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Renormalization-prescription dependence of the quantum-chromodynamic coupling constant

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Massless quantum chromodynamics cannot be renormalized on-shell; various possible off-shell renormalization prescriptions yield different definitions of a scale-dependent coupling constant g. We show how to relate physical predictions computed in different renormalization schemes. In particular, we compute the dimensionally regularized two- and three-point functions at the symmetric point in momentum space through one-loop order, and deduce the relation between $g_{\rm min}$ defined by minimal subtraction and $g_{\rm mom}$ defined by momentum-space subtraction. We find that $g_{\rm mom}$ is fairly insensitive to which vertex one chooses to define it, and only weakly gauge dependent. $g_{\rm min}$ is shown to depend strongly on the dimensional-regularization procedure, and can therefore differ quite dramatically from $g_{\rm mom}$. The scale dependence of g is conventionally parametrized by a scale-invariant mass Λ; the ratio of Λ's defined by any two renormalization schemes is a pure number which we show is exactly deducible from our one-loop results.

I. INTRODUCTION

In an asymptotically free theory the scaling behavior of Green's functions in the deep Euclidean region can be computed in perturbation theory. This was first used to obtain the now classic predictions of quantum chromodynamics (QCD)[1] for scale breaking in deep-inelastic lepton-hadron scattering, which are in substantial agreement with experiment. The domain of applicability of perturbation theory has since been shown to include e^+e^- annihilation to hadrons, deep-inelastic photon-photon scattering, various properties of heavy-quark systems, and, through the proof of factorization, a number of inclusive single-particle and jet cross sections.[1] All of these predictions should be testable in the near future.

Typical of the predictions of perturbative QCD are those for the structure functions of deep-inelastic scattering processes. The nth moment of a nonsinglet structure function can be expanded in a power series in the scale-dependent QCD coupling constant g as follows:

$$M_n = g^{a_n}[1 + b_n g^2 + O(g^4)]A_n \,. \qquad (1)$$

a_n and b_n are calculable numbers. The overall normalization A_n cannot be calculated within the framework of perturbation theory, but is scale independent. The next-to-leading-order coefficients have recently been computed for deep-inelastic scattering off hadrons and photons, and this development has sparked interest in the question of how predictions such as (1) depend on the way in which QCD is renormalized.[2-4] To exhibit explicitly the renormalization-prescription dependence of Eq. (1), consider a different renormalization scheme from the one we shall suppose has been used to compute the coefficients in (1). This scheme will lead to a different definition g' of the coupling constant. We have

$$g = g'[1 + ag'^2 + O(g'^4)] \,, \qquad (2)$$

where a is a calculable constant. Physically measurable quantities such as M_n are, of course, independent of a renormalization scheme. Therefore, since g is the only free parameter in massless QCD, the prediction for M_n got by using the second scheme is obtained by substituting Eq. (2) in Eq. (1):

$$\begin{aligned} M_n &= (g')^{a_n}[1 + (b_n + aa_n)g'^2 + O(g'^4)]A_n \\ &\equiv (g')^{a_n}[1 + b_n' g'^2 + O(g'^4)]A_n \,. \end{aligned} \qquad (3)$$

Thus, the coefficients b_n are renormalization-prescription dependent. It should be emphasized that if the perturbation series in Eqs. (1) and (3) were known to all orders, the renormalization convention dependence of g would have absolutely no physical consequences. In practice, however, one can only compute the first two or three orders of the expansion. The truncated series therefore differ from each other by terms of the first uncomputed order in g. Since $g^2/4\pi \sim 0.3-0.6$ at currently accessible energies, these uncomputed terms are almost as important as the leading terms and different renormalization schemes yield quantitatively different predictions for physical processes.

This convention dependence of the coupling does not generally arise in QED for two reasons. First, because $\alpha_{\rm QED} \ll 1$ the convergence properties of QED expansions do not much depend on the convention used to define the coupling. Second, there is a "natural definition" of $\alpha_{\rm QED}$ based on a low-energy theorem for Compton scattering. That theorem tells us that $\alpha_{\rm QED}$ can be measured "directly."[5] Unfortunately, in QCD there are no convenient low-energy theorems, and the running coupling constant vanishes as $q^2 \to \infty$ [hence an expansion in $g(\infty)$ is not possible], so it follows that there is no natural definition of g. It thus comes as no surprise to find in the literature that not all physicists define g in the same way[6]—different definitions are convenient for different purposes. Clearly, it is important to be able to relate the various definitions to one another.

We will in what follows describe a number of ways in which massless QCD can be renormalized. One class of methods is known as momentum-space subtraction. This kind of renormalization defines (in a way to be made precise later) g_{mom} by incorporating into it certain radiative corrections of a given vertex. The definition depends (among other things) on which choice of vertex is made. For instance, one can (and we do) consider either the trigluon, the ghost-ghost-gluon, or the fermion-fermion-gluon vertex. Each of these is related to the others through Ward identities[7,8] which we shall explicitly verify in the course of the paper. Each vertex leads to a different g_{mom}, although, as we shall see, the dependence turns out to be weak. It has been conjectured[2,9,10] (and checked in some examples[10]) that because momentum-space subtraction is a "physical" method of renormalization we should expect reasonable convergence from expansions of physical quantities in terms of g_{mom}. This feature is very important because at currently accessible energies $g^2/4\pi$ is not very small; therefore, low-order predictions are meaningless unless coefficients of a g expansion are small (as they are conjectured to be for momentum-space subtraction). Most high-order results have for reasons of technical simplicity been computed[3,4] by renormalizing via "minimal subtraction." This technique depends on how the field theory is regularized. For dimensional regularization,[11] which we will be using, the minimal-subtraction renormalization is done by subtraction of poles in $(N-4)$. There is some arbitrariness depending on how diagrams are analytically continued away from four dimensions, but since minimal subtraction is not physical it is not expected[9,10] that expansions in g_{min} will converge well. Another possible disadvantage of using g_{min} is that there may be some cases where it is not convenient to regularize dimensionally. Such a situation may occur when computing processes in the presence of instantons[12] or in a curved background manifold.[13]

We see that although minimal subtraction is usually convenient for calculating, there may be some advantages to writing the results in terms of g_{mom} rather than g_{min}. By means of a one-loop calculation we shall derive relations of the form

$$g_{\text{mom}}^2 = g_{\text{min}}^2 \left(1 + c\,\frac{g_{\text{min}}^2}{4\pi} + O(g_{\text{min}}^4)\right).$$

Then by imitating Eqs. (1)–(3) it is possible for any prediction made through next-to-leading order in g_{min} to be rewritten as an expansion in g_{mom}. The methods described can of course be generalized to higher orders.

The scale dependence of the renormalized coupling constant is usually explicitly exhibited by writing QCD predictions as expansions in $1/\ln(Q^2/\Lambda^2)$, where Λ is a scale-invariant mass parameter. Since Λ is a well-defined function (to be specified later) of g, it too depends on the renormalization scheme. However, a particularly attractive feature of the Λ parameter is that different renormalizations of it can be related to one another through *all orders* by means of one-loop calculations. We will show how that comes about and will compute the ratios of Λ's defined by different schemes.

II. PROCEDURE

Here we shall detail the procedure to be followed for relating g's and illustrate it by performing subtractions on the gluon propagator and trigluon vertex.

A. Bare and renormalized parameters: definitions and identities

It is necessary to begin by defining the QCD Lagrangian expressed first in terms of unrenormalized quantities (subscripted with a B) and then in terms of renormalized quantities and renormalization constants Z. Written in the general covariant gauge it is[7,8,14]

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}\,\vec{F}_B^{\mu\nu}\cdot\vec{F}_{B\mu\nu} - \frac{1}{2\alpha_B}(\partial_\mu \vec{A}_B^\mu)^2 \\ &-(\partial_\mu \vec{\eta}^\dagger)\cdot(\partial^\mu \vec{\eta}_B + g_B \vec{A}_B^\mu \times \vec{\eta}_B) \\ &+ \bar{\Psi}_B(i\not{\partial} + g_B \vec{A}_B \cdot \vec{T})\Psi_B\,, \end{aligned} \tag{4}$$

where

$$\vec{F}_B^{\mu\nu} = \partial^\mu \vec{A}_B^\nu - \partial^\nu \vec{A}_B^\mu + g_B \vec{A}_B^\mu \times \vec{A}_B^\nu\,.$$

We have used the notation $\vec{A}_\mu \cdot \vec{A}_\nu = A_\mu^a A_\nu^a$, $(\vec{A}_\mu \times \vec{A}_\nu)^a = f^{abc} A_\mu^b A_\nu^c$, where f^{abc} are the structure constants of SU(3). g_B is the bare coupling constant, A_B is an SU(3) color field, $\vec{\eta}_B$ are ghost fields, ψ_B represents n_F flavors of massless quarks, and α_B is the gauge parameter. [We will often state our results for a general group rather than for SU(3).] This Lagrangian is rewritten in terms of renormalized quantities (*without* the subscript B) and multiplicative factors Z_i:

$$\mathcal{L} = -\tfrac{1}{4}Z_3(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu)^2 - \tfrac{1}{2}Z_1 g(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu)\cdot(\vec{A}^\mu \times \vec{A}^\nu) - \tfrac{1}{4}Z_4 g^2(\vec{A}_\mu \times \vec{A}_\nu)^2 - \frac{1}{2\alpha}(\partial_\mu \vec{A}^\mu)^2 + \tilde{Z}_3 \vec{\eta}^\dagger \cdot \partial^2 \vec{\eta}$$
$$+\tilde{Z}_1 g \vec{\eta}^\dagger \cdot \partial_\mu(\vec{A}^\mu \times \vec{\eta}) + iZ_2 \bar{\psi}\not{\partial}\psi + gZ_1^F \bar{\psi}\vec{\not{A}}\cdot\vec{T}\psi\,. \tag{5}$$

By equating Eqs. (4) and (5) we find that the parameters are constrained by (Ward identities)

$$\frac{Z_1^F}{Z_2} = \frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3}\,, \quad \frac{Z_4}{Z_3} = \left(\frac{Z_1}{Z_3}\right)^2,$$
$$g = (Z_1^{-1}Z_3^{3/2})g_B\,, \quad \alpha_B = Z_3\alpha\,. \tag{6}$$

Apart from these constraints, the Z's are in principle completely arbitrary. In practice, because QCD needs regularizing, the arbitrariness of the Z's is *only* in the finite parts. To be specific, we choose to regularize QCD dimensionally following the rules of 't Hooft and Veltman.[15] In that case the infinities of perturbation theory are manifest as poles in $(N-4)$ and these are canceled by choosing some of the Z_i to also have poles (thus Z's are made N dependent). Those poles are not arbitrary but the finite parts are (up to Ward identities). It is this arbitrariness which leads to the different possible definitions of the coupling constant. To demonstrate that, consider defining Z_1' and Z_3' by

$$Z_1' = Z_1(1+ag^2)$$
and
$$Z_3' = Z_3(1+bg^2)\,. \tag{7}$$

From Eq. (6)

$$\begin{aligned} g &= Z_1^{-1}Z_3^{3/2}g_B \\ &= Z_1^{-1}Z_3^{3/2}(Z_3'^{-3/2}Z_1')g' \\ &= (1+g^2a)(1+g^2b)^{-3/2}g'\,. \end{aligned}$$

So

$$g = [1+g'^2(a-\tfrac{3}{2}b)]g' + O(g'^5)\,. \tag{8}$$

Equations (7) and (8) are then the prototypes of the relations to be derived. Differences in renormalization prescription simply amount to differences in the definitions of the Z's [Eq. (7)] and hence in the definition of the coupling constant [Eq. (8)].

B. The gluon propagator and Z_3's

In order to define Z_3 we calculate, in one loop [since we are only going to be interested in Z's through $O(g^2)$], the propagator $\Pi_{ab}^{\mu\nu}(p)$. The Feynman diagrams of Fig. 1—these include the (Z_3-1) counterterm through $O(g^2)$—are computed in $N=4+\epsilon$ dimensions[15] to give

$$\Pi_{ab}^{\mu\nu}(p) = -i\delta_{ab}\left\{\left[\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2}\right]h(p^2) + \alpha\frac{p^\mu p^\nu}{p^2} + O(\epsilon)\right\}\mu^\epsilon\,, \tag{9}$$

where

$$\begin{aligned} h(p^2) = 1 + \frac{g^2C_2(G)}{16\pi^2}&\left\{-\frac{13}{6}\left[\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln\left(\frac{-p^2}{\mu^2}\right)\right] + \tfrac{97}{36}\right. \\ &\left. + \alpha\left(\frac{1}{2}\left[\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln\left(\frac{-p^2}{\mu^2}\right)\right] + \tfrac{1}{2}\right) + \alpha^2\left(\frac{1}{4}\right)\right\} \\ &+ \frac{g^2}{16\pi^2}T(R)n_F\left\{\frac{4}{3}\left[\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln\left(\frac{-p^2}{\mu^2}\right)\right] - \tfrac{20}{9}\right\} - (Z_3\mu^{-\epsilon} - 1)\,. \end{aligned} \tag{10}$$

μ is a mass parameter chosen arbitrarily,[16] $C_2(G)$ is the (adjoint representation) Casimir operator [for SU(3), $C_2(G)=3$], $T(R)=\frac{1}{2}$ for SU(3) (Ref. 17), and γ_E is Euler's constant $=0.577\,215\,664\cdots$. Notice that Z_3 is not yet defined. In Eq. (10) we see that the longitudinal piece of the propagator $(\alpha p^\mu p^\nu/p^2)$ has no $O(g^2)$ corrections. This is a well-known consequence of the Ward identities.[7,8]

Now we define Z_3. For the propagator to be finite the counterterm must cancel the pole part of Π. Minimal subtraction defines the renormalization constants so that they cancel *only* the pole part

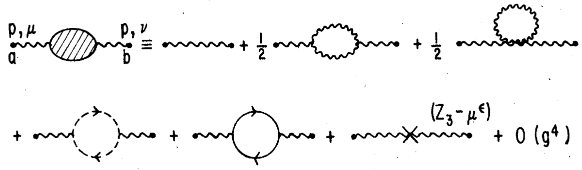

FIG. 1. Graphs contributing to $\Pi^{\mu\nu}_{ab}(p)$.

of the relevant Green's function. Hence

$$Z_3^{\min}(\epsilon)=\mu^\epsilon\left\{1+\frac{g_{\min}{}^2}{16\pi^2}\left[C_2(G)\left(-\frac{13}{3\epsilon}+\frac{\alpha}{\epsilon}\right)+n_F T(R)\frac{8}{3\epsilon}\right]\right\}. \tag{11}$$

It should be observed that there is an explicit dependence on the arbitrary mass μ and, in fact, if $Z_3^{\min}(\epsilon)$ were rewritten as $(\mu')^\epsilon[1+g^2 f(\epsilon)]$ where f is dimensionless, then f would involve a nonpole part. We see thus that $Z_3^{\min}$ is actually μ dependent and so should be written $Z_3^{\min}(\epsilon,\mu)$.[17] Likewise, g should be written as $g_{\min}(\mu)$.

Momentum-space subtraction is implemented by prescribing the values of divergent propagators and vertices at some fixed configuration of external momenta characterized by a scale M. More precisely, a fully dressed, renormalized propagator with momentum p is defined to be equal to the bare propagator at $p^2=-M^2$. In the case of vertices (which will be discussed in detail in the following sections) a decomposition into invariant amplitudes is first made. Renormalizability implies that divergences (poles) occur only in the amplitude multiplying the bare vertex; this amplitude is defined to be equal to 1 at scale M.

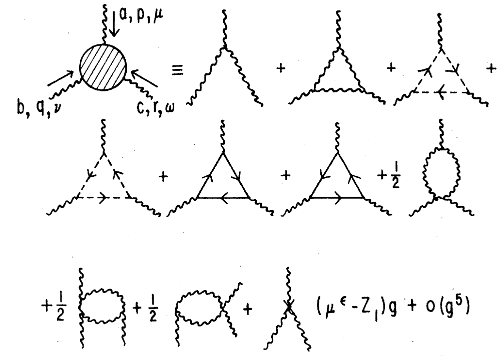

FIG. 2. Graphs contributing to $\Gamma^{abc}_{\mu\nu\omega}(p,q,r)$.

Referring to Eq. (10) we see that we must choose the counterterm $Z_3^{\text{mom}}(\epsilon,\mu,M/\mu)$ so that $h(-M^2)=1$. [Notice that terms of $O(\epsilon)$ are dropped in the definition of Z^{mom}.] We note that the renormalized amplitude

$$h(-p^2)=1+g_{\text{mom}}{}^2\times\text{const}\times\ln\left(\frac{-p^2}{M^2}\right)+O(g^4) \tag{12}$$

depends only on M and not on μ. This can easily be seen to be true of the other Green's functions renormalized by momentum-space subtraction. Since the set of renormalized Green's functions defines the theory completely, and these depend only on g_{mom} and M, we write $g_{\text{mom}}=g_{\text{mom}}(M)$ when we wish to discuss its behavior under scale transformations. The Z_i^{mom} depend explicitly on the ratio M/μ, and so, in general, will the relation between $g_{\min}(\mu)$ and $g_{\text{mom}}(M)$. This will be discussed further in subsection E.

C. The trigluon vertex and Z_1's

The vertex function $\Gamma^{abc}_{\mu\nu\omega}(p,q,r)$ is defined to be the value of the trigluon vertex at the symmetric point $p^2=q^2=r^2=-M^2$. Feynman diagrams (Fig. 2) are computed to give

$$\Gamma^{abc}_{\mu\nu\omega}(p,q,r)=gf_{abc}\{[g_{\mu\nu}(p-q)_\omega+g_{\nu\omega}(q-r)_\mu+g_{\omega\mu}(r-p)_\nu](G_0(-M^2)+Z_1\mu^{-\epsilon}) - (q-r)_\mu(r-p)_\nu(p-q)_\omega G_1(-M^2)-(r_\mu p_\nu q_\omega - r_\nu p_\omega q_\mu)G_2(-M^2)\}\mu^\epsilon+O(\epsilon), \tag{13}$$

where

$$G_0(-M^2)=\frac{g^2}{16\pi^2}\left\{C_2(G)\left[\left(-\tfrac{3}{8}+\tfrac{23}{72}I+\frac{17}{6\hat\epsilon}\right)+\alpha\left(-\tfrac{3}{8}-\tfrac{3}{8}I-\frac{3}{2\hat\epsilon}\right)+\alpha^2(-\tfrac{5}{8}+\tfrac{1}{12}I)+\alpha^3(\tfrac{1}{24})\right]+T(R)n_f\left(2-\tfrac{8}{9}I-\frac{8}{3\hat\epsilon}\right)\right\}, \tag{14a}$$

$$M^2G_1(-M^2)=\frac{g^2}{16\pi^2}\{C_2(G)[(-\tfrac{97}{216}-\tfrac{67}{216}I)+\alpha(\tfrac{3}{4}-\tfrac{1}{12}I)+\alpha^2(-\tfrac{5}{24}+\tfrac{1}{8}I)+\alpha^3(\tfrac{1}{18}-\tfrac{1}{36}I)]+T(R)n_F(-\tfrac{8}{27}+\tfrac{16}{27}I)\}, \tag{14b}$$

$$M^2G_2(-M^2)=\frac{g^2}{16\pi^2}\{C_2(G)[(-\tfrac{77}{36}-\tfrac{41}{36}I)+\alpha(\tfrac{5}{4}+\tfrac{5}{12}I)+\alpha^2(\tfrac{3}{4}-\tfrac{1}{6}I)+\alpha^3(-\tfrac{1}{12})]+T(R)n_F(\tfrac{4}{9}+\tfrac{16}{9}I)\}. \quad (14c)$$

In these equations[18]

$$I=-2\int_0^1\frac{\ln x}{x^2-x+1}=2.3439072\ldots,$$

$$\frac{1}{\hat{\epsilon}}=\frac{1}{\epsilon}+\tfrac{1}{2}[\gamma_E-\ln(4\pi)+\ln(M^2/\mu^2)],$$

and Z_1 (as well as g) is yet to be defined.

Minimal subtraction as before defines $Z_1^{\rm min}$ to cancel only the poles in Eq. (14a) (and as for $Z_3^{\rm min}$ the definition depends on the choice of scale, μ):

$$Z_1^{\rm min}(\epsilon,\mu)=\mu^\epsilon\left\{1-\frac{g_{\rm min}{}^2(\mu)}{16\pi^2}\left[C_2(G)\left(\frac{17}{6\epsilon}-\frac{3\alpha}{2\epsilon}\right)+T(R)n_F\left(-\frac{8}{3\epsilon}\right)\right]\right\}. \quad (15)$$

We will define momentum-space subtraction by choosing $Z_1^{\rm mom}$ to cancel G_0 at scale M [again we drop the terms of $O(\epsilon)$]:

$$Z_1^{\rm mom}(\epsilon,\mu,M/\mu)=\mu^\epsilon(1-G_0(-M^2)) \quad (16)$$

with $g\to g_{\rm mom}(M)$ in the expression for G_0.

Observe that G_1 and G_2 are nonzero so there is no way to remove all g^2 corrections to the vertex. Because of that, there are alternative reasonable ways to define the vertex counterterms (hence Z_1) by momentum-space subtraction and in Sec. III we will explore some of these options.

D. A comment on the calculation of the propagators and vertices

The computation of the diagrams in Fig. 1 and especially in Fig. 2 is algebraically complex. It was done by computer using the algebraic manipulation program SCHOONSCHIP.[19] Intermediate steps in the calculation typically involved several thousand terms. The three-point functions to be discussed in Sec. III, as with the trigluon vertex just described, were evaluated at the symmetric Euclidean point $p^2=q^2=r^2=-M^2$. At the symmetric point it was found possible to reduce all Feynman parametric integrals analytically to the single integral I defined in the previous subsection. These integrals were of the form

$$\int_0^1 dx\int_0^1 dy\,\frac{\theta(1-x-y)x^{m_1}y^{m_2}}{[x(1-x)+y(1-y)-xy]^m},\quad m=0,1,2,3,4$$

or in some cases the numerator of the above integrand was multiplied by $\ln[x(1-x)+y(1-y)-xy]$. In the Appendix we describe methods used to reduce these integrals and in Table IV we tabulate those that were encountered in our calculation.

The transversality of the gluon propagator and the known tensor structure of the trigluon vertex were not used in the calculation (although momentum conservation was), hence their emergence in the output of the program provided a check on the algebra. As a further nontrivial check it was verified that our results satisfied the relevant Ward identities (this will be demonstrated in Sec. III).

E. Relating $g_{\rm mom}$ to $g_{\rm min}$

Returning to the results of subsections A–C we can now relate $g_{\rm mom}(M)$ to $g_{\rm min}(\mu)$. To do that, we must first relate μ, the arbitrary mass introduced in continuing Feynman integrals to $4+\epsilon$ dimensions, and M, the subtraction point in the momentum-space subtraction scheme. We note that the predictions of perturbative QCD such as Eq. (1) are written as power series in a scale-dependent coupling constant $g_{\rm min}(\mu)$ or $g_{\rm mom}(M)$. Since there are no other dimensional parameters available, the scales μ^2 and M^2 are clearly proportional to the "Q^2" at which the experiment is done [e.g., in electroproduction, $Q=(-q^2)^{1/2}$ is the mass of the virtual photon]. We choose $M^2=Q^2$ in order that all of the *indeterminate* radiative corrections in perturbation theory are absorbed into the definition of the coupling constant at the scale at which the experiment is performed. In the case of minimal subtraction, we refer to standard practice in which results are written in terms of the coupling $g_{\rm min}(Q)$. Thus, in the following we shall set $M=\mu$.

Following Eq. (8) we can relate $g_{\rm mom}$ to $g_{\rm min}$. We use for the Z's the results in Eqs. (11), (12), (15), and (16) and find that

$$g_{\mathrm{mom}}^2(\mu)=g_{\mathrm{min}}^2(\mu)\left(1+\frac{g_{\mathrm{min}}^2(\mu)}{4\pi}A(\alpha,n_F)+O(g_{\mathrm{min}}^4(\mu))\right), \tag{17}$$

where

$$A(\alpha,n_F)=\frac{1}{2\pi}\left(C_2(G)\{[-\tfrac{11}{6}(\gamma_E-\ln 4\pi)+\tfrac{11}{3}+\tfrac{23}{72}I]+\alpha(\tfrac{3}{8}-\tfrac{3}{8}I)+\alpha^2(-\tfrac{1}{4}+\tfrac{1}{12}I)+\alpha^3(\tfrac{1}{24})\}\right.$$
$$\left.+T(R)n_F[\tfrac{2}{3}(\gamma_E-\ln 4\pi)-\tfrac{4}{3}-\tfrac{8}{9}I]\right). \tag{18}$$

Notice that for notational convenience we suppress the dependence of g_{mom} on α and n_F. A tabulation of $A(\alpha,n_F)$ for various values of α and n_F is given in Table I. Of particular interest are the Landau and Feynman gauge results for four flavors: $A(0,4)=2.32$, $A(1,4)=2.07$.

F. Defining Λ and finding $\Lambda_{\mathrm{mom}}/\Lambda_{\mathrm{min}}$

At this stage we should explain how our $g(\mu')$ compares to the usual running coupling constant[17] $\bar{g}(g,\ln(\mu'/\mu))$. In fact, they are the same. $\bar{g}$ is defined to be the value of the coupling constant renormalized at μ' [what we call $g(\mu')$] if it is known to have the value g when renormalized at μ [what we call $g(\mu)$]. We simply continue, however, to refer to the running coupling constant as $g(\mu)$. It satisfies[16,20] the renormalization-group equation

$$\mu\frac{dg(\mu)}{d\mu}=\beta(g(\mu))=-\beta_0 g^3-\beta_1 g^5+O(g^7). \tag{19}$$

Through the renormalization group the q^2 dependence of Green's functions is given implicitly by the dependence on the momenta of the (running) coupling constant $g(|q|)$. We can determine $g(Q_0)$ (for some chosen Q_0) by measuring the momentum dependence of the calculated physical quantity and using Eq. (19) to relate $g(|q|)$ to $g(Q_0)$.

One convenient way to keep track of the momentum dependence of $g(\mu)$ is to integrate Eq. (19) in terms of a μ-independent mass parameter Λ,

$$\int^{g(\mu)}\frac{dg'}{\beta(g')}=\ln(\mu/\Lambda), \tag{20}$$

and solve for g as an explicit function of (μ/Λ). To do this one must first fix the arbitrary integration constant in Eq. (20). There is at present no universally agreed upon convention for doing so. One could, for instance, use some particular fixed numerical value of the coupling constant as a lower limit on the integral in Eq. (20), but this leads to an awkward expression for $g(\mu)$. We shall follow the convention used, for example, in Refs. 3 and 4 by fixing the integration constant in Eq. (20) so that

$$\ln\left(\frac{\mu}{\Lambda}\right)=\frac{1}{2\beta_0 g^2(\mu)}+\frac{\beta_1}{2\beta_0^2}\ln[\beta_0 g^2(\mu)]$$
$$+O(g^2(\mu)). \tag{21}$$

This equation can be solved for $g(\mu)$ when $\mu\gg\Lambda$:

$$g^2(\mu)=\frac{1}{\beta_0\ln(\mu^2/\Lambda^2)}-\frac{\beta_1\ln\ln(\mu^2/\Lambda^2)}{\beta_0^3\ln^2(\mu^2/\Lambda^2)}$$
$$+O\left(\frac{1}{\ln^3(\mu^2/\Lambda^2)}\right) \tag{22}$$

and it is easily verified that $g(\mu)$ satisfies Eq. (19) provided that $d\Lambda/d\mu=0$. We see that the particular choice of integration constant implicit in Eq. (21) leads to an expression for $g(\mu)$ that contains no term of the form const/$\ln^2(\mu^2/\Lambda^2)$. The phenomenological implications of this choice are discussed, for example, in Refs. 3, 4, and 10. Clearly any other choice of integration constant leads to a definition of Λ which differs from the one used here by a multiplicative constant.

Using Eq. (21) to define Λ_{mom} and Λ_{min} in terms of $g_{\mathrm{mom}}(\mu)$ and $g_{\mathrm{min}}(\mu)$, respectively, we derive that

TABLE I. Values of $A(\alpha,n_F)$ and $\Lambda_{\mathrm{mom}}/\Lambda_{\mathrm{min}}$. These quantities are defined in Sec. II [Eqs. (17) and (23)].

α	n_F	$A(\alpha,n_F)$	$\Lambda_{\mathrm{mom}}/\Lambda_{\mathrm{min}}$
0	0	3.82	8.86
0	1	3.44	8.11
0	2	3.07	7.34
0	3	2.69	6.55
0	4	2.32	5.73
0	5	1.94	4.91
1	0	3.57	7.69
1	3	2.44	5.51
1	4	2.07	4.76
1	5	1.69	4.01
3	3	2.27	4.89
3	4	1.90	4.18
−2	4	2.53	6.76

$$\frac{\Lambda_{\text{mom}}}{\Lambda_{\text{min}}}=\exp\left\{-\frac{1}{2\beta_0}\left[\frac{1}{g_{\text{mom}}^{\,2}(\mu)}-\frac{1}{g_{\text{min}}^{\,2}(\mu)}\right]+O(g^2)\right\}$$
$$=\exp\left[\frac{1}{8\pi\beta_0}A(\alpha,n_F)+O(g^2)\right], \qquad (23)$$

where $A(\alpha,n_F)$ is given by Eqs. (17) and (18). Note that the ratio $\Lambda_{\text{mom}}/\Lambda_{\text{min}}$ is independent of the choice of integration constant in Eq. (20), provided that the same convention for relating Λ to g is used in both renormalization schemes. Since Λ is scale invariant, we can let $\mu\to\infty$ in Eq. (23) so that

$$\Lambda_{\text{mom}}/\Lambda_{\text{min}}=\lim_{\mu\to\infty}\exp\left[\frac{A(\alpha,n_F)}{8\pi\beta_0}+O(g_{\text{min}}^{\,2}(\mu))\right]$$
$$=\exp\left[\frac{A(\alpha,n_F)}{8\pi\beta_0}\right], \qquad (24)$$

since $\lim_{\mu\to\infty}g_{\text{min}}(\mu)=0$ by asymptotic freedom. This is just the assertion that $\Lambda_{\text{mom}}/\Lambda_{\text{min}}$ is computable through all orders by means of a one-loop calculation. The importance of that observation is the following: QCD predictions for the high-energy limit of cross sections are often written as functions of the single parameter Λ, e.g., in deep-inelastic scattering,

$$M_n\sim[\ln(Q^2/\Lambda^2)]^{a_n}\left[1+\sum_{t=1}^{\infty}\sum_{s=0}^{t}\frac{b_{st}^{(n)}[\ln\ln(Q^2/\Lambda^2)]^s}{[\ln(Q^2/\Lambda^2)]^t}\right],$$

where $b_{st}^{(n)}$ are calculable numbers. Most of the $b_{st}^{(n)}$ are renormalizable-prescription dependent [as was described by Eqs. (1)–(3) of the Introduction]. For technical convenience the calculation of the b's is probably best done in the minimal subtraction scheme. But since we know $\Lambda_{\text{mom}}/\Lambda_{\text{min}}$ through all orders, the above series for M_n can be completely written (with almost no effort) in terms of quantities renormalized by momentum-space subtraction. All we need to know is the one-loop value of $\Lambda_{\text{mom}}/\Lambda_{\text{min}}$. For various gauges and flavors these are given in Table I alongside the values of $A(\alpha,n_F)$. With four flavors in the Landau gauge $\Lambda_{\text{mom}}/\Lambda_{\text{min}}=5.73$; in the Feynman gauge $\Lambda_{\text{mom}}/\Lambda_{\text{min}}=4.76$.

III. DEPENDENCE OF RESULTS ON VARIOUS RENORMALIZATION PRESCRIPTIONS: RELATIONS AMONG THREE-POINT FUNCTIONS

In this section we shall compute the ghost and fermion propagators, ghost-ghost-gluon vertices, and the fermion-fermion-gluon vertex. It will be shown how these obey Ward identities which we shall describe. Next we shall define the renormalization constants according to subtraction procedures appropriate to those vertices. The resulting values of g_{mom} will be related to one another (and to the g_{mom} defined in Sec. II) and to g_{min}. Finally, we shall describe the dependence of g_{min} on the regularization scheme used and comment on differences between g_{mom} and g_{min}.

A. Ward identities

Checking the Ward identities[8,21] provided a check on our computations (see Sec. II, part D). We shall discuss these identities very briefly following the notation of Kluberg-Stern and Zuber[8] and refer the reader to their paper for a more detailed discussion.

Following Ref. 8 we add a term $J_\mu^a(x)D_{ab}^\mu\eta^b(x)$ to the Lagrangian where J_μ^a is a classical anticommuting source. Let Γ be the generating functional of connected one-particle irreducible (1PI) Green's functions. The transverse part of the gluon propagator can be written

$$\frac{\delta^2\Gamma}{\delta A_\mu^a\delta A_\nu^b}=\delta_{ab}(p^\mu p^\nu-g^{\mu\nu}p^2)I(p^2)\mu^{-\epsilon}. \qquad (25)$$

Using Bose symmetry and color-charge conjugation invariance it can be shown that the three-gluon vertex at $p^2=q^2=r^2=-\mu^2$ can be parametrized [cf. Eq. (13)]:

$$\frac{\delta^3\Gamma}{\delta A_\mu^a\delta A_\nu^b\delta A_\rho^c}=-igf^{abc}\{[g_{\mu\nu}(p-q)_\rho+g_{\nu\rho}(q-r)_\mu+g_{\rho\mu}(r-p)_\nu]G(p^2)$$
$$-(q-r)_\mu(r-p)_\nu(p-q)_\rho G_1(p^2)-(r_\mu p_\nu q_\rho-q_\mu r_\nu p_\rho)G_2(p^2)\}\mu^\epsilon. \qquad (26)$$

The three-point function with one J vertex and one external gluon and ghost [see Fig. 3(a)] can be parametrized at the symmetric point as follows:

$$\frac{\delta^3\Gamma}{\delta A_\nu^c\delta\eta^b\delta J_\mu^a}=-gf^{abc}[g_{\mu\nu}H(p^2)+p_\mu p_\nu H_1(p^2)+p_\mu q_\nu H_2(p^2)+p_\nu q_\mu H_3(p^2)+q_\mu q_\nu H_4(p^2)]\mu^\epsilon. \qquad (27)$$

Finally, we need the inverse ghost propagator

$$\frac{\delta^2\Gamma}{\delta\eta_i\delta\bar\eta_j}=-p^2\delta_{ij}\tilde I(p^2)\mu^\epsilon. \qquad (28)$$

The Ward identity relating the invariant functions defined above is

$$[H + \tfrac{1}{2}p^2(H_3 + H_4)]I = [G - \tfrac{1}{2}p^2 G_2]\tilde{I}. \tag{29}$$

We list below our results for the invariants through order g^2 [for simplicity we shall sometimes omit writing the counterterms and we shall drop all terms of $O(\epsilon)$]:

$$I(-\mu^2) = 1 + \frac{g^2}{16\pi^2}\left\{C_2(G)\left[\frac{13}{3\hat{\epsilon}} - \tfrac{97}{36} + \alpha\left(-\tfrac{1}{2} - \frac{1}{\hat{\epsilon}}\right) - \frac{\alpha^2}{4}\right] + T(R)n_F\left(\tfrac{20}{9} - \frac{8}{3\hat{\epsilon}}\right)\right\}. \tag{30}$$

This result is just Eq. (10) rewritten; $1/\hat{\epsilon}$ is defined after Eq. (14) to be $1/\hat{\epsilon} = 1/\epsilon + \frac{1}{2}[\gamma_E - \ln(4\pi)]$ (as discussed in Sec. II E, μ^2 and $-p^2$ have been set equal in all these equations). $G(-\mu^2) = 1 + G_0(-\mu^2)$, $G_1(-\mu^2)$, and $G_2(-\mu^2)$ are all given in Eqs. (14). The remaining invariants are

$$H(-\mu^2) = 1 + \frac{g^2 C_2(G)}{16\pi^2}\left[\left(\tfrac{3}{8} + \tfrac{1}{24}I\right) + \alpha\left(\tfrac{3}{4} - \tfrac{7}{24}I - \frac{1}{\hat{\epsilon}}\right) - \frac{\alpha^2}{8}\right], \tag{31a}$$

$$\mu^2 H_1(-\mu^2) = \frac{g^2 C_2(G)}{16\pi^2}\left[\left(-\tfrac{1}{12} + \tfrac{1}{24}I\right) + \alpha\left(\tfrac{1}{3} - \tfrac{1}{4}I\right) + \alpha^2\left(-\tfrac{1}{4} + \tfrac{1}{24}I\right)\right], \tag{31b}$$

$$\mu^2 H_2(-\mu^2) = \frac{g^2 C_2(G)}{16\pi^2}\left[\left(\tfrac{1}{3} + \tfrac{7}{24}I\right) + \alpha\left(-\tfrac{1}{3} + \tfrac{1}{12}I\right) + \alpha^2\left(-\tfrac{1}{24}I\right)\right], \tag{31c}$$

$$\mu^2 H_3(-\mu^2) = \frac{g^2 C_2(G)}{16\pi^2}\left[\left(\tfrac{1}{12} + \tfrac{5}{24}I\right) + \alpha\left(\tfrac{1}{6} - \tfrac{1}{4}I\right) + \alpha^2\left(-\tfrac{1}{4} + \tfrac{1}{24}I\right)\right], \tag{31d}$$

$$\mu^2 H_4(-\mu^2) = \frac{g^2 C_2(G)}{16\pi^2}\left[\left(\tfrac{1}{6} + \tfrac{3}{8}I\right) + \alpha\left(-\tfrac{1}{6}\right) + \alpha^2\left(-\tfrac{1}{24}I\right)\right], \tag{31e}$$

where I was defined in Sec. II. Finally,

$$\tilde{I}(-\mu^2) = 1 + \frac{g^2 C_2(G)}{16\pi^2}\left(\frac{3}{2\hat{\epsilon}} - 1 - \frac{\alpha}{2\hat{\epsilon}}\right). \tag{32}$$

These above equations are inserted into the two sides of the Ward identity, Eq. (29), and it is easily verified that the equality is satisfied through $O(g^2)$. Although we have not written the counterterms, it can be immediately checked that because of the Ward identities for the counterterms [Eq. (6)], these also satisfy Eq. (29). [This is provided, of course, that the Z's are chosen to satisfy Eq. (6). If they are not then we would not be considering the Lagrangian of Eq. (4).]

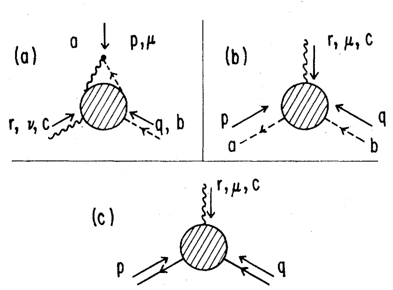

FIG. 3. Vertices: p,q,r are momenta; a, b, and c are color indices: (a) gluon-source-ghost vertex, (b) ghost-gluon vertex, (c) quark-gluon vertex.

B. Alternative momentum-space subtraction procedures

The three-gluon vertex discussed in Sec. II is not the only Green's function that can be used to define the coupling constant at a fixed point in momentum space. One could just as well use the quark-gluon vertex, the ghost-gluon vertex, or the four-gluon vertex for this purpose. Since there is only one coupling constant in the theory, one can in general specify the value of only one of these Green's functions at the subtraction point; the others are then determined by the Ward identities. (The pole terms automatically satisfy the Ward identities,[11] which is one of the technical advantages of the minimal prescription.)

We have computed the quark-gluon and ghost-gluon three-point functions at the symmetric point [Figs. 3(b) and 3(c)] as well as the quark and ghost propagators [the ghost propagator was given in the preceding subsection as $\tilde{I}^{-1}(p^2)$]. In order to define the counterterms we follow the procedure described in Sec. II. $\tilde{Z}_3^{\text{mom}}$ and Z_2^{mom} are defined to cancel all radiative corrections to the propagator at $p^2 = -\mu^2$. $\tilde{Z}_1^{\text{mom}}$ and Z_{1F}^{mom} are defined to cancel all radiative corrections to the term in the three-point function proportional to the 0th order (skeletal) vertex. (Remember, however, that because of the Ward identities it is not possible to define Z_1's independently of one another. More will be said about this shortly.) There is some arbitrariness as to how the ghost-gluon vertex is parametrized at the symmetric point. Following Ref. 8 we define

$$(\tilde{Z}_1^{\rm mom}\mu^{-\epsilon})^{-1} \equiv \tilde{G}(-\mu^2) = H(-\mu^2) - \mu^2 H_1 + \tfrac{1}{2}\mu^2 H_3 ,$$

where H's are given by Eqs. (27) and (31).

The computed values of the $Z^{\rm mom}$'s are given below. In these formulas $C_2(R)$ is the Casimir operator in the fundamental representation [$C_2(R) = (1/2n)(n^2-1)$ for SU(n), for SU(3), $C_2(R)=\frac{4}{3}$]. [Also, we do not label the renormalized g's which appear. This does not matter since any distinction between two renormalizations of g appears only in $O(g^4)$ in the equations below.] We find that

$$Z_2^{\rm mom}(\mu) = \left[1 + \frac{g^2}{16\pi^2} C_2(R)\alpha\left(\frac{2}{\hat{\epsilon}} - 1\right)\right]\mu^{\epsilon} , \tag{33}$$

$$Z_{1F}^{\rm mom}(\mu) = \left(1 + \frac{g^2}{16\pi^2}\left\{C_2(R)\left[(2 - \tfrac{1}{3}I) + \alpha\left(-2 + \tfrac{2}{3}I + \frac{2}{\hat{\epsilon}}\right)\right] + C_2(G)\left[\left(-\tfrac{13}{4} + \tfrac{13}{24}I + \frac{3}{2\hat{\epsilon}}\right) + \alpha\left(\tfrac{1}{6}I + \frac{1}{2\hat{\epsilon}}\right) + \alpha^2\left(\tfrac{1}{4} - \tfrac{1}{24}I\right)\right]\right\}\right)\mu^{\epsilon} , \tag{34}$$

$$\tilde{Z}_3^{\rm mom}(\mu) = \left[1 + \frac{g^2}{16\pi^2} C_2(G)\left(1 - \frac{3}{2\hat{\epsilon}} + \frac{\alpha}{2\hat{\epsilon}}\right)\right]\mu^{\epsilon} , \tag{35}$$

$$\tilde{Z}_1^{\rm mom}(\mu) = \left\{1 + \frac{g^2}{16\pi^2} C_2(G)\left[\left(-\tfrac{1}{2} - \tfrac{5}{48}I\right) + \alpha\left(-\tfrac{1}{2} + \tfrac{1}{6}I + \frac{1}{\hat{\epsilon}}\right) + \alpha^2\left(\frac{I}{48}\right)\right]\right\}\mu^{\epsilon} . \tag{36}$$

We emphasize once again that the renormalization constants above *cannot* be chosen all together. If, for instance, we have defined $Z_3^{\rm mom}$, $Z_1^{\rm mom}$, and $\tilde{Z}_3^{\rm mom}$ then we are not free to choose Z_1 since it is restricted by the Ward identities

$$Z_1 = Z_3^{\rm mom}(\tilde{Z}_1^{\rm mom}/\tilde{Z}_3^{\rm mom}) . \tag{37}$$

It is easy to see by comparing the right-hand side of Eq. (37) with $Z_1^{\rm mom}$ [Eqs. (14a) and (16)] that $Z_1 \neq Z_1^{\rm mom}$. (Notice that if we had used minimal subtraction to define all Z's it would turn out that $Z^{\rm min}$'s do satisfy the Ward identities, since the pole terms do. Thus, $g_{\rm min}$ is independent of the vertex chosen in doing the subtraction.) There is no inconsistency in the fact that the $Z^{\rm mom}$'s do not satisfy Ward identities among themselves. It simply means that only *some* of the Z's may be set equal to $Z^{\rm mom}$'s. We see now the arbitrariness to which we previously alluded. Momentum-space subtraction can be done in a number of ways depending on which of the renormalization constants $Z^{\rm mom}$ we choose to use. These allow new definitions of the coupling constant. Following Eq. (6) we define $g'_{\rm mom}$ and $g''_{\rm mom}$ (the implicit dependence on μ is suppressed here):

$$g'_{\rm mom} = Z_3^{\rm mom\ 1/2}\frac{Z_2^{\rm mom}}{Z_{1F}^{\rm mom}} g_B , \tag{38}$$

$$g''_{\rm mom} = Z_3^{\rm mom\ 1/2}\frac{\tilde{Z}_3^{\rm mom}}{\tilde{Z}_1^{\rm mom}} g_B . \tag{39}$$

In analogy with Eq. (17) we define the constants $A'(\alpha, n_F)$ and $A''(\alpha, n_F)$ which relate $g'_{\rm mom}$ and $g''_{\rm mom}$ to $g_{\rm min}$. $\Lambda'_{\rm mom}$ and $\Lambda''_{\rm mom}$ are defined by analogy with $\Lambda_{\rm mom}$. Expressions for all of these are easily obtained from our equations for $Z^{\rm mom}$'s [cf. II E and II F] and in Table II we list some numerical values for these quantities. We can also relate $g'_{\rm mom}$ and $g''_{\rm mom}$ to $g_{\rm mom}$. For instance,

$$g'_{\rm mom} = \frac{Z_2^{\rm mom} Z_1^{\rm mom}}{Z_3^{\rm mom} Z_{1F}^{\rm mom}} g_{\rm mom} .$$

The ratio of Z's measures, in effect, the degree to which $Z^{\rm mom}$'s do not satisfy Ward identities among themselves. We will write

$$g'^{\,2}_{\rm mom} = g_{\rm mom}^{\,2}\left(1 + \frac{B(\alpha, n_F)}{4\pi} g_{\rm mom}^{\,2} + O(g^4)\right)$$

and

$$g''^{\,2}_{\rm mom} = g_{\rm mom}^{\,2}\left(1 + \frac{C(\alpha, n_F)}{4\pi} g_{\rm mom}^{\,2} + O(g^4)\right)$$

(g, g', and g'' are taken at the same α and n_F). Some of these values as well as their corresponding Λ values are listed in Table III. We also show in that table ratios of $\Lambda/[\Lambda_{\rm mom}(\alpha=0)]$ for four flavors and various Λ's. Figure 4 displays the

TABLE II. Values of $A'(\alpha, n_F)$, $A''(\alpha, n_F)$, $\Lambda'_{mom}/\Lambda_{min}$, and $\Lambda''_{mom}/\Lambda_{min}$. These quantities are defined in Sec. III B.

α	n_F	$A'(\alpha, n_F)$	$\Lambda'_{mom}/\Lambda_{min}$	$A''(\alpha, n_F)$	$\Lambda''_{mom}/\Lambda_{min}$
0	0	3.04	5.68	3.19	6.17
0	3	2.46	5.59	2.61	6.19
0	4	2.27	5.55	2.42	6.19
0	5	2.08	5.50	2.23	6.20
1	3	2.26	4.86	2.82	7.15
1	4	2.07	4.77	2.63	7.24
1	5	1.88	4.67	2.43	7.35
−2	4	2.59	7.06	2.22	5.33
3	4	1.60	3.33	3.26	11.68

gauge dependence of g_{mom} for $n_f = 4$.

Some features of Table III are particularly worth noting. First, we see that the g_{mom}'s (or Λ_{mom}'s) are fairly insensitive to which vertex was used in the subtraction. The ratio $\Lambda'_{mom}/\Lambda_{mom}$ is typically much closer to 1 than the ratio $\Lambda_{mom}/\Lambda_{min}$. For instance, for four flavors in the Landau gauge, $\Lambda'_{mom}/\Lambda_{mom} = 0.97$ and $\Lambda''_{mom}/\Lambda_{mom} = 1.08$, whereas $\Lambda_{min}/\Lambda_{mom} = 0.17$. Second, we observe that the gauge dependence [see Fig. 4 and Table III (c)] of Λ's is quite weak in the vicinity of $\alpha = 0$. Of course, because $A(\alpha, n_F)$ depends cubically on α, and A' and A'' depend quadratically on α, it must eventually be the case that for large enough values of α the dependence on gauge becomes rather dramatic. The fact that for small α there are no such dramatic dependences bolsters (somewhat) the assumption[10] that for an "optimal" subtraction prescription one ought to use a small gauge parameter.

TABLE III. (a) Relationships between coupling constants at equal α and n_F: $B(\alpha, n_F)$ and $C(\alpha, n_F)$, respectively, relate g'^2_{mom} to g^2_{mom} and g''^2_{mom} to g^2_{mom}. As before, A relates g^2_{mom} to g^2_{min} (see Sec. III B), (b) ratios of Λ's at equal α and n_F, (c) gauge dependence: ratios of Λ's to Λ_{mom} $(\alpha = 0) \equiv \tilde{\Lambda}$, $n_F = 4$.

(a)

α	n_F	$B(\alpha, n_F)$	$C(\alpha, n_F)$	$A(\alpha, n_F)$ [a]
0	0	−0.78	−0.63	3.82
0	3	−0.23	−0.08	2.69
0	4	−0.04	0.10	2.32
1	3	−0.18	0.37	2.44
1	4	0.00	0.56	2.07
−2	4	0.06	−0.31	2.53
3	4	−0.30	1.36	1.90

(b)

α	n_F	$\Lambda'_{mom}/\Lambda_{mom}$	$\Lambda''_{mom}/\Lambda_{mom}$	$\Lambda_{min}/\Lambda_{mom}$ [a]
0	0	0.64	0.70	0.11
0	3	0.85	0.94	0.15
0	4	0.97	1.08	0.17
1	3	0.88	1.30	0.18
1	4	1.00	1.52	0.21
−2	4	1.05	0.79	0.15
3	4	0.80	2.80	0.24

(c)

α	$\Lambda_{mom}/\tilde{\Lambda}$	$\Lambda'_{mom}/\tilde{\Lambda}$	$\Lambda''_{mom}/\tilde{\Lambda}$
−2	1.18	1.23	0.93
0	1.00	0.97	1.08
1	0.83	0.83	1.26
3	0.73	0.58	2.04

[a]These values also appear in Table I.

C. Arbitrariness of the minimal-subtraction prescription

In this subsection we wish to discuss the fact that minimal subtraction leads to a coupling constant that depends crucially on the regularization procedure, i.e., on how one chooses to continue

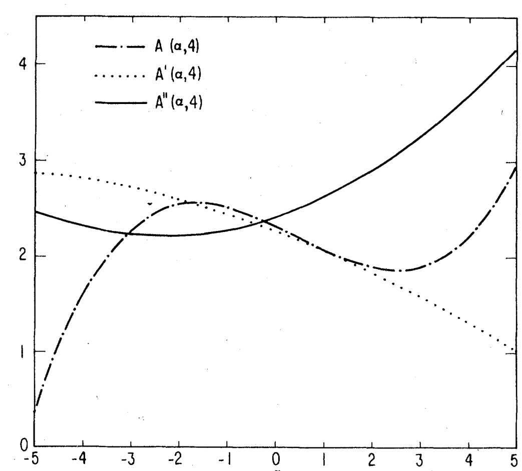

FIG. 4. Gauge dependence of g_{mom}, g'_{mom}, and g''_{mom} defined by subtracting the trigluon, quark-gluon, and ghost-gluon vertices, respectively, for four quark flavors. The functions A, A', and A'' measure the deviation of g, g', and g'' from g_{min} [see Eqs. (17), (38), and (39)]. $\alpha = 0$ corresponds to the Landau gauge.

Feynman integrals away from four-dimensional space. This is to be contrasted with momentum-space subtraction which as we have seen leads to a coupling constant that is completely independent of the regularization procedure. In minimal subtraction the artifacts of the regularization procedure are shared between the renormalization constants and the renormalized Green's functions. In momentum-space subtraction all of these artifacts are relegated to the renormalization constants while renormalized quantities depend only on the point in momentum space at which the subtraction has been performed. While it is eminently reasonable (from the point of view of choosing a coupling constant which optimizes the convergence of S-matrix expansions) to identify the momentum-space subtraction point with the Q^2 of the physical process that one is computing, and while it may be argued[10] that there is a preferred value of the gauge parameter ($\alpha = 0$), there can surely be no *a priori* argument in favor of any particular convention of dimensional regularization.

We first discuss the observation of Bardeen *et al.*[3,4] that factors of $\ln 4\pi$ and γ_E appear to be largely responsible for the rather large second-order coefficients in their deep-inelastic moment predictions. They note that a rescaling of $\Lambda_{\min} \to \overline{\Lambda}_{\min} = \Lambda_{\min} \exp[\frac{1}{2}\ln(4\pi - \gamma_E)]$ removes the factors of $\ln 4\pi - \gamma_E$ from their results and leads to smaller coefficients. One way in which this rescaling of Λ can be accomplished is to analytically continue away from dimension 4 by letting, for instance (in the dimensional continuation that we have been using),

$$\frac{d^N k}{(2\pi)^N} \to \frac{d^N k}{(2\pi)^N}\left[\exp\left(\frac{N-4}{2}(\ln 4\pi - \gamma_E)\right)\right].$$

Then minimal subtraction with respect to this new method of regularization yields $\overline{\Lambda}$ rather than Λ. We note that since $\overline{\Lambda}_{\min} = 2.66\ \Lambda_{\min}$, the factors of $\ln 4\pi$ and γ_E account for roughly half the difference between minimal and momentum-space subtraction. Although with some suitable analytic continuation we can arrive at a Λ equal to Λ_{mom}, we would not know which continuation scheme to use unless we had in fact computed Λ_{mom}.

One dimensional continuation technique which does in fact lead to a Λ roughly equal to Λ_{mom} is the one used originally by 't Hooft and Veltman.[11] They differ from our prescription in that the measure of their loop momentum integral is $d^N k/(2\pi)^4$ rather than $d^N k/(2\pi)^N$ which we use (following Refs. 3 and 4). This, as in the example above, can be implemented trivially by multiplying all the pole terms in our one-loop results by a factor $(2\pi)^{N-4} = 1 + \epsilon \ln 2\pi + O(\epsilon^2)$, and results in a change of scale $\Lambda_{\min} \to \Lambda'_{\min} = 2\pi\Lambda_{\min} \sim \Lambda_{\text{mom}}$ (see Table II). It should be noted that in these examples we are describing only the modification that occurs in one loop. As explained in Sec. II E the renormalization method need only be defined through one loop in order to determine through all orders any physical quantity expanded in terms of Λ. However, for expansions in the coupling constant it would in general be necessary to show how to relate in higher orders the different methods of dimensional continuation.

As a final example let us define[22] the trace of the unit Dirac matrix in N dimensions to be N instead of 4. This modification only affects diagrams with fermion loops and is implemented by multiplying the pole terms associated with these diagrams by appropriate factors of $(1 + \frac{1}{4}\epsilon)$. This leads to a change $\Lambda_{\min} \to \Lambda''_{\min} = \Lambda_{\min}\exp[n_F/(66 - 4n_F)]$ one which now depends explicitly on the number of fermion flavors.

IV. CONCLUSIONS

The question of renormalization-prescription dependence in QCD is a very important one because there does not yet exist a *natural* prescription. We have shown in this paper how to relate some frequently used renormalization techniques. The methods described can be used to relate other conventions and other expansions.

Specifically, we found the relation between various momentum-space subtractions at the symmetric Euclidean point and we further related these to the minimal-subtraction scheme employed, for instance in Bardeen *et al.*[3,4] As an example of our results we computed in the Landau gauge and four flavors

$$g_{\text{mom}}{}^2 = g_{\min}{}^2\left[1 + 2.32\left(\frac{g_{\min}{}^2}{4\pi}\right) + O(g_{\min}{}^2)\right],$$

where g_{mom} is the coupling derived from subtraction at the trigluon vertex. This can be compared to

$$g'_{\text{mom}}{}^2 = g_{\min}{}^2\left[1 + 2.27\left(\frac{g_{\min}{}^2}{4\pi}\right) + O(g^2)\right]$$

and

$$g''_{\text{mom}}{}^2 = g_{\min}{}^2\left[1 + 2.42\left(\frac{g_{\min}{}^2}{4\pi}\right) + O(g^2)\right],$$

where g' is the coupling derived from the quark-gluon vertex and g'' is the coupling associated with the ghost-gluon vertex. We see that the coupling is very insensitive to which vertex was used to define the momentum-space subtraction but is *very* dependent on whether momentum or minimal subtraction is used. Furthermore, as we discussed in Sec. III C, $g_{\min}$ depends critically on the particular dimensional continuation method used in the regularization scheme. In Table III we tabulate a number of comparisons between different g's, in particular, between g_{mom}'s. The fact that for the Landau (and Feynman) gauge and four flavors all g_{mom}'s are so similar is very striking. Also, the weak dependence on gauge for small gauge parameters is noteworthy. We have no explanation for these weak dependences other than the suggestion that this feature is related to the conjecture made in the Introduction (see also Ref. 10) that momentum-space subtraction is somewhat "physical" and so tends to optimize perturbation expansions. (In this way of thinking, the Landau gauge is the preferred covariant gauge because the propagator does not, in that gauge, involve the unphysical longitudinal degree of freedom.) We have also presented our results in terms of Λ_1/Λ_2 where Λ is the mass used to parametrize the running coupling constant and depends on the renormalization prescription only in lowest order (this was described in Sec. II F). From the tables we see that whereas $\Lambda_{\text{mom}}/\Lambda_{\text{min}}=5.73$, $\Lambda_{\text{mom}}/\Lambda'_{\text{mom}}=1.03$. This just further emphasizes the points made above.

In deriving our results we calculated analytically the finite parts (as well as the poles) of a number of propagators and vertices at the symmetric point. This was done in general covariant gauge and the results were checked by verifying, in Sec. III, the Ward identities.

One immediate use of our calculation is to apply it to two-loop calculations done using minimal subtractions (which is technically simpler than doing a direct calculation using momentum-space subtraction). These can, with our results, be immediately rewritten as expansions in g_{mom}. As explained above, such expansions are presumed to converge quicker than those in g_{min}. Examples of that investigation are discussed in another paper (Ref. 10).

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APPENDIX: Parametric integrals involved in the evaluation of one-loop vertices at the symmetric point

We here discuss the integrals

$$I(m_1,m_2,m,\lambda)=\int_0^1 dx\int_0^1 dy\,\frac{\theta(1-x-y)x^{m_1}y^{m_2}}{(\Delta(\lambda))^m}$$

and

$$J(m_1,m_2)=\int_0^1 dx\int_0^1 dy\,\theta(1-x-y)x^{m_1}y^{m_2}\ln(\Delta)\,,$$

where $\Delta(\lambda)=x(1-x)+y(1-y)-xy+\lambda$ and $\Delta=\Delta(0)$. We are specifically interested in $0<\lambda\ll 1$. In fact, the sums of I's that actually occur in the evaluation of vertices are convergent so that λ, which is just a regularizing parameter in these parametric integrals, is ultimately taken to be 0. In what follows we describe some of the techniques used in evaluating I and J. Table IV lists the results for those integrals we encountered.

(A) *Integrals* $I(m_1,m_2,1,0)\equiv I(m_1,m_2,1)$.

(1) We have

$$I(0,0,1)=\int_0^1\int_0^{1-x}\frac{1}{[x(1-x)+y(1-y)-xy]}$$
$$=\int_0^1 dx\frac{1}{R_+-R_-}\left[\ln\frac{(1-x)-R_+}{-R_+}+\ln\frac{-R_-}{(1-x)-R_-}\right]\,,$$

where $R_\pm$ are roots of $x(1-x)+y(1-x-y)=0$. Change the variables to

$$z=\left(\frac{(1+3x)^{1/2}+(1-x)^{1/2}}{(1+3x)^{1/2}-(1-x)^{1/2}}\right)^{-1}.$$

It can then be shown that

$$I(0,0,1)=-\int_0^1\frac{2\ln(z)}{z^2-z+1}$$
$$=2\times(1.17195361934\cdots)\equiv I\,.$$

The latter equality is to be found in Ref. 18, p. 533.

(2) $I(1,0,1)$: Note that

$$\frac{d}{dx}\Delta=1-2x-y\,,$$

$$\frac{d}{dy}\Delta=1-2y-x\,,$$

so

$$y=-\frac{1}{2}\frac{d}{dy}\Delta+\frac{(1-x)}{2}\,. \qquad \text{(A1)}$$

Notice (this is a trick we use over and over again) that by symmetry between x and y, $I(1,0,1)=I(0,1,1)$, so

$$\int\int\frac{y}{\Delta}=\int\int\frac{x}{\Delta}$$

and by (A1)

$$-\int\int\frac{1}{2\Delta}+\int\int\frac{y+\frac{1}{2}x}{\Delta}=-\tfrac{1}{2}\int\int\frac{d\Delta/dy}{\Delta}\,.$$

TABLE IV. Values of the parametric integrals $I(m_1, m_2, m, \lambda)$ and $J(m_1, m_2)$ encountered in the calculation of the three-point functions at the symmetric point. These integrals are defined in the Appendix. We only list values for $m_1 \leqq m_2$ since the integrals are symmetric in m_1 and m_2.

$I(m_1, m_2, m, \lambda) = k_1 \ln\lambda + k_2 I + k_3 + O(\lambda)$

m_1	m_2	m	k_1	k_2	k_3
0	0	1		1	
0	1	1		$\frac{1}{3}$	
0	2	1		$\frac{1}{3}$	$-\frac{1}{3}$
0	3	1		$\frac{7}{27}$	$-\frac{8}{27}$
1	1	1			$\frac{1}{6}$
1	2	1		$\frac{1}{27}$	$-\frac{1}{54}$
0	1	2	-1		
0	2	2	-1	$-\frac{2}{3}$	
0	3	2	-1	$-\frac{2}{3}$	$-\frac{2}{3}$
0	4	2	-1	$-\frac{8}{9}$	$-\frac{5}{9}$
0	5	2	-1	$-\frac{80}{81}$	$-\frac{50}{81}$
1	1	2		$\frac{1}{3}$	
1	2	2			$\frac{1}{3}$
1	3	2		$\frac{1}{9}$	$-\frac{1}{18}$
1	4	2		$\frac{4}{81}$	$\frac{5}{162}$
2	2	2		$-\frac{2}{9}$	$\frac{11}{18}$
2	3	2		$-\frac{8}{81}$	$\frac{22}{81}$
1	1	3	-1		$-\frac{1}{2}$
1	2	3	$-\frac{1}{2}$		$-\frac{1}{2}$
1	3	3	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{3}$
1	4	3	$-\frac{1}{2}$	$-\frac{2}{9}$	$-\frac{8}{9}$
1	5	3	$-\frac{1}{2}$	$-\frac{10}{27}$	$-\frac{79}{108}$
2	2	3		$\frac{1}{3}$	$-\frac{1}{3}$
2	3	3		$\frac{1}{9}$	$-\frac{1}{18}$
2	4	3		$\frac{8}{27}$	$-\frac{61}{108}$
3	3	3		$-\frac{4}{27}$	$\frac{11}{27}$
2	2	4	$-\frac{2}{3}$		$\frac{7}{6}$
2	3	4	$-\frac{1}{3}$		$\frac{1}{2}$
2	4	4		$\frac{7}{27}$	$-\frac{8}{27}$
3	3	4	$-\frac{1}{3}$	$-\frac{8}{27}$	$\frac{22}{27}$

$J(m_1, m_2) = l_1 I + l_2$

m_1	m_2	l_1	l_2
0	0	$\frac{1}{3}$	$-\frac{3}{2}$
0	1	$\frac{1}{9}$	$-\frac{1}{2}$
0	2	$\frac{2}{27}$	$-\frac{65}{216}$
1	2	$\frac{1}{54}$	$-\frac{43}{432}$

Therefore

$$\tfrac{3}{2}\int\int \frac{y}{\Delta} - \int\int \frac{1}{2\Delta} = -\tfrac{1}{2}\int\int \frac{d\Delta/dy}{\Delta}. \quad \text{(A2)}$$

The right-hand side is just

$$-\tfrac{1}{2}\int_0^1 dy\,(\ln\Delta)\Big|_0^{1-y} = 0$$

so

$$I(1,0,1) = I/3.$$

(3) $I(1,1,1)$: In imitation of $I(1,0,1)$ rewrite xy as

$$x\left(-\frac{1}{2}\frac{d}{dy}(\Delta) + \frac{(1-x)}{2}\right),$$

then integrate the term involving $d\Delta/dy$ with respect to y. This, as above, gives 0, so we get

$$I(1,1,1) = \frac{I(1,0,1)}{2} - \tfrac{1}{2}I(2,0,1). \quad \text{(A3)}$$

(4) $I(2,0,1)$: $x^2 = -\Delta + (1-x)y + x - y^2$. Using x-y symmetry and (A3) we deduce that

$$\tfrac{3}{2}I(2,0,1) = -\int\int\frac{\Delta}{\Delta} + \tfrac{1}{2}\int\int\frac{1}{\Delta},$$

and so finally $I(2,0,1) = \frac{1}{3}(I-1)$. Therefore by (A3), $I(1,1,1) = \frac{1}{6}$.

(5) Other integrals, $I(m,n,1)$, can be evaluated using tricks similar to those in (1)–(4) above.

(B) *An example of evaluating a* $J(m_1, m_2)$. We evaluate $J(0,0)$ as follows:

$$\int\int \ln\Delta = \int_0^1 dx\, y\ln\Delta\Big|_0^{1-x} - \int\int\frac{y\,d\Delta/dy}{\Delta}$$

$$= \int_0^1 (1-x)\ln(x(1-x))$$

$$-\int\int\frac{-2y^2 + y - xy}{\Delta}.$$

The one-dimensional integral is easily calculated and the other one is just a sum of $I(m_1, m_2, 1)$ which we computed in (A) above. The result is $J(0,0) = -\frac{3}{2} + \frac{1}{3}I$.

(C) *Integrals* $I(0,0,m,\lambda)$ *where* $m > 1$. We briefly outline a proof of the following theorem:

$$\int\int\frac{1}{\Delta^n(\lambda)} = \frac{3(n-2)/(n-1)}{1+3\lambda}\int\int\frac{1}{(\Delta(\lambda))^{n-1}} + \frac{3}{(n-1)(1+3\lambda)}\int_0^1 dx\,\frac{(1-x)}{[x(1-x)+\lambda]^{n-1}}.$$

Proof (outline). Basically we use the tricks shown in (A):

(a) We find that

$$\int\int \frac{x}{\Delta^n} = -\tfrac{1}{3}\int\int \frac{(-3x+1-1)}{\Delta^n}$$

$$= -\tfrac{1}{3}\int\int \frac{d\Delta/dx}{\Delta^n} + \tfrac{1}{3}\int\int \frac{1}{\Delta^n}$$

$$= \tfrac{1}{3}\int\int \frac{1}{\Delta^n}$$

since

$$\int_0^1 dx \; \frac{f(x)}{\Delta^n}\Bigg|_{y=0}^{y=1-x} = 0\,,$$

where we have abbreviated the notation for $\Delta(\lambda)$ and call it now Δ.

(b) By writing x^2 as $-x[d\Delta/dy + (-1+2y)]$ we can show that

$$\int\int \frac{x^2}{\Delta^n} = \tfrac{1}{9}\int\int \frac{1}{\Delta^n} + \frac{2}{3(n-1)}\int_0^1 dx\left(\frac{y}{\Delta^{n-1}}\Bigg|_0^{1-x}\right) - \frac{2}{3(n-1)}\int\int \frac{1}{\Delta^{n-1}}\,.$$

(c) Similarly,

$$\int\int \frac{xy}{\Delta^n} = \tfrac{1}{9}\int\int \frac{1}{\Delta^n} - \frac{1}{3(n-1)}\int_0^1 dx\left(\frac{y}{\Delta^{n-1}}\Bigg|_0^{1-x}\right) + \frac{1}{3(n-1)}\int\int \frac{1}{\Delta^{n-1}}\,.$$

(d) Also,

$$\int\int \frac{\Delta}{\Delta^n} = \int\int \frac{2x-2x^2-xy+\lambda}{\Delta^n}$$

$$= \tfrac{1}{3}\int\int \frac{1}{\Delta^n} - \frac{1}{n-1}\int_0^1 \frac{1-x}{[x(1-x)+\lambda]^{n-1}} + \frac{1}{n-1}\int\int \frac{1}{\Delta^{n-1}} + \int\int \frac{\lambda}{\Delta^n}$$

by (a), (b), and (c). But the left-hand side is $\int\int 1/\Delta^{n-1}$ so the theorem follows.

As an example, the above theorem gives

$$\int\int \frac{1}{\Delta^2} = \left(3\int_0^1 \frac{(1-x)}{[x(1-x)+\lambda]}\right)\frac{1}{1+3\lambda}\,.$$

Then by iteration of the theorem $I(0,0,m,\lambda)$ (for $n>1$) is a sum of one-dimensional integrals of the form $\int_0^1 (1-x)/[x(1-x)+\lambda]^n$.

(D) *Integrals* $\int\int [xyf(x,y)/(\Delta(\lambda))^m]$, *where f is symmetric in x and y.*

Theorem:

$$\int\int \frac{xyf(x,y)}{\Delta^m} = \frac{1}{3}\left[\int\int \frac{1}{\Delta^m}\left(\frac{d}{dy}\Delta\right)f(x,y) + \frac{xf(x,y)}{\Delta^m}\right]\,.$$

Proof. Write

$$x = -(-x+1-2y)+1-2y$$

and

$$y = -\tfrac{1}{2}(-x+1-2y)+\tfrac{1}{2}-\tfrac{1}{2}x\,.$$

xy can be written first using the substitution for x, then that for y. It is then easy to prove the theorem by using the fact that

$$\int\int \frac{x^a}{\Delta^b}f(x,y) = \int\int \frac{y^a}{\Delta^b}f(x,y)\,.$$

(E) *The rest of the integrals.* In what follows we say that an integral is of lower rank than $I(a',b',m',\lambda)$ if it is a one-dimensional integral of simply evaluated rational functions or if it is of the form $I(a,b,m,\lambda)$ with $a+b<a'+b'$ and $m\leq m'$. Our technique is to iteratively reduce $I(a,b,m,\lambda)$ to integrals of the form $I(a',b',1)$, or $I(0,0,m')$, or one-parameter simple integrals. All of these are evaluated easily either directly or by (A) and (C) above. As an example of how we do this, suppose our integral is

$$I(a,a,m,\lambda) = \int\int \frac{xy(x^{a-1}y^{a-1})}{(\Delta(\lambda))^m}\,.$$

By the theorem of (D) this is reduced to a sum of integrals of lower rank. If, on the other hand, we have $I(a,b,m,\lambda)$ with $b>a$, then it is possible, by rewriting x and x^2 in terms of derivatives of Δ or of Δ itself, to express this in terms of $I(a+1, b-1,m,\lambda)$ and integrals of lower rank. By doing this we can eventually reduce such integrals to symmetric ones, i.e., $I(a,a,m,\lambda)$. [The third possibility is $I(a+1,a,m,\lambda)$ but by x-y symmetry it turns out to be easily shown directly that this reduces to integrals of lower rank.]

The procedure described is cumbersome and we found it helpful to do the iteration above using the symbolic manipulation program MACSYMA, of the Mathlab group at the MIT Laboratory for Computer Science.

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[15]We use a slight variation of the rules given in Ref. 7. Where those authors have $d^N k/(2\pi)^4$ we have $d^N k/(2\pi)^N$.

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