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Effective potential for non-Abelian gauge theories and spontaneous symmetry breaking

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We derive a formal expression for the one-loop corrected effective potential of non-Abelian gauge theory in the covariant gauge. This expression is checked, for arbitrary gauge group, by graphically calculating the one-loop renormalization constants. The effective potential for a restricted range of fields is explicitly evaluated for the gauge groups SU(2) and SU(3) in the Landau gauge. We extend the range of validity of these results using the renormalization group to improve the effective potential. Extension to larger, simple gauge groups follows immediately from our formalism. The possibility, and physical meaning, of a nonzero vacuum expectation value is discussed. A nonzero vacuum is shown to behave as a regulator for infrared divergences.

I. INTRODUCTION

The effective potential for a field theory[1] has been used extensively in studies of spontaneous symmetry breaking. One of the more interesting applications is that of Coleman and Weinberg, who examined one-loop radiative corrections to the effective potential in massless scalar electrodynamics.[2] This theory was thought to be plagued by logarithmic, infrared divergences in its Green's functions on the scalar mass shell. The result of the above analysis was, however, that the theory escapes this problem by cleverly growing, at the one-loop level, an effective *wrong-sign* mass term. A new nonzero vacuum is thereby introduced which breaks the original U(1) symmetry, gives mass to the scalar field, and avoids an on-shell infrared catastrophe. The gauge dependence of the effective potential clouds this issue but the above conclusions survive analysis in this regard.[3] The starting point of the present investigation is the anticipation that pure, non-Abelian, Yang-Mills theory, being both massless and gauge invariant, could escape its well-known infrared difficulties by a mechanism similar to that found in the Coleman-Weinberg example.

Formal expressions for the one-loop effective action of non-Abelian gauge theory have been obtained by several authors[4] using Schwinger's proper-time formalism.[5] These expressions have been evaluated for certain simple background fields.[4,6] However, a direct evaluation of the one-loop effective potential (rather than the effective action) is more in keeping with the Coleman-Weinberg approach. We denote the one-loop effective potential by $V[\vec{\phi}_\mu]$, where $\vec{\phi}_\mu$ are space-time constant gauge fields. In Sec. III of this paper we present a formal expression for $V[\vec{\phi}_\mu]$ in the covariant gauge (α gauge) for an arbitrary, simple gauge group. This expression is checked in Sec. IV by graphically calculating the one-loop wave-function and coupling renormalization constants (Z_A and Z_g, respectively) in the Feynman gauge ($\alpha = 1$).

In Sec. II we examine the classical (tree) potential for non-Abelian gauge theories and are led to fields of the form $\vec{r}v_\mu$ ($\vec{r}$ is a gauge group vector and v_μ a space-time constant Lorentz vector) as possible candidates for nonzero vacuums. These vacuums clearly do not satisfy the boundary conditions that are normally imposed on gauge fields and are, therefore, somewhat pathological. A careful discussion of these pathologies is given in Sec. II. However, since these vacuums appear only as background fields in the relevant path integral, we do not consider them a source of difficulty in the calculation of the effective potential. What is more, we will show that once one is willing to allow such fields in the domain of the theory they act, at least to the one-loop level, as a regulator for infrared divergences that would otherwise occur. In Sec. II it is shown that these vacuums are simply gauge translates of the zero field and are, therefore, with respect to any gauge-invariant quantity, equivalent to the zero field. However, for a non-Abelian gauge theory, neither the effective potential nor the S matrix is strictly gauge invariant. When calculated to the one-loop level, logarithms appear in the effective potential which explicitly break gauge invariance. These always accompany massless fields and appear in exact analogy with the Coleman-Weinberg example. We emphasize that these logarithms break gauge invariance in a different, and more fundamental way than the trivial breaking of this invariance caused by gauge fixing (e.g., α dependence). These logarithms limit the region of field space where the one-loop approximation is valid. In combination with the known asymptotic freedom on non-Abelian gauge theory[7] they imply that this approximation is only valid for fields far from the zero field. This conclusion is strengthened and made precise by analysis of the renormalization-group-improved effective potential introduced in Sec. VII.

We are unable to calculate the entire one-loop effective potential graphically. It is easier to evaluate directly the determinants occurring in the formal expression for $V[\vec{\phi}_\mu]$. We have done this, restricting ourselves to fields in the neighborhood of $\vec{r}\phi_\mu$, for the gauge groups SU(2) and SU(3). Extension to larger simple groups is obvious from the formalism. These calculations appear in Secs. V and VI, respectively, and are carried out in the $\alpha=0$ (Landau) gauge. Following Coleman and Weinberg, we use the renormalization group to extend the range of validity of our results. This analysis makes the role played by asymptotic freedom precise and is presented in Sec. VII.[8]

N-fold differentiation of the renormalization-group-improved (RGI) potential, evaluated at the vacuum (for now, assume arbitrary $\vec{r}v_\mu$), yields the RGI one-loop, on-shell, N-point one-particle-irreducible (1PI) vertex function. The above analysis carries over for these vertices. It follows (see Sec. VII) that the one-loop approximation to the N-point vertex is valid if and only if v^2 is much larger than zero. That is, in order for the *one-loop approximation* for the N-point vertices to be valid it is necessary (no matter how small the coupling constant) to expand the theory around a *nonzero* vacuum of the form $\vec{r}v_\mu$, where $v^2 \gtrsim \mu^2 > 0$ (μ is the renormalization parameter). If one attempted to evalutate these vertices at $v^2=0$, one would find the famous infrared divergences of non-Abelian gauge theory. Traditionally one thinks of the S matrix as gauge invariant which if true would imply that all $\vec{r}v_\mu$ vacuums are equivalent. But we have just seen that they are not equivalent at the one-loop level. What has gone wrong? The answer is that the gauge invariance of the S matrix is *broken* by the logarithms (or, stated another way, the S matrix is gauge invariant but the running coupling constant is not). The part of the S matrix *independent* of logarithms *is* gauge invariant and, presumably, any dependence on $\vec{r}v_\mu$ *drops out.* We therefore do not expect any violation of either Lorentz or internal symmetry. However, the logarithms, by breaking gauge invariance, do select between various $\vec{r}v_\mu$ vacuums. The theory, therefore, can and does demand a nonzero vacuum at the one-loop level. This vacuum acts as an *infrared* (IR) *regulator* and prevents IR divergences on-shell. We emphasize that the requirement that $v^2 \gtrsim \mu^2 > 0$ at the one-loop level follows from the renormalization-group analysis and is *not put in by hand.* It would appear then that the supposed one-loop, on-shell IR divergences of non-Abelian gauge theory are merely artifacts of having chosen the *wrong vacuum* ($\vec{0}_\mu$). These divergences can be regulated by choosing the nonzero vacuum(s) $\vec{r}v_\mu$ demanded by the renormalization group.

II. THE ZERO-LOOP POTENTIAL

We consider a pure Yang-Mills theory with simple, compact, connected gauge group G (dim $G=N$). The classical Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a ,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g c^a{}_{jk} A^j_\mu A^k_\nu , \tag{2.1}$$

where group indices are raised and lowered with respect to δ_{ij} (δ^{ij}). The N A^a fields transform as the adjoint representation under the *global* action of G. Denote a basis of the Lie algebra of G by T_i ($i=1,\ldots, N$) and note that

$$[T_j, T_k] = ic^a{}_{jk} T_a ,$$
$$\mathrm{Tr} T_i = 0 . \tag{2.2}$$

We can (and do) demand that the T_i's satisfy the normalization condition

$$\mathrm{Tr}(T_i T_j) = a\delta_{ij} , \tag{2.3}$$

where a is chosen to be 2 for SO(N) and $\frac{1}{2}$ for SU(N). Finally, note that the Jacobi identity implies that c_{ijk} is completely antisymmetric. The bare (tree graph) effective potential $V_0[\vec{\phi}_\mu]$ is simply the potential part of the Lagrangian density in (2.1) with the fields taken to be space-time constants ϕ^a_μ. That is,

$$V_0[\vec{\phi}_\mu] = \tfrac{1}{4} g^2 c^a{}_{bc} c_a{}^{de} \phi^b_\mu \phi^c_\nu \phi^\mu_d \phi^\nu_e . \tag{2.4}$$

The first and second partial derivatives of $V[\vec{\phi}_\mu]$ are of interest and are given by

$$\frac{\partial V_0[\vec{\phi}_\mu]}{\partial \phi^i_\alpha} = g^2 c^{ai}{}_c c_a{}^{de} \phi^c_\nu \phi^\nu_e \phi_{\alpha d} , \tag{2.5}$$

$$\frac{\partial^2 V_0[\vec{\phi}_\mu]}{\partial \phi^j_\beta \partial \phi^i_\alpha} = g^2 [c^{ai}{}_c c_a{}^{je} (g_{\alpha\beta} \phi^c_\nu \phi^\nu_e - \phi_{\alpha e} \phi^c_\beta) + c^{aij} c_a{}^{de} \phi_{\alpha d} \phi_{\beta e}] . \tag{2.6}$$

Any vacuum must extremize the potential. We therefore look for solutions of the equation obtained by setting (2.5) equal to zero. This equation is a complicated function of the group structure constants and may, in general, have many solutions. However, one solution

$$\vec{\phi}_\mu = \vec{r}\phi_\mu \tag{2.7}$$

is both obvious (from the antisymmetry of c_{abc}) and physically important. [Note that $\vec{\phi}_\mu = \vec{0}$ is a degenerate form of (2.7).] The effective potential clearly vanishes for all such solutions. Therefore, a nonzero vacuum of the above form is energetically as probable as the zero vacuum. For such fields the second partial derivative becomes

$$gc^{ai}{}_{c}r^{c}c_{a}{}^{je}r_{e}(g_{\alpha\beta}\phi^{2}-\phi_{\alpha}\phi_{\beta}). \tag{2.8}$$

The form of (2.8) is dependent on the explicit values of the structure constants and is difficult to treat generally. For simplicity, consider the group SU(2). In this case the group space matrix of partial derivatives is diagonal and given by

$$g^{2}(g_{\alpha\beta}\phi^{2}-\phi_{\alpha}\phi_{\beta})\begin{pmatrix}1 & & \\ & 1 & \\ & & 0\end{pmatrix}. \tag{2.9}$$

Now diagonalize the Lorentz index matrix. Going to a frame in which $\phi_2=\phi_3=0$, we find the diagonal matrix to be

$$\begin{bmatrix}0 & & & \\ & -(\phi_0{}^2+\phi_1{}^2) & & \\ & & -\phi^2 & \\ & & & -\phi^2\end{bmatrix}. \tag{2.10}$$

Note that the second term is always negative (or zero). The closest we can get to a local minimum is when ϕ is spacelike. For the rest of this paper we will always assume this to be the case.

Fields of the form $\vec{r}\phi_\mu$ clearly do not satisfy the boundary conditions normally imposed on gauge fields. Furthermore, as we will now show, such fields are "illegal" gauge transforms of the zero vacuum. Under a finite gauge transformation $U(\theta)$ the fields $\vec{\phi}_\mu$ transform as

$$\vec{T}\cdot\vec{\phi}'_{\mu}=U(\theta)\left(\vec{T}\cdot\vec{\phi}_{\mu}-\frac{i}{g}U^{-1}(\theta)\partial_{\mu}U(\theta)\right)U^{-1}(\theta). \tag{2.11}$$

Start with the zero vacuum $\vec{\phi}_\mu=\vec{0}$ and let

$$\vec{\theta}=-x^{\mu}\vec{v}_{\mu}g, \tag{2.12}$$

where $\vec{v}_\mu$ is a space-time constant. We find that

$$\vec{\phi}'_{\mu}=\vec{v}_{\mu} \tag{2.13}$$

if and only if $\vec{v}_\mu=\vec{r}v_\mu$. Therefore, the $\vec{r}\phi_\mu$ fields are simply gauge transforms of zero. Moreover, these gauge transformations are illegal in the sense that $\vec{\theta}$ in (2.12) grows without bound for large x^μ. The unorthodox boundary conditions and the "illegality" of the gauge transformations are obviously one and the same issue. Such gauge transformations spell danger in the canonical path-integral approach to quantization. Note, for instance, that both zero and $\vec{r}\phi_\mu$ satisfy the Landau gauge condition $\partial^\mu\vec{A}_\mu=0$. It follows that if we allow $\vec{r}\phi_\mu$ fields in the theory, the usual Faddeev-Popov procedure is not sufficient to fix the Landau gauge. This is faintly reminiscent of the problem of Gribov ambiguities. Should fields of the form $\vec{r}\phi_\mu$ therefore be disallowed? We would like to argue that, at least as regards issues discussed in this paper, such fields are at worst innocuous and should be retained. We have three reasons for believing this to be the case. First, in the evaluation of the one-loop effective potential (see Sec. III) the gauge fields in the domain of $V[\vec{\phi}_\mu]$ enter only as background fields around which the path integral is evaluated. To the one-loop approximation this path integral is a simple Gaussian whose evaluation is straightforward and independent of the nature of the background field. The check on the renormalization constants Z_A and Z_g in Sec. IV gives us confidence that this reasoning is correct. Second, it is not too hard to convince oneself that if the gauge fields $\vec{A}_\mu$ in the bare Lagrangian are written as $\vec{A}_\mu=\vec{A}'_\mu+\vec{r}v_\mu$, and the theory is quantized in the Landau gauge, perfectly well behaved Green's functions can be obtained. In fact, on-shell Green's functions are not well behaved only when $\vec{r}v_\mu=0$. Thus, in perhaps a naive but operationally clear sense, the pathologies in $\vec{r}v_\mu$ vacuums do not effect Green's functions and these are the objects of interest in this paper. Lastly, we *strongly emphasize* that if fields of the form $\vec{r}\phi_\mu$ are retained in the theory, they act, at least to the one-loop level, as a regulator for the on-shell, infrared divergences that would otherwise plague the theory. Therefore, we feel one has two choices: to disallow fields of the form $\vec{r}\phi_\mu$ because of gauge-fixing ambiguities and pathological boundary conditions, thus allowing on-shell, infrared divergences, or to allow such fields, thus, at least to one loop, regulating infrared divergences, though perhaps leaving oneself open to problems of definition of the path integral. Since it is the purpose of this paper to discuss the Coleman-Weinberg mechanism as it relates to infrared behavior, we feel justified in choosing the latter approach.

The one-loop corrected effective potential in the Landau gauge is explicitly calculated in Secs. V and VI for the groups SU(2) and SU(3), respectively. The qualitative features of these results can be guessed in advance. Invoking gauge invariance, we expect the loop-corrected effective action to be a power series in invariants formed from $F^a_{\mu\nu}$. The effective potential $V[\vec{\phi}_\mu]$ is proportional to the effective action, with the fields taken to be space-time constants. The nth derivative of the potential, evaluated at fields $\vec{v}_\mu$ which extremize the action, is precisely the 1PI Green's function with n vanishing external momenta. For $\vec{v}_\mu=\vec{0}$ these Green's functions in the Landau gauge are known to diverge logarithmically. It follows that $V[\vec{\phi}_\mu]$ contains logarithms that diverge as $\vec{\phi}_\mu$ vanishes. On dimensional grounds these must be multiples of $\ln(\vec{\phi}^2/\mu^2)$, where μ is some renormalization point with dimensions of mass. We can now guess the form of the first term in the one-loop correction to $V[\vec{\phi}_\mu]$ in the Landau gauge. It is

$$g_R{}^2\left(A + B\ln\frac{\vec{\phi}^2}{\mu^2}\right)\frac{g_R{}^2}{4}c^a{}_{bc}c_a{}^{de}\phi^b_\mu\phi^c_\nu\phi^\mu_d\phi^\nu_e\,, \qquad (2.14)$$

where g_R $(\vec{\phi}_\mu)$ is the renormalized coupling constant (gauge field) and A, B are constants independent of $\vec{\phi}_\mu$. This form is borne out by explicit calculation. Higher-loop corrections to $V[\vec{\phi}_\mu]$ involve higher powers of g_R and $\frac{1}{2}g_R\ln(\vec{\phi}^2/\mu^2)$. It is clear that the one-loop approximation is valid if and only if *both* g_R and $\frac{1}{2}g_R\ln(\vec{\phi}^2/\mu^2)$ are small. For nonzero g_R, $\frac{1}{2}g_R\ln(\vec{\phi}^2/\mu^2)$ is *never* small near the origin. Furthermore, *loosely* changing g_R to the running effective coupling constant $g(\mu)$, we know from the usual asymptotic freedom arguments that $g(\mu)$ is small for large μ. Therefore $\frac{1}{2}g(\mu)\ln(\vec{\phi}^2/\mu^2)$ is small only for

$$\vec{\phi}^2 \approx \mu^2 \gg 0\,. \qquad (2.15)$$

This result is verified and strengthened by the precise calculations of Sec. VII.

We conclude that the one-loop approximation to $V[\vec{\phi}_\mu]$ in the Landau gauge is valid for fields far from the zero field. Note that $\vec{r}v_\mu$, $v^2\approx\mu^2$ extremizes $V[\vec{\phi}_\mu]$ and $V[\vec{r}v_\mu]$ vanishes. Remembering that for field configurations $\vec{r}\phi_\mu$ the inverse of the transformation defined by (2.12) does *not* take us out of the Landau gauge, we see that it cannot be invoked to gauge the vacuum $\vec{r}v_\mu$ away if we wish to stay in the one-loop approximation. As v_μ goes to zero it enters regions where first two-, three-, and finally infinite-loop corrections to $V[\vec{\phi}_\mu]$ cannot be overlooked. Since order by order $\ln(\vec{\phi}^2/\mu^2)$ terms break the gauge invariance of the effective action it is unlikely that the fully summed action is gauge invariant. Therefore, a transformation such as (2.12) cannot be carried out with impunity. In the following section we define and derive the effective potential for Yang-Mills fields.

III. THE ONE-LOOP EFFECTIVE POTENTIAL

We quantize the theory using the path-integral formulation. The sourceless vacuum-to-vacuum amplitude is given by

$$Z[0]=N\int[dA_\mu][d\xi][d\eta]\exp(iS_{\text{eff}})\,, \qquad (3.1)$$

where

$$\mathcal{L}_{\text{eff}}=\mathcal{L}+\mathcal{L}_{\text{gauge fixing}}+\mathcal{L}_{\text{ghost}}\,. \qquad (3.2)$$

The ghost fields $\vec{\eta}$ and $\vec{\xi}$ are *anticommuting* Lorentz scalars and transform under G as the defining representation and its conjugate, respectively. In this paper we will work in the α gauges defined by

$$\mathcal{L}_{\text{gauge fixing}}=-\frac{1}{2\alpha}(\partial^\mu\vec{A}_\mu)^2\,,$$
$$\mathcal{L}_{\text{ghost}}=-\xi^a\partial_\mu(\partial^\mu\delta_{ab}-gc^c{}_{ab}A^\mu_c)\eta^b\,. \qquad (3.3)$$

Let $\vec{J}_\mu$, $\vec{\beta}^\dagger$, and $\vec{\beta}$ be sources for $\vec{A}_\mu$, $\vec{\eta}$, and $\vec{\xi}$, respectively. We want to compute $W[J_\mu]$ (the *connected* generating functional for *Yang-Mills* fields) defined by

$$Z[\vec{J}_\mu]=e^{iW[\vec{J}_\mu]}=\lim_{\vec{\beta},\vec{\beta}^\dagger\to 0}Z[\vec{J}_\mu,\vec{\beta},\vec{\beta}^\dagger]\,, \qquad (3.4)$$

where

$$Z[J_\mu,\vec{\beta},\vec{\beta}^\dagger]=N\int[d\vec{A}_\mu][d\vec{\xi}][d\vec{\eta}]\exp\left\{i\left[S_{\text{eff}}+\int d^4x(-\vec{J}_\mu\cdot\vec{A}^\mu+\vec{\xi}\cdot\vec{\beta}+\vec{\beta}^\dagger\cdot\vec{\eta})\right]\right\}\,. \qquad (3.5)$$

An expansion in loops is obtained by expanding the argument of the exponential in (3.5) around fields $\vec{A}_{\mu(0)}$, $\vec{\xi}_{(0)}$, and $\vec{\eta}_{(0)}$ which satisfy the classical equations of motion

$$\left.\frac{\delta S_{\text{eff}}}{\delta\vec{A}_\mu}\right|_{(0)}=\vec{J}_\mu\,,\quad\left.\frac{\vec{\delta}S_{\text{eff}}}{\delta\vec{\xi}}\right|_{(0)}=-\vec{\beta}\,,\quad\left.\frac{S_{\text{eff}}\overleftarrow{\delta}}{\delta\vec{\eta}}\right|_{(0)}=-\vec{\beta}^\dagger\,. \qquad (3.6)$$

Note that the anticommuting nature of the ghost fields necessitates introducing left- and right-acting functional derivatives of these fields. The notation for such derivatives is introduced in (3.6) and is obvious. The left-right nature of these derivative seems to make a series expansion in the ghost variables ambiguous. For example, the second derivative of S_{eff}, with respect to $\vec{\xi}$ and then $\vec{\eta}$, can be carried out in four inequivalent ways. However, this ambiguity does not really exist since careful integration by parts (respecting the directional nature of the partial derivatives) will restore equality. In Appendix A we present the calculation of $W[\vec{J}_\mu]$. The result is

$$W[\vec{J}_\mu]=S[\vec{A}_{\mu(0)}]-\int d^4x\,\vec{J}_\mu\cdot\vec{A}^\mu_{(0)}-i\ln\det\{\delta^e{}_f\delta(x-y)-gc^e{}_{fc}[A^c_{\alpha(0)}\partial^\alpha_x+(\partial^\alpha A^c_{\alpha(0)})]D_F(x-y)\}$$
$$+\frac{i}{2}\ln\det\{g^\beta_\mu\delta^e{}_f\delta(x-y)+[g\hat{G}_f{}^e|_\gamma{}^\beta-g^2K_f{}^e|_\gamma{}^\beta+((2\partial_\gamma A^{\beta c}_{(0)})-(\partial^\beta A^c_{(0)\gamma})-(\partial^\alpha A^c_{(0)\alpha})g^\beta{}_\gamma)gc_f{}^e{}_c]D^\gamma_{F\mu}(x-y\,|\,\alpha)\}\,, \qquad (3.7)$$

where

$$\hat{G}_f^g\big|^{\gamma\beta} = 2c_f{}^g{}_c[A_{(0)}^{c(\gamma\partial\beta)} - A_{(0)}^c \cdot \partial g^{\gamma\beta}],$$
$$K_f^g\big|^{\gamma\beta} = c^a{}_f{}^g c_a{}^{dc} A^\gamma_{c(0)} A^\beta_{d(0)} + c^a{}_f{}^c c_a{}^{dg}[A^\gamma_{c(0)} A^\beta_{d(0)} - A_{c(0)} \cdot A_{d(0)} g^{\gamma\beta}]. \tag{3.8}$$

$D_F(x-y)$ and $D^\gamma_{F\mu}(x-y\,|\,\alpha)$ are the usual propagators for ghosts and gluons in the α gauge. Note that the ghost contribution has a coefficient which is -2 times the gluon coefficient. The minus sign follows from the fermionic statistics of $\vec{\xi}$, $\vec{\eta}$, and the factor 2 from the fact that the path integral is over *two independent* ghost fields.

The effective action of the Yang-Mills theory is given by the functional

$$\Gamma[\vec{\Phi}_\mu] = W[\vec{J}_\mu] + \int d^4x\, \vec{J}_\mu \cdot \vec{\Phi}^\mu, \tag{3.9}$$

where $\vec{\phi}_\mu(x)$ is defined by

$$\vec{\Phi}_\mu(x) = \frac{-\delta W[\vec{J}]}{\delta \vec{J}_\mu(x)}. \tag{3.10}$$

To the one-loop approximation $\Gamma[\vec{\Phi}_\mu]$ is obtained from Eq. (3.9) with $W[\vec{J}_\mu]$ given by (3.7), where $\vec{A}^\mu_{(0)}$ has been everywhere replaced by $\vec{\Phi}^\mu$. Now

$$\left.\frac{-\delta W[\vec{J}]}{\delta \vec{J}_\mu(x)}\right|_{\vec{J}=\vec{0}} = \vec{v}_\mu, \tag{3.11}$$

where $\vec{v}_\mu$ is a constant vector field, not necessarily zero. It follows that

$$\left.\frac{\delta\Gamma[\vec{\Phi}]}{\delta\vec{\Phi}_\mu}\right|_{\vec{\Phi}=\vec{v}} = 0. \tag{3.12}$$

This is the fundamental equation for probing the vacuum structure of the theory. Any solution $\vec{v}_\mu$ of (3.12) represents a possible vacuum expectation value of the fields. For theories without derivative couplings, Eq. (3.12) is most easily solved by considering the effective potential defined by

$$V[\vec{\phi}_\mu] = -\Gamma[\vec{\phi}_\mu]\left(\int d^4x\right)^{-1}, \tag{3.13}$$

where $\vec{\phi}_\mu$ is a constant vector field. $V[\vec{\phi}_\mu]$ is a function (not a functional) of $\vec{\phi}_\mu$. Equation (3.12) is equivalent to the equation

$$\frac{dV[\vec{\phi}]}{d\vec{\phi}_\mu} = 0. \tag{3.14}$$

For a theory with derivative couplings, the equivalence of Eq. (3.12) to (3.14) is no longer obvious. In the Yang-Mills case there appear to be contributions to (3.12) from derivative couplings, but in Appendix B we show that such terms systematically cancel. We therefore base our investigation of symmetry breaking on $V[\vec{\phi}_\mu]$ and Eq. (3.14). In momentum space

$$V[\vec{\phi}_\mu] = \frac{g^2}{4} c^a{}_{bc} c_a{}^{de} \phi^b_\mu \phi^c_\nu \phi^\mu_d \phi^\nu_e + i\int \frac{d^4k}{(2\pi)^4} \ln\det\left(\delta_f{}^g - igc^g{}_{fc}\phi^c_\alpha \frac{k^\alpha}{k^2}\right)$$
$$- \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ln\det(\delta_f{}^g \delta^\beta_\mu - [igG_f{}^g\big|_\gamma{}^\beta + g^2 K_f{}^g\big|_\gamma{}^\beta] D^\gamma_{F\mu}(k\,|\,\alpha)), \tag{3.15}$$

$$G_f{}^{g|\gamma\beta} = 2c_f{}^g{}_c(\phi^{c(\gamma}k^{\beta)} - \phi^c \cdot k g^{\gamma\beta}),$$
$$K_f{}^{g|\gamma\beta} = c^a{}_f{}^g c_a{}^{dc}\phi^\gamma_c \phi^\beta_d + c^a{}_f{}^c c_a{}^{dg}(\phi^\gamma_c\phi^\beta_d - \phi_c \cdot \phi_d g^{\gamma\beta}), \tag{3.16}$$
$$D^\gamma_{F\mu}(k\,|\,\alpha) = \frac{1}{k^2}\left(-g^\gamma{}_\mu + (1-\alpha)\frac{k^\gamma k_\mu}{k^2}\right).$$

$V[\vec{\phi}_\mu]$ depends upon enormously complicated determinants over mixed group and Lorentz indices. The evaluation of these determinants for *arbitrary* groups and field configurations has eluded us. However, it follows from the discussion in Sec. II that field configurations of physical interest lie infinitesimally near fields of the form $\vec{\phi}_\mu = \vec{r}\phi_\mu$. We have succeeded in evaluating $V[\vec{\phi}_\mu]$ in this region for the groups SU(2) and SU(3) by direct calculation of the determinants. These calculations are somewhat esoteric and do not particularly elucidate the structure of the loop expansion. Insight into this structure, as well as an important check on the validity of (3.15), can be obtained by calculating the ***ultraviolet-divergent part*** of $V[\vec{\phi}_\mu]$ by direct expansion of (3.15) in Feynman diagrams. This calculation can be performed *for any gauge group*.

IV. ONE-LOOP GRAPHICAL EXPANSION

An expansion in one-loop Feynman diagrams is obtained from (3.15) by converting the ln det to Tr ln and expanding the logarithm. The Feynman rules are given in Fig. 1. Graphs contributing to the ultraviolet divergence of $V[\vec{\phi}_\mu]$ are shown in

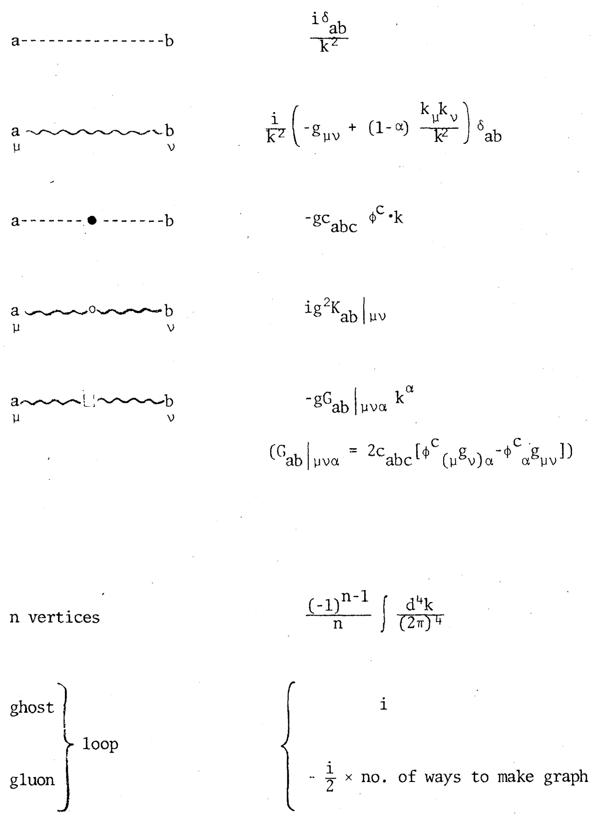

FIG. 1. Feynman rules for the one-loop expansion of the effective potential in non-Abelian gauge theory. The group structure constants are denoted c_{abc}.

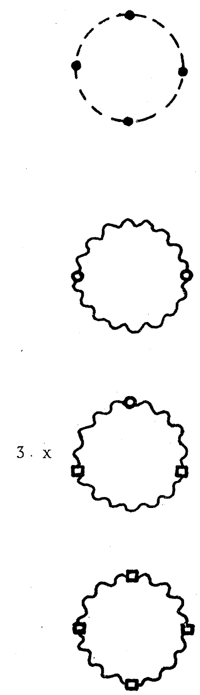

FIG. 2. The four ultraviolet-divergent graphs in the one-loop expansion of the effective potential. The ghost and gluon contributions are denoted by (a) and (b), respectively.

Fig. 2 where we explicitly display the extra factor involved in gluon loops. The calculation of these graphs vastly simplifies in the $\alpha=1$ gauge. We work in this gauge for the remainder of this section. Using dimensional regularization[9] the ghost graph (a) is found to be

$$\frac{-g_R{}^4\mu^{4-n}}{256\pi^2}\left[\frac{1}{n-4}+\ln\left(\frac{m}{2\sqrt{\pi}\mu}\right)+\frac{\gamma}{2}\right]c_f{}^g{}_b\,c_{gh(d}\,c^{hi}{}_c\,c_{e)}{}^f{}_i \times \phi^b_\mu\phi^c_\nu\phi^{\mu d}\phi^{\nu e}, \quad (4.1)$$

where g_R is the renormalized coupling constant, γ is the Euler constant, and m is a ghost "mass" inserted by hand into the propagator. Equation (4.1) displays a $1/(n-4)$ ultraviolet divergence. There is also a logarithmic infrared divergence as m vanishes. The ghost loop is not proportional to the counterterm [first term in (3.15)] since it is completely symmetric under the interchange of d, c, e, whereas the counterterm manifestly is not. Therefore the ghost loop taken alone cannot be ultravioletly renormalized. The gluon loops share this property. When ghost and gluon graphs are added, surprising cancellations occur producing a final result proportional to the counterterm. This clearly displays the need for ghosts in renormalizable non-Abelian gauge theory.

The difficulty encountered summing ghost and gluon loops is in the reduction of structure constants to a single expression proportional to the counterterm. After repeated use of the Jacobi identity and symmetry properties, the number of these constants in each product is reduced from four to two using group-theoretical projection operators. In this we follow Cvitanović.[10] The sum of graphs (a) and (b) is, after a long calculation,

$$\frac{-g_R{}^4\mu^{4-n}}{4}\left(\frac{C_2(G)}{24\pi^2}\right)\left[\frac{1}{n-4}+\ln\left(\frac{m}{2\sqrt{\pi}\mu}\right)+\frac{\gamma}{2}\right]c^a{}_{bc}\,c_a{}^{de}\,\phi^b_\mu\phi^c_\nu\phi^\mu_d\phi^\nu_e, \quad (4.2)$$

where $C_2(G)$ is the value of the quadratic Casimir operator for the adjoint representation of the gauge group. The Casimir invariants for SO(N) and SU(N), respectively, are

$$C_2(\mathrm{SO}(N))=2(N-2),\qquad C_2(\mathrm{SU}(N))=N. \quad (4.3)$$

The ultraviolet divergence in (4.2) can be eliminated by *both coupling-constant and wave-func-*

tion renormalization. The counterterm may be written

$$\tfrac{1}{4}g_R{}^2\mu^{4-n}Z_g{}^2Z_A{}^2c^a{}_{bc}c_a{}^{de}\phi^b_\mu\phi^c_\nu\phi^\mu_d\phi^\nu_e\,, \tag{4.4}$$

where the coupling constant and the wave function are renormalized by Z_g and $\sqrt{Z_A}$, respectively. The $1/(n-4)$ divergence will cancel if

$$Z_g{}^2Z_A{}^2=1+\frac{C_2(G)}{24\pi^2}\frac{1}{n-4}. \tag{4.5}$$

This expression is satisfied by

$$\begin{aligned} Z_A&=1-\frac{g_R{}^2}{16\pi^2}(\tfrac{10}{3}C_2(G))\frac{1}{n-4},\\ Z_g&=1+\frac{g_R{}^2}{16\pi^2}(\tfrac{11}{3}C_2(G))\frac{1}{n-4}, \end{aligned} \tag{4.6}$$

which are the well-known renormalization constants of Yang-Mills theory in the $\alpha=1$ gauge.[11] Thus the effective potential (3.15) and the loop expansion obtained from it are consistent with known properties of non-Abelian gauge theory. Note that the decomposition (4.6) cannot be obtained directly in our approach. Equation (4.2) retains its infrared divergence after renormalization. It is assumed (but not shown) that this divergence will cancel against similar terms in ultraviolet *finite* loop graphs. The complexity of summing just these four graphs of Fig. 2 mitigates against an attempt to evaluate $V[\vec{\phi}_\mu]$ by graphical techniques. We therefore turn to direct evaluation of the determinants in (3.15).

V. EFFECTIVE POTENTIAL FOR SU(2)

We begin by reexpressing (3.15) in a more useful form. Ignoring constants independent of $\vec{\phi}_\mu$ we have

$$\begin{aligned} V[\vec{\phi}_\mu]=&\frac{g^2}{4}c^a{}_{bc}c_a{}^{de}\phi^b_\mu\phi^c_\nu\phi^\mu_d\phi^\nu_e\\ &+i\int\frac{d^4k}{(2\pi)^4}\ln\det(\delta_f{}^gk^2-igc^g{}_{fc}\phi^c_\alpha k^\alpha)\\ &-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\det[\delta_f{}^gD^{-1\beta}_{F\gamma}(k\,|\,\alpha)\\ &-(igG_f{}^g|_\gamma{}^\beta+g^2K_f{}^g|_\gamma{}^\beta)]\,, \end{aligned} \tag{5.1}$$

where

$$D^{-1\beta}_{F\gamma}(k\,|\,\alpha)=k^2\left[-g^\beta_\gamma+\left(1-\frac{1}{\alpha}\right)\frac{k_\gamma k^\beta}{k^2}\right]. \tag{5.2}$$

For $G=\mathrm{SU}(2)$ the structure constants are given by the alternating tensor ϵ_{abc}. The calculation proceeds in two steps. First we evaluate $V[\vec{\phi}_\mu]$ at fields given by

$$\vec{\phi}_\alpha=\vec{r}\phi_\alpha\,, \tag{5.3}$$

where, by virtue of transitivity of the SU(2) action on its adjoint representation, we may take $\vec{r}$ to be

$$\vec{r}=\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{5.4}$$

without loss of generality. The ghost determinant in (5.1) is then given by

$$\det\begin{pmatrix}k^2 & -ig\phi\cdot k & \\ +ig\phi\cdot k & k^2 & \\ & & k^2\end{pmatrix}=\det\begin{pmatrix}(k-g\phi)\cdot k & & \\ & (k+g\phi)\cdot k & \\ & & k^2\end{pmatrix}=\prod_{p=k,\,k\pm g\phi}p\cdot k\,, \tag{5.5}$$

where the matrix diagonalizes under unitary transformation

$$U=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ & & 1\end{pmatrix}. \tag{5.6}$$

The entire ghost contribution to $V[\vec{r}\phi_\mu]$ is

$$\sum_{p=k,\,k\pm g\phi}i\int\frac{d^4k}{(2\pi)^4}\ln p\cdot k\,. \tag{5.7}$$

Similarly the gluon matrix in group space is

$$\begin{pmatrix}D^{-1\beta}_{F\gamma}+A_\gamma{}^\beta & -iX_\gamma{}^\beta & \\ iX_\gamma{}^\beta & D^{-1\beta}_{F\gamma}+A_\gamma{}^\beta & \\ & & D^{-1\beta}_{F\gamma}\end{pmatrix}, \tag{5.8}$$

where

$$A_\gamma{}^\beta=g^2(\phi_\gamma\phi^\beta-\phi^2g_\gamma{}^\beta)\,,$$

$$X^{\gamma\beta}=2g(\phi^{(\gamma}k^{\beta)}-\phi\cdot kg^{\gamma\beta})\,.$$

This matrix is also diagonalized by (5.6) and yields

$$\begin{pmatrix}F(k-g\phi)_\gamma{}^\beta & & \\ & F(k+g\phi)_\gamma{}^\beta & \\ & & F(k)_\gamma{}^\beta\end{pmatrix}, \tag{5.9}$$

where $F(p)_\gamma{}^\beta$ is defined by

$$F(p)_\gamma{}^\beta = -p^2 g_\gamma{}^\beta + p_\gamma p^\beta - \frac{1}{\alpha} k_\gamma k^\beta \,. \tag{5.10}$$

$F(p)_\gamma{}^\beta$ is arrived at by attempting to "square" the eigenvalues. It follows that ln det of (5.8) is given by

$$\sum_{p=k,\, k\pm g\phi} \ln \det F(p)_\gamma{}^\beta \,, \tag{5.11}$$

where the determinant here is over Lorentz indices only. This determinant is easily evaluated as follows:

$$\det F(p)_\gamma{}^\beta = \frac{1}{\alpha}(p\cdot k)^2 p^4 \,. \tag{5.12}$$

The entire gluon contribution to $V[\vec{\mathrm{r}}\phi_\mu]$ is

$$\sum_{p=k,\, k\pm g\phi} \left(-i\int \frac{d^4k}{(2\pi)^4} \ln p\cdot k - i\int \frac{d^4k}{(2\pi)^4} \ln p^2 \right) , \tag{5.13}$$

where we have ignored the constant proportional to $\ln\alpha$. The second integral in (5.13) can be evaluated by using the Euclidean form for k. Note that the pure ϕ terms cancel. In the large-k limit the integral approaches

$$\int \frac{d^4k}{(2\pi)^4} \ln k^2 \,, \tag{5.14}$$

which, being independent of ϕ, can be disregarded. Thus, adding (5.13) to (5.7) we find that the one-loop contribution to $V[\vec{\mathrm{r}}\phi_\mu]$ *vanishes*. It is easy to see that this result is correct. If the ghost and gluon contributions did not cancel then this sum would have an ultraviolet divergence. However, the counterterm vanishes when $\vec{\phi}_\mu = \vec{\mathrm{r}}\phi_\mu$. We would be unable to subtract the infinity in violation of the known renormalizability of gauge theory.

Proceeding to step two, we evaluate $V[\vec{\phi}_\mu]$ at fields

$$\vec{\phi}_\mu = \vec{\mathrm{r}}\phi_\mu + \epsilon\vec{\mathrm{K}}\psi_\mu \,, \tag{5.15}$$

where $\vec{\mathrm{r}}$ is given by (5.4), ϵ is infinitesimally small, and $\vec{\mathrm{K}}$ is chosen (without loss of generality) to be

$$\vec{\mathrm{K}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} . \tag{5.16}$$

The ghost matrix is given by

$$\begin{pmatrix} k^2 & -iX & \\ iX & k^2 & \\ & & k^2 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -iY \\ 0 & iY & 0 \end{pmatrix} , \tag{5.17}$$

where $X = g\phi\cdot k$, $Y = g\psi\cdot k$. Denote the left (right) matrix by X_g (Y_g). X_g is of course the step-one ghost matrix. Define $\tilde{X}_g$ by

$$\tilde{X}_g = UX_gU^\dagger = \begin{pmatrix} (k-g\phi)\cdot k & & \\ & (k+g\phi)\cdot k & \\ & & k^2 \end{pmatrix} . \tag{5.18}$$

Then

$$\det(X_g + \epsilon Y_g) = \det(\tilde{X}_g + \epsilon \tilde{Y}_g) \,, \tag{5.19}$$

where

$$\tilde{Y}_g = UY_gU^\dagger = \frac{g(\psi\cdot k)}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{pmatrix} . \tag{5.20}$$

Converting ln det to Tr ln and expanding the logarithm to order ϵ^2 we find

$$\ln\det\tilde{X}_g + \epsilon\,\mathrm{Tr}(\tilde{Y}_g\tilde{X}_g^{-1}) - \tfrac{1}{2}\epsilon^2\,\mathrm{Tr}(\tilde{Y}_g\tilde{X}_g^{-1})^2 \,. \tag{5.21}$$

The first term will cancel against the ϵ^0 gluon contribution and is ignored. $\mathrm{Tr}(\tilde{Y}_g\tilde{X}_g^{-1})$ vanishes. The final result for the ghost contribution to $V(\vec{\mathrm{r}}\phi_\mu + \epsilon\vec{\mathrm{K}}\psi_\mu)$ (to order ϵ^2) is

$$-\epsilon^2\frac{g^2}{2} \sum_{p=k\pm g\phi} i\int \frac{d^4k}{(2\pi)^4} \frac{(\psi\cdot k)^2}{k^2(p\cdot k)} \,. \tag{5.22}$$

The gluon determinant in group space, after diagonalization of the ϵ^0 matrix, is

$$\det(\tilde{X} + \epsilon\tilde{Y}) \,, \tag{5.23}$$

where $\tilde{X}$ is given by expression (5.9) and

$$\tilde{Y} = \frac{1}{2} \begin{pmatrix} \epsilon B_\gamma{}^\beta & i\epsilon B_\gamma{}^\beta & \sqrt{2}Q_\gamma{}^\beta \\ -i\epsilon B_\gamma{}^\beta & \epsilon B_\gamma{}^\beta & -i\sqrt{2}R_\gamma{}^\beta \\ \sqrt{2}Q^\beta{}_\gamma & i\sqrt{2}R^\beta{}_\gamma & 2\epsilon B_\gamma{}^\beta \end{pmatrix} , \tag{5.24}$$

$$\begin{aligned} B_\gamma{}^\beta &= g^2(\psi_\gamma\psi^\beta - \psi^2 g_\gamma{}^\beta) \,, \\ Q^{\gamma\beta} &= g[-(\psi\cdot p_-)g^{\gamma\beta} - (\psi\cdot k)g^{\gamma\beta} + p_-^\gamma\psi^\beta \\ &\qquad + \psi^\gamma k^\beta - 2g\phi^{[\gamma}\psi^{\beta]}] \,, \\ R^{\gamma\beta} &= g[-(\psi\cdot p_+)g^{\gamma\beta} - (\psi\cdot k)g^{\gamma\beta} + p_+^\gamma\psi^\beta \\ &\qquad + \psi^\gamma k^\beta + 2g\phi^{[\gamma}\psi^{\beta]}] \,, \\ p_\pm &= k \pm g\phi \,. \end{aligned} \tag{5.25}$$

Ignoring the ϵ^0 term (which cancels) we find the entire gluon contribution to $V(\vec{\mathrm{r}}\phi_\mu + \epsilon\vec{\mathrm{K}}\psi_\mu)$ (order ϵ^2) to be

$$\begin{aligned} -\frac{\epsilon^2}{2}\left(\frac{i}{2}\right) \int \frac{d^4k}{(2\pi)^4} \{ &\mathrm{Tr}B[F^{-1}(p_-) + F^{-1}(p_+) + 2F^{-1}(k)] \\ &- \mathrm{Tr}[QF^{-1}(k)Q^tF^{-1}(p_-) \\ &+ RF^{-1}(k)R^tF^{-1}(p_+)]\} \,, \end{aligned} \tag{5.26}$$

where

$$F^{-1}(p)^{\mu\beta}=-\frac{1}{p^2}\left(g^{\mu\beta}+(k^2+\alpha p^2)\frac{p^\mu p^\beta}{(p\cdot k)^2}-\frac{2p^{(\mu}k^{\beta)}}{(p\cdot k)}\right) \tag{5.27}$$

is the inverse of $F(p)^{\mu\beta}$. Clearly a long calculation is in the offing so, for simplicity, we take $\alpha=0$ (Landau gauge). We then have the useful property

$$F^{-1}(p)_\mu{}^\beta k_\beta=0 \tag{5.28}$$

for any p. This choice will also simplify the renormalization-group equation for the effective potential (see Sec. VII). Note that

$$F^{-1}(k)^\beta_\mu=\frac{1}{k^2}\left(-g^\beta_\mu+\frac{k_\mu k^\beta}{k^2}\right) \tag{5.29}$$

is the usual Landau gauge gluon propagator. Expand Eq. (5.26) using Eqs. (5.25), (5.27), (5.28), and (5.29). Collect terms so as to maximally reduce powers in the denominators. Integrals that are functions of k only all vanish. The remaining integrals naturally divide into two types, those that have nonzero powers of $p_\pm\cdot k$ in their denominators and those that do not. The ghost term (5.22) and the first trace in (5.26) have the first type of integral only. The second trace in (5.26) has both types of integrals. We find after integration that the terms of the first type *exactly cancel* each other. The remaining terms can be dimensionally integrated and summed to yield

$$\frac{g_R{}^4\mu^{4-n}}{16\pi^2}\left(\frac{-g_R{}^2\phi^2}{4\pi\mu^2}\right)^{n/2-2}\left[-4\Gamma(2-n/2)+\frac{8}{n}(n-1)\Gamma(3-n/2)\right]B(n/2,n/2)\epsilon^2[\phi^2\psi^2-(\phi\cdot\psi)^2], \tag{5.30}$$

where $B(n/2,n/2)$ is the β function. As in Sec. IV, surprising cancellations ensured that this term would be proportional to the counterterm. In Appendix C we evaluate a typical integral contributing to (5.30) to show how the β function and the important $\phi^2\mu^{-2}$ term arise. Using

$$B(n/2,n/2)=\tfrac{1}{6}-\tfrac{5}{36}(n-4)+O((n-4)^2), \tag{5.31}$$

we can expand (5.30) and find

$$\frac{g_R{}^4\mu^{4-n}}{16\pi^2}\left[\frac{4}{3(n-4)}+\tfrac{2}{3}\gamma-\tfrac{1}{9}+\tfrac{2}{3}\ln\left(\frac{-g_R{}^2\phi^2}{4\pi\mu^2}\right)\right]\epsilon^2[\phi^2\psi^2-(\phi\cdot\psi)^2]. \tag{5.32}$$

The counterterm is given by

$$\tfrac{1}{2}g_R{}^2\mu^{4-n}Z_g{}^2Z_A{}^2\epsilon^2[\phi^2\psi^2-(\phi\cdot\psi)^2], \tag{5.33}$$

where the fields have been renormalized. The ultraviolet divergence in (5.32) can clearly be removed by taking

$$Z_A=1-\frac{g_R{}^2}{16\pi^2}[\tfrac{13}{3}C_2(\mathrm{SU}(2))]\frac{1}{n-4},\qquad Z_g=1+\frac{g_R{}^2}{16\pi^2}[\tfrac{11}{3}C_2(\mathrm{SU}(2))]\frac{1}{n-4} \tag{5.34}$$

$[C_2(\mathrm{SU}(2))=2]$ the well-known renormalization constants of Yang-Mills theory in the $\alpha=0$ gauge. Putting everything together we find the *renormalized* effective potential in the Landau gauge to be

$$V[\vec{\phi}_\mu]=\frac{g_R{}^2}{4}\left(1+\frac{g_R{}^2}{8\pi^2}\left\{\tfrac{2}{3}\ln\left(\frac{-\phi^2}{\mu^2}\right)+\left[\tfrac{2}{3}\ln\left(\frac{g_R{}^2}{4\pi}\right)+\tfrac{2}{3}\gamma-\tfrac{1}{9}\right]\right\}\right)c^a{}_{bc}c_a{}^{de}\phi^b_\mu\phi^c_\nu\phi^\mu_d\phi^\nu_e \tag{5.35}$$

for fields of the form (5.15) to second order in ϵ. In the next section we extend this result to the gauge group SU(3).

VI. EFFECTIVE POTENTIAL FOR SU(3)

The relevant expression for the effective potential for $G=\mathrm{SU}(3)$ is still given by (5.1). As in the preceding section we can evaluate the determinants for fields of the form (5.15) only. However, for SU(3) the direction of vector $\vec{r}$ becomes relevant. It is obvious that SU(3) *does not act transitively* on its adjoint representation. This permits the appearance of several little groups and, in fact, it is known that there are precisely two: U(1) $\times$U(1) and SU(2)$\times$U(1).[12] Vectors having the former little group are dense on a sphere in octet space. The remaining vectors with little group SU(2) $\times$U(1) lie on two isolated, closed orbits of which the $\vec{8}$ direction is a member. Since we are interested in possible applications to unified gauge models, SU(2)$\times$U(1) is of more interest. We therefore choose $\vec{r}$ to have this little group and, without loss of generality, to lie in the $\vec{8}$ direction. As before, we can take its magnitude to be unity. Consider the first term in (5.1) for fields of the form (5.15) with $\vec{K}$ unit norm, orthogonal to $\vec{r}$ but otherwise arbitrary. It is clear from the

SU(3) structure constants that any component of $\vec{K}$ in the $\vec{1}$, $\vec{2}$, $\vec{3}$, or $\vec{8}$ direction does not contribute to this term. We must take $\vec{K}$ to be a linear combination of the remaining four directions. For simplicity we choose $\vec{K}$ in the $\vec{4}$ direction. The counterterm then is given by

$$\tfrac{1}{4}g^2(\tfrac{3}{2})[\phi^2\psi^2-(\phi\cdot\psi)^2]\epsilon^2\,. \tag{6.1}$$

Note that this becomes identical in form to the SU(2) counterterm if we set

$$g'=(\tfrac{3}{2})^{1/2}g\,,\qquad \phi'=\frac{1}{\sqrt{2}}\phi\,. \tag{6.2}$$

We now evaluate the second and third terms in (5.1). As before, the contributions of zeroeth order in ϵ cancel. It appears, after a great deal of algebra, that the ghost and gluon terms expressed in terms of g' and ϕ' of (6.2) are precisely the SU(2) expressions (5.22) and (5.26), respectively. The renormalization constants in the $\alpha=0$ gauge can then be read off from (5.34):

$$Z_A=1-\frac{g_R^{\,2}}{16\pi^2}\{\tfrac{13}{3}[\tfrac{3}{2}C_2(\mathrm{SU}(2))]\}\frac{1}{n-4}\,,\qquad Z_g=1+\frac{g_R^{\,2}}{16\pi^2}\{\tfrac{11}{3}[\tfrac{3}{2}C_2(\mathrm{SU}(2))]\}\frac{1}{n-4}\,, \tag{6.3}$$

since

$$\tfrac{3}{2}C_2(\mathrm{SU}(2))=C_2(\mathrm{SU}(3))\,. \tag{6.4}$$

This result is consistent with Yang-Mills theory. For SU(3), the renormalized effective potential for fields of the above form in the Landau gauge, to second order in ϵ can be read off directly from (5.35) and is

$$V[\vec{\phi}_\mu]=\frac{g_R^{\,2}}{4}\left(1+\frac{g_R^{\,2}}{8\pi^2}\left\{\ln\left(\frac{-\phi^2}{\mu^2}\right)+\left[\ln\left(\frac{3g_R^{\,2}}{8\pi}\right)+\gamma-\tfrac{1}{6}\right]\right\}\right)c^a{}_{bc}c_a{}^{de}\phi^b_\mu\phi^c_\nu\phi^\mu_d\phi^\nu_e\,, \tag{6.5}$$

where the fields have been renormalized. Notice that nothing in this solution prevents us from taking $\psi_\mu=\phi_\mu$. In this case

$$\vec{\phi}_\mu=(\vec{r}+\epsilon\vec{K})\phi_\mu \tag{6.6}$$

and $V[\vec{\phi}_\mu]$ vanishes. It follows from this and the discussion at the beginning of the section that there is nothing in the theory to single out the $\vec{8}$ direction. The vacuum direction $\vec{r}+\epsilon\vec{K}$ *for every* $\vec{K}$ is just as good as $\vec{r}$. However, most vacuums $\vec{r}+\epsilon\vec{K}$ lie off the isolated SU(2)×U(1) orbit and have little group U(1)×U(1). Therefore symmetry breaking in non-Abelian gauge theory *does not naturally single out the little group.* This result is due to gauge invariance which prevents terms cubic in the adjoint field from appearing in the effective potential.

VII. THE RENORMALIZATION GROUP

In Sec. II the one-loop approximation to $V[\vec{\phi}_\mu]$ was found to be valid only when *both* g_R and $\frac{1}{2}g_R\ln(\phi^2/\mu^2)$ were small. This restriction on the second term played a major role in the physical interpretation of the nonzero vacuum. Following Coleman and Weinberg, we know that the range of validity of the logarithmic term can be extended using the renormalization-group-improved effective potential. The renormalization-group equation for the effective action in the $\alpha=0$ gauge[13] is given by

$$\left(\mu\frac{\partial}{\partial_\mu}+\beta(g_R)\frac{\partial}{\partial g_R}+\gamma(g_R)\int d^4x\,A^a_\mu(x)\frac{\delta}{\delta A^a_\mu(x)}\right)\Gamma[\vec{A}_\mu]=0\,. \tag{7.1}$$

For the effective potential this equation becomes

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma\phi^a_\mu\frac{\partial}{\partial\phi^a_\mu}\right)V[\vec{\phi}_\mu]=0\,. \tag{7.2}$$

It follows from Secs. II, V, and VI that to one-loop approximation for fields near $\vec{r}\phi_\mu$

$$V[\vec{\phi}_\mu]=c^a{}_{bc}c_a{}^{de}\phi^b_\mu\phi^c_\nu\phi^\mu_d\phi^\nu_e V^{(4)}\left(g_R,\tfrac{1}{2}\ln\left(\frac{-\phi^2}{\mu^2}\right)\right)\,, \tag{7.3}$$

where

$$V^{(4)}=\frac{g_R^{\,2}}{4}\left\{1+g_R^{\,2}\left[A+\frac{B}{2}\ln\left(\frac{-\phi^2}{\mu^2}\right)\right]\right\}\,. \tag{7.4}$$

A and B are independent of $\vec{\phi}_\mu$ and depend on the gauge group. If we let

$$t=\tfrac{1}{2}\ln\left(\frac{-\phi^2}{\mu^2}\right)\,,\qquad \bar{\beta}=\beta(1-\gamma)^{-1}\,,\qquad \bar{\gamma}=\gamma(1-\gamma)^{-1}\,, \tag{7.5}$$

and change variable μ to t we find that

$$\left(-\frac{\partial}{\partial t}+\bar{\beta}\frac{\partial}{\partial g_R}+\bar{\gamma}\phi^a_\mu\frac{\partial}{\partial\phi^a_\mu}\right)V[\vec{\phi}_\mu]=0\,. \tag{7.6}$$

Differentiating four times with respect to ϕ^a_μ and noting that V^4 is a function of g_R and t only, we arrive at the renormalization-group equation

$$\left(-\frac{\partial}{\partial t}+\bar{\beta}\frac{\partial}{\partial g_R}+4\bar{\gamma}\right)V^{(4)}(g_R,t)=0\,. \tag{7.7}$$

The general solution for this equation is known and is given by

$$V^{(4)}(g_R,t)=V^{(4)}(g'(t,g_R),0)\times\exp\left[4\int_0^t dt\,\bar{\gamma}(g'(t,g_R))\right], \quad (7.8)$$

where $g'(t,g_R)$ is an effective coupling constant satisfying the differential equation

$$\frac{dg'}{dt}=\bar{\beta}(g'), \quad (7.9)$$

with boundary condition

$$g'(0,g_R)=g_R. \quad (7.10)$$

To one-loop approximation, consistent with the renormalization constants in (5.34) and (6.3), the β and γ functions in the $\alpha=0$ gauge are[11]

$$\beta=-\frac{g_R^3}{16\pi^2}\,\tfrac{11}{3}\,C_2(G),\qquad \gamma=+\frac{g_R^2}{16\pi^2}\,\tfrac{13}{3}\,C_2(G). \quad (7.11)$$

Note that this approximation is valid if and only if g_R is small. If we assume that g' is small for some range of t then the functions in (7.11), with g' substituted for g_R, can be used to find a valid approximate solution of Eq. (7.9). This solution to order g_R^3 is given by

$$g'(t,g_R)=g_R\left[1+\frac{g_R^2}{8\pi^2}\,(\tfrac{11}{3}C_2(G))\,t\right]^{-1/2}. \quad (7.12)$$

Now $g'(t,g_R)$ must be small for *any* small g_R. This will be true if and only if

$$t\gtrsim 0. \quad (7.13)$$

As predicted, the renormalization group has lifted the restriction that g_Rt be small. It is important to note that t must be positive (as opposed to the negative range for t in the Coleman-Weinberg example) precisely because of the famous (asymptotic freedom) minus sign in the β function for non-Abelian gauge theories. Therefore, for such theories the renormalization-group-improved potential to one-loop approximation is valid for $g'(t,g_R)$ small and

$$-\phi^2\gtrsim\mu^2, \quad (7.14)$$

but not near the zero field. This strengthens the conclusion of Sec. II and makes the role played by asymptotic freedom precise. The exponential factor in (7.8) can be evaluated to order g_R^2 by substituting (7.12) into (7.11) and evaluating the integral. The result is

$$\left[1+\frac{g_R^2}{8\pi^2}\,(\tfrac{11}{3}\,C_2(G))\,t\right]^{26/11}=\left(\frac{g_R^2}{g'^2}\right)^{26/11} \quad (7.15)$$

Putting everything together we have from (7.8) and (7.4) that

$$V^{(4)}(g_R,t)=\frac{g'^2}{4}\,(1+g'^2A)\left(\frac{g_R^2}{g'^2}\right)^{26/11}. \quad (7.16)$$

The effective potential is determined from (7.3). For example, if $G=SU(3)$ then it follows from (7.3), (6.5), (7.12), and (7.16) that

$$V[\vec{\phi}_\mu]=c^a{}_{bc}\,c_a{}^{de}\,\phi^b_\mu\phi^\mu_d\phi^\nu_e\,\frac{g_R^2}{4}\left[1+\frac{g_R^2}{8\pi^2}\,\tfrac{11}{2}\ln\left(\frac{-\phi^2}{\mu^2}\right)\right]^{15/11}\times\left\{1+\frac{g_R^2}{8\pi^2}\left[1+\frac{g_R^2}{8\pi^2}\,\tfrac{11}{2}\ln\left(\frac{-\phi^2}{\mu^2}\right)\right]^{-1}\left[\ln\left(\frac{3g_R^2}{8\pi^2}\right)+\gamma-\tfrac{1}{6}\right]\right\} \quad (7.17)$$

is the renormalization-group-improved, one-loop effective potential near fields of the form $\vec{r}\phi_\mu$. Its region of validity is given by (7.14).

VIII. DISCUSSION

Examination of the tree level potential led us to consider constant gauge fields of the form $\vec{r}v_\mu$ as possible candidates for the vacuum state of a non-Abelian gauge theory. It was shown that such fields are simply gauge translates of the zero field and are, therefore, with respect to any *gauge-invariant* quantity, equivalent to the zero field. However, by direct computation, it was found that for a non-Abelian gauge theory the effective potential and (it follows) N-point, 1PI, on-shell vertex functions are not strictly gauge invariant at the *one-loop level*. In the one-loop effective potential gauge invariance is broken in two ways. First it is broken by terms due to gauge fixing (α dependent) and second (and much more important) by logarithms $\ln(\phi^2/\mu^2)$. These logarithms, in conjunction with asymptotic freedom, imply that the one-loop approximation to the effective potential is valid only far from the zero field. Therefore, at the *one-loop level* the fields $\vec{r}v_\mu$ are no longer equivalent to $\vec{0}^\mu$ (as far as the effective potential is concerned), and a renormalization-group analysis shows that one must expand around vacuum(s) $\vec{r}v_\mu$ with the property that $v^2\gtrsim\mu^2>0$. This is true even though all fields of the form $\vec{r}v$ are *energetically* equivalent at one loop ($V[\vec{r}v_\mu] = V[\vec{0}_\mu] = 0$). The effective potential is notoriously difficult to interpret physically. It is perhaps more enlightening to consider the one-loop, N-point, 1PI, on-shell vertex functions of the theory. These can be obtained by calculating the Nth derivative of the one-loop effective potential and evaluating it at the vacuum (for now, arbitrary $\vec{r}v_\mu$). Since the 1PI vertices are derived from the effective potential it is clear that the one-loop approximation to these vertices is valid only if we choose nonzero vacuums(s) $\vec{r}v_\mu$ such that $v^2 \gtrsim \mu^2 > 0$ as above. This has the effect of regulating the logarithm terms *which would diverge if we tried (incorrectly) to use the zero field as the vacuum*. Traditionally one thinks of the S matrix as gauge independent. This is true if one ignores the logarithmic terms. However, these logarithms break the gauge invariance of the S matrix and therefore the theory can, and does, distinguish between $\vec{r}v_\mu$ and $\vec{0}_\mu$. If one wants the one-loop approximation to be valid, then *the theory demands* (no matter how small the coupling constant) a nonzero vacuum $\vec{r}v_\mu$ such that $v^2 \gtrsim \mu^2 > 0$. This acts as an infrared (IR) regulator preventing disastrous logarithmic divergences. In the sector of the S matrix independent of logarithms gauge invariance demands that $\vec{r}v_\mu$ be indistinguishable from $\vec{0}_\mu$. We therefore do not expect that our nonzero vacuums will break either Lorentz or internal symmetries.

To conclude, it would appear that at least at the one-loop level the supposed IR divergences of non-Abelian gauge theory are merely artifacts of having chosen the *wrong vacuum* ($\vec{0}_\mu$). In fact, these divergences are regulated by choosing the nonzero vacuum(s) $\vec{r}v_\mu$ demanded by the renormalization group. These nonzero vacuums are energetically equivalent to the zero field. It is therefore not energetics, but the theories need to regulate its one-loop divergences, that forces nonzero vacuums upon us. Such vacuums, *outside of logarithms*, are equivalent to $\vec{0}^\mu$ and therefore spontaneous symmetry breaking, in the manner of Coleman and Weinberg, does not occur.

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APPENDIX A: THE LOOP EXPANSION

The zero- and one-loop contributions to the generating functional $Z[\vec{J}_\mu, \vec{\beta}, \vec{\beta}^\dagger]$ are given by

$$Z[\vec{J}_\mu, \vec{\beta}, \vec{\beta}^\dagger] = N \exp\left\{ i \left[S[\vec{A}_{\mu(0)}, \vec{\xi}_{(0)}, \vec{\eta}_{(0)}] + \int d^4x (-\vec{J}_\mu \cdot \vec{A}^\mu_{(0)} + \vec{\xi}_{(0)} \cdot \vec{\beta} + \beta^\dagger \cdot \vec{\eta}_{(0)}) \right] \right\}$$
$$\times \int [\vec{d}A_\mu][\vec{d}\xi][\vec{d}\eta] \exp\left(\frac{i}{2} \int d^4x\, d^4y\, \tilde{\psi}_i(x) \frac{\vec{\delta}}{\delta\psi_i(x)} S \frac{\vec{\delta}}{\delta\psi_j(y)} \bigg|_0 \tilde{\psi}_j(y) \right), \quad \text{(A1)}$$

where $\tilde{\psi}_i(x) = \psi_i(x) - \psi_{i(0)} - \psi_{i(0)}(x)$. The second functional derivative of S (for the directional derivative shown) can be evaluated. The entire space-time integrand of the second term in (A1) is

$$(\tilde{A}^b_\nu(x),\ \tilde{\xi}^c(x)) \begin{bmatrix} \dfrac{\delta^2 S}{\delta A^\nu_b(x) \delta A^\alpha_e(y)} \bigg|_0 & 2gc_{dab}(\partial^\nu \xi^a_{(0)}(y))\delta(x-y) \\ -2gc_{cae}\eta^a_{(0)}(y)\partial^\alpha\delta(x-y) & -2\partial^2\delta_{dc} + 2gc_{cda}A^a_{\alpha(0)}\partial^\alpha + 2gc_{cda}(\partial^\alpha A^a_{\alpha(0)})\delta(x-y) \end{bmatrix} \begin{bmatrix} \tilde{A}^e_\alpha(y) \\ \eta^{-d}(y) \end{bmatrix}. \quad \text{(A2)}$$

Iteratively solving the field equations (3.6) for $\vec{\xi}_{(0)}$ and $\vec{\eta}_{(0)}$ we find that as $\vec{\beta}$ and $\vec{\beta}^\dagger$ go to zero, $\vec{\eta}_{(0)}$ vanishes and $\vec{\eta}_{(0)}$ vanishes and $\vec{\xi}_{(0)}$ approaches an arbitrary constant. Therefore, in this limit the off-diagonal terms in (A2) vanish. The gluon and ghost modes thus decouple and the path integral can be evaluated in the usual way. The result is

$$W[\vec{J}_\mu] = S[\vec{A}_{\mu(0)}] - \int d^4x\, \vec{J}_\mu \cdot \vec{A}^\mu_{(0)} - i \ln\det((-2\partial^2\delta^a{}_b\, 2gc_b{}^a cA^c_\alpha(0)\partial^\alpha + 2gc_b{}^a{}_c(\partial^\alpha A^c_{\alpha(0)}))\delta(x-y)$$
$$+ \frac{i}{2} \ln\det\left(\frac{\delta^2 S}{\delta A^b_\nu(x) \delta A^\alpha_c(y)} \bigg|_0 \right). \quad \text{(A3)}$$

Evaluating the second derivative using (2.1) and throwing away infinite constants we arrive at Eqs. (3.7) and (3.8).

APPENDIX B: DERIVATIVE COUPLINGS

To the one-loop approximation $\Gamma[\Phi]$ is given by (3.7). The dangerous terms in the graphical expansion of $\Gamma[\Phi]$ are those which involve $\Phi\partial_\mu\Phi$. A typical example is

$$c_f{}^g{}_c\, c_g{}^f{}_d \int d^4x\, d^4y(\partial_x^\alpha \Phi_\alpha^c(x)) \times D_{F\mu}^\beta(x-y)\Phi^{d\mu}(y)\partial_{\gamma y} D_{F\beta}^\gamma(y-x)\ . \qquad \text{(B1)}$$

Functionally differentiating this expression with respect to Φ_ξ^e we find

$$-c_f{}^{ge} c_g{}^f{}_d \int d^4y\, \partial_{z\xi}(D_{F\mu}^\beta(z-y)\Phi^{d\mu}\partial_{y\gamma} D_{F\beta}^\gamma(y-z)) + \text{term proportional to } \partial^\alpha\Phi_\alpha\ . \qquad \text{(B2)}$$

Taking Φ to be a space-time constant ϕ the last term in (B2) vanishes and the first term becomes

$$-c_f{}^{ge} c_g{}^{fd}\,\phi^{d\mu}\int d^4y[(\partial_{z\xi} D_{F\mu}^\xi(z-y))(\partial_{y\gamma} D_{F\beta}^\gamma(y-z)) + D_{F\mu}^\beta(z-y)\partial_{z\xi}\partial_{y\gamma} D_{F\beta}^\gamma(y-z)] \qquad \text{(B3)}$$

In the first term the derivatives act on the first variable in the propagator, making that term, in momentum space, proportional to $-k_\xi k_\gamma$. The second term is proportional to $+k_\xi k_\gamma$ and therefore (B3) vanishes. This result follows from the trace over space-time variables. Clearly, every such term vanishes for the same reason and therefore they do not contribute to calculation of the vacuum expectation value.

APPENDIX C: INTEGRALS

A typical integral contributing to (5.30) is

$$g^2\int \frac{d^4k}{(2\pi)^4}\,\frac{k^\alpha k^\beta}{(k-g\phi)^2 k^4}\ . \qquad \text{(C1)}$$

Combining the denominator we have

$$\int_0^1 (x-1)2g_R{}^2\mu^{4-n}\int \frac{d^nk}{(2\pi)^n}\,\frac{k^\alpha k^\beta}{(m^2-2p\cdot k-k^2)^3} \qquad \text{(C2)}$$

in n dimensions, where

$$m^2=\phi^2 g^2 x\ ,$$
$$p=-\phi g x\ , \qquad \text{(C3)}$$
$$m^2+p^2=\phi^2 g^2 x(x-1)\ .$$

Note that $m^2>0$ if and only if ϕ is spacelike (our condition from Sec. II). The k integral can be evaluated in the usual way[6] and the result is

$$g_R{}^2\,\frac{i}{16\pi^2}\left(\frac{1}{4\pi\mu^2}\right)^{n/2-2}\frac{1}{(m^2+p^2)^{2-n/2}}\left(\Gamma(3-n/2)\,\frac{p_\alpha p_\beta}{m^2+p^2}-\Gamma(2-n/2)\,\frac{g_{\alpha\beta}}{2}\right)\ . \qquad \text{(C4)}$$

The term $(-g_R{}^2\phi^2)^{n/2-2}$ can be extracted from $(m^2+p^2)^{n/2-2}$ and combined with $(4\pi\mu^2)^{2-n/2}$ to yield the important factor

$$\left(\frac{-g_R{}^2\phi^2}{4\pi\mu^2}\right)^{n/2-2}\ . \qquad \text{(C5)}$$

The remaining x integrals are precisely the definition of the β function. The final result is

$$g_R{}^2\,\frac{i}{16\pi^2}\left(\frac{-g_R{}^2\phi^2}{4\pi\mu^2}\right)^{n/2-2}\left[\Gamma(3-n/2)\,\frac{\phi_\alpha\phi_\beta}{\phi^2}\,B(n/2,n/2-1)+\Gamma(2-n/2)\,\frac{g_{\alpha\beta}}{2}\,B(n/2-1,\ n/2)\right]\ . \qquad \text{(C6)}$$

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their calculation to arbitrary, compact gauge groups and gives an important check on the functional method by graphically calculating the renormalization constants.

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