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Fock-space description of the $1/N_c$ expansion of quantum chromodynamics

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We present a Fock-space formulation of the expansion of quantum chromodynamics in a power series in $1/N_c$, where N_c is the number of colors. We hope this formulation will aid spectrum calculations in the $N_c \to \infty$ limit.

I. INTRODUCTION

Among the many approaches toward spectrum calculations in quantum chromodynamics (QCD), which have evolved during the past few years, 't Hooft's $1/N_c$ expansion[1] (N_c is the number of colors) remains one of the most attractive. In this article we present a formulation of the $1/N_c$ expansion which seems particularly suited for spectrum calculations.

't Hooft has shown[1] that for a specific process the $N_c \to \infty$ limit with $N_c g^2$ fixed is obtained by summing the subset of all Feynman graphs which are planar in both color and momentum flow and which have a minimum number of quark loops. For example, the $N_c \to \infty$ limit for the process $e^+e^- \to$ anything is the sum of all planar diagrams with precisely one quark loop. This graphical approach is awkward for spectrum calculations: One has to search for poles in appropriate Green's functions. As in the derivation of the Bethe-Salpeter equation, the locations of these poles are found by solving the eigenvalue problem for some homogeneous integral equation. We feel it is useful to observe that the integral equations appropriate to the $N_c \to \infty$ limit can be obtained directly without the intermediate step of writing down Green's functions as sums over a specific class of graphs.

The subject of this article is just such a direct approach. We begin in Sec. II by introducing the usual Fock space for QCD without quarks. We then restrict ourselves to the color-singlet subspace. The action of the Hamiltonian on this color-singlet subspace simplifies in the $N_c \to \infty$ limit and a simple algorithm for taking the large-N_c limit emerges.

In an ordinary reference frame, the bare vacuum is not an eigenstate of the Hamiltonian. The fact that the energy density of the vacuum blows up like $N_c^{\,2}$ then introduces spurious divergences in the Schrödinger equation. In the infinite-momentum frame these divergences are not present inasmuch as the bare vacuum is then an eigenstate of the complete Hamiltonian. Our final equations are therefore written in the infinite-momentum frame. In Sec. III we discuss the structure of the equations to be solved and remind the reader of a previous attempt to extract some information from these equations.

II. THE FOCK-SPACE FORMULATION OF THE $N_c \to \infty$ LIMIT

In this article we restrict our attention to the $SU(N_c)$ Yang-Mills theory without quarks. This is not an essential restriction because the $1/N_c$ expansion singles out the pure gluon theory in leading order: Quark effects come in only in nonleading order.

We begin by introducing the usual Fock space. The creation operator for a bare gluon will be denoted

$$a^\dagger_{i\alpha\beta}(\vec{p}), \tag{2.1}$$

where α, β are color indices each running from 1 to N_c, $\vec{p}$ is the three-momentum in an ordinary frame or the pair $(p_\perp, P^+)$ in the infinite-momentum frame $[V^\pm \equiv (1/\sqrt{2})(V^0 \pm V^3)]$ and i labels the two physical polarizations of the gluon. We shall be working in a physical gauge: the axial gauge $A_3 = 0$ in an ordinary frame, and the light-cone gauge $A^+ = 0$ in the infinite-momentum frame. In these gauges the 0 and 3 components of the gluon field are explicitly eliminated.

The relationship between the $a^\dagger$'s and the 1, 2 components of the gluon field is as follows ($i = 1, 2$):

$$A_{i\alpha\beta}(\vec{x}) = \int \frac{d^3k}{(2|\vec{k}|)^{1/2}} \frac{1}{(2\pi)^{3/2}} \left[\delta_{ij} + \frac{k_i k_j}{k_\perp^{\,2}} \left(1 + \frac{|\vec{k}|}{k^3}\right)\right] \left[a_{j\alpha\beta}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} - a^\dagger_{j\beta\alpha}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}\right], \tag{2.2a}$$

$$F_{0i\alpha\beta}(\vec{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{(2|\vec{k}|)^{1/2}}(-i|\vec{k}|)\left[\delta_{ij}+\frac{k_i k_j}{k_\perp^2}\left(1+\frac{k_3}{|\vec{k}|}\right)\right]\left[a_{j\alpha\beta}(\vec{k})e^{i\vec{k}\cdot\vec{x}}+a^\dagger_{j\beta\alpha}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right] \tag{2.2b}$$

for the axial gauge in an ordinary reference frame. $A_{i\alpha\beta}(\vec{x})$ and $-F_{0i\beta\alpha}(\vec{x})$ are canonically conjugate, provided

$$[a_{i\alpha\beta}(\vec{p}),\ a^\dagger_{j\gamma\delta}(\vec{p}\,')]=\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta^3(\vec{p}\,'-\vec{p}\,')\ . \tag{2.3}$$

The bare Hamiltonian in the axial gauge is

$$H_0^{\text{axial}}=\int d^3p\,|\vec{p}|\sum_{i\alpha\beta}a^\dagger_{i\alpha\beta}(\vec{p})a_{i\alpha\beta}(\vec{p})\ , \tag{2.4}$$

where we have dropped the zero-point energy.

In the light-cone gauge, the situation is simpler:

$$A_{i\alpha\beta}(\underline{x}_\perp,x^-)=\frac{1}{(2\pi)^{3/2}}\int_0^\infty\frac{dP^+}{(2P^+)^{1/2}}\int d\underline{p}_\perp\left[a_{i\alpha\beta}(\underline{p}_\perp,P^+)e^{i(\underline{x}_\perp\cdot\underline{p}_\perp-x^-P^+)}-a^\dagger_{i\beta\alpha}(\underline{p}_\perp,P^+)e^{-i(\underline{x}_\perp\cdot\underline{p}_\perp-x^-P^+)}\right], \tag{2.5}$$

and the light-cone commutation relations are guaranteed, provided

$$[a_{i\alpha\beta}(\underline{p}_\perp,P^+),\ a^\dagger_{j\gamma\delta}(\underline{p}_\perp',P^{+\prime})]=\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\,\delta^2(\underline{p}_\perp-\underline{p}_\perp')\delta(P^+-P^{+\prime})\ . \tag{2.6}$$

The bare Hamiltonian in the light-cone gauge is

$$P_0^-=\int d\underline{p}_\perp\int_0^\infty dP^+\,\frac{\underline{p}_\perp^{\,2}}{2P^+}\sum_{i,\alpha,\beta}a^\dagger_{i\alpha\beta}(\underline{p}_\perp,P^+)a_{i\alpha\beta}(\underline{p}_\perp,P^+)\ . \tag{2.7}$$

The bare vacuum is defined by the condition

$$a_{i\alpha\beta}(\vec{p})|0\rangle_B=0\ , \tag{2.8}$$

and the Fock space is the space of states generated by letting all possible polynomials of the $a^\dagger$'s act on $|0\rangle_B$.

Our next task is to characterize the color-singlet subspace. This is easily done: Any color-singlet state will be a linear combination of states of the form

$$|\psi\rangle=\left(\frac{1}{\sqrt{N_c}}\right)^{l_1+l_2\cdots+l_n}\mathrm{Tr}[a^\dagger_{11}a^\dagger_{12}\cdots a^\dagger_{1l_1}]\,\mathrm{Tr}[a^\dagger_{21}a^\dagger_{22}\cdots a^\dagger_{2l_2}]\cdots\mathrm{Tr}[a^\dagger_{n1}a^\dagger_{n2}\cdots a^\dagger_{nl_n}]|0\rangle_B\ , \tag{2.9}$$

where the $a^\dagger$'s are regarded as matrices in the color indices. The subscripts in Eq. (2.9) signify that each $a^\dagger$ has a different momentum and polarization, in general.

Consider first the N_c dependence of the norm of the state (2.9). The evaluation of the norm proceeds by contracting the a's in the bra in all possible ways with the $a^\dagger$'s in the ket. Factors of N_c will arise through the formula $\sum_\alpha\delta_{\alpha\alpha}=N_c$. The contraction scheme which yields the most factors of N_c is essentially unique. The maximum number of factors of N_c is $l_1+l_2+\cdots+l_n$, and in such a contraction scheme a trace in the bra with l a's must completely contract against a trace in the ket with precisely l $a^\dagger$'s. The scheme of contractions between two such corresponding traces is unique up to cyclic permutation (in leading order as $N_c\to\infty$): There are l ways to make the first contraction, but then the rest are uniquely given. The only contractions which yield a factor of N_c are those where the a and $a^\dagger$ have a common color index. A simple example will make this clear:

$$\langle 0|\,\mathrm{Tr}[a_1a_2a_3]\,\mathrm{Tr}[a^\dagger_{1'}a^\dagger_{2'}a^\dagger_{3'}]\,|0\rangle=\delta_{11'}\langle 0|\,\mathrm{Tr}[a_2a_3a^\dagger_{2'}a^\dagger_{3'}]\,|0\rangle+\delta_{21'}\langle 0|\,\mathrm{Tr}[a_3a_1a^\dagger_{2'}a^\dagger_{3'}]\,|0\rangle+\delta_{31'}\langle 0|\,\mathrm{Tr}[a_1a_2a^\dagger_{2'}a^\dagger_{3'}]\,|0\rangle$$

but, e.g.,

$$\langle 0|\,\mathrm{Tr}[a_2a_3a^\dagger_{2'}a^\dagger_{3'}]\,|0\rangle=N_c\,\delta_{32'}\langle 0|\,\mathrm{Tr}[a_2a^\dagger_{3'}]\,|0\rangle+\delta_{33'}\langle 0|\,\mathrm{Tr}[a_2]\,\mathrm{Tr}[a^\dagger_{2'}]\,|0\rangle$$
$$=N_c^{\,3}\delta_{32'}\delta_{23'}+N_c\,\delta_{33'}\delta_{22'}\underset{N_c\to\infty}{\sim}N_c^{\,3}\delta_{32'}\delta_{23'}\ .$$

Thus

$$\langle 0|\, \mathrm{Tr}[a_1 a_2 a_3]\, \mathrm{Tr}[a_{1'}^\dagger a_{2'}^\dagger a_{3'}^\dagger]\, |0\rangle \underset{N_c \to \infty}{\sim} N_c{}^3(\delta_{11'}\delta_{32'}\delta_{23'} + \delta_{21'}\delta_{12'}\delta_{33'} + \delta_{31'}\delta_{22'}\delta_{13'}) \,.$$

The factor $(1/\sqrt{N_c}\,)^{l_1+\cdots+l_n}$ in (2.9) was chosen so that $|\psi\rangle$ has a finite norm in the limit $N_c \to \infty$. The overlap of two states of the form (2.9) will go to zero as $N_c \to \infty$ unless the two states have identical partitions of $a^\dagger$'s among the different traces. Thus *states of the form (2.9) form an orthonormal (in this sense) basis of the color-singlet Hilbert space in the $N_c \to \infty$ limit.*

Now we are in a position to consider the structure of the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle \tag{2.10}$$

in the large-N_c limit. For our discussion of counting powers of N_c we may suppress the dependence of the equation on momentum and gluon polarization. Thus as far as dependence on N_c is concerned, H has the generic structure

$$\begin{aligned} H \sim\ & \mathrm{Tr}[a^\dagger a] + \frac{1}{\sqrt{N_c}}\{\mathrm{Tr}[a_1^\dagger a_2^\dagger a_3] + \mathrm{Tr}[a_3^\dagger a_2 a_1]\} + \frac{1}{N_c}\{\mathrm{Tr}[a_1^\dagger a_2^\dagger a_3^\dagger a_4] + \mathrm{Tr}[a_4^\dagger a_3 a_2 a_1]\} \\ & + \frac{1}{N_c}\{\mathrm{Tr}[a_1^\dagger a_2^\dagger a_3 a_4] + :\mathrm{Tr}[a_1^\dagger a_2 a_3^\dagger a_4]\} + \frac{1}{\sqrt{N_c}}\{\mathrm{Tr}[a_1^\dagger a_2^\dagger a_3^\dagger] + \mathrm{Tr}[a_3 a_2 a_1]\} + \frac{1}{N_c}\{\mathrm{Tr}[a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger] + \mathrm{Tr}[a_4 a_3 a_2 a_1]\}\,. \end{aligned} \tag{2.11}$$

Terms of the sort contained in the last two sets of braces are not present in the infinite-momentum-frame Hamiltonian P^-. In the matrix products we adopt the convention that

$$(a^\dagger)_{\alpha\beta} \equiv (a_{\beta\alpha})^\dagger \,. \tag{2.12}$$

Now consider how each type of term in (2.11) acts on a state of the type (2.9). Terms of the type $\mathrm{Tr}[a^\dagger a]$, $(1/\sqrt{N_c})\,\mathrm{Tr}[a^\dagger a^\dagger a]$, $(1/N_c)\mathrm{Tr}[a^\dagger a^\dagger a^\dagger a]$ merely substitute $a^\dagger$, $(1/\sqrt{N_c})a^\dagger a^\dagger$, or $(1/N_c)a^\dagger a^\dagger a^\dagger$ for a single $a^\dagger$ in the state. The number of traces is not disturbed; the number of $a^\dagger$'s in a single trace is increased by zero, one, or two, but the explicit factors of $1/\sqrt{N_c}$ ensure that each contribution remains of order 1. Terms of the type $(1/\sqrt{N_c}\,)\mathrm{Tr}[a^\dagger aa]$ and $(1/N_c)a^\dagger a^\dagger aa]$ involve two contractions. The first substitutes $(1/\sqrt{N_c})a^\dagger a$ or $(1/N_c)a^\dagger aa$ for a single $a^\dagger$. The second contraction will remove another $a^\dagger$ from the state. Counting leftover factors of $1/N_c$, we see that these contractions will yield a state of order $1/N_c$ unless the contraction itself supplies an extra factor of N_c. There is only one contraction which does this, namely the a contracting against the only $a^\dagger$ with a common color index. We may characterize this verbally by saying that the only terms which survive the $N \to \infty$ limit are those in which the a's *contract against nearest neighbors on the same trace*. A similar conclusion applies to the term $(1/N_c)\mathrm{Tr}[a^\dagger aaa]$: The only surviving contractions are those in which all three a's contract against three nearest neighbors on the same trace.

It is amusing to observe what the suppressed contractions do to the structure of the state. If the two contractions are against non-nearest neighbors on the same trace, the result of the contraction is to split that single trace into two traces. If the two contractions are against two a's from different traces, the two traces are tied together into a single trace, similarly for the term involving three contractions. If the contractions are against three non-nearest neighbors on the same trace, the trace splits into three, or if one is against one nearest neighbor and one non-nearest neighbor, splits into two. The various possibilities are illustrated pictorially in Fig. 1. A term of the form: $(1/N_c)\mathrm{Tr}[a^\dagger a a^\dagger a]$: is always nonleading and either splits a trace or joins two traces. In general we have the result that changes in the color "topology" are suppressed by powers of $1/N_c$ in the large-N_c limit.

Finally, we come to the terms in H which involve only $a^\dagger$'s or only a's. The presence of these terms means that the bare vacuum is not an eigenstate of H. Their action on states of the form (2.9) yields states of order N_c as $N_c \to \infty$. For example,

$$\frac{1}{\sqrt{N_c}}\,\mathrm{Tr}[a_1^\dagger a_2^\dagger a_3^\dagger]\,|0\rangle_B = N_c\left(\frac{1}{\sqrt{N_c}}\right)^3 \mathrm{Tr}[a_1^\dagger a_2^\dagger a_3^\dagger]\,|0\rangle_B \,. \tag{2.13}$$

The presence of these terms complicates enormously the Fock-space description of the large-N_c limit. The large-N_c limit of Feynman graphs makes it evident that these terms which blow up as $N_c \to \infty$ are all associated with the fact that the energy density of the vacuum is of order $N_c{}^2$. To see this, refer to Fig. 2. If one expands the U matrix in perturbation theory a term such as

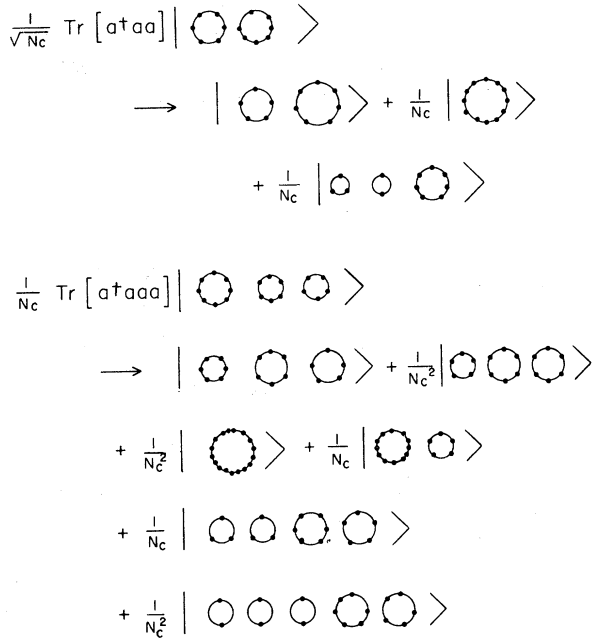

FIG. 1. Pictorial representation of the action of various terms in the Hamiltonian (2.11) on states of the form (2.9). Each trace is represented by a ring and each $a^\dagger$ in each trace by a dot on the appropriate ring. Only one typical example of each type of contraction scheme is included on the right-hand side of each equation.

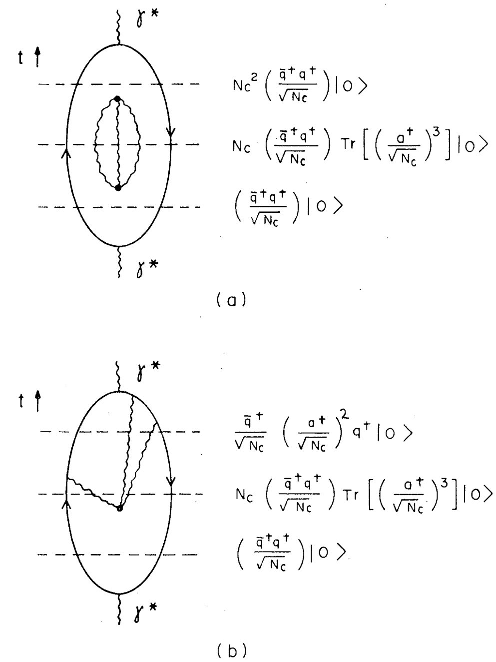

FIG. 2. Two possible time evolutions of a vacuum fluctuation for e^+e^- annihilation. In (a) the fluctuation has no effect, whereas in (b) the fluctuation modifies the measurable amplitude. We also show the Fock-space representation of the intermediate states at three successive instants of time.

(2.13) corresponds to the production of three gluons in the vacuum. The time evolution of these produced gluons can either lead to their subsequent annihilation in the vacuum [Fig. 2(a)] or their absorption by the part of the graph connected to external lines [Fig. 2(b)]. The first of these is just a disconnected bubble and contributes to the energy density of the vacuum a piece of order $N_c^{\,2}$: The creation amplitude is of order N_c and the annihilation amplitude is of order N_c. The second process [Fig. 2(b)] is physically measurable, but the amplitude for absorption is of order $1/N_c$) (it involves a change of color topology) so the whole process is of order (1) in the $N_c \to \infty$ limit.

In an analogous way other processes which are *a priori* suppressed such as the splitting of a trace into two traces can be enhanced if the gluons in one of the two traces are ultimately annihilated in the vacuum. The terms in the Hamiltonian responsible for this enhancement are the ones with only annihilation operators. For example, the term $(1/\sqrt{N_c})\mathrm{Tr}[aaa]$ will yield a net factor of

$$\frac{1}{\sqrt{N_c}}\left(\frac{1}{\sqrt{N_c}}\right)^3 N_c^{\,3} = N_c$$

when contracted against a term $[1/(\sqrt{N_c})^3]$ $\times\,\mathrm{Tr}[a^\dagger a^\dagger a^\dagger]$ in the ket. Contractions of this term with $a^\dagger$'s in larger traces will be at most or order one.

To make progress in the presence of these diverging contributions, it is clear that one must, at the very least, begin with an approximate vacuum state which is dramatically different from the bare vacuum. In order to get an inkling about the structure of such an approximate vacuum, consider the simple quantum-mechanical model,

$$H_{\mathrm{QM}} = E_0\,\mathrm{Tr}[a^\dagger a] + \frac{1}{\sqrt{N}}\,\{\mathrm{Tr}[a^{\dagger 3}] + \mathrm{Tr}[a^3]\}\,,$$

which has parallel structure to our Hamiltonian. Let us try a state (for $N_c \to \infty$) of the form

$$|\psi\rangle = \sum_n c_n \left\{\mathrm{Tr}\left[\left(\frac{a^\dagger}{\sqrt{N}}\right)^3\right]\right\}|\,0\rangle\,.$$

One obtains the recursion relation

$$3nc_nE_0 + Nc_{n-1} + 3(n+1)Nc_{n+1} = Ec_n\,.$$

Setting $c_n = K^n/n!$ we obtain

$$3E_0 + \frac{N}{K} + \frac{3NK}{n} = E\,\frac{1}{n}\;,$$

which has the solution

$$K=-\frac{N}{3E_0}\ ,$$
$$E=-N^2E_0\ ,$$
$$|\psi\rangle=\exp\left\{-\frac{N}{3E_0}\ \mathrm{Tr}\left[\left(\frac{a^\dagger}{\sqrt{N}}\right)^3\right]\right\}|\,0\rangle\ .$$

Of course, the above analogy can only be taken as a schematic indication of what the field-theory vacuum looks like, since we have ignored completely the dependence on $\vec{p}$ and i. But the example does indicate that a trial vacuum must describe some kind of condensate of rings of gluons. In the real problem a trial vacuum must be found for which the terms in the last two sets of braces in (2.11) are replaced by numbers. Then, using the new vacuum, one can attempt to form states such as (2.9) and attempt an approximate treatment of the Schrödinger equation for excited states.

An alternative approach, which holds the promise of a more systematic treatment, is to formulate the whole problem in an infinite-momentum frame. Then the troublesome terms are not present and one can completely remove the explicit N_c dependence from the problem. For in this case, in leading order, the Hamiltonian P^- acts independently on each trace, so, without loss of generality, we may consider only states with a single trace:

$$|\,\psi\rangle=\sum_{n,\{\vec{p}_n,i_n\}}\psi_n(\vec{p}_1,i_1;\vec{p}_2,i_2;\cdots;\vec{p}_n,i_n)\frac{1}{(\sqrt{N_c})^n}\ \mathrm{Tr}[a^\dagger_{i_1}(\vec{p}_1)a^\dagger_{i_2}(\vec{p}_2)\cdots a^\dagger_{i_n}(\vec{p}_n)]|0\rangle_B\ . \qquad (2.14)$$

Furthermore, when we write out the Schrödinger equation,

$$P^-|\psi\rangle=p^-|\psi\rangle\ , \qquad (2.15a)$$

to which we add the constraint,

$$P^+|\psi\rangle=p^+|\psi\rangle\ , \qquad (2.15b)$$

we can make the replacement

$$P^-|\psi\rangle\to(P^-|\psi\rangle)_{\text{nearest-neighbor contractions}}\ . \qquad (2.16)$$

Constraint (2.15b) is easily enforced by requiring

$$\sum_{i=1}^{n}\ P_i^+=p^+\ \ \text{for each } n\ . \qquad (2.17)$$

We may similarly restrict ourselves to the "transverse center-of-mass" system by imposing

$$\sum_{i=1}^{n}\ \underline{p}_{i\perp}=0\ \ \text{for each } n\ . \qquad (2.18)$$

Then solving the eigenvalue problem (2.15a) will give us directly the (mass)^2 values of the gluonic particle spectrum:

$$M^2=2p^+p^-\ . \qquad (2.19)$$

It is a straightforward matter to work out the nearest-neighbor contractions required in (2.16). Then, since the states $\mathrm{Tr}[a_1^\dagger\cdots a_n^\dagger]|0\rangle_B$ are orthogonal for different values of n, (2.15a) becomes a set of coupled integral equations for the amplitudes $\psi_n(\vec{p}_1,i_1,\cdots;\vec{p}_n,i_n)$. Since we are not going to do any detailed analysis in this article, there is no point in presenting explicitly these equations. Instead we will simply describe their qualitative structure. The general form of the equation is (very symbolically)

$$\begin{aligned}p^-\psi_n(1\cdots n)=&\left(\sum_{i=1}^{n}\frac{{p_{\perp i}}^2}{2P_i^+}\right)\psi_n(1\cdots n)+\lambda^2\sum_{i=1}^{n}K^1(i)\psi_n(1\cdots n)+\lambda^2\sum_{i=1}^{n}K_0{}^2(i,i+1)\psi_n(1\cdots n)\\&+\lambda\sum_{i=1}^{n}\sum_{\{j\}}K^{-1}(i,j)\psi_{n+1}(1,\cdots,i-1,\{j\},\{j\}-\{i\},i+1,\ldots,n)+\lambda\sum_{i=1}^{n}K^{+1}(i,i+1)\psi_{n-1}(1,\ldots,i,i+2,\ldots,n)\\&+\lambda^2\sum_{i=1}^{n}\sum_{\{j,k\}}K^{-2}(i;j,k)\psi_{n+2}(1,\ldots,i-1,\{j\},\{k\},\{i\}-\{j\}-\{k\},i+1,\ldots,n)\\&+\lambda^2\sum_{i=1}^{n}K^{+2}(i,i+1,i+2)\psi_{n-1}(1,\ldots,i,i+3,\ldots,n)\ .\end{aligned} \qquad (2.20)$$

The K's are integral operators in the momentum variables which are inferred by performing the contractions in (2.16), and $\lambda=g\sqrt{N_c}$ is the fixed coupling constant.

Thus the combination of the $N_c\to\infty$ limit and the choice of infinite-momentum frame has reduced the problem to a many-body problem of a special type. The particles (gluons) are ordered on a ring and only nearest neighbors on the ring can interact. However, the number of particles is not fixed: A gluon can fission into 2 or 3 gluons, and 2 or 3 gluons can fuse into one. In the next section we shall discuss attempts to gain some insight into the implications of Eq. (2.20).

III. CONCLUDING REMARKS

In this article we have presented a Fock-space formalism adapted to spectrum calculations in the large-N_c limit. We have not addressed here the difficult problem of how to solve Eq. (2.20). Some ideas on approaching this problem have been presented by us in a previous article,[2] where we tried to exploit the only simplification of the large-N_c limit which is the presence of only nearest-neighbor interactions in Eq. (2.20). We attempted a type of Tamm-Dancoff[3] approximation where we assumed a fixed mean number $\bar{n}$ of particles which fluctuates only by occasional single-particle exchange between nearest neighbors. Replacing this single-particle exchange by an effective potential, we were able to gain some insight into the dynamics of the system for $\bar{n}\to\infty$, in which the level spacing approaches that of the relativistic string model.[4] In particular, a formula relating the slope of Regge trajectories to the microscopic dynamics described by the effective gluon-gluon potential was obtained.

Our argument that the large-n components of the amplitude ψ_n are important was very heuristic and crude. We argued that the fact that the effective gluon-gluon potential was attractive, at least at large distances, might make it energetically favorable for many gluons to be present. Now it is clear that the attractive Coulomb potential will only overcome the kinetic energy of massless particles if its strength is larger than a critical value:

$$E_{2\,\text{gluons}} \sim \frac{1}{R} - \frac{\alpha_{\text{eff}}}{R}$$

is negative only for $\alpha_{\text{eff}} > 1$. Because of asymptotic freedom, which says that α_{eff} grows with R, this requirement can be translated into the statement $R > R_c$. Deep-inelastic scattering indicates that R_c is a characteristic scale in hadronic physics, say a half a fermi. Thus our large-$\bar{n}$ ansatz should be applicable only to the small momentum components of the hadronic wave functions where "small" means small compared to 100 MeV. These components of the hadronic wave function certainly play a dominant role in the structure of "big" hadrons, e.g., the hadrons that contribute to high-energy low-momentum-transfer scattering and hadrons which have a high angular momentum. But the success of simple "valence-quark" models of low-lying hadrons indicates that the structure of low-lying hadrons is dominated by the small-n components of ψ_n.

Indeed, if one tries to apply our particular large-$\bar{n}$ ansatz[2] to the lowest-lying hadrons, the result is that the lightest hadrons are tachyons. We have given a simple physical explanation of this tachyon instability.[2] In our large-$\bar{n}$ ansatz we assumed that each of the many constituent gluons had the same P^+, b. Our approximation to Eq. (2.20) then became a nonrelativistic many-body chain problem in the two transverse dimensions. It is a general feature of nearest-neighbor interactions that the ground-state energy of a single closed chain is larger than the sum of the ground-state energies of two smaller closed chains with the same total number of particles. Thus our $N_c\to\infty$ closed string can decay with amplitude of order $1/N_c$ into two smaller closed strings. This explanation suggests an obvious cure to the instability: The bare vacuum should be replaced by a condensate of closed gluonic chains at some equilibrium density of order N_c.

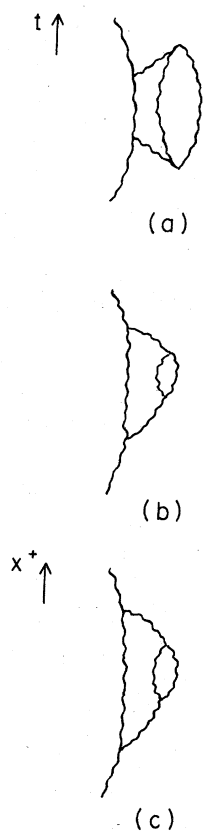

FIG. 3. (a) One particular time ordering of the Feynman graph in (b) which describes the scattering of a vacuum fluctuation on a gluon. In the infinite-momentum frame only one x^+ ordering occurs as shown in (c).

As we have seen in Sec. II, when the large-N_c limit is formulated in an ordinary reference frame, the notion of a closed gluonic ring condensate is forced on one from the beginning. The skeptical reader might ask what has happened to this condensate in the infinite-momentum frame. It is conceivable that in going to the infinite-momentum frame we have lost some essential ingredient in the dynamics. A partial answer to this question is that at least some aspects of the condensate dynamics are present in the infinite-momentum frame. For example, consider the elementary process shown in Fig. 3. In Fig. 3(a) we draw the old-fashioned perturbation-theory graph which describes a ring of three gluons in the condensate scattering against a gluon. Of course, it is well known that this is only one of many time orderings which contribute to the single Feynman graph drawn in Fig. 3(b), and it is also well known that in the infinite-momentum frame this complete Feynman graph is obtained by the single x^+ ordering of Fig. 3(c). Thus what are condensate effects in an ordinary reference frame are disguised as self-energy corrections in an infinite-momentum frame.

For consistency of this interpretation we must ask whether the approximations to Eq. (2.20) which led to the tachyon instability do not include these self-energy effects: If they are included and the instability remains, the infinite-momentum frame description is inadequate. Fortunately, the fixed-$\bar{n}$ single-particle-exchange ansatz does exclude the self-energy contributions in Fig. 3(c). Thus, the instability we discovered could go away if the infinite-momentum-frame dynamics is handled more exactly. In this better treatment it would be essential that ψ_n's for all values of n ranging from a few to many be included: The true hadronic wave function is a superposition of states of all numbers of gluons. The ψ_n's for low values of n are probed in short-distance experiments while the ψ_n's for high values of n are probed in long-distance experiments (e.g., peripheral high-energy experiments).

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