefinition of Λ will lead to a different numerical value of Λ extracted from experiment, but the fits to the data will be consistent with each other up to corrections of $O(\bar{g}^4)$. Here we shall discuss in detail only the $\overline{MS}$ scheme for Λ introduced in Ref. 6 which corresponds to the absorption of the $(\ln 4\pi - \gamma_E)$ terms as in Eq. (4.20). In the case of deep-inelastic scattering off hadronic targets, this scheme minimizes next-to-the-leading-order corrections for the $n=2$ moment. A similar scheme has been discussed by the authors of Ref. 9 in which the next-to-the-leading-order corrections for $n=3$ are minimized. The value of $\bar{\Lambda}$ which we have found[6] by fitting the asymptotic-freedom formulas to the moments of F_3 was $\bar{\Lambda} = 0.5$ GeV. We shall use this value in our formulas for the photon structure function.

In Fig. 2 we have plotted the quantity $F^{\gamma}_{2n}/(\alpha^2 \ln Q^2/\bar{\Lambda}^2)$ for the parton model, asymptotic freedom in the leading order, and for asymptotic freedom with higher-order corrections. The quantity in question is independent of Q^2 for the cases of the parton model and asymptotic freedom in the leading order. Higher-order corrections, on the other hand, introduce the Q^2 dependence as follows:

$$\frac{F^{\gamma}_{2,n}}{\alpha^2 \ln(Q^2/\bar{\Lambda}^2)} = a_n + \tilde{a}_n \frac{\ln\ln(Q^2/\bar{\Lambda}^2)}{\ln(Q^2/\bar{\Lambda}^2)} + \frac{\bar{b}_n}{\ln(Q^2/\bar{\Lambda}^2)} \,. \tag{5.2}$$

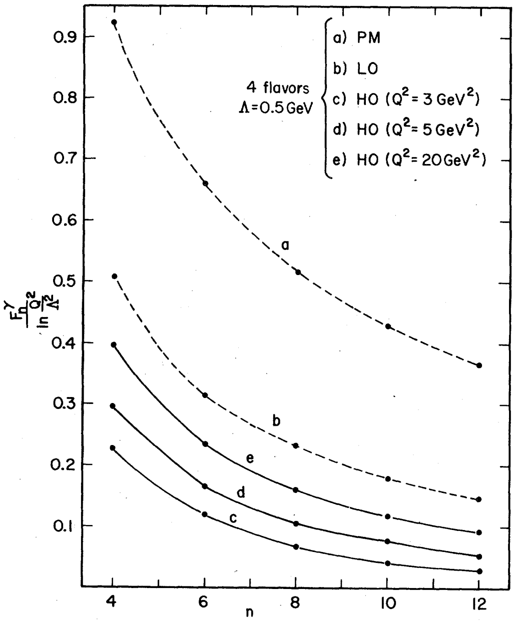

FIG. 2. Moments of the photon structure functions in units of α^2 as predicted by the parton model (a), asymptotic freedom in the leading order (b), and asymptotic freedom with higher-order corrections (c, d, e). The predictions are for $\Lambda = 0.5$ GeV and 4 flavors.

Asymptotically the last two terms in Eq. (5.2) will go to zero, and the leading-order result will be obtained. All the effects discussed in connection with Table III are seen in Fig. 2. The formulas (4.19) and (3.34) can be inverted and the result written as follows:

$$F^{\gamma}_2(x,Q^2) = \alpha^2\left[a(x)\ln\frac{Q^2}{\bar{\Lambda}^2} + \tilde{a}(x)\ln\ln\frac{Q^2}{\bar{\Lambda}^2} + \bar{b}(x)\right] \tag{5.3}$$

and

$$F^{\gamma}_2(x,Q^2)\big|_{\rm PM} = \alpha^2 p(x)\ln\frac{Q^2}{\bar{\Lambda}^2} \tag{5.4}$$

for the parton model. The formula (3.34) can be inverted analytically, and one obtains

$$p(x) = 4\delta_\gamma[x^2 + (1-x)^2]\,x\,, \tag{5.5}$$

which is the familiar expression of Ref. 17. The formula (4.19) has a complicated n dependence and must be inverted numerically. We have found, however, approximate analytic expressions for the functions $a(x)$, $\tilde{a}(x)$, and $\bar{b}(x)$ which for certain ranges of x are good representations of the exact inversion. For $f=4$ they are given as follows:

$$a(x) = x[1.52x^{1.32} + 4.38(1-x)^{0.97}] \quad \text{for } 0.3 \le x \le 0.9\,, \tag{5.6}$$

$$\tilde{a}(x) = x[1.12x^{1.32} + 3.24(1-x)^{0.97}] \quad \text{for } 0.4 \le x \le 0.9\,, \tag{5.7}$$

$$\bar{b}(x) = -x[5.29x^{1.49} + 19.99(1-x)^{4.19}] \quad \text{for } 0.4 \leq x \leq 0.9\,. \tag{5.8}$$

The structure functions $F_2^\gamma(x, Q^2)$ as given by the leading-order prediction and higher-order calculations are shown in Fig. 3. We observe that the largest effects of higher-order corrections are at large values of x.

So far we have used the same value of Λ in the leading-order and higher-order calculations, and we have found that the higher-order corrections were large. It has been demonstrated in Ref. 6 that in the case of deep-inelastic scattering off hadrons the asymptotic-freedom formulas with higher-order corrections included and $\bar{\Lambda} = 0.5$ GeV could be very well approximated for $2 \leq Q^2 \leq 30$ GeV^2 by the leading-order (LO) expression with $\Lambda_{LO} = 0.73$ GeV and with the unknown hadronic matrix elements of gluon and quark operators suitably modified relative to the higher-order case. In the photon-photon scattering the modification can be done only in the value of Λ, and it is of interest to see whether we can find a Λ_{LO} defined by

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x, Q^2) = \alpha^2 a_n \ln \frac{Q^2}{\Lambda_{LO}^{\;2}}\,, \tag{5.9}$$

so that the leading-order expression is approximately equal to the full expression (4.19) which is calculated with $\bar{\Lambda} = 0.5$. The result of such an exercise is shown in Fig. 4. The following lessons can be taken from this figure:

(i) It is impossible to find Λ_{LO} which would reproduce the formula (4.19) with the accuracy found in the case of deep-inelastic scattering off hadronic targets. For fixed Q^2 the formula (4.19) predicts faster drop of the moments with increasing n that is given by the leading-order formula (5.9). Also Q^2 dependence is slightly different in the two cases.

(ii) The effects of the next-to-the-leading-order corrections in photon-photon scattering are larger than in the deep-inelastic scattering. $\Lambda_{LO} \approx 1$ GeV in the present case as compared to $\Lambda_{LO} \approx 0.73$ found in Refs. 25 and 6. We recall that in both cases the higher-order formulas are calculated with $\bar{\Lambda} = 0.5$.

(iii) If one is interested only in 10–20% accuracy, then we can conclude that the leading-order formula, (5.9) can mimic the higher-order expression (4.19), but Λ_{LO} is *not the same* as the one found in the deep-inelastic scattering off hadronic targets. This illustrates the fact first pointed out by Bace[23] that it is *incorrect* to use the same value of Λ in two different processes when next-to-the-leading-order corrections are not explicitly included.

Another way to compare higher-order effects in photon-photon scattering and in the deep-inelastic scattering off hadronic targets is to cast the formula (3.20) and the corresponding formula for the moments of $F_3^{\nu,\bar{\nu}}$ in the following form:

$$M_n^\gamma(Q^2) = \frac{a_n}{\beta_0}\left(\frac{\bar{g}^2}{16\pi^2}\right)^{-1}\left[1 + \frac{\bar{g}^2}{16\pi^2}\left(\beta_0\,\frac{\bar{b}_n}{a_n}\right)\right], \tag{5.10}$$

and

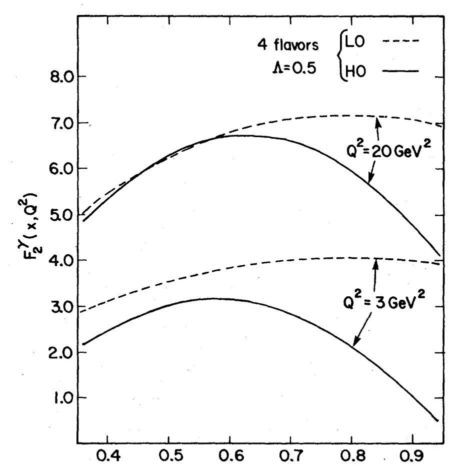

FIG. 3. Photon structure function in units of α^2 as predicted by asymptotic freedom in the leading order (dashed lines) and asymptotic freedom with higher-order corrections (solid lines) for $\Lambda = 0.5$ GeV and various values of Q^2. The curves correspond to the formulas of (5.6)–(5.8). The curves for leading order agree within a few percent of those of Ref. 11 over the range of x plotted in this figure.

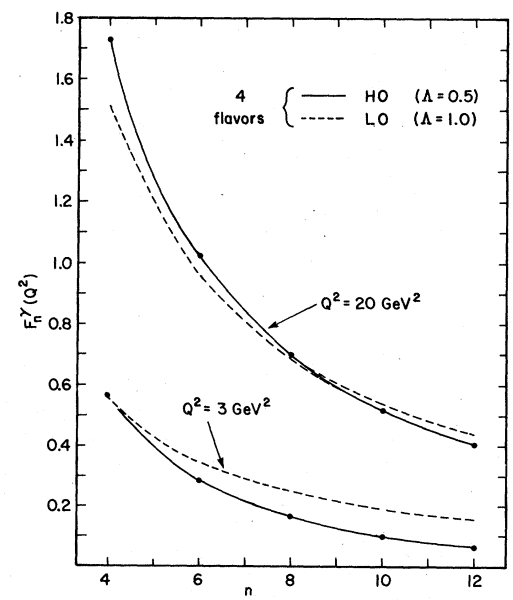

FIG. 4. Moments of the photon structure function in units of α^2 as predicted by asymptotic freedom in the leading order with $\Lambda = 1.0$ GeV (dashed lines) and asymptotic freedom with higher-order corrections for $\Lambda = 0.5$ GeV (solid lines).

$$(M_n^{(3)})^{2\beta_0/\gamma_0^n} = A_n\left(\frac{\bar{g}^2}{16\pi^2}\right)\left(1+\frac{\bar{g}^2}{16\pi^2}\frac{2\beta_0}{\gamma_0^n}(\bar{B}_n+P_n)\right), \tag{5.11}$$

with $\bar{B}_n$ and P_n given in Ref. 6. The plots of the coefficients of $\bar{g}^2/(16\pi^2)$ in Eqs. (5.10) and (5.11) as functions of n are shown in Fig. 5. The n dependence is similar, but the effect of the next-to-the-leading-order term in the photon-photon scattering is significantly larger as already noted in the analysis above.

VI. LONGITUDINAL PHOTON STRUCTURE FUNCTION

So far we have considered only the structure function F_2^γ. Witten[11] has also calculated the longitudinal photon structure function F_L^γ and for completeness we quote his result written in our notation.

In order to derive an asymptotic-freedom formula for F_L^γ, one proceeds along the steps of Sec. III with the only changes being in the coefficient functions $C_n^j(1,\bar{g}^2,\alpha)$ which are now replaced by the following expressions:

$$C_{L,n}^j(1,\bar{g}^2,\alpha)=\begin{cases} e^2\delta_\psi \dfrac{\bar{g}^2}{16\pi^2} B_{\psi,L}^n, & j=\psi, \\ e^2\delta_\psi \dfrac{\bar{g}^2}{16\pi^2} B_{G,L}^n, & j=G, \\ e^2\delta_{NS} \dfrac{\bar{g}^2}{16\pi^2} B_{NS,L}^n, & j=NS, \\ \dfrac{e^4}{16\pi^2}\delta_\gamma B_{\gamma,L}^n, & j=\gamma, \end{cases} \tag{6.1}$$

where[4,6,25]

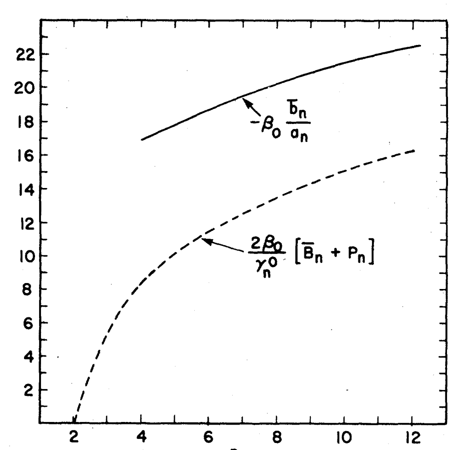

FIG. 5. Comparison of the coefficients of $\bar{g}^2/16\pi^2$ in photon-photon scattering (solid line) and in deep-inelastic scattering (dashed line).

$$B_{\psi,L}^n = B_{NS,L}^n = \frac{4}{3}\frac{4}{n+1}, \tag{6.2}$$

$$B_{G,L}^n = \frac{8f}{(n+1)(n+2)}, \tag{6.3}$$

$$B_{\gamma,L}^n = \frac{2B_{G,L}^n}{f}, \tag{6.4}$$

and δ_ψ, δ_{NS}, and δ_γ are defined in Sec. IV.

The moments of the longitudinal photon structure function are then given as follows:

$$\int_0^1 dx\,x^{n-2}F_L^\gamma(x,Q^2) = \alpha^2\left[\frac{1}{2\beta_0}(K_\psi^{0,n}\delta_\psi R_{\psi,L}^n + K_{NS}^{0,n}\delta_{NS}R_{NS,L}^n) + \delta_\gamma B_{\gamma,L}^n\right] + O(\bar{g}^2), \tag{6.5}$$

where

$$R_{\psi,L}^n = \frac{B_{\psi,L}^n}{d}\left(1+\frac{\gamma_{GG}^{0,n}}{2\beta_0}\right) - \frac{B_{G,L}^n}{d}\frac{\gamma_{G\psi}^{0,n}}{2\beta_0}, \tag{6.6}$$

$$R_{NS,L}^n = \frac{B_{NS,L}^n}{(1+\gamma_{NS}^{0,n}/2\beta_0)}, \tag{6.7}$$

and d is given by Eq. (3.26). Notice that Eqs. (6.5)–(6.7) can be obtained directly from Eqs. (3.25), (3.27), and (3.28) by putting there all two-loop contributions to zero and replacing the parameters B_j^n by $B_{j,L}^n$.

For comparison we quote the parton-model prediction

$$\int_0^1 dx\,x^{n-2}F_L^\gamma(x,Q^2)|_{PM} = \alpha^2\delta_\gamma B_{\gamma,L}^n. \tag{6.8}$$

We observe that both asymptotic freedom and the parton model predict scaling for the longitudinal structure function, although the scaling functions are different in these two cases. As shown by Witten the renormalization effects as given by the first two terms in Eq. (6.5) are small and, consequently, the longitudinal structure function as predicted by asymptotic freedom is very similar to that obtained in the parton model. This is to be contrasted with the predictions for F_2^γ where the renormalization effects are large.

VII. SUMMARY AND CONCLUSIONS

In this paper we have calculated the photon structure function F_2^γ in asymptotically free gauge theories up to and including next-to-the-leading-order corrections. Our result is a straightforward generalization of Witten's analysis where the photon structure function was calculated in the leading order of asymptotic freedom. The next-to-the-leading-order corrections found here are large at reasonable values of Q^2. We have compared our results with the higher-order corrections to deep-inelastic structure functions and concluded that the higher-order corrections calculated here are larger than those found in deep-inelastic scattering.

We have shown that not all of the higher-order corrections to photon-photon scattering can be absorbed by redefining the parameter Λ. Therefore, the shape of the photon structure function found in the leading order is modified by higher-order effects particularly for large n or correspondingly for large values of x. This is to be contrasted with deep-inelastic scattering off hadronic targets where the higher-order corrections not absorbed into Λ could be, in the range of Q^2 available, absorbed in the hadronic matrix elements of quark and gluon operators. We do not have this freedom in the photon-photon scattering. This makes the process in question particularly suitable for the theoretical study of higher-order corrections. Unfortunately, the measurements of the photon structure functions are much more involved than those needed for hadronic structure functions. However, it is possible that the ideas discussed in this paper will be tested at PETRA, PEP, and LEP.

In our analysis we have neither included mass effects due to heavy quarks nor discussed the contributions in which photon behaves similar to a hadron; the last sum in Eq. (5.1). Both give small effects at large values of Q^2, but at Q^2 of 0 (few GeV^2) both could give non-negligible effect. In particular, mass effects due to charm production could be important. In our paper we have made calculations for three and four flavors with all quark masses zero. At low values of Q^2, 0 (few GeV^2), the massless approximation is probably justified for the light quarks but certainly not justified for the charm quark contributions.

At these low values of Q^2 the charm-quark contribution is expected to be small, but for large values of Q^2 our predictions for four flavors become valid. The study of this transition is beyond the scope of the present paper.

Recently Hill and Ross[22] have studied mass effects in the photon-photon scattering in the leading order of asymptotic freedom. They find sensitivity of their results to the small values of p^2. In particular, Hill and Ross claim that for $p^2 \geq 300$ MeV^2 Witten's result should hold for light quarks, whereas for smaller values of p^2 other (nonleading) contributions could be important. We would like to remark only that both the matrix element $\langle\gamma|O_\gamma|\gamma\rangle$ and the coefficient $C_n^\gamma(Q^2/\mu^2, g^2, \alpha)$, which constitute Witten's and our results, are independent of the value of p^2. The p^2 dependence which the authors study is related to the *perturbative calculation* of the matrix elements of hadronic operators such as $\langle\gamma|O_\psi|\gamma\rangle$, $\langle\gamma|O_G|\gamma\rangle$, etc. We do not expect these matrix elements to be sensitive to p^2 for small p^2 since this dependence should be dictated by the appropriate *hadronic* singularities. If our analysis applies at $p^2 \sim 300$ MeV^2, it should also apply at $p^2 = 0$ independent of light quark masses. We agree, however, with Hill and Ross that the mass effects due to production of heavy quarks should be included in a detailed comparison with the experimental data to be obtained in the future.

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