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Coherent states for general potentials. III. Nonconfining one-dimensional examples

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We apply our minimum-uncertainty coherent-states (MUCS) formalism to two one-dimensional systems that have continua: the symmetric Rosen-Morse potential and the Morse potential. The coherent states are discussed analytically in great detail, and the connections to annihilation-operator and displacement-operator coherent states are given. For the Rosen-Morse system the existence of a continuum does not prevent one from obtaining the coherent states in analytic, closed form. The Morse system, with its energy-dependent natural classical variable X_c, has a natural quantum operator X which is Hamiltonian-dependent. This Hamiltonian dependence is complicated and prevents an easy analytic solution for the MUCS. However, approximate MUCS can be obtained by analytic approximation techniques.

I. INTRODUCTION

In this paper we apply our minimum-uncertainty coherent-states (MUCS) formalism[1,2] to two nonconfining one-dimensional potentials (potentails that have both a discrete and a continuous spectrum). Even though there are continua, the coherent states can be found in the same manner as in the case of confining potentials.[3] In the discussion that follows we shall make extensive references to the results of the preceding two papers,[2,3] referencing equations from them as Eq. (I-x.x) and Eq. (II-x.x).

Our first example, in Sec. II, is the symmetric Rosen-Morse (RM) potential[1,4-7]

$$V(x)=U_0 \tanh^2 ax\ . \tag{1.1}$$

This potential has a finite number of bound states whose eigenenergies are proportional to a quadratic function of n, and it also has a continuous spectrum. In Sec. III we study the one-dimensional Morse potential[8-12]

$$V(x)=U_0(1-e^{-ax})^2\ . \tag{1.2}$$

This potential is asymmetric. It rises to infinity at the left, but has a continuum since it only rises to U_0 on the right. Here, there are also a finite number of bound states proportional to a different quadratic function of n.

For each of these potentials we discuss (a) the classical motion and the natural classical variables, (b) the natural quantum operators and the MUCS, (c) the natural quantum operators in terms of the n-dependent raising and lowering operators, (d) the time dependence of the MUCS and of the natural quantum operators, (e) the limits in which these systems reduce to the harmonic oscillator, and (f) n-independent raising and lowering operators and their use in defining annihilation-operator coherent states (AOCS) and displacement-operator coherent states (DOCS). These AOCS-DOCS are inequivalent to our MUCS and are not in unitary-exponential-operator DOCS form. (Partial results on our MUCS have already been given elsewhere for the Rosen-Morse[7] and Morse[12] systems.)

The Morse potential has the feature that the natural classical variable X_c is a function of the energy. Therefore, its quantum counterpart X is a function of the quantum Hamiltonian. Unfortunately this function is not simple. As a result the MUCS for the Morse potential are obtained only in a certain analytic approximation. However, this does provide a specific example of how our MUCS techniques can be used approximately.

Finally, we remark on the choice of the raising and lowering operators that occur in our formalism. The well-known factorization method of Schrödinger,[13] as expounded by Infeld and Hull,[14] produces a factorization of a second-order differential operator as a product of two first-order differential operators. These first-order differential operators are raising and lowering operators.

However, as has been observed for both the Rosen-Morse[6] and Morse[10] potentials, the standard Infeld-Hull factorization yields raising and lowering operators that affect *both* the potential and the eigenstate number n. For instance, in the Rosen-Morse potential the Infeld-Hull raising and lowering operators take one from the nth eigenstate in a potential labeled by s [see Eq. (2.2)] to the $(n\pm1)$ eigenstate in a potential labeled by $(s\pm1)$. Such operators do not have a simple physical interpretation. In contrast, the raising and lowering operators that occur in our formalism affect only the eigenstates in a fixed potential and have a simple connection to the natural quantum operators. The choice of the correct raising and lowering operators will also be crucial in dealing with the multidimensional examples of paper IV.

II. ROSEN-MORSE POTENTIAL

A. Classical motion and the natural classical variables

The symmetric Rosen-Morse (RM) potential is

$$V(x)=U_0 \tanh^2 z\ ,\quad z\equiv ax\ , \tag{2.1}$$

$$U_0 = \mathcal{E}_0 s(s+1), \quad \mathcal{E}_0 = \frac{\hbar^2 a^2}{2m}, \tag{2.2}$$

where our use of $\mathcal{E}_0$ and s presage definitions which are usual for the quantum system. The natural classical variables which satisfy Eqs. (I-3.2) to I-3.4) are

$$X_c = \sinh z = A(E)\sin\omega_c t, \tag{2.3}$$

$$P_c = ap\cosh z = m\omega_c A(E)\cos\omega_c t = a[2mE]^{1/2}\cos\omega_c t, \tag{2.4}$$

$$A(E) = \left(\frac{E}{U_0 - E}\right)^{1/2}, \tag{2.5}$$

$$\omega_c = \left[\frac{2a^2}{m}(U_0 - E)\right]^{1/2}. \tag{2.6}$$

The classical equations of motion are

$$\dot{X}_c = P_c/m, \tag{2.7}$$

$$\dot{P}_c = -2a^2(U_0 - E)X_c = -m\omega_c^2 X_c. \tag{2.8}$$

Comparing Eqs. (II-3.1)–(II-3.8) for the Pöschl-Teller (PT) potential with (2.1)–(2.8), the change from the confining $\tan^2 z$ potential to the $\tanh^2 z$ potential manifests itself in the change of (U_0+E) to (U_0-E). For the RM potential, if E is greater than U_0, the classical particle is unconfined. The equations of motion are still the same, however, and the solutions (2.3) and (2.4) for X_c and P_c become

$$X_c(E > U_0) = \left(\frac{E}{E - U_0}\right)^{1/2}\sinh\left[\frac{2a^2}{m}(E - U_0)\right]^{1/2} t, \tag{2.9}$$

$$P_c(E > U_0) = a[2mE]^{1/2}\cosh\left[\frac{2a^2}{m}(E - U_0)\right]^{1/2} t. \tag{2.10}$$

B. Natural quantum operators and MUCS

The quantum operator analogs of (2.3) and (2.4) are

$$X = \sinh z, \tag{2.11}$$

$$P = \frac{\hbar a^2}{2i}\left(\cosh z\frac{d}{dz} + \frac{d}{dz}\cosh z\right). \tag{2.12}$$

Their equations of motion are

$$\dot{X} = \frac{1}{i\hbar}[X, H] = P/m, \tag{2.13}$$

$$\dot{P} = \frac{1}{i\hbar}[P, H] = -a^2\{U_0 - H - \tfrac{1}{4}\mathcal{E}_0, X\}. \tag{2.14}$$

As for the PT case in paper II, again we have a zero-point contribution $+\frac{1}{4}\mathcal{E}_0$.

Taking the factor $(\hbar a^2)$ out of the definition (2.12) to make P dimensionless,

$$[X, P] = i\cosh^2 z, \tag{2.15}$$

yielding the uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \geq \tfrac{1}{4}\langle\cosh^2 z\rangle^2. \tag{2.16}$$

The normalized states which satisfy the equality in (2.16) are

$$\psi_{\text{MUS}} = N(C,B)\phi_{\text{RM}}(z), \tag{2.17a}$$

$$N(C,B) = \left[\frac{a\Gamma(B + \frac{1}{2} + iu)\Gamma(B + \frac{1}{2} - iu)}{\pi^{1/2}\Gamma(B)\Gamma(B + \frac{1}{2})}\right]^{1/2}, \tag{2.17b}$$

$$\phi_{\text{RM}}(z) = (\cosh z)^{-B}\exp[C\sin^{-1}(\tanh z)], \tag{2.17c}$$

$$B = \frac{1}{2}\left[1 + \frac{\langle\cosh^2 z\rangle}{(\Delta\sinh z)^2}\right], \tag{2.18}$$

$$C \equiv u + iv = B\langle\sinh z\rangle + \left\langle\cosh z\frac{d}{dz}\right\rangle. \tag{2.19}$$

An amusing mathematical resemblance to the Pöschl-Teller minimum-uncertainty states (II-3.15) can be obtained by observing that

$$\exp[C\sin^{-1}(\tanh z)] = \left(\frac{1 - i\sinh z}{1 + i\sinh z}\right)^{iC/2}. \tag{2.20}$$

The bound-state eigenfunctions and eigenenergies of the RM potential are[5,6]

$$\psi_n = N(n,s)\phi_n(z), \quad n \leq [s] \tag{2.21a}$$

$$N(n,s) = \left[\frac{a(s-n)\Gamma(2s-n+1)}{\Gamma(n+1)}\right]^{1/2}, \tag{2.21b}$$

$$\phi_n(z) = P_s^{n-s}(\tanh z), \tag{2.21c}$$

$$E_n = \mathcal{E}_0(2ns - n^2 + s). \tag{2.22}$$

In (2.21), [ ] is the greatest integer function and P_s^{n-s} is the associated Legendre function.[15] [We use the notation $N($, $)$ for all normalization constants, but the context will make clear which is meant.] For $C=0$ the $n=0$ ground state is the special case of the states (2.17) given by $B=s$. Thus, according to the prescription of paper I, our normalized MUCS are

$$\psi_{\text{RM}} = \psi_{\text{MUS}}(B=s) \tag{2.23a}$$

$$= N(C,s)\phi_{\text{RM}}(z, B=s). \tag{2.23b}$$

One can verify that for these coherent states

$$\langle X\rangle = \frac{u}{s - \frac{1}{2}}, \tag{2.24}$$

$$\langle X^2\rangle = \frac{1}{2(s-1)} + \frac{u^2}{(s-1)(s-\frac{1}{2})}, \tag{2.25}$$

$$\langle P\rangle = v, \tag{2.26}$$

$$\langle P^2\rangle = \frac{[(s-\frac{1}{2})^2+u^2]}{2(s-1)} + v^2 , \qquad (2.27)$$

so that

$$(\Delta X)^2 = \frac{[(s-\frac{1}{2})^2+u^2]}{2(s-1)(s-\frac{1}{2})^2} , \qquad (2.28a)$$

$$(\Delta P)^2 = \frac{[(s-\frac{1}{2})^2+u^2]}{2(s-1)} , \qquad (2.28b)$$

$$(\Delta X)^2(\Delta P)^2 = \tfrac{1}{4}\langle \cosh^2 z\rangle^2 . \qquad (2.28c)$$

Also,

$$\langle H\rangle/\mathcal{E}_0 = \left[\frac{s(s+\frac{1}{2})}{(s+\frac{1}{2})^2+u^2}\right](u^2+v^2)+s . \qquad (2.29)$$

A new aspect of this problem is the continuum contribution. The continuum is reached when $n \geqslant s$. This changes the form of the superscript of the associated Legendre function in (2.21). In units of $\mathcal{E}_0$, $(s-n)^2$ is the difference in energy from the bottom of the continuum to the nth eigenstate. Therefore, $|n-s|$ is proportional to the wave number. In the continuum region the superscript becomes complex, and the solutions to the Schrödinger equation are $P_s^{\pm ik}(\tanh z)$, where $k = [(E-U_0)/\mathcal{E}_0]^{1/2}$. This result is physically correct since, from p. 166 of Ref. 15, $P_s^{\pm ik} \sim \exp(\pm ikz)$ as $z \to \pm\infty$.

Thus, the continuum solutions to the problem properly behave as a plane wave at $\pm\infty$, and for large times their time development will be similar to that of a plane wave. Therefore, the MUCS overlap with the bound states will give the long-time coherence properties. With time the continuum contributions will ultimately disperse as plane waves. The pieces of interest are the bound-state contributions, so we will just designate the rest as "continuum."

The MUCS can be decomposed into the number states as

$$\psi_{\rm RM} = \frac{1}{a} N(C,s) \sum_{n=0}^{[s]} N(n,s)\Theta(n,C,s)\psi_n + \text{continuum} \qquad (2.30)$$

$$\Theta(n,C,s) = a\langle \phi_n | \phi_{\rm RM}\rangle , \qquad (2.31)$$

$$\Theta(n=\text{even},C,s) = \frac{\pi}{\Gamma(s+1-n)2^{3s-2n-1}} \sum_{j=0}^{n/2} \frac{(s+\frac{1}{2}-\frac{1}{2}n)_j(-\frac{1}{2}n)_j}{(s+1-n)_j(j!)(2s+2j-n)2^{2j}} \times [B(s+j+\tfrac{1}{2}-\tfrac{1}{2}n-i\tfrac{1}{2}C, s+j+\tfrac{1}{2}-\tfrac{1}{2}n+i\tfrac{1}{2}C)]^{-1} , \qquad (2.32)$$

$$\Theta(n=\text{odd},C,s) = \frac{\pi C}{\Gamma(s+1-n)2^{3s-2n}} \sum_{j=0}^{n/2-1/2} \frac{(s+1-\frac{1}{2}n)_j(-\frac{1}{2}n+\frac{1}{2})_j}{(s+1-n)_j(j!)(2s+2j-n)2^{2j}} \times [(s+j-\tfrac{1}{2}n+i\tfrac{1}{2}C)B(s+j+1-\tfrac{1}{2}n-i\tfrac{1}{2}C, s+j-\tfrac{1}{2}n+i\tfrac{1}{2}C)]^{-1} . \qquad (2.33)$$

The symbol $(a)_k = \Gamma(a+k)/\Gamma(a)$ is Pochhammer's symbol.

C. Natural quantum operators as n-dependent raising and lowering operators

For the RM system, the n-dependent raising and lowering operators are

$$A_n^{\pm} = (\sinh z)(s-n) \mp \cosh z \frac{d}{dz} , \qquad (2.34)$$

with the properties

$$A_n^{\pm}\psi_n = (s-n)D(n-\tfrac{1}{2}\pm\tfrac{1}{2}, s)\psi_{n\pm1} , \qquad (2.35)$$

$$D(n,s) = \left[\frac{(n+1)(2s-n)}{(s-n)(s-n-1)}\right]^{1/2} . \qquad (2.36)$$

Equations (2.34)–(2.36) follow from the standard recursion relations for associated Legendre functions, which can be combined to yield[15]

$$\left[\frac{(s-n)x}{(1-x^2)^{1/2}} \mp (1-x^2)^{1/2}\frac{d}{dx}\right] P_s^{n-s}(x) = \begin{cases} P_s^{n+1-s}(x) & (2.37a) \\ \frac{1}{2}n(2s-n+1)P_s^{n-1-s}(x) . & (2.37b) \end{cases}$$

From (2.34), X and P can be written as

$$X = \frac{1}{4(s-n)} \{[A_n^- + (A_n^+)^\dagger] + [A_n^+ + (A_n^-)^\dagger]\} , \qquad (2.38)$$

$$P = \frac{1}{4i} \{[A_n^- + (A_n^+)^\dagger] - [A_n^+ + (A_n^-)^\dagger]\} . \qquad (2.39)$$

This implies that

$$X\psi_n = \tfrac{1}{2}D(n,s)\psi_{n+1} + \tfrac{1}{2}D(n-1,s)\psi_{n-1} , \qquad (2.40)$$

$$P\psi_n = -\frac{1}{2i}(s-n-\tfrac{1}{2})D(n,s)\psi_{n+1} + \frac{1}{2i}(s-n+\tfrac{1}{2})D(n-1,s)\psi_{n-1} . \qquad (2.41)$$

Again, observe that, comparing the defining equation (I-A4) for the MUCS to (2.34), subject to the appropriate restriction on $\Delta X/\Delta P$, the MUCS can be defined as eigenstates of the ground-state destruction operator A_0^-.

D. Time dependence

The time evolution of the coherent-state wave packets is

$$\rho(x,t)=\Psi^*_{\mathrm{RM}}(x,t)\Psi_{\mathrm{RM}}(x,t), \tag{2.42}$$

$$\Psi_{\mathrm{RM}}(x,t)=e^{-iHt/\hbar}\psi_{\mathrm{RM}}(x)$$
$$=\frac{1}{a}N(c,s)\sum_{n=0}^{[s]}N(n,s)\Theta(n,C,s)\psi_n \times\exp[-i\mathcal{E}_0(2ns-n^2+s)t/\hbar]$$
$$+\text{continuum}\,. \tag{2.43}$$

These packets have been studied numerically for many cases, as will be described in paper V.

Defining h by

$$h\equiv(U_0-H)/\mathcal{E}_0\,, \tag{2.44}$$

the quantum equations of motion (2.13) and (2.14) can be written as

$$[h,X]=q\,, \tag{2.45}$$

$$[h,q]=X(4h-1)+2q\,, \tag{2.46}$$

$$q=\sinh z+2\cosh z\frac{d}{dz}=2iP\,. \tag{2.47}$$

These can then be used as before to obtain $X(t)$ and $P(t)$ as the sums of series of nested commutators as was done in the example in paper II. The results are

$$X(t)\equiv e^{iHt/\hbar}Xe^{-iHt/\hbar}$$
$$=Xe^{-i\omega_0 t}\left[\cos\omega_H t+i\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t\right]$$
$$+Pe^{-i\omega_0 t}2\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t\,, \tag{2.48}$$

$$P(t)=Pe^{-i\omega_0 t}\left[\cos\omega_H t-i\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t\right]$$
$$-Xe^{-i\omega_0 t}(h-\tfrac{1}{4})2\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t\,, \tag{2.49}$$

where ω_0 is defined as

$$\omega_0=\mathcal{E}_0/\hbar\,, \tag{2.50a}$$

and ω_H has the form

$$\omega_H=\left[\frac{2a^2}{m}(U_0-H)\right]^{1/2}=2[\mathcal{E}_0(U_0-H)]^{1/2}/\hbar\,. \tag{2.50b}$$

This is the functional form of the classical angular velocity (2.7), with $E\to H$.

Actually, for all our one-dimensional examples one can similarly calculate other operators, such as $X^2(t)$, by the same techniques. Here $X^2(t)$ turns out to be

$$X^2(t)=\left[X^2-\frac{1}{2}\left(\frac{H+\mathcal{E}_0}{U_0-H-\mathcal{E}_0}\right)\right]\exp(-i4\omega_0 t)\cos(2\omega_H t)$$
$$+\left[\{X,P\}+i(2X^2+1)+\frac{iU_0}{U_0-H-\mathcal{E}_0}\right]\left(\frac{\omega_0}{\omega_H}\right)$$
$$\times\exp(-i4\omega_0 t)\sin(2\omega_H t)+\frac{1}{2}\left(\frac{H+\mathcal{E}_0}{U_0-H-\mathcal{E}_0}\right)\,. \tag{2.51}$$

This expression can be compared to the harmonic-oscillator result (I-2.37) for $x^2(t)$.

The expectation values of the time-dependent operators $X(t)$, $P(t)$, $X^2(t)$, and $P^2(t)$ can be calculated by using the decomposition of the MUCS into eigenstates. The results are listed in Appendix A.

E. Harmonic-oscillator limit

This time, taking the harmonic-oscillator (HO) limit as

$$\lim_{\mathrm{HO}}\equiv\lim\begin{cases}s\to\infty\\ a\to 0\\ sa^2\to m\omega/h\,,\end{cases}$$

all the results of this section again go over to the harmonic-oscillator results discussed in paper I. The techniques for showing this are the same as for the PT case of paper II. As will also be discussed in the numerical results of paper V, the coherence time for our MUCS is increased by decreasing the parameter $\langle H\rangle/U_0$ and by increasing the number of eigenstates that have a significant overlap with the coherent state.

F. n-independent raising and lowering operators and other coherent states

Applied to the RM system, our n-independent raising and lowering operators are obtained by making the transformation

$$(\sinh z)(s-n)\to(\sinh z)h^{1/2}\,, \tag{2.52}$$

so that

$$A_n^\pm\to A^\pm=(\sinh z)h^{1/2}\mp\cosh z\frac{d}{dz}\,. \tag{2.53}$$

For the discrete spectrum one can then write the Hamiltonian as

$$H\to\mathcal{H}=\tfrac{1}{2}\mathcal{E}_0(A^+A^-+A^-A^+)\,, \tag{2.54}$$

since it gives

$$\mathcal{H}|n\rangle = E_n|n\rangle \,. \tag{2.55}$$

The discrete spectrum contribution to the annihilation-operator coherent states (AOCS) can be constructed by demanding that

$$A^-\psi_\alpha = \alpha\psi_\alpha \,, \tag{2.56}$$

$$\psi_\alpha \equiv N_\alpha\left(\sum_{n=0}^{[s]} a_n\psi_n + \text{continuum}\right). \tag{2.57}$$

Equations (2.56) and (2.57) yield

$$\frac{a_{n+1}}{a_n} = \alpha\left[\frac{(s-n)}{(n+1)(2s-n)(s-n-1)}\right]^{1/2}, \tag{2.58}$$

so that the AOCS can be written as

$$\psi_\alpha = N_\alpha\left[\sum_{n=0}^{[s]} \alpha^n\left(\frac{1}{n!}\,\frac{\Gamma(2s)}{\Gamma(2s-n)}\,\frac{s}{s-n}\right)^{1/2}\psi_n + \text{continuum}\right]. \tag{2.59}$$

These states are not as easy to handle analytically as our MUCS, to which they are inequivalent. We will discuss a numerical example of these states in paper V.

By observing that

$$\psi_n = \left[\frac{(s-n)}{n(s-n+1)(2s-n+1)}\right]^{1/2} A^+\psi_{n-1} \tag{2.60a}$$

$$= \left[\frac{(s-n)}{s}\,\frac{\Gamma(2s)}{\Gamma(2s-n)n!}\right]^{1/2}(A^+)^n\psi_0 \,, \tag{2.60b}$$

Eq. (2.59) can be put in the DOCS form

$$\psi_\alpha = N_\alpha\left[\Gamma(2s)\left(\sum_{n=0}^{[s]}\frac{(\alpha A^+)^n}{n!\,\Gamma(2s-n)}\right)\psi_0 + \text{continuum}\right]. \tag{2.61}$$

However, because the sum goes only to $[s]$ and also because the gamma function is $\Gamma(2s-n)$ instead of $\Gamma(2s+n)$, the above DOCS operator cannot be put in the simple modified Bessel function form of Eq. (II-3.68) for the PT-DOCS.

III. ONE-DIMENSIONAL MORSE POTENTIAL

A. Classical motion and the natural classical variables

The one-dimensional Morse (M) potential is

$$V(x) = U_0(1-e^{-z})^2, \quad z \equiv ax \,, \tag{3.1}$$

$$U_0 = \lambda^2\mathcal{E}_0, \quad \mathcal{E}_0 = \frac{\hbar^2 a^2}{2m} \,. \tag{3.2}$$

The natural classical variables which satisfy Eqs. (I-3.2)–(I-3.4) are

$$X_c = e^z - U_0/(U_0-E) = A(E)\sin\omega_c t \,, \tag{3.3}$$

$$P_c = ape^z = m\omega_c A(E)\cos\omega_c t = a\left(\frac{2mU_0E}{U_0-E}\right)^{1/2}\cos\omega_c t \,, \tag{3.4}$$

$$A(E) = \frac{(U_0E)^{1/2}}{U_0-E} \,, \tag{3.5}$$

$$\omega_c = \left[\frac{2a^2(U_0-E)}{m}\right]^{1/2}. \tag{3.6}$$

The classical equations of motion are thus[16]

$$\dot{X}_c = P_c/m \,, \tag{3.7}$$

$$\dot{P}_c = -2a^2(U_0-E)X_c = -m\omega_c^2 X_c \,. \tag{3.8}$$

As with the RM potential, when $E > U_0$ the equations of motion remain the same, but the sin and cos in (3.3) and (3.4) become sinh and cosh of $\{t[2a^2(E-U_0)/m]^{1/2}\}$. This represents the motion of a particle which is free to travel to the right, but is confined at the left by the exponentially rising potential.

B. Natural quantum operators and minimum-uncertainty coherent states

The Morse potential is a more complicated system to discuss since X_c depends upon E. Therefore, when going to the quantum system, one makes the transformation $E \to$ (H plus some zero-point contribution). This zero-point contribution turns out to be $\mathcal{E}_0/4$, as we will see in Eq. (3.42). For now we simply give the natural quantum operators

$$X = e^z - \frac{U_0}{U_0 - H - \mathcal{E}_0/4} \,, \tag{3.9}$$

$$P = \frac{1}{2i}\left(e^z\frac{d}{dz} + \frac{d}{dz}e^z\right)(\hbar a^2) \,. \tag{3.10}$$

Their equations of motion are

$$\dot{X} = \frac{1}{i\hbar}[X, H] = P/m \,, \tag{3.11}$$

$$\dot{P} = -a^2\left(\{U_0 - H - \mathcal{E}_0/4, X\} + \frac{\mathcal{E}_0/2}{U_0 - H - \mathcal{E}_0/4}\right). \tag{3.12}$$

Again dropping $(\hbar a^2)$ in P, we observe that since X functionally involves H in a denominator, the operator G in the commutator

$$[X, P] = iG \tag{3.13}$$

is not easy to calculate analytically. Since this operator is needed to calculate the exact MUCS, we revert to an approximate analytic technique. This technique[12] is to consider the operators P and

$$\overline{X} = e^z - \frac{U_0}{U_0 - \overline{E} - \mathcal{E}_0/4} \,, \tag{3.14}$$

where $\overline{E} = \langle H\rangle$. This expectation value of the Hamiltonian in the approximate MUCS we obtain below.

With

$$[\overline{X}, P] = ie^{2z} \equiv i\overline{G} \tag{3.15}$$

implying the uncertainty relation

$$(\Delta \overline{X})^2(\Delta P)^2 \geq \tfrac{1}{4}\langle \overline{G}\rangle^2 , \tag{3.16}$$

the MUS that satisfy the equality in (3.16) are

$$\psi_{\text{MUS}} = \hat{N}(C,B)\hat{\phi}_{\text{M}}(z) , \tag{3.17}$$

$$\hat{\phi}_{\text{M}}(z) = e^{-(B+\frac{1}{2})z}\exp(-Ce^{-z}) , \tag{3.18}$$

$$B = \frac{\langle e^{2z}\rangle}{2(\Delta\overline{X})^2} , \tag{3.19}$$

$$C = B\langle e^{z}\rangle + i\langle P\rangle \equiv u + iv . \tag{3.20}$$

The normalized number eigenstates and eigenenergies for the Morse potential are[11,12]

$$\psi_n = N(n,\lambda)\phi_n(y) , \quad 0 \leq n \leq [\lambda - \tfrac{1}{2}] \tag{3.21a}$$

$$= \left[\frac{a(2\lambda - 2n - 1)\Gamma(n+1)}{\Gamma(2\lambda - n)}\right]^{1/2} \times y^{\lambda-1/2-n}e^{-y/2}L_n^{(2\lambda-2n-1)}(y) , \tag{3.21b}$$

$$y \equiv 2\lambda\, e^{-z} , \tag{3.22}$$

$$E_n = \mathcal{E}_0[2\lambda(n+\tfrac{1}{2}) - (n+\tfrac{1}{2})^2] , \tag{3.23}$$

where $L_n^{(\alpha)}$ are the generalized Laguerre polynomials.[15] Comparing (3.21) to (3.18), one sees that for the MUS to contain the ground state as a special case, one needs the special value for the complex parameter $C = B\lambda/(\lambda-1)$ and one has to restrict the parameter B to be $(\lambda - 1)$. Changing variables to y,[16] and with $\hat{N}(C,\lambda-1) \equiv \overline{N}(C,\lambda) = N(C,\lambda)(2\lambda)^{\lambda-1/2}$, our MUCS are

$$\psi_{\text{M}} = N(C,\lambda)\phi_M , \tag{3.24}$$

$$\phi_M = y^{\lambda-1/2}\exp(-\tfrac{1}{2}Cy/\lambda) , \tag{3.25a}$$

$$N(C,\lambda) = a^{1/2}[u/\lambda]^{\lambda-1/2}/[\Gamma(2\lambda-1)]^{1/2} , \tag{3.25b}$$

$$C \equiv (\lambda-1)\langle 2\lambda/y\rangle + i\langle P\rangle \equiv u + iv . \tag{3.26}$$

One can verify that

$$\langle \overline{X}\rangle = \frac{u}{\lambda - 1} , \tag{3.27}$$

$$\langle \overline{X}^2\rangle = \frac{u^2}{(\lambda-1)(\lambda-3/2)} , \tag{3.28}$$

$$\langle P\rangle = v , \tag{3.29}$$

$$\langle P^2\rangle = \frac{u^2}{2(\lambda-\frac{3}{2})} + v^2 , \tag{3.30}$$

$$(\Delta\overline{X})^2(\Delta P)^2 = \left[\frac{u^2}{2(\lambda-1)^2(\lambda-\frac{3}{2})}\right]\left[\frac{u^2}{2(\lambda-\frac{3}{2})}\right] = \tfrac{1}{4}\langle\overline{G}\rangle^2 . \tag{3.31}$$

The Hamiltonian expectation value in these states, which also gives $\overline{E}$, is

$$\langle H\rangle/\mathcal{E}_0 \equiv \overline{E}/\mathcal{E}_0 = \lambda\left(\lambda - \frac{1}{2}\right)\left[\frac{v^2}{u^2} + \left(\frac{\lambda}{u} - 1\right)^2\right] + (\lambda - \tfrac{1}{4}) . \tag{3.32}$$

For the ground state $C = u + iv = \lambda$, yielding the ground-state energy $\mathcal{E}_0(\lambda - \frac{1}{4})$.

The MUCS can be decomposed into number states as

$$\psi_{\text{M}} = \frac{1}{a}N(C,\lambda)\sum_{n=0}^{[\lambda-1/2]} N(n,\lambda)\Theta(n,C,\lambda)\psi_n + \text{continuum} , \tag{3.33}$$

$$\Theta(n,C,\lambda) = a\langle\phi_n|\phi_M\rangle = \sum_{j=0}^{n}\frac{(-1)^j}{j!}\binom{2\lambda-n-1}{n-j}\left(\frac{2}{1+C/\lambda}\right)^{2\lambda-n-1+j} \times \Gamma(2\lambda - n - 1 + j) . \tag{3.34}$$

C. Natural quantum operators as n-dependent raising and lowering operators

For the Morse system, the n-dependent raising and lowering operators are[12]

$$A_n^{\pm} = \frac{1}{2\lambda}\left[e^z(\lambda - n - \tfrac{1}{2}) \mp e^z\frac{d}{dz} - \frac{\lambda^2}{\lambda - (n+\frac{1}{2}) \mp \frac{1}{2}}\right] \tag{3.35}$$

$$= \left[\frac{\lambda - n - \frac{1}{2}}{y} \pm \frac{d}{dy} - \frac{\lambda}{2[\lambda - (n+\frac{1}{2}) \mp \frac{1}{2}]}\right] , \tag{3.36}$$

with the properties

$$A_n^{\pm}\psi_n = \frac{(\lambda - n - \frac{1}{2})}{2\lambda}D(n - \tfrac{1}{2} \pm \tfrac{1}{2}, \lambda)\psi_{n\pm1} , \tag{3.37}$$

$$D(n,\lambda) = \left(\frac{\lambda}{\lambda - n - 1}\right) \times \left[\frac{(n+1)(2\lambda - n - 1)}{(\lambda - n - \frac{1}{2})(\lambda - n - \frac{3}{2})}\right]^{1/2} . \tag{3.38}$$

Equations (3.35)–(3.38) follow from the standard recurrence relations for generalized Laguerre polynomials, which can be combined to yield[15]

$$(n+1)(n+\alpha)L_{n+1}^{(\alpha-2)}(t) = [\alpha(\alpha-1) - t(\alpha+n)]L_n^{(\alpha)}(t) + (\alpha-1)t\left(\frac{d}{dt}L_n^{(\alpha)}(t)\right) , \tag{3.39}$$

$$tL_{n-1}^{(\alpha+2)}(t) = -nL_n^{(\alpha)}(t) - (\alpha+1)\left(\frac{d}{dt}L_n^{(\alpha)}(t)\right) . \tag{3.40}$$

Calculating the natural quantum operators in terms of the n-dependent raising and lowering operators,

$$X_n = \frac{(2\lambda)}{4(\lambda - n - \frac{1}{2})}\{[A_n^- + (A_n^+)^\dagger] + [A_n^+ + (A_n^-)^\dagger]\} \tag{3.41}$$

$$= e^z - \frac{U_0}{U_0 - E_n - \mathcal{E}_0/4} , \tag{3.42}$$

$$P = \frac{(2\lambda)}{4i}\{[A_n^- + (A_n^+)^\dagger] - [A_n^+ + (A_n^-)^\dagger]\} . \tag{3.43}$$

Equation (3.42) shows why the zero-point contribution $\mathcal{E}_0/4$ was chosen in the definition (3.9) of X. It is only with this choice that one obtains agreement between (3.9) and the raising and lowering operator construction of X_n in (3.41).

Using the forms (3.41) and (3.43) for X and P, one has

$$X\psi_n=\tfrac{1}{2}D(n,\lambda)\psi_{n+1}+\tfrac{1}{2}D(n-1,\lambda)\psi_{n-1}\,, \tag{3.44}$$

$$P\psi_n=\frac{-1}{2i}(\lambda-n-1)D(n,\lambda)\psi_{n+1}+\frac{1}{2i}(\lambda-n)D(n-1,\lambda)\psi_{n-1}\,. \tag{3.45}$$

The ground-state destruction operator A_0^- can again be used as a basis for defining the approximate MUCS, with the proper value for $\Delta\overline{X}/\Delta P$.

D. Time dependence

The time evolution of the coherent-state wave packets is

$$\rho(x,t)=\Psi_M^*(x,t)\Psi_M(x,t)\,, \tag{3.46}$$

$$\begin{aligned}\Psi_M(x,t)&=e^{-iHt/\hbar}\psi_M(x)\\&=\frac{1}{a}N(C,\lambda)\sum_{n=0}^{[\lambda-1/2]}N(n,\lambda)\Theta(n,C,\lambda)\psi_n\\&\quad\times\exp\{-i\mathcal{E}_0[2\lambda(n+\tfrac{1}{2})-(n+\tfrac{1}{2})^2]\}\\&\quad+\text{continuum}\,.\end{aligned} \tag{3.47}$$

These packets have also been studied numerically for many cases, as will be described in paper V.

Defining h to be

$$h=(U_0-H)/\mathcal{E}_0\,, \tag{3.48}$$

the quantum equations of motion (3.11) and (3.12) can be written as

$$[h,X]=q\,, \tag{3.49}$$

$$[h,q]=X(4h-1)+2q\,, \tag{3.50}$$

$$q=2iP\,. \tag{3.51}$$

As before, these equations can be used to solve the nested sums of commutators for the time-dependent operators $X(t)$ and $P(t)$. The results are

$$X(t)=Xe^{-i\omega_0t}\left[\cos\omega_Ht+i\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_Ht\right]+Pe^{-i\omega_0t}\left[2\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_Ht\right], \tag{3.52}$$

$$P(t)=Pe^{-i\omega_0t}\left[\cos\omega_Ht-i\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_Ht\right]-Xe^{-i\omega_0t}\left[(h-\tfrac{1}{4})2\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_Ht\right], \tag{3.53}$$

$$\omega_0=\mathcal{E}_0/\hbar\,, \tag{3.54}$$

$$\omega_H=\frac{2\mathcal{E}_0}{\hbar}\left[\frac{U_0-H}{\mathcal{E}_0}\right]^{1/2}=\left[\frac{2a^2}{m}(U_0-H)\right]^{1/2}. \tag{3.55}$$

We emphasize that if the definition (3.9) for X had not had the zero-point factor $\mathcal{E}_0/4$ in it, then the result (3.52) would not have had the now standard form. This is another verification that the insertion of this zero-point factor in Eq. (3.9) is physically correct.

The expectation values of the time-dependent operators $X(t)$, $P(t)$, $X^2(t)$, and $P^2(t)$ in the MUCS are given in Appendix A.

E. Harmonic-oscillator limit

As discussed in Ref. 12, in the limit defined by Eq. (II-3.53), all our results once again go over to the harmonic-oscillator results given in paper I. The techniques for showing this are similar to the PT case of paper II, except that one uses the relation

$$H_n(x/\sqrt{2})=(-1)^n2^{n/2}\Gamma(n+1)\lim_{\alpha\to\infty}[\alpha^{-n/2}L_n^{(\alpha)}(\sqrt{\alpha}x+\alpha)] \tag{3.56}$$

on p. 251 of Ref. 15 to show that the generalized Laguerre polynomial eigenfunctions go over to the harmonic-oscillator eigenfunctions.

Once again, as will be discussed in the numerical results of paper V, the coherence time for these approximate MUCS is increased by decreasing the parameter $\langle H\rangle/U_0$ and by increasing the number of eigenstates that have a significant overlap with the coherent state.

F. n-independent raising and lowering operators and other coherent states

The Morse potential provides an interesting example of how the $A_n^\pm$ operators used to provide the "simplest" X_n in Eq. (3.42) need not be the "simplest" operators to use to construct AOCS from the n-independent raising and lowering operators.

In particular, multiplying Eq. (3.36) for $A_n^\pm$ by the denominator of the last term, we have

$$\mathcal{A}_n^\pm\equiv A_n^\pm 2[\lambda-(n+\tfrac{1}{2})\mp\tfrac{1}{2}]\,, \tag{3.57}$$

yielding

$$\mathcal{A}_n^\pm\to\mathcal{A}^\pm=\frac{2}{y}(h\mp\tfrac{1}{2}h^{1/2})\pm2\frac{d}{dy}h^{1/2}-\frac{d}{dy}-\lambda\,. \tag{3.58}$$

One chooses these n-independent operators because

$$H\to\mathcal{H}=\tfrac{1}{2}\mathcal{E}_0(\mathcal{A}^+\mathcal{A}^-+\mathcal{A}^-\mathcal{A}^++\tfrac{1}{2})\,, \tag{3.59}$$

$$\mathcal{H}|n\rangle = E_n|n\rangle . \qquad (3.60)$$

Note that using the $\mathcal{A}_n^{\pm}$ on the ψ_n amounts to using the standard[15] raising and lowering operators on the generalized Laguerre polynomials.

Employing $\mathcal{A}_n^{\pm}$, one defines the AOCS and DOCS by proceeding exactly as in the last section for the RM case, except that everywhere $(\lambda - \frac{1}{2})$ is substituted for s. Writing out the results,

$$A^- \psi_\alpha = \alpha \psi_\alpha , \qquad (3.61)$$

$$\psi_\alpha = N_\alpha \left[\sum_{n=0}^{[\lambda-1/2]} \alpha^n \left(\frac{1}{n!} \frac{\Gamma(2\lambda-1)}{\Gamma(2\lambda-1-n)} \frac{\lambda-\frac{1}{2}}{\lambda-\frac{1}{2}-n} \right)^{1/2} \psi_n + \text{continuum} \right], \qquad (3.62)$$

which gives the DOCS form

$$\psi_\alpha = N_\alpha \left[\Gamma(2\lambda-1) \left(\sum_{n=0}^{[\lambda-1/2]} \frac{(\alpha A^+)^n}{n!\,\Gamma(2\lambda-1-n)} \right) \psi_0 + \text{continuum} \right]. \qquad (3.63)$$

The same comments hold as for the RM case that the finite sum in (3.63) does not yield the closed form Bessel function for the DOCS operator as in the PT case.

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APPENDIX A. TIME-DEPENDENT EXPECTATION VALUES FOR THE PÖSCHL-TELLER, ROSEN-MORSE AND MORSE COHERENT STATES

Let

$$\gamma = (\lambda \text{ or } s), \qquad (A1)$$

where λ and s are the strength parameters explicitly defined in Eqs. (II-2.2), (2.2), and (3.2) for the PT, RM, and Morse potentials, respectively. Then the expectation values $\langle X(t)\rangle$, $\langle P(t)\rangle$, $\langle X^2(t)\rangle$, and $\langle P^2(t)\rangle$ in the MUCS can all be written in the same forms. Noting that for the RM and Morse potentials the sums are finite (up to $n = [s-1]$ and $n = [\lambda - \frac{3}{2}]$ for $\langle X(t)\rangle$ and $\langle P(t)\rangle$, for example) and that for these potentials we are leaving out the continuum contributions, we have

$$\langle X(t)\rangle = \frac{|N(C,\gamma)|^2}{2a^2} \sum_{n=0} N(n+1,\gamma) N(n,\gamma) D(n,\gamma) \times \left[\mathcal{O}^*(n+1,C,\gamma)\mathcal{O}(n,C,\gamma) \times \exp(i2b_n\omega_0 t) + (\text{H.c.}) \right], \qquad (A2)$$

$$\langle P(t)\rangle = \frac{i|N(C,\gamma)|^2}{2a^2} \sum_{n=0} b_n N(n+1,\gamma) N(n,\gamma) D(n,\gamma) \times \left[\mathcal{O}(n+1,C,\gamma)\mathcal{O}(n,C,\gamma) \times \exp(i2b_n\omega_0 t) - (\text{H.c.}) \right], \qquad (A3)$$

$$\langle X^2(t)\rangle = \frac{|N(C,\gamma)|^2}{4a^2} \sum_n Y_n^* Y_n , \qquad (A4)$$

$$Y_n = N(n-1,\gamma)\mathcal{O}(n-1,C,\gamma)D(n-1,\gamma)\exp(i2b_n\omega_0 t) + N(n+1,\gamma)\mathcal{O}(n+1,C,\gamma)D(n,\gamma)\exp(-i2g_n\omega_0 t), \qquad (A5)$$

$$\langle P^2(t)\rangle = \frac{|N(C,\gamma)|^2}{4a^2} \sum_{n=0} Z_n^* Z_n , \qquad (A6)$$

$$Z_n = g_n N(n-1,\gamma)\mathcal{O}(n-1,C,\gamma)D(n-1,\gamma)\exp(i2b_n\omega_0 t) - b_n N(n+1,\gamma)\mathcal{O}(n+1,C,\gamma)D(n,\gamma)\exp(-i2g_n\omega_0 t), \qquad (A7)$$

where

$$b_n = \begin{cases} \lambda + n + \frac{1}{2}, & \text{PT} \\ s - n - \frac{1}{2}, & \text{RM} \\ \lambda - n - 1, & \text{M} \end{cases} \qquad (A8)$$

and

$$g_n = \begin{cases} \lambda + n - \frac{1}{2} = b_n - 1, & \text{PT} \\ s - n + \frac{1}{2} = b_n + 1, & \text{RM} \\ \lambda - n = b_n + 1, & \text{M}. \end{cases} \qquad (A9)$$

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[16]The factor 2 in Eq. (3.8) was unfortunately dropped in the type setting of the proofs in Eq. (4.6) of Ref. 12. Also, below Eq. (2.7) of Ref. 12 one should have $(-s^2) = \epsilon_n - \lambda^2$. Finally, $N(C,\lambda)$ was given two meanings in Ref. 12. From Eq. (4.21) to one line below Eq. (4.23) it has the meaning of the present $\bar{N}(C,\lambda)$ used in this paper.