Multilegged propagators in strong-coupling expansions

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(Received 8 May 1979)

This paper is a continuation of a previous paper on strong-coupling expansions in quantum field theory. We are concerned here with one-dimensional quantum field theories (quantum-mechanical models). Our general approach is to derive graphical rules for constructing the strong-coupling expansion from a Lagrangian path integral in the presence of external sources. After reviewing the normalization of one-dimensional path integrals, we examine in detail the model Hamiltonian $H = |p| + |q|$. We show that in the strong-coupling expansion the graphs are constructed from multilegged propagators attached to multilegged vertices. We use these graphical rules to calculate the ground-state energy for this Hamiltonian. One motivation for examining expansions involving multilegged propagators is provided by the Lagrangian for quantum chromodynamics whose strong-coupling expansion also involves multilegged propagators.

I. INTRODUCTION

In a previous paper[1] we showed how to construct the strong-coupling expansion of a quantum field theory from its Lagrangian path-integral representation. The models we examined (self-interacting scalar boson field theories) have graphical strong-coupling expansions whose graphs are constructed from propagators joined to vertices. Interactions such as $g\phi^4$, $g\phi^{2N}$, $g\cos\phi$, $g(1+\phi^2)^{-1}$, $gP(\phi)$, where $P(x)=\sum a_n x^{2n}$, and so on, give rise to graphs which have vertices having any even number of connections and propagators having two legs. Thus, propagators are represented by lines whose two ends (legs) are connected to vertices.

In this paper, we consider models having a more general graphical strong-coupling expansion. In this more general expansion, propagators may have more than two legs. We represent a multilegged propagator as a head (circle) from which many legs (lines) emerge. These lines are joined to the multivertices of the theory (see Fig. 1).

To develop the theory of multilegged propagators we study a simple quantum-mechanical model characterized by the Hamiltonian

$$H = |p| + |q| . \tag{1.1}$$

This Hamiltonian is of interest because it represents a massless particle in a linearly rising potential (a massless quark on a string). We have chosen to study this particular Hamiltonian because its graphical strong-coupling expansion consists of propagators having any even number of legs and vertices having any even number of connections. After developing the graphical rules for this theory, we use them to calculate the ground-state energy to ninth order and obtain fair numerical results.

In a separate paper[2] we examine the large-order behavior of this strong-coupling perturbation expansion. Using methods similar to those of Lipatov[3] we obtain this behavior from the functional integral representation of the theory. Lipatov showed that the large-order behavior of a weak-coupling perturbation series is controlled by an instanton. We show that the large-order behavior of the strong-coupling series is also controlled by an instantonlike object. However, since the strong-coupling expansion is necessarily defined on a lattice, this object is a discrete function defined on the integers.

One motivation for investigating these multilegged propagators in such detail is that they occur

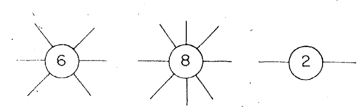

FIG. 1. Graphical representation of multilegged propagators. The number in each circle indicates the number of legs on that propagator.

naturally in the strong-coupling expansion of quantum chromodynamics. At the end of this paper we make some brief remarks explaining why this is so.

This paper is primarily concerned with functional integrals in one-dimensional space-time. In Sec. II we discuss the normalization of such functional integrals. (In our previous paper[1] we left the normalization unspecified.) In Sec. III we develop the graphical rules for the strong-coupling expansion of the theory defined by (1.1). In Sec. IV we use these rules to calculate the first nine terms in the expansion of the ground-state energy. From these terms we obtain a numerical approximation to the ground-state energy which is in reasonable agreement with the exact value. Finally, in Sec. V we comment on why multilegged propagators occur in the strong-coupling expansion of quantum chromodynamics.

II. NORMALIZATION OF FUNCTIONAL INTEGRALS

In this section we review the normalization of functional integrals.

The vacuum persistence function of a one-dimensional quantum theory in the presence of an external source $J(t)$ can be represented as a Euclidean functional integral (in which time t is replaced by it):

$$\langle 0_+|0_-\rangle_J = \lim_{T\to\infty} \int Dq \exp\left[-\int_{-T/2}^{T/2} L(t)dt\right]. \tag{2.1}$$

If the Hamiltonian for the theory is

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}m^2q^2 + gF(q), \tag{2.2}$$

then the Lagrangian L in the q representation is given by

$$L = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}m^2q^2 + gF(q) - J(t)q, \tag{2.3}$$

where we have inserted the external source J.

If the functional integral is properly normalized (that is, if Dq is properly defined), then a direct evaluation of the integral in (2.1) with $J=0$ gives the ground-state energy E,

$$E = \lim_{T\to\infty} -\frac{1}{T}\ln\langle 0_+|0_-\rangle|_{J=0}. \tag{2.4}$$

A. Normalization-independent calculation of E

In our previous paper[1] we were able to calculate the derivative of the ground-state energy of a ϕ^4 theory with respect to the coupling constant g without ever specifying the normalization of the functional integral. For this theory the formula for dE/dg is

$$4g\frac{dE}{dg} = \int \frac{dp}{2\pi} G^{-1}(p)[W_2(p) - G(p)],$$

where $G(p)$ is the free propagator and $W_2(p)$ is the exact two-point Green's function. In the general case where $F(q)$ in (2.2) and (2.3) is a monomial

$$F(q) = |q|^N, \tag{2.5}$$

dE/dg satisfies the formula

$$Ng\frac{dE}{dg} = \int \frac{dp}{2\pi} G^{-1}(p)[W_2(p) - G(p)]. \tag{2.6}$$

The result in (2.6) is easily derived from the field equation satisfied by the operator $q(t)$,

$$-\ddot{q} + m^2q + gF'(q) - J = 0, \tag{2.7}$$

which is obtained by varying the Lagrangian in (2.3). If we take the expectation value of (2.7) between the states ${}_J\langle 0_+|$ and $|0_-\rangle_J$ and divide by $\langle 0_+|0_-\rangle_J$ then we obtain

$$\left(-\frac{d^2}{dt^2} + m^2\right)\frac{\langle 0_+|q(t)|0_-\rangle_J}{\langle 0_+|0_-\rangle_J} + g\frac{\langle 0_+|F'(q(t))|0_-\rangle_J}{\langle 0_+|0_-\rangle_J} = J(t). \tag{2.8}$$

Next, we take a functional derivative of (2.8) with respect to $J(t')$ and set $J=0$:

$$\left(-\frac{d^2}{dt^2} + m^2\right)W_2(t,t') + g\frac{\delta}{\delta J(t')}\left(\frac{\langle 0_+|F'(q(t))|0_-\rangle_J}{\langle 0_+|0_-\rangle_J}\right)\Bigg|_{J=0} = \delta(t-t'), \tag{2.9}$$

where $W_2(t,t')$ is the connected two-point Green's function of the theory.

Now we assume that

$$\langle 0_+|q(t)|0_-\rangle|_{J=0} = 0.$$

In higher space-time dimensions this is the analog of ruling out symmetry breaking. In one dimension it assumes merely that F is an even function of q. Noting that the free propagator $G(t,t')$ in Euclidean space satisfies

$$\left(-\frac{d^2}{dt^2} + m^2\right)G(t,t') = \delta(t-t'),$$

$$G(t,t') = \frac{1}{2m}\exp(-m|t-t'|),$$

$$G^{-1}(t,t') = \left(-\frac{d^2}{dt^2} + m^2\right)\delta(t-t'),$$

we set $t=t'$ and write (2.9) as

$$\int dt'\, G^{-1}(t,t')[W_2(t',t) - G(t',t)] = -g\frac{\langle 0_+|F'(q(t))q(t)|0_-\rangle|_{J=0}}{\langle 0_+|0_-\rangle|_{J=0}}. \tag{2.10}$$

Finally, (2.5) yields $F'(q(t))q(t) = NF(q(t))$ and translation invariance implies that

$$\langle 0_+|F(q(t))|0_-\rangle|_{J=0}=\frac{\langle 0_+|\int dt\,F(q(t))|0_-\rangle|_{J=0}}{T}\;.$$

Thus, using (2.4), (2.10) becomes

$$\int dt'\,G^{-1}(t,t')[W_2(t',t)-G(t',t)]=gN\frac{dE}{dg}\;,$$

which in momentum space is (2.6).

We can calculate $W_2(p)$ using graphical techniques. Since the graphical rules do not depend on the normalization of the functional integral in (2.1), the formula in (2.6) shows that we can calculate dE/dg without specifying this normalization.

B. Normalization of functional integrals

However, to calculate E directly we must know the normalization. We now proceed to derive the normalization. To do this we merely observe that the normalization of (2.1), that is, the definition of Dq, is independent of J and q. Indeed, we can reduce the integral in (2.1) to a functional differential operator acting on a much simpler functional integral:

$$\int Dq\exp\left[-\int_{-T/2}^{T/2}L(t)dt\right]=\exp\left[-g\int_{-T/2}^{T/2}F\left(\frac{\delta}{\delta J(t)}\right)dt\right]\int Dq\exp\left[-\int_{-T/2}^{T/2}(\tfrac{1}{2}\dot q^2+\tfrac{1}{2}m^2q^2-Jq)dt\right]$$

$$=\exp\left[-g\int_{-T/2}^{T/2}F\left(\frac{\delta}{\delta J(t)}\right)dt\right]\exp\left[\tfrac{1}{2}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt'\,J(t)G(t,t')J(t')\right]$$

$$\times\int Dq\exp\left[-\int_{-T/2}^{T/2}(\tfrac{1}{2}\dot q^2+\tfrac{1}{2}m^2q^2)dt\right].$$

Thus, to determine the normalization associated with Dq we may examine the right side of this expression with g and J set equal to zero. This reduces the problem to the harmonic oscillator. The ground-state energy E in this case is known to be $m/2$.

We will show that if we define the functional integral (2.1) on a lattice, then the proper normalization is obtained by the transcription

$$\int Dq\to\prod_{k=1}^{n}\int_{-\infty}^{\infty}\frac{dq_k}{(2\pi a)^{1/2}}\;, \tag{2.11}$$

where a is the lattice spacing, n is the number of lattice points, and $an=T$ is the volume of space.

On such a lattice we may write

$$-\int_{-T/2}^{T/2}dt\frac{m^2}{2}q^2\to-\sum_{i=1}^{n}\frac{m^2a}{2}q_i^{\,2}=-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{m^2a}{2}q_i\delta_{ij}q_j$$

and

$$-\int_{-T/2}^{T/2}dt\,\tfrac{1}{2}\dot q^2=\int_{-T/2}^{T/2}dt\,\tfrac{1}{2}q\frac{d^2}{dt^2}q$$

$$\to\frac{1}{2a}\sum_{i=1}^{n}\sum_{j=1}^{n}q_i(\delta_{i+1,j}-2\delta_{i,j}+\delta_{i,j+1})q_j\,.$$

With these transcriptions, the functional integral in (2.1) with $g=J=0$ becomes

$$\prod_{k=1}^{n}\int_{-\infty}^{\infty}\frac{dq_k}{(2\pi a)^{1/2}}\exp\left(-\frac{1}{2a}\sum_{i=1}^{n}\sum_{j=1}^{n}q_iA_{i,j}q_j\right), \tag{2.12}$$

where $A_{i,j}$ is the tridiagonal matrix $-\delta_{i+1,j}-\delta_{i,j+1}+x\delta_{i,j}$ and $x=2+m^2a^2$. Evaluating the Gaussian integral in (2.12) gives $1/(\mathrm{Det}A)^{1/2}$.

By expanding in minors, it is easy to show that the determinant D_n of the $n\times n$ matrix A satisfies the constant-coefficient difference equation

$$D_n=xD_{n-1}-D_{n-2},\quad D_1=x,\quad D_2=x^2-1\,.$$

The exact solution of this difference equation is

$$D_n=\frac{1}{(x^2-4)^{1/2}}\left[\left(\frac{x+(x^2-4)^{1/2}}{2}\right)^{n+1}-\left(\frac{x-(x^2-4)^{1/2}}{2}\right)^{n+1}\right].$$

Substituting $x=2+m^2a^2$ and treating a as small gives

$$D_n\simeq\frac{1}{2am}\left[e^{(n+1)\ln(1+am)}-e^{(n+1)\ln(1-am)}\right]$$

$$\simeq\frac{1}{2am}\,e^{(n+1)am}\,,$$

for large n. Thus,

$$\frac{1}{(\mathrm{Det}A)^{1/2}}\simeq e^{-mT/2}\,,$$

and therefore we have shown that

$$\lim_{T\to\infty}-\frac{1}{T}\ln\int Dq\exp\left[-\int_{-T/2}^{T/2}dt\left(\tfrac{1}{2}\dot q^2+\frac{m^2}{2}q^2\right)\right]=\frac{m}{2}\,,$$

which is indeed the ground-state energy of the harmonic oscillator. This verifies that our choice of normalization in (2.11) was correct. Moreover, it shows that the normalization in (2.11) is correct even if J and g in (2.3) are nonvanishing.

C. Strong-coupling expansion of E for the harmonic oscillator

Now that the normalization is specified correctly, we will try to calculate the ground-state energy E for the harmonic oscillator from (2.1) using the strong-coupling expansion techniques developed in Ref. 1. We will treat the mass m as a large parameter and expand in powers of $1/m$. Of course, the answer we hope to get is $E=m/2$.

In the presence of an external source J, (2.1) reads

$$\langle 0_+|0_-\rangle_J = \int Dq \exp\left[\int\left(\frac{\dot{q}^2}{2} - \frac{m^2}{2}q^2 + Jq\right)dt\right]$$

$$= \exp\left[\tfrac{1}{2}\int\int dt\, dt' \frac{\delta}{\delta J(t)}\frac{d^2}{dt^2}\delta(t-t')\frac{\delta}{\delta J(t')}\right]\int Dq \exp\left[\int\left(-\frac{m^2}{2}q^2+Jq\right)dt\right]. \tag{2.13}$$

Following Ref. 1 we evaluate the remaining functional integral in (2.13) on the lattice as follows:

$$\int Dq \exp\left[\int\left(-\frac{m^2}{2}q^2+Jq\right)dt\right] \to \prod_{i=1}^{n}\int_{-\infty}^{\infty}\frac{dq}{(2\pi a)^{1/2}}\exp\left(-\frac{m^2 a}{2}+J_i q a\right) = \prod_{i=1}^{n}\frac{1}{ma}\exp\left(\frac{J_i^2 a}{2m^2}\right)$$

$$\to e^{-n\ln(ma)}\exp\left[\frac{1}{2m^2}\int J^2(x)dx\right].$$

Thus, using $T=na$, we have

$$\langle 0_+|0_-\rangle = e^{-(T/a)\ln(ma)}\exp\left[\tfrac{1}{2}\int\int dt\, dt' \frac{\delta}{\delta J(t)}\frac{d^2}{dt^2}\delta(t-t')\frac{\delta}{\delta J(t')}\right]\exp\left[\frac{1}{2m^2}\int J^2(x)dx\right]. \tag{2.14}$$

From (2.14) we can read off the graphical rules for this theory. For each line we have $(d^2/dt^2)\delta(t-t')$. The vertices are all two-point vertices and we associate the number $1/m^2$ with each vertex.

Setting $J=0$ in (2.14) we obtain

$$\langle 0_+|0_-\rangle_{J=0} = e^{-(T/a)\ln(ma)}(1+\text{all graphs having no external lines})$$

$$= e^{-(T/a)\ln(ma)}\exp(\text{all connected graphs having no external lines}). \tag{2.15}$$

The connected graphs in this expansion are closed polygons as shown in Fig. 2. The polygon having k vertices has a symmetry number $1/(2k)$ and a factor of m^{-2k} for the vertices. The integral becomes a sum when evaluated on the lattice:

$$\sum_{i_1=1}^{n} a \sum_{i_2=1}^{n} a \cdots \sum_{i_k=1}^{n} a\left(\frac{\delta_{i_1+1,i_2}-2\delta_{i_1,i_2}+\delta_{i_1,i_2+1}}{a^3}\right)\cdots\left(\frac{\delta_{i_k+1,i_1}-2\delta_{i_k,i_1}+\delta_{i_k,i_1+1}}{a^3}\right) = n(-1)^k\frac{(2k)!\,a^{-2k}}{(k!)^2}$$

$$= \frac{T}{a}(-1)^k\frac{(2k)!\,a^{-2k}}{(k!)^2}.$$

Combining these results (2.15) becomes

$$\langle 0_+|0_-\rangle_{J=0} = e^{-(T/a)\ln(ma)}\exp\left[\sum_{k=1}^{\infty}\frac{1}{2k}(ma)^{-2k}\frac{(2k)!}{(k!)^2}(-1)^k\frac{T}{a}\right]. \tag{2.16}$$

Taking the logarithm of both sides of (2.16) and dividing by T gives

$$\frac{\ln\langle 0_+|0_-\rangle_{J=0}}{T} = -\frac{1}{a}\ln(ma) + \frac{1}{a}\sum_{k=1}^{\infty}\frac{1}{2k}\frac{(-1)^k}{(ma)^{2k}}\frac{(2k)!}{(k!)^2}. \tag{2.17}$$

In the limit of zero lattice spacing ($a\to 0$) the right side of (2.17) should approach the exact value of $-E=-m/2$.

Following the approach of Ref. 1, we introduce a dimensionless parameter x defined by

$$x=m^{-2}a^{-2}.$$

The right side of (2.17) then becomes

$$-\frac{m}{2}\left[-\sqrt{x}\ln x - \sqrt{x}\sum_{k=1}^{\infty}\frac{1}{k}(-1)^k x^k\frac{(2k)!}{(k!)^2}\right]. \tag{2.18}$$

Ultimately, since we will take $a\to 0$, the parameter x will tend to ∞. However, for the moment, we

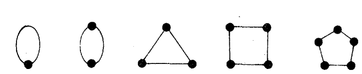

FIG. 2. The connected graphs contributing to the strong-coupling expansion of the ground-state energy of the harmonic oscillator [see (2.15)].

treat x as a small parameter because m is large and a is fixed.

Treating x as a small parameter, the series in (2.18) may be summed exactly in closed form and the limit $x\to\infty$ may be taken. If this is done we find that (2.18) approaches $-m/2$, as desired.

Suppose, however, that only a few terms in this series are known. Then how well do the extrapolation procedures of Ref. 1 work? To answer this question we replace $\ln x$ by $\ln(1+x)$. (No error is introduced here because x will eventually tend to ∞.) We then expand

$$\ln(1+x)=x-x^2/2+x^3/3-\cdots.$$

The right side of (2.18) now becomes $-\frac{1}{2}mA$, where

$$A=x^{3/2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(n+1)}\left[\frac{(2n+2)!}{(n+1)!\,(n+1)!}-1\right]. \quad (2.19)$$

Proceeding as in Ref. 1, we extrapolate to the value of $A^{2/3}=x/(\Sigma)^{-2/3}$ by truncating the Taylor expansion of the $-\frac{2}{3}$ power of the series after two terms and then taking the limit $x\to\infty$. This gives $A_1^{2/3}$, the first extrapolant to $A^{2/3}$. Next, we square this equation, truncate the series after three terms, and let $x\to\infty$. The square root of this ratio gives $A_2^{2/3}$, the second approximant to $A^{2/3}$. Proceeding in this manner, we have calculated the first 12 approximants to $A^{2/3}$. They are listed in Table I. While these extrapolants appear to converge to the correct limit 1, the convergence is agonizingly slow.

TABLE I. The first 12 approximants to the value of $A^{2/3}$, where A is represented by the series in (2.19). Observe the slow convergence to the correct answer $A^{2/3}=1$.

n	nth approximant to $A^{2/3}$
1	0.6000
2	0.8847
3	0.7937
4	0.8812
5	0.8681
6	0.9039
7	0.9037
8	0.9221
9	0.9246
10	0.9353
11	0.9382
12	0.9449

We use a simple technique for improving the rate of convergence of the approximants to the ground-state energy. Instead of computing the ground-state energy E directly we calculate $m(dE/dm)$. To do this, we return to the formula in (2.17) and take md/dm. We obtain the much simpler series

$$-\frac{1}{a}+\frac{1}{a}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(ma)^{2k}}\,\frac{(2k)!}{(k!)^2}\;. \quad (2.20)$$

Eliminating a in this series and replacing it with $m^{-1}x^{-1/2}$ gives

$$-m\sqrt{x}\sum_{n=0}^{\infty}\frac{(2n)!\,x^n(-1)^n}{(n!)^2}\,.$$

This series should converge to $-m(dE/dm)=-m/2$ as $x\to\infty$. Using the extrapolation procedure we have just described, we find that every approximant gives *exactly* $-m/2$. This represents a vast improvement in the information that can be retrieved from the series in (2.17).

Why does the operator md/dm so improve the summability of the series in (2.17)? A heuristic answer is suggested by a theorem in Hille[4] which we paraphrase: If a linear method of summability M sums the geometric series $1-z+z^2-z^3+\cdots$ to the limit $(1+z)^{-1}$ for every z in a domain S, starlike with respect to the origin and containing the disk $|z|<1$, and the limit exists uniformly on compact subsets of S, then M also sums the series $f(z)=\sum_{n=0}^{\infty}C_nZ^n$, which must be regular at the origin, to the function $f(z)$ in the star $SA[f]$, uniformly with respect to z on compact sets.

The method of summability we have been using is not linear so the theorem does not directly apply. However, it may be that the crucial restriction is whether or not f is regular at the origin. Observe that the operator md/dm removes the branch cut at the origin from (2.17) and replaces this expression by (2.20) which is regular at $x=0$ after it is squared. (Squaring is the first step in our extrapolation procedure.)

The above remarks suggest that whether or not the normalization of the functional integral in (2.1) is known, the formula (2.6) that we used in Ref. 1 is likely to be superior to the direct calculation of E in (2.4). Because of this, when we calculate the ground-state energy E of the Hamiltonian $H=|p|+g|q|$ in Sec. IV we will extrapolate the series for $g(dE/dg)$ rather than the series for E.

D. Strong-coupling expansion of E for the anharmonic oscillator

Before concluding this section we show how to incorporate the normalization of the functional integral into a direct strong-coupling expansion for the ground-state energy of the anharmonic oscillator. Following Ref. 1 we write

$$\int Dq \exp\left[-\int_{-T/2}^{T/2} (\tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}m^2q^2 + \tfrac{1}{4}gq^4 - Jq)dt\right]$$

$$= \exp\left[-\tfrac{1}{2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} dt\, dt' \frac{\delta}{\delta J(t)} G^{-1}(t,t') \frac{\delta}{\delta J(t')}\right] \int Dq \exp\left[-\int_{-T/2}^{T/2} (\tfrac{1}{4}gq^4 - Jq)dt\right],$$

where $G^{-1}(t,t') = (-d^2/dt^2 + m^2)\delta(t-t')$.

Next, we evaluate the remaining functional integral on a lattice:

$$\prod_{i=1}^{n} \frac{dq}{(2\pi a)^{1/2}} \exp\left(-\frac{a}{4} gq^4 + aJ_i q\right) = \prod_{i=1}^{n} \frac{\Gamma(1/4)}{2\sqrt{\pi}\, a^{3/4} g^{1/4}} \frac{F(J_i a^{3/4} g^{-1/4})}{F(0)}$$

$$= \exp\left[n \ln\left(\frac{\Gamma(1/4)}{2\sqrt{\pi}\, a^{3/4} g^{1/4}}\right)\right] \exp\left[\frac{1}{a}\int_{-T/2}^{T/2} dt \ln \frac{F(a^{3/4} g^{-1/4} J(t))}{F(0)}\right],$$

where $F(x) = \int_{-\infty}^{\infty} dt \exp(-t^4/4 + xt)$ and $F(0) = \Gamma(1/4)/\sqrt{2}$.

Using the graphical rules derived in Ref. 1, we obtain from this result that

$$E = \lim_{a\to 0} \frac{1}{a}\left\{\ln[2\sqrt{\pi}\, a^{3/4} g^{1/4} \Gamma(1/4)] - \frac{1}{T}(\text{all connected graphs having no external lines})\right\},$$

where we have not needed to calculate the two-point Green's function as an intermediate step.

III. THE MODEL $H = |p| + |q|$

In this section we develop the strong-coupling (large g) expansion of the model defined by the Hamiltonian

$$H = |p| + g|q|. \tag{3.1}$$

At the end of the calculation we set $g = 1$ to recover the symmetrical Hamiltonian in (1.1).

Following the prescription in Euclidean space that

$$L = ip\dot{q} - H,$$

we define the vacuum persistence function for this model as

$$\langle 0_+|0_-\rangle_J = \int\int Dp\, Dq \exp\left[\int_{-T/2}^{T/2} dt\, (ip\dot{q} - |p| - g|q| + Jq)\right], \tag{3.2}$$

where the normalization is chosen so that on a lattice

$$\int\int Dp\, Dq \to \prod_{i=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dp_i dq_i}{2\pi}.$$

Note that this choice of normalization is consistent with that in (2.11) for Hamiltonians in which p appears quadratically. For example, for the harmonic oscillator the p integral is Gaussian and can be evaluated in closed form

$$\langle 0_+|0_-\rangle = \int\int Dp\, Dq \exp\left[\int_{-T/2}^{T/2} dt\, (ip\dot{q} - p^2/2 - m^2q^2/2)\right]$$

$$\to \prod_{i=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dp_i dq_i}{2\pi} \exp(iap_i\dot{q}_i - ap_i^2/2 - am^2 q_i^2/2) = \prod_{i=1}^{n} \frac{dq_i}{(2\pi a)^{1/2}} \exp(-a\dot{q}_i^2/2 - am^2 q_i^2/2),$$

which agrees with the normalization used in (2.11).

Returning to (3.2) we evaluate the p integral (which now is not Gaussian) in closed form on the lattice:

$$\langle 0_+|0_-\rangle_J \to \prod_{i=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dp_i dq_i}{2\pi} \exp(iap_i\dot{q}_i - a|p_i| - ga|q_i| + aJ_iq_i)$$

$$= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{dq_i}{\pi a} \frac{\exp(-ga|q_i| + aJ_iq_i)}{1+\dot{q}_i^{\,2}}$$

$$= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{dq_i}{\pi a} \exp\left[\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \dot{q}_i^{\,2k}\right] \exp(-ga|q_i| + aJ_iq_i)$$

$$\to \int Dq \exp\left\{\int_{-T/2}^{T/2} dt \left[\frac{1}{a} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \dot{q}(t)^{2k} - g|q(t)| + J(t)q(t)\right]\right\}, \tag{3.3}$$

where in the final step we have returned to the continuum representation.

As we did in Ref. 1 we factor out the derivative terms from the functional integral and replace them by derivatives with respect to J. To do this we use the following identity:

$$\dot{q}(t)^{2k} = \int dt_1 \cdots dt_{2k} \delta'(t_1 - t) \cdots \delta'(t_{2k} - t) q(t_1) \cdots q(t_{2k}),$$

$$\langle 0_+|0_-\rangle_J = \exp\left\{\int_{-T/2}^{T/2} dt \left[\frac{1}{a} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \prod_{l=1}^{2k} \int dt_l\, \delta'(t_l - t) \frac{\delta}{\delta J(t_l)}\right]\right\} \int Dq \exp\left[\int_{-T/2}^{T/2} dt\,(-g|q| + Jq)\right].$$

Now we can evaluate the remaining functional integral easily because it is a product of ordinary integrals on the lattice:

$$\prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{dq}{\pi a} e^{-g|q|a + J_i qa} = \prod_{i=1}^{n} \frac{2}{\pi a^2 g} \frac{1}{1 - J_i^{\,2}/g^2} = \left(\frac{2}{\pi a^2 g}\right)^n \prod_{i=1}^{n} \exp\left(\sum_{k=1}^{\infty} \frac{J_i^{\,2k} g^{-2k}}{k}\right).$$

Thus, in continuum language,

$$\langle 0_+|0_-\rangle = \lim_{T\to\infty} e^{-(T/a)\ln(\pi a^2 g/2)} \exp\left\{\int_{-T/2}^{T/2} dt \left[\frac{1}{a} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \prod_{l=1}^{2k} \int dt_l \delta'(t_l - t) \frac{\delta}{\delta J(t_l)}\right]\right\}$$
$$\times \exp\left(\frac{1}{a} \int_{-T/2}^{T/2} \sum_{k=1}^{\infty} \frac{J^{2k} g^{-2k}}{k} dt\right). \tag{3.4}$$

From this result we can immediately read off the graphical rules of the theory:

Propagators. Multilegged propagators having any even number of legs can occur. The $2k$th propagator is

$$\frac{(2k)!(-1)^k}{ak} \int dt \prod_{l=1}^{2k} \delta'(t_l - t). \tag{3.5}$$

Vertices. Vertices with any even number of connections occur. The $2k$th vertex has associated with it the factor

$$\frac{(2k)!}{akg^{2k}}. \tag{3.6}$$

We must integrate over the position of every vertex.

We observe that the two-legged propagator is just proportional to

$$\int dt\, \delta'(t_1 - t)\delta'(t_2 - t) = -\delta''(t_1 - t_2),$$

which is just $G^{-1}(t_1, t_2)$ with $m^2 = 0$. Thus, our multilegged propagators are natural, totally symmetric generalizations of the ordinary propagator that was used in Ref. 1.[5]

Using (2.4) we can now express the ground-state energy E of H in (3.1):

$$E = \lim_{T\to\infty} \frac{1}{a} \ln(\pi a^2 g/2) - \frac{1}{T}\begin{pmatrix}\text{all connected graphs having} \\ \text{no external lines}\end{pmatrix}. \tag{3.7}$$

The specific calculation of E in (3.1) will be postponed to Sec. IV.

We conclude this section with some remarks on the lattice representation of the multilegged propagators in this theory. On the lattice the $2k$th propagator has the form

$$\frac{(2k)!(-1)^k}{ka^{4k}} \sum_i \prod_{l=1}^{2k} (\delta_{i+1,j_l} - \delta_{i,j_l}), \tag{3.8}$$

where for simplicity we have chosen to use a forward difference definition of the derivative of a δ function. It is interesting that the forward difference scheme used here is consistent with the symmetric difference scheme used in Ref. 1 for the two-legged propagator. To see this, we calculate the sum over i in (3.8) for the case $k=1$:

$$\frac{2}{a^4}(\delta_{j_1+1,j_2} - 2\delta_{j_1,j_2} + \delta_{j_1,j_2+1}).$$

Observe that this object is a *symmetric* difference representation of a second derivative of a δ function.

IV. COMPUTATION OF THE GROUND-STATE ENERGY

In this section we evaluate (3.7). To do this we must draw every diagram having no external lines in each order in powers of $1/g^2$ [see (3.5)]. For each diagram we must compute the symmetry number, the graphical integral, and multiply by the numerical weights for each multilegged propagator and vertex. Summing over all diagrams in Nth order gives the Nth term in the perturbation series for the ground-state energy.

A. First-order calculation

In first order there is only one diagram having no external lines. It consists of a two-vertex tied to a two-propagator (see Fig. 3).

The symmetry number for this graph is $\frac{1}{2}$. The numerical weight for the two-vertex is $2a^{-1}g^{-2}$ and the numerical weight for the two-propagator is $-2a^{-4}$ [see (3.8)]. The lattice sum to be performed is a double sum, one sum over the head of the propagator and one sum over the position of the vertex:

$$\sum_{j=1}^{n} a \sum_{i=1}^{n} (\delta_{i+1,j} - \delta_{i,j})^2 = a \sum_{i=1}^{n} \sum_{j=1}^{n} (\delta_{i+1,j} + \delta_{i,j})$$
$$= a \sum_{i=1}^{n} 2 = 2na = 2T.$$

Multiplying all of these numerical results together, we obtain

$$(\tfrac{1}{2})(2a^{-1}g^{-2})(-2a^{-4})(2T) = -4Tg^{-2}a^{-5}. \qquad (4.1)$$

B. Second-order calculation

There are four connected graphs in second order. These are shown on Fig. 4.

The symmetry numbers for the graphs are
(a) $\frac{1}{24}$,
(b) $\frac{1}{4}$,
(c) $\frac{1}{8}$,
(d) $\frac{1}{8}$.

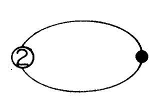

FIG. 3. The only graph contributing to the leading-order ($1/g^2$) term in the expansion of the ground-state energy of $H = |p| + g|q|$.

The numerical weights for the graphs are
(a) $144a^{-9}g^{-4}$,
(b) $16a^{-10}g^{-4}$,
(c) $48a^{-10}g^{-4}$,
(d) $48a^{-9}g^{-4}$.

The results of evaluating lattice integrals are
(a) $2T$,
(b) $6Ta$,
(c) $4Ta$,
(d) $4T$.

Multiplying all of these numerical results together for each graph, we obtain
(a) $12Tg^{-4}a^{-9}$,
(b) $12Tg^{-4}a^{-9}$,
(c) $24Tg^{-4}a^{-9}$,
(d) $24Tg^{-4}a^{-9}$.

Observe from Fig. 4 that graphs (c) and (d) are dual in the sense that they interchange under the operation of exchanging heads of propagators for vertices having the same number of connections. Graphs (a) and (b) are self-dual. In any order all dual pairs of graphs contribute equally to the ground-state energy.

The final step is to sum over all graphs. The result is

$$84Tg^{-4}a^{-9}. \qquad (4.2)$$

C. Third-, fourth-, and fifth-order calculations

There are 10 connected graphs in third order. These are displayed along with their symmetry numbers in Fig. 5. The sum over these graphs gives

$$\frac{-12\,832}{3} Tg^{-6}a^{-13} \qquad (4.3)$$

for the third-order contribution to the ground-state energy.

There are 45 connected graphs in fourth order

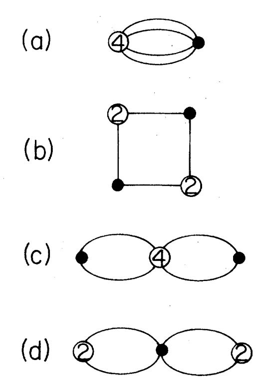

FIG. 4. The four connected graphs contributing to the second-order ($1/g^4$) term in the expansion of the ground-state energy of $H = |p| + g|q|$.

and 177 graphs in fifth order. Their respective contributions to the expansion of the ground-state energy are

$$365\,096\,Tg^{-8}a^{-17} \tag{4.4}$$

and

$$-\frac{228\,603\,264}{5}\,Tg^{-10}a^{-21}\,. \tag{4.5}$$

D. Sixth- through ninth-order calculations

After calculating the first five orders by hand, we programmed a computer to draw all distinct connected diagrams in a given order, compute the symmetry number for each diagram, evaluate the lattice sums for each diagram, and sum over all diagrams to obtain the final result. Using the DEC-10 computer at the University of Pittsburgh we reproduced the first five perturbation coefficients in (4.1)–(4.5) in 2 seconds. We then used the computer to calculate the next four coefficients. We find that in sixth order there are 995 distinct graphs. The computer took a total of 9 seconds to obtain the sixth-order coefficient:

$$7\,851\,727\,232\,Tg^{-12}a^{-25}\,. \tag{4.6}$$

In seventh, eighth, and ninth orders there are 5785, 39 758, and 297 345 distinct graphs, respectively. To find and evaluate these graphs the computer required 72 seconds, 11.5 minutes, and 4 hours; the perturbation coefficients are

$$-\frac{12\,406\,189\,231\,104}{7}\,Tg^{-14}a^{-29}\,, \tag{4.7}$$

$$509\,842\,931\,459\,904\,Tg^{-16}a^{-33}\,, \tag{4.8}$$

$$-\frac{1\,641\,635\,326\,219\,761\,664}{9}\,Tg^{-18}a^{-37}\,. \tag{4.9}$$

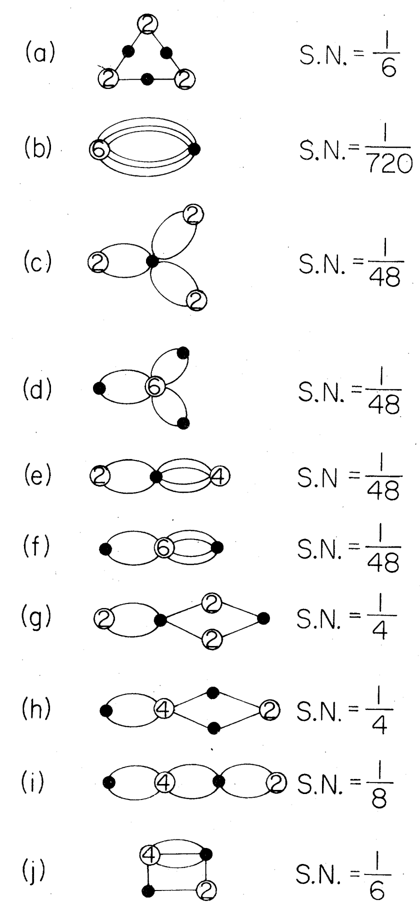

FIG. 5. All connected graphs contributing to the third-order ($1/g^6$) term in the expansion of the ground-state energy. To the right of each graph is given the symmetry number for the graph. Graphs (a), (b), (i), and (j) are self-dual, and (c) and (d), (e) and (f), and (g) and (h) are dual pairs of graphs.

The perturbation coefficients in (4.1)–(4.9) are exact. No approximations have been made. Substituting (4.1)–(4.9) into (3.7) and canceling T gives the following expansion for E:

$$E=\lim_{a\to 0}\frac{1}{a}\left[\ln(\pi a^2 g/2)+\frac{4}{g^2a^4}-\frac{84}{g^4a^8}+\frac{12\,832}{3g^6a^{12}}-\frac{365\,096}{g^8a^{16}}+\frac{228\,603\,264}{5g^{10}a^{20}}-\frac{7\,851\,727\,232}{g^{12}a^{24}}+\frac{12\,406\,189\,231\,104}{7g^{14}a^{28}}-\frac{509\,842\,931\,459\,904}{g^{16}a^{32}}+\frac{1\,641\,635\,326\,21\,9\,761\,664}{9g^{18}a^{36}}-\cdots\right]. \tag{4.10}$$

We will shortly extrapolate this formula to zero lattice spacing.

E. Some comments on graph counting

Before we proceed with the calculation of E, we make two remarks which are very useful for checking that no graphs have been omitted and that the symmetry numbers have been correctly calculated.

The first is that if we take the set of all graphs of order k (disconnected, as well as connected), sum over all symmetry numbers and weights, but set every lattice integral contribution equal to 1, and also set the lattice spacing $a=1$, we obtain in kth order

$$\frac{(2k)!}{g^{2k}}(-1)^k. \qquad (4.11)$$

Second, for the same class of graphs, if we sum over all symmetry numbers in kth order (and ignore all other weight factors except for g), we obtain a perfect square integer divided by $(2k)!$. For example, for the first five orders we have

$$\frac{1^2}{2!g^2},\ \frac{4^2}{4!g^4},\ \frac{31^2}{6!g^6},\ \frac{379^2}{8!g^8},\ \frac{6556^2}{10!g^{10}}. \qquad (4.12)$$

To prove the first property of graphs, we merely restrict the integral in (3.3) to one lattice point and set $J=0$ and $a=1$. This reduces the quantum theory to a zero-space-time-dimensional theory and effectively replaces all lattice integrals by 1. If there is just one lattice point q_1, then $\dot{q}_1^2=q_1^2$. Thus, (3.3) becomes

$$\langle 0_+|0_-\rangle_{1\ \text{lattice point},\ J=0}=\int_{-\infty}^{\infty}\frac{dq}{\pi a}\frac{e^{-g|q|}}{1+q^2}=\frac{2}{\pi g}\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{g^{2k}}.$$

The factor $2/\pi g$ corresponds to the $\ln(\pi a^2 g/2)$ term in (4.10). The remaining series is precisely (4.11). This verifies the first property.

To prove the second property we argue that if all vertices and multilegged propagators have weights one, then in zero space-time dimension the vacuum persistence function has the form

$$\exp\left(\frac{1}{2!}\frac{d^2}{dx^2}+\frac{1}{4!}\frac{d^4}{dx^4}+\frac{1}{6!}\frac{d^6}{dx^6}+\cdots\right)\exp\left(\frac{x^2}{2!g^2}+\frac{x^4}{4!g^4}+\frac{x^6}{6!g^6}+\cdots\right)\bigg|_{x=0}=\exp\left(-1+\cosh\frac{d}{dx}\right)\exp\left(-1+\cosh\frac{x}{g}\right)\bigg|_{x=0}.$$

If we Taylor expand $\exp(-1+\cosh x)$ we obtain

$$\exp(-1+\cosh x)=1+\sum_{n=1}^{\infty}a_n x^{2n}/(2n)!,$$

where $a_1=1$, $a_2=4$, $a_3=31$, $a_4=379$, $a_5=6556$. Thus, the vacuum persistence function is

$$\left[1+\sum_{n=1}^{\infty}\frac{a_n}{(2n)!}\left(\frac{d}{dx}\right)^{2n}\right]\left[1+\sum_{n=1}^{\infty}\frac{a_n}{(2n)!}\left(\frac{x}{g}\right)^{2n}\right]\bigg|_{x=0}=1+\sum_{n=1}^{\infty}\frac{a_n^2}{(2n)!^2}\left(\frac{d}{dx}\right)^{2n}\left(\frac{x}{g}\right)^{2n}$$

$$=1+\sum_{n=1}^{\infty}\frac{a_n^2}{(2n)!g^{2n}}.$$

The coefficients in this series are precisely those in (4.12).

F. Extrapolation to zero lattice spacing

We argued in Sec. II C that the rate at which the extrapolants converge to E is vastly enhanced by converting the strong-coupling expansion for E into the strong-coupling expansion for $g\,dE/dg$. Performing this operation on (4.10) gives

$$g\frac{dE}{dg}=\lim_{a\to 0}\frac{1}{a}\left(1-\frac{8}{g^2a^4}+\frac{336}{g^4a^8}-\frac{25\,664}{g^6a^{12}}+\frac{2\,920\,768}{g^8a^{16}}-\frac{457\,206\,528}{g^{10}a^{20}}+\frac{94\,220\,726\,784}{g^{12}a^{24}}\right.$$
$$\left.-\frac{24\,812\,378\,462\,208}{g^{14}a^{28}}+\frac{8\,157\,486\,903\,358\,464}{g^{16}a^{32}}-\frac{3\,283\,270\,652\,439\,523\,328}{g^{18}a^{36}}+\cdots\right). \qquad (4.13)$$

By a purely dimensional analysis we know that $E(g)=C\sqrt{g}$, where C is a numerical constant. Furthermore, it is convenient to define $x=g^{-2}a^{-4}$. Thus, in terms of the dimensionless parameter x, we have from (4.13) the following series representation for C:

$$C=\lim_{x\to\infty}2x^{1/4}(1-8x+336x^2-25\,664x^3+\cdots-3\,283\,270\,652\,439\,523\,328x^9+\cdots). \qquad (4.14)$$

Although this series is rapidly divergent, the coefficients appear to be all integers and to oscillate in sign.

To use the extrapolation procedure described in Ref. 1, we raise (4.14) to the fourth power:

$$C^4=\lim_{x\to\infty}16x(1-32x+1728x^2-136\,960x^3+15\,086\,336x^4-2\,243\,933\,184x^5+440\,649\,545\,728x^6$$
$$-111\,689\,038\,635\,008x^7+35\,708\,919\,509\,594\,112x^8-14\,091\,209\,469\,426\,417\,664x^9+\cdots). \qquad (4.15)$$

To obtain the first extrapolant, we rewrite (4.15) as

$$C^4 = \lim_{x\to\infty} \frac{16x}{1+32x+\cdots}.$$

Truncating the denominator after two terms and letting $x\to\infty$, we obtain the first extrapolant C_1:

$$C_1 = (\tfrac{1}{2})^{1/4} = 0.840\,896\,42. \tag{4.16}$$

Unfortunately, the second extrapolant obtained by using this procedure is complex. We believe that this happens because the coefficients grow so rapidly. Indeed, in Ref. 2 we show that the kth coefficient grows roughly like $(2k+1)!$.

Therefore, we resort to the technique of converting (4.15) into the $\binom{k}{k+1}$ Padé sequence and letting $x\to\infty$ for each element of the sequence. Because (4.15) has 10 terms we can calculate four more extrapolants, C_2 from the $\binom{1}{2}$ Padé, C_3 from the $\binom{2}{3}$ Padé, and so on. We find that

$$\begin{aligned} C_2 &= 0.907\,223\,72\,, \\ C_3 &= 0.936\,779\,64\,, \\ C_4 &= 0.954\,129\,03\,, \\ C_5 &= 0.965\,786\,31\,. \end{aligned} \tag{4.17}$$

The extrapolants $C_1, C_2, C_3, \ldots$ appear to be slowly and monotonically converging to the exact value of C which is[6]

$$C \doteq 1.1041\,. \tag{4.18}$$

The rate of convergence of the sequence of extrapolants $C_1, C_2, C_3, \ldots$ can be improved by the use of extrapolation procedures such as Richardson extrapolation.[7] However, the rate of convergence of the sequence $C_1, C_2, C_3, \ldots$ is still much slower than we had anticipated on the basis of our work on the anharmonic oscillator.[1] We believe that the reason for this sluggish convergence is that the power series in (4.14) is so rapidly divergent. The corresponding power series for the anharmonic oscillator is convergent.

V. MULTILEGGED PROPAGATORS IN QUANTUM CHROMODYNAMICS

In this section we suggest that multilegged propagators will play a role in quantum chromodynamics. For simplicity we consider an SU(2) non-Abelian gauge theory in the absence of fermion fields. The sourceless Lagrangian for this model is

$$L = -\tfrac{1}{4}(F^a_{\mu\nu})^2\,, \tag{5.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu\,, \tag{5.2}$$

and where we have set $g=1$.

We will express the vacuum persistence function for this model in a gauge in which $A_0=0$ and ignore any complications associated with gauge fixing. In this gauge we have

$$\begin{aligned} L = +\tfrac{1}{2}\{& A^a_i D_{ij} A^a_j - \epsilon^{abc} A^b_i A^c_j (\partial_i A^a_j - \partial_j A^a_i) \\ &+ \tfrac{1}{2}[(A^a_i A^a_j)^2 - (A^a_i A^a_i)^2] + 2J^a_i A^a_i\}\,, \end{aligned}$$

where J^a_i is an external source and D_{ij} is the differential operator $\delta_{ij}\partial^2 - \partial_i\partial_j$.

The vacuum persistence function in Minkowski space is

$$\int DA\, e^{iL}\,.$$

This may be rewritten as

$$\begin{aligned} \exp\left[-\frac{i}{2}\frac{\delta}{\delta J^a_i} D_{ij} \frac{\delta}{\delta J^a_j} + \epsilon^{abc}\frac{\delta}{\delta J^b_i}\frac{\delta}{\delta J^c_j}\left(\partial_i \frac{\delta}{\delta J^a_j} - \partial_j \frac{\delta}{\delta J^a_i}\right)\right] \\ \times \int DA\, \exp\{\tfrac{1}{4} i[(A^a_i A^a_j)^2 - (A^a_i A^a_i)^2] + iJ^a_i A^a_i\}\,. \end{aligned} \tag{5.3}$$

Even in this very naive approach we see evidence of multilegged propagators. The first term in the functional differential operator is a two-legged propagator and the second term is a three-legged propagator. We conjecture that if the gauge-fixing is correctly carried through, then all multilegged propagators will occur.

ACKNOWLEDGMENTS

We wish to thank J. E. Mandula and R. Coifman for many stimulating discussions. Two of us, C. M. B. and G. S. G., wish to thank P. Carruthers and the Los Alamos Scientific Laboratory and two of us, C. M. B. and R. R., wish to acknowledge the Aspen Center for Physics for their hospitality. Four of us, C. M. B., F. C., G. S. G., and D. H. S., wish to acknowledge the Department of Energy and two of us, M. L. S. and R. R., wish to acknowledge the National Science Foundation for partial financial support for this project. We are also indebted to the Laboratory for Computer Science at M. I. T. for allowing us the use of MACSYMA to perform algebraic manipulation.

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