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Coherent states for general potentials. II. Confining one-dimensional examples

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We apply our minimum-uncertainty coherent-states formalism, which is physically motivated by the classical motion, to two confining one-dimensional systems: the harmonic oscillator with centripetal barrier and the symmetric Pöschl-Teller potentials. The minimum-uncertainty coherent states are discussed in great detail, and the connections to annihilation-operator coherent states and displacement-operator coherent states are given. The first system discussed provides an excellent bridge between the harmonic oscillator and more general potentials because, even though it is a nonharmonic potential, its energy eigenvalues are equally spaced. Thus, its coherent states have many, but not all, of the properties of the harmonic-oscillator coherent states.

I. INTRODUCTION

In paper I of this series,[1] we presented the classical motion physical basis and the formalism of our minimum-uncertainty coherent states (MUCS) method for obtaining coherent states for general potentials.[2] These states are a particular subset of the minimum-uncertainty states for generalized X and P operators, rather than for the usual x-p operators.[3] Here we apply our formalism to two confining systems and exhibit the results in detail.

Our first example, discussed in Sec. II, provides an illuminating bridge between the one-dimensional harmonic oscillator and more general one-dimensional potentials. The potential

$$V(x)=U_0\left(ax-\frac{1}{ax}\right)^2,\ x\geq 0 \tag{1.1}$$

has equally spaced eigenvalues, just as does the harmonic oscillator. But because this potential is not symmetric about its minimum at $x=1/a$, the coherence properties are not as complete. The eigenvalues are equally spaced because by changes of variables this problem can be transformed into the eigenvalue problem for the wave function $\chi_{nl}=(R_{nl}r)$ of the three-dimensional harmonic oscillator, with l not restricted to be an integer. For this reason we call Eq. (1.1) the harmonic oscillator with centripetal barrier (HOCB) potential. This potential has many of the features of the harmonic-oscillator system, but not all of them. The similarities and differences to the harmonic oscillator provide deep insights into the coherent states for more general potentials.

In Sec. III we study the symmetric Pöschl-Teller (PT) potential,[4,5]

$$V(x)=U_0\tan^2 ax\,. \tag{1.2}$$

This is a confining potential contained within an infinite square well. It has a countable number of bound states whose eigenenergies are proportional to a quadratic function of n.

For both of the above potentials we discuss a number of properties: the classical motion and the natural classical variables; the quantum problem leading to the natural quantum operators and the minimum-uncertainty coherent states; the relation of the above to the n-dependent raising and lowering operators; the time dependence of the minimum-uncertainty coherent states (MUCS) and the natural quantum operators; the limit in which these systems reduce to the harmonic oscillator; and n-independent raising and lowering operators leading to other[6-8] coherent states. These other, generally inequivalent states, are the annihilation-operator coherent states (AOCS) and the displacement-operator coherent states (DOCS).

The HOCB system is an example in which the classical variable X_c is energy dependent. Therefore, the quantum operator X is Hamiltonian dependent. Even so, one can solve the problem exactly. We also find n-independent raising and lowering operators $A^{\pm}$ that can be used to define annihilation-operator or displacement-operator coherent states equivalent to the MUCS. But these states are *not* equivalent to Perelomov DOCS,[7,8] defined by a unitary exponential operator. The $A^{\pm}$ are not the "simplest" in the sense that the Hamiltonian cannot be written as ($A^{\pm}\neq \mathfrak{A}^{\pm}$)

$$\mathcal{H}=\text{const}\times(\mathfrak{A}^+\mathfrak{A}^-+\mathfrak{A}^-\mathfrak{A}^++\text{const})\,. \tag{1.3}$$

But related operators $\mathfrak{A}^{\pm}$ do satisfy Eq. (1.3), and these $\mathfrak{A}^{\pm}$ can be used to define AOCS-DOCS that are of the unitary-exponential-operator DOCS form.

For the Pöschl-Teller system, on the other hand, the "simplest" MUCS and the "simplest" AOCS-DOCS are obtained from the same n-independent raising and lowering operators, but are not equivalent. Further, these DOCS do not have the unitary-exponential-operator DOCS form.

The differences in the above two cases occur because the HOCB eigenvalues are equally spaced and those for the PT system are not. Further explanation of this is given in the relevant subsections of the text.

The discussion that follows relies heavily on the ideas and formalism developed[1] in paper I of this series. Equations from that paper are denoted by Eq. (I-x.x).

II. HARMONIC OSCILLATOR WITH "CENTRIPETAL BARRIER"

A. Classical motion and the natural classical variables

The potential that we term the harmonic oscillator with "centripetal barrier" (HOCB) can be written as

$$V(x) = U_0\left(\frac{1}{z} - z\right)^2, \quad z = ax, \tag{2.1}$$

$$U_0 = \mathcal{E}_0\nu^2 \equiv \mathcal{E}_0\lambda(\lambda+1), \quad \mathcal{E}_0 = \hbar^2a^2/2m, \tag{2.2}$$

where the definitions $\mathcal{E}_0$, ν, and λ are useful for the quantum problem.

The origin of our terminology can be seen as follows. In the three-dimensional classical Hamiltonian for the harmonic oscillator, interpret r and p_r as one-dimensional variables x and p. Let $U_0^2 = \omega^2L^2/2$ and $a^2 = m\omega^2/U_0$, where L is the angular momentum. Then, apart from a shift in the zero of energy, the effective potential is given by Eq. (2.1).

The natural classical variables that satisfy Eqs. (I-3.2) to (I-3.4) for this problem are

$$X_c = z^2 - \left(1 + \frac{E}{2U_0}\right) = A(E)\sin\omega_c t, \tag{2.3}$$

and

$$P_c = 2apz = m\omega_c A(E)\cos\omega_c t$$
$$= 2a\left[2mE\left(1 + \frac{E}{4U_0}\right)\right]^{1/2}\cos\omega_c t, \tag{2.4}$$

where

$$A(E) = \left[\frac{E}{U_0}\left(1 + \frac{E}{4U_0}\right)\right]^{1/2} \tag{2.5}$$

and

$$\omega_c = \left(\frac{8U_0a^2}{m}\right)^{1/2} = 4\mathcal{E}_0\nu/\hbar. \tag{2.6}$$

As for the harmonic oscillator, ω_c is independent of energy. The classical equations of motion are

$$\dot{X}_c = P_c/m, \tag{2.7}$$

$$\dot{P}_c = -8U_0a^2X = -m\omega_c^2X. \tag{2.8}$$

B. The quantum eigenvalue problem

This example is of special interest because it is a one-dimensional nonharmonic potential with equally spaced eigenvalues and also because it gives us a bridge between one-dimensional systems and multidimensional systems. The reason for both statements is that the one-dimensional Schrödinger equation for the potential of Eq. (2.1) can be mapped into the radial equation for the quantity

$$\chi_{Nl} \equiv rR_{Nl} \tag{2.9}$$

of the three-dimensional harmonic oscillator by a transformation similar to that used in the classical case. The standard radial solutions of the three-dimensional oscillator are given as[9]

$$R_{Nl} = \left(\frac{2a_0^3\Gamma(\tfrac{1}{2}N - \tfrac{1}{2}l + 1)}{\Gamma(\tfrac{1}{2}N + \tfrac{1}{2}l + \tfrac{3}{2})}\right)^{1/2} \times \rho^l e^{-\rho^2/2} L_{N/2-l/2}^{(l+1/2)}(\rho^2), \tag{2.10}$$

$$\rho = r\left(\frac{m\omega}{\hbar}\right)^{1/2} \equiv ra_0, \tag{2.11}$$

$$E_N = \hbar\omega(N + \tfrac{3}{2}), \tag{2.12}$$

$$N = l, l+2, l+4, \ldots . \tag{2.13}$$

The mapping that connects the two problems is

$$l \to \lambda, \tag{2.14}$$

$$\left(\frac{N-l}{2}\right) \to n, \tag{2.15}$$

$$\left(\frac{m\omega}{\hbar}\right) r^2 \to y, \tag{2.16a}$$

$$y \equiv \nu z^2 = [\lambda(\lambda+1)]^{1/2}z^2, \tag{2.16b}$$

$$\psi_n = N(n,\lambda)\phi_n \tag{2.17a}$$

$$= \left(\frac{2a\nu^{1/2}\Gamma(n+1)}{\Gamma(\lambda + \tfrac{3}{2} + n)}\right)^{1/2} \times e^{-y/2}y^{(\lambda+1)/2}L_n^{(\lambda+1/2)}(y), \tag{2.17b}$$

$$E_n = \mathcal{E}_0[\nu(4n + 2\lambda + 3) - 2\nu^2], \quad n = 0, 1, 2, \ldots . \tag{2.18}$$

Now a number of interesting observations can be made. First, this is a system where the levels are equally spaced, as for the standard harmonic oscillator. As has already been pointed out, the reason is that the potential of Eq. (2.1) represents a harmonic oscillator, but in three dimensions. This is also the reason the classical ω_c is independent of energy.

Next, the quantum condition for the three-dimensional radial equation is that $(N-l)/2$ be an integer. However, for the three-dimensional problem l is an integer, so N must be an integer. In the one-dimensional interpretation λ (the substitute for l) need not be an integer, but the quantization condition that $(N-\lambda)/2$ be an integer remains.

Finally, we make an important point on the raising and lowering operators

$$A_n^{\pm}=\frac{1}{2}y\mp y\frac{d}{dy}-\frac{1}{2}(2n+\lambda+\tfrac{3}{2}\pm\tfrac{1}{2}) \tag{2.19a}$$

$$=\tfrac{1}{2}\nu z^2\mp\frac{1}{2}z\frac{d}{dz}-\frac{1}{2}(2n+\lambda+\tfrac{3}{2}\pm\tfrac{1}{2}) \tag{2.19b}$$

with the properties

$$A_n^{\pm}\psi_n=-D(n-\tfrac{1}{2}\pm\tfrac{1}{2},\lambda)\psi_{n\pm1}, \tag{2.20}$$

$$D(n,\lambda)=[(n+1)(n+\tfrac{3}{2}+\lambda)]^{1/2}. \tag{2.21}$$

One naively would hope that these operators would also be appropriate for the three-dimensional radial problem. However, this is not correct, as will be discussed in paper IV. The operators of (2.19) raise and lower $n=(N-l)/2$ by one unit in the function χ_{Nl}, but have a very different effect on the radial wave function R_{Nl}. We will see in paper IV that the appropriate raising and lowering operators, those that yield X and P operators that vary properly between apsidal distances, are the "l" raising and lowering operators.

C. Natural quantum operators and MUCS

From the natural classical variables (2.3) and (2.4), we obtain the natural quantum operators

$$X=z^2-\left(1+\frac{H}{2U_0}\right), \tag{2.22}$$

and

$$P=\frac{\hbar a^2}{i}\left(z\frac{d}{dz}+\frac{d}{dz}z\right). \tag{2.23}$$

Now observe that, in this equally spaced eigenenergy system, $(A_n^+)^\dagger=A_n^-$. Therefore,

$$X_n=z^2-\left(1+\frac{E_n}{2U_0}\right)=\frac{1}{\nu}(A_n^-+A_n^+), \tag{2.24a}$$

and

$$P=\frac{2}{i}(A_n^--A_n^+)(\hbar a^2). \tag{2.24b}$$

Thus, our natural quantum operators are in agreement with those obtained from the raising and lowering operators.

Even though X is Hamiltonian-dependent we can still proceed to solve the problem exactly. The quantum equations of motion are

$$\dot{X}=\frac{1}{i\hbar}[X,H]=P/m, \tag{2.25}$$

$$\dot{P}=\frac{1}{i\hbar}[P,H]=-8U_0a^2X$$

$$=-m\omega_c{}^2X. \tag{2.26}$$

In the remaining discussion we drop $(\hbar a^2)$ in P. We have

$$[X,P]=iG, \tag{2.27}$$

$$G=4\left(1+\frac{H}{2U_0}\right). \tag{2.28}$$

From Appendix A of paper I, the minimum-uncertainty states (MUS) are the solutions to the eigenvalue equation

$$\left(X+\frac{i\langle G\rangle}{2(\Delta P)^2}P\right)\psi_{\text{MUS}}=C\psi_{\text{MUS}}, \tag{2.29}$$

$$C\equiv u+iv=\langle X\rangle+\frac{i\langle G\rangle\langle P\rangle}{2(\Delta P)^2}. \tag{2.30}$$

These states give

$$(\Delta X)^2(\Delta P)^2=\tfrac{1}{4}\langle G\rangle^2. \tag{2.31}$$

One can verify that for the ground state

$$\frac{\langle G\rangle}{2(\Delta P)^2}\equiv\frac{\Delta X}{\Delta P}=\frac{1}{2\nu}, \tag{2.32}$$

and that the ground state is a minimum-uncertainty state with $C=0$. According to the criterion established in I, we demand that our MUCS satisfy Eq. (2.29) subject to Eq. (2.32); that is, these states are the solutions to

$$0=\left[y\frac{d^2}{dy^2}+(\tfrac{1}{2}+y)\frac{d}{dy}+\frac{1}{4}\left(y-\frac{\nu^2}{y}+1\right)-\frac{\nu}{2}C\right]\psi_{\text{HOCB}}. \tag{2.33}$$

Taking the sample solution

$$\psi_{\text{HOCB}}=N\phi_0 g=Ne^{-y/2}y^{(\lambda+1)/2}g \tag{2.34}$$

gives the equation for g

$$0=yg''+(\lambda+\tfrac{3}{2})g'-(\nu/2)Cg. \tag{2.35}$$

A trial power-series solution yields the result

$$g=\sum_{n=0}^{\infty}\frac{(\nu C/2)^n y^n}{n!\,(\lambda+\frac{3}{2})_n}. \tag{2.36}$$

Using the definition of the modified Bessel function $I_\rho(y)$ given below in Eq. (3.66), $y^{(\lambda+1/2)/2}g$ is proportional to $I_{\lambda+1/2}(2(\nu Cy/2)^{1/2})$. Therefore, the normalized MUCS are

$$\psi_{\text{HOCB}}=N(C,\lambda)\phi_{\text{HOCB}}, \tag{2.37}$$

$$\phi_{\text{HOCB}}=e^{-y/2}y^{1/4}I_{\lambda+1/2}(2(\nu Cy/2)^{1/2}), \tag{2.38}$$

$$N(C,\lambda)=[2a\nu^{1/2}]^{1/2}e^{-\nu u/2}[I_{\lambda+1/2}(\nu|C|)]^{-1/2}. \tag{2.39}$$

One of the standard generating functions for the Laguerre polynomials is

$$e^z(zx)^{-\alpha/2}J_\alpha(2(zx)^{1/2})=\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)}{\Gamma(\alpha+n+1)}z^n. \tag{2.40}$$

Changing the J_α to an I_α allows our MUCS to be decomposed into the number states as

$$\psi_{\mathrm{HOCB}} = \left(\frac{e^{i\nu v}(\beta)^{\lambda+1/2}}{I_{\lambda+1/2}(2|\beta|)^{1/2}}\right)^{1/2} \times \sum_{n=0}^{\infty} \frac{(-\beta)^n \psi_n}{[\Gamma(n+1)\Gamma(\lambda+\frac{3}{2}+n)]^{1/2}}, \tag{2.41}$$

where

$$\beta \equiv \nu C/2\,. \tag{2.42}$$

This decomposition, and the relations

$$\nu X\psi_n = -[n(n+\tfrac{1}{2}+\lambda)]^{1/2}\psi_{n-1} - [(n+1)(n+\tfrac{3}{2}+\lambda)]^{1/2}\psi_{n+1}\,, \tag{2.43}$$

$$(i/2)P\psi_n = -[n(n+\tfrac{1}{2}+\lambda)]^{1/2}\psi_{n-1} + [(n+1)(n+\tfrac{3}{2}+\lambda)]^{1/2}\psi_{n+1}\,, \tag{2.44}$$

obtained from Eq. (2.24), allow expectation values of interest to be calculated. In particular, defining

$$R = \frac{I_{\lambda+3/2}(2|\beta|)}{I_{\lambda+1/2}(2|\beta|)}, \tag{2.45}$$

$$\kappa = (\lambda+\tfrac{3}{2}) + 2|\beta|R\,, \tag{2.46}$$

one has

$$\frac{\langle H\rangle}{\mathcal{E}_0\nu} = (2\lambda+3-2\nu) + 4|\beta|R\,, \tag{2.47}$$

$$\langle X\rangle = u\,, \tag{2.48}$$

$$\langle X^2\rangle = \frac{1}{\nu^2}\kappa + u^2\,, \tag{2.49}$$

$$\langle P\rangle = 2\nu v\,, \tag{2.50}$$

$$\langle P^2\rangle = 4\nu^2 v^2 + 4\kappa\,, \tag{2.51}$$

so that

$$(\Delta X)^2(\Delta P)^2 = \tfrac{1}{4}\langle G\rangle^2\,, \tag{2.52}$$

$$\langle G\rangle = \frac{4}{\nu}\kappa\,. \tag{2.53}$$

D. Time dependence of the MUCS, X, and P

Because the eigenenergies are equally spaced and ψ_{HOCB} has the form (2.41), up to a phase factor one has

$$\Psi(x,t)_{\mathrm{HOCB}} = e^{-iHt/\hbar}\psi_{\mathrm{HOCB}} = \exp[-i\mathcal{E}_0\nu(2\lambda+3-2\nu)t/\hbar] \times \psi_{\mathrm{HOCB}}(C \to C\exp(-i\mathcal{E}_0 4\nu t/\hbar))\,. \tag{2.54}$$

Thus, the wave packet evolves from that of a coherent state labeled by C, to that of a coherent state labeled by $C\exp(-i\mathcal{E}_0 4\nu t/\hbar) = C\exp(-i\omega_c t)$. Further, $\langle H\rangle$, $(\Delta X)^2$, and $(\Delta P)^2$ depend only on $|C|$, and thus are constant in time for our evolving MUCS. Our MUCS remain coherent states for all time.

However, this does not mean that our MUCS are as fully coherent as the MUCS for the simple harmonic oscillator. By full coherence we mean that (i) the wave packet remains localized in position and momentum, (ii) it does not change shape with time, (iii) its mean oscillation in position has the classical frequency, and (iv) the amplitude of the oscillation is the classical amplitude. For the harmonic oscillator all the above are true. For the HOCB (i) is satisfied since $(\Delta X)^2$ is constant in time. As we shall see below, property (iii) is also satisfied but properties (ii) and (iv) only hold approximately.

For a system with equally spaced levels, any wave packet will return to its original shape after a time corresponding to one classical oscillation. That is simply because the eigenfunctions in the decomposition have time dependence $\exp(-in\omega_c t)$, which is unity at the end of every classical oscillation. One wants to minimize the change of shape in the meantime. The harmonic-oscillator coherent states do not change shape at all. But for the HOCB system the wave packets must change shape because the HOCB potential is asymmetric about its minimum. A wave packet which could retain its shape near the right-hand turning point would be distorted near the left-hand turning point.

This is also true for our MUCS. Even though $(\Delta X)^2$ is constant with time, $(\Delta x)^2$ is not.

$$a^2\langle x^2\rangle = u + \frac{1}{\nu} + \frac{|C|I'_{\lambda+1/2}(\nu|C|)}{I_{\lambda+1/2}(\nu|C|)}, \tag{2.55}$$

$$a\langle x\rangle = [\nu^{1/2}e^{\nu u}I_{\lambda+1/2}(\nu|C|)]^{-1} \times \frac{|\beta|^{\lambda+1/2}\Gamma(\lambda+2)}{\Gamma^2(\lambda+\frac{3}{2})}\psi_2\,, \tag{2.56}$$

$$\psi_2 = \sum_{\substack{n=0\\ m=0}}^{\infty} \frac{(\beta^*)^n(\beta)^m(\lambda+2)_{n+m}}{n!\,m!\,(\lambda+\frac{3}{2})_n(\lambda+\frac{3}{2})_m}, \tag{2.57}$$

ψ_2 the confluent hypergeometric series of two variables. For small C, the first-order expansion shows that

$$a^2(\Delta x)^2 = \left(\frac{\lambda+\frac{3}{2}}{\nu}+u\right)\left(1-\frac{\Gamma^2(\lambda+2)}{\Gamma(\lambda+\frac{3}{2})\Gamma(\lambda+\frac{5}{2})}\right), \tag{2.58}$$

exhibiting a larger width on the right (positive u) and a smaller width on the left (negative u). However, $(\Delta X)^2$ being a constant is our criterion for asserting that these states satisfy property (ii) about as well as can be expected.

Property (iii) is easy to show. Because the eigenvalues are equally spaced [and hence the equations of motion (2.7) and (2.8) are similar to those for the harmonic oscillator], $X(t)$ and $P(t)$ can be calculated as were $x(t)$ and $p(t)$ for the harmonic oscillator [see Eqs. (I-2.35) and (I-2.36)]. In fact the answers are identical,

$$X(t) = X\cos\omega_c t + \frac{P}{m\omega_c}\sin\omega_c t\,, \tag{2.59}$$

$$P(t)=P\cos\omega_c t - m\omega_c X \sin\omega_c t\,, \tag{2.60}$$

where in this paragraph alone we reintroduce $(\hbar a^2)$ in P. Taking the expectation value of (2.59) in a coherent state gives

$$\langle X(t)\rangle = u\cos\omega_c t + \left(\frac{2\nu v}{m\omega_c}\right)\sin\omega_c t \tag{2.61a}$$

$$= |C|\sin(\omega_c t+\phi)\,, \tag{2.61b}$$

$$\phi=\tan^{-1}\left[u\left(\frac{m\omega_c}{2\nu v}\right)\right]. \tag{2.62}$$

Comparing this to the classical equation (2.3), we see that a coherent state has the classical angular velocity ω_c.

Finally we inquire about property (iv): Is the quantum analog of the classical amplitude $A(E)$ in Eq. (2.5) equal to $|C|$ in Eq. (2.61b)? The quantum analog of $A(E)$ is obtained by replacing E with $\langle H\rangle - E_0$, where E_0 is the ground-state energy. Doing this one has

$$A(\langle H\rangle)=\left[\frac{\langle H\rangle - E_0}{U_0}\left(1+\frac{\langle H\rangle - E_0}{4U_0}\right)\right]^{1/2} \tag{2.63a}$$

$$=\left[2|C|R\left(1+\frac{|C|R}{2}\right)\right]^{1/2}. \tag{2.63b}$$

In the limit of large C, Eq. (2.63) is exactly equal to $|C|$. This holds because for large C, R given by Eq. (2.45) is equal to unity.

However, even for small $|C|$, the equality holds almost exactly. To show this, we give some numerical examples. Let $\mathcal{E}_0$ be given by the value[10] for the HF-Morse system that Walker and Preston[10,11] studied numerically, and the value of $(E_n - E_{n-1})$ be given by the Walker-Preston value for $(E_1-E_0)_{\text{Morse}}$. Then $\lambda=10.94$. For ease of comparison with tables,[12] we modify this to $\lambda=9.5$. Also define an effective n by

$$n_{\text{eff}}=\frac{\langle H\rangle - E_0}{4\nu\mathcal{E}_0}. \tag{2.64}$$

Then numerical calculations yield

$$A(\langle H\rangle)=\begin{cases}0.957\ |C| \\ 0.965\ |C| \\ 0.977\ |C|\,,\end{cases}\qquad n_{\text{eff}}=\begin{cases}0.489 \\ 1.950 \\ 5.98\,.\end{cases} \tag{2.65}$$

E. Harmonic-oscillator limit

It is not too surprising that one can define a limit in which this system reduces to the simple harmonic oscillator (HO). It is

$$\lim_{\text{HO}} = \lim \begin{cases}\lambda\to\infty \\ a\to 0 \\ \lambda a^2 \to m\omega/\hbar\,.\end{cases} \tag{2.66}$$

Then all the quantities we have discussed approach those for the harmonic oscillator. In the next section we will discuss this limit extensively for the Pöschl-Teller potential.

F. n-independent raising and lowering operators and other coherent states

By defining

$$h=\frac{H+2U_0}{4\nu\mathcal{E}_0}=\frac{\nu}{2}\left(1+\frac{H}{2U_0}\right), \tag{2.67}$$

the n-dependent raising and lowering operators of Eq. (2.19) can be changed into the n-independent operators,

$$A^{\pm}=\left(\tfrac{1}{2}y \mp y\frac{d}{dy} - (h\pm\tfrac{1}{4})\right). \tag{2.68}$$

Defining the AOCS by

$$A^-\psi_\beta=\beta\psi_\beta=\beta\sum_{n=0}^{\infty}a_n\psi_n\,, \tag{2.69}$$

one finds the solution is given by

$$\frac{a_{n+1}}{a_n}=\frac{-\beta}{[(n+1)(n+\frac{3}{2}+\lambda)]^{1/2}}. \tag{2.70}$$

But this is exactly the relation satisfied by our MUCS of (2.41) with β given by Eq. (2.42). Therefore, for this equally-spaced-eigenvalues example, the AOCS defined by the n-independent operator A^- are the same as our MUCS. [The reason is simply that the operator A^- given in Eq. (2.68) is identical to $(\nu/2)(X+iP/2\nu)$ of Eqs. (2.29) and (2.32).] Further, by repeated use of (2.20), ψ_n can be written as

$$\psi_n=\left(\frac{\Gamma(\frac{3}{2}+\lambda)}{n!\,\Gamma(n+\frac{3}{2}+\lambda)}\right)^{1/2}(-A^+)^n\psi_0\,, \tag{2.71}$$

meaning our MUCS-AOCS can be put in the DOCS form

$$\psi_{\text{HOCB}}=\left(\frac{e^{-i\nu v}\beta^{\lambda+1/2}\Gamma(\frac{3}{2}+\lambda)}{I_{\lambda+1/2}(2|\beta|)}\right)^{1/2} \times(\beta A^+)^{-(\lambda+1/2)}I_{\lambda+1/2}(2(\beta A^+)^{1/2})\psi_0\,. \tag{2.72}$$

Observe that our coherent states are not in the Perelomov unitary-exponential-operator DOCS form, although one expects this could be done for an equally spaced eigenvalue problem. The resolution is that there are other coherent states, defined by related operators, which can be put in the Perelomov form. The key is to find those n-independent raising and lowering operators which can yield the results

$$\mathcal{H}=2\nu\mathcal{E}_0[\mathcal{A}^+\mathcal{A}^- + \mathcal{A}^-\mathcal{A}^+ + (\lambda+\tfrac{1}{2}-\nu)]\,, \tag{2.73}$$

$$\mathcal{H}|n\rangle = E_n|n\rangle\,. \tag{2.74}$$

These turn out to be

$$\mathcal{A}^{\pm} = \left((h \pm \tfrac{1}{4}) - \tfrac{1}{2}y \pm y\frac{d}{dy}\right)\frac{1}{[h + \frac{1}{2}(\lambda + \frac{1}{2} \pm 1)]^{1/2}}. \quad (2.75)$$

They are obtained from

$$\mathcal{A}_n^{\pm} = -A_n^{\pm}\left(n + \lambda + \frac{2 \pm 1}{2}\right)^{-1/2}, \quad (2.76)$$

just as (2.68) is obtained from (2.20).

Because these operators satisfy

$$\mathcal{A}^{\pm}|n\rangle = (n + \tfrac{1}{2} \pm \tfrac{1}{2})^{1/2}|n \pm 1\rangle, \quad (2.77)$$

the formalism used for the simple harmonic oscillator in paper I goes through exactly for this system. Therefore, we can define AOCS-DOCS which are eigenstates of $\mathcal{A}^-$ given by

$$\mathcal{A}^-\psi_\alpha = \alpha\psi_\alpha, \quad (2.78)$$

$$\psi_\alpha = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\psi_n \quad (2.79)$$

$$= \exp(\alpha\mathcal{A}^+ - \alpha^*\mathcal{A}^-)\psi_0. \quad (2.80)$$

However, because these ψ_n are not those of the harmonic oscillator, the reader should not be confused into thinking the ψ_α are the ordinary coherent states.

Thus, we have found that for HOCB the "simplest" MUCS are not the same as the "simplest" AOCS-DOCS, as they were for the harmonic oscillator. In our next (PT) example they are the same, but then they do not satisfy the unitary-exponential-operator DOCS form. This is because there the eigenvalues are not equally spaced. With unequally spaced eigenvalues any $\mathcal{A}^{\pm}$ which can yield a Hamiltonian equation like (2.74) cannot also yield a raising-lowering equation like (2.77), with a c-number on the right varying as $n^{1/2}$.

III. PÖSCHL-TELLER POTENTIAL

A. Classical motion and the natural classical variables

The symmetric Pöschl-Teller potential is

$$V(x) = U_0\tan^2 z, \quad z \equiv ax, \quad (3.1)$$

$$U_0 \equiv \mathcal{E}_0\lambda(\lambda - 1), \quad \mathcal{E}_0 = \frac{\hbar^2 a^2}{2m}, \quad (3.2)$$

where $\mathcal{E}_0$ and λ are useful for the quantum system. This is a confining potential contained within an infinite square well with sides at $x = \pm d = \pm\pi/(2a)$.

The natural classical variables that satisfy Eqs. (I-3.2) to (I-3.4) are

$$X_c = \sin z = A(E)\sin\omega_c t, \quad (3.3)$$

$$P_c = ap\cos z = m\omega_c A(E)\cos\omega_c t$$
$$= a[2mE]^{1/2}\cos\omega_c t, \quad (3.4)$$

$$A(E) = \left(\frac{E}{U_0 + E}\right)^{1/2}, \quad (3.5)$$

$$\omega_c = \left[\frac{2a^2(U_0 + E)}{m}\right]^{1/2}. \quad (3.6)$$

The classical equations of motion are thus

$$\dot{X}_c = P_c/m, \quad (3.7)$$

$$\dot{P}_c = -2a^2(U_0 + E)X_c = -m\omega_c^2 X_c. \quad (3.8)$$

B. Natural quantum operators and MUCS

The operators that are the quantum analogs of (2.3) and (2.4) are[2]

$$X = \sin z, \quad (3.9)$$

$$P = \frac{1}{2i}\left(\frac{d}{dz}\cos z + \cos z\frac{d}{dz}\right)(\hbar a^2). \quad (3.10)$$

Their equations of motion are

$$\dot{X} = \frac{1}{i\hbar}[X, H] = P/m, \quad (3.11)$$

$$\dot{P} = -a^2\{U_0 + H - \tfrac{1}{4}\mathcal{E}_0, X\}. \quad (3.12)$$

Because (3.12) is the symmetrized version of (3.8) with $E \to (H - \frac{1}{4}\mathcal{E}_0)$, $-\frac{1}{4}\mathcal{E}_0$ being a zero-point contribution, these are the quantum analogs of the classical equations of motion (3.7) and (3.8).

Taking the factor $(\hbar a^2)$ out of the definition (3.10) to make P dimensionless, one has

$$[X, P] = i\cos^2 z, \quad (3.13)$$

implying the uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \geq \tfrac{1}{4}\langle\cos^2 z\rangle^2. \quad (3.14)$$

The normalized states that satisfy the equality in this uncertainty relation are

$$\psi_{\text{MUS}}(C, B; z) \equiv N(C, B)\phi_{\text{PT}}(z) \quad (3.15a)$$

$$N(C, B) = \left(\frac{a}{\pi^{1/2}}\frac{\Gamma(B + \frac{1}{2})\Gamma(B + 1)}{\Gamma(B + \frac{1}{2} + u)\Gamma(B + \frac{1}{2} - u)}\right)^{1/2}, \quad (3.15b)$$

$$\phi_{\text{PT}}(z) = (\cos z)^B\left(\frac{1 + \sin z}{1 - \sin z}\right)^{C/2}, \quad (3.15c)$$

$$B \equiv \tfrac{1}{2}[-1 + \langle\cos^2 z\rangle/(\Delta\sin z)^2], \quad (3.16)$$

$$C \equiv u + iv = B\langle\sin z\rangle + \langle(\cos z)(d/dz)\rangle. \quad (3.17)$$

The eigenfunctions and eigenenergies of the PT potential are[4,5,13]

$$\psi_n \equiv N(n, \lambda)\phi_n(z), \quad (3.18a)$$

$$= \left(\frac{a(\lambda + n)\Gamma(2\lambda + n)}{\Gamma(n + 1)}\right)^{1/2}(\cos z)^{1/2}$$
$$\times P_{n+\lambda-1/2}^{1/2-\lambda}(\sin z), \quad (3.18b)$$

$$E_n = \mathcal{E}_0(2n\lambda + n^2 + \lambda). \quad (3.19)$$

So, for $C = 0$, the $n = 0$ ground state is a special case of the states (3.15) with $B = \lambda$. Thus, our MUCS, complete with normalization constants, are

$$\psi_{\rm PT} = \psi_{\rm MUS}(C, B=\lambda; z) \tag{3.20a}$$
$$= N(C,\lambda)\phi_{\rm PT}(z). \tag{3.20b}$$

One can verify that for the MUCS,[14]

$$\langle X\rangle = \frac{u}{\lambda+\frac{1}{2}}, \tag{3.21}$$

$$\langle X^2\rangle = \left(\frac{1}{2(\lambda+1)} + \frac{u^2}{(\lambda+1)(\lambda+\frac{1}{2})}\right), \tag{3.22}$$

$$\langle P\rangle = v, \tag{3.23}$$

$$\langle P^2\rangle = \frac{(\lambda+\frac{1}{2})^2-u^2}{2(\lambda+1)} + v^2, \tag{3.24}$$

so that

$$(\Delta X)^2(\Delta P)^2 = \frac{(\lambda+\frac{1}{2})^2-u^2}{2(\lambda+1)(\lambda+\frac{1}{2})^2} \times \frac{(\lambda+\frac{1}{2})^2-u^2}{2(\lambda+1)} \tag{3.25a}$$

$$= \tfrac{1}{4}\langle \cos^2 z\rangle^2. \tag{3.25b}$$

Also,

$$\langle H\rangle/\mathcal{E}_0 = \left(\frac{\lambda(\lambda-\frac{1}{2})}{(\lambda-\frac{1}{2})^2-u^2}\right)(u^2+v^2)+\lambda. \tag{3.26}$$

The MUCS can be decomposed into the number states by the use of integrals involving special functions.[15] The result is

$$\psi_{\rm PT} = \frac{1}{a}N(C,\lambda)\sum_{n=0}^{\infty} N(n,\lambda)\mathcal{O}(n,C,\lambda)\psi_n, \tag{3.27}$$

$$\mathcal{O}(n,C,\lambda) \equiv a\langle\phi_n|\phi_{\rm PT}\rangle$$
$$= \frac{1}{\Gamma(\lambda+\frac{1}{2})2^{n-\lambda-1/2}} \times \sum_{k=0}^{n} \frac{(-(n+\lambda-\frac{1}{2}))_k(-n)_k(-1)^k}{(\lambda+\frac{1}{2})_k(k!)}$$
$$\times B(\lambda+\tfrac{1}{2}+(\tfrac{1}{2}C-k), \lambda+\tfrac{1}{2}-(\tfrac{1}{2}C-k)). \tag{3.28}$$

C. Natural quantum operators as n-dependent raising and lowering operators

That the operators X and P obtained from the "natural quantum variables" X_c and P_c should be considered the "natural quantum operators" can also be seen by starting from the quantum system. Recall that for the harmonic oscillator x and p can be written as the sum and difference of the lowering and raising operators a^- and $a^+ = (a^-)^\dagger$. [See Eqs. (I-2.13) to (I-2.15).]

For the PT system the raising and lowering operators are n dependent, and are

$$A_n^{\pm} = (\sin z)(n+\lambda) \mp \cos z\frac{d}{dz}, \tag{3.29}$$

with the properties

$$A_n^{\pm}\psi_n = (n+\lambda)D(n-\tfrac{1}{2}\pm\tfrac{1}{2},\lambda)\psi_{n\pm1}, \tag{3.30}$$

$$D(n,\lambda) \equiv \left(\frac{(n+1)}{(\lambda+n)}\frac{(2\lambda+n)}{(\lambda+n+1)}\right)^{1/2}. \tag{3.31}$$

Equations (3.29)–(3.31) follow from the standard recursion relations for associated Legendre polynomials, which can be combined to yield[16]

$$(1-t^2)\frac{d}{dt}P_\nu^\mu(t) = (\nu+1)tP_\nu^\mu(t) - (\nu-\mu+1)P_{\nu+1}^\mu(t) \tag{3.32}$$
$$= -\nu t P_\nu^\mu(t) + (\nu+\mu)P_{\nu-1}^\mu(t). \tag{3.33}$$

From (3.29) X and P can be written as the sum and difference of the quantities $[A_n^- + (A_n^+)^\dagger]$ and $[A_n^+ + (A_n^-)^\dagger]$, which are the adjoints of each other. Specifically,

$$X = \frac{1}{4(n+\lambda)}\{[A_n^- + (A_n^+)^\dagger] + [A_n^+ + (A_n^-)^\dagger]\}, \tag{3.34}$$

$$P = \frac{1}{4i}\{[A_n^- + (A_n^+)] - [A_n^+ + (A_n^-)]\}. \tag{3.35}$$

This implies that

$$X\psi_n = \tfrac{1}{2}D(n,\lambda)\psi_{n+1} + \tfrac{1}{2}D(n-1,\lambda)\psi_{n-1}, \tag{3.36}$$

$$P\psi_n = -\frac{1}{2i}(n+\lambda+\tfrac{1}{2})D(n,\lambda)\psi_{n+1} + \frac{1}{2i}(n+\lambda-\tfrac{1}{2})D(n-1,\lambda)\psi_{n-1}. \tag{3.37}$$

Finally, observe that by writing the defining Eq. (I-A4) for the MUCS in terms of these operators, again subject to the restriction on $\Delta X/\Delta P$ that emerges from $B=\lambda$, the MUCS can be defined as eigenstates of the ground-state destruction operator, A_0^-.

D. Time dependence

The time dependence of these states can be studied by following the evolution of the coherent-state wave packets:

$$\rho(x,t) = \Psi_{\rm PT}^*(x,t)\Psi_{\rm PT}(x,t), \tag{3.38}$$

$$\Psi_{\rm PT}(x,t) = e^{-iHt/\hbar}\psi_{\rm PT}(x)$$
$$= \frac{1}{a}N(C,\lambda)\sum_{n=0}^{\infty}N(n,\lambda)\mathcal{O}(n,C,\lambda)\psi_n \times \exp[-i\mathcal{E}_0(n^2+2n\lambda+\lambda)t/\hbar]. \tag{3.39}$$

This has been done numerically for many cases, as will be described in paper V.

The time-dependent operators are also of interest. Defining h and τ by

$$h \equiv (H+U_0)/\mathcal{E}_0, \tag{3.40}$$

$$\tau \equiv \mathcal{E}_0 t/\hbar, \tag{3.41}$$

one has, in the notation of Sec. IV of paper I,

$$X(t) = e^{iHt/\hbar} X e^{-iHt/\hbar}$$
$$= \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} ([h, \ )^n X(])^n \equiv \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} X_n . \tag{3.42}$$

By rewriting the quantum equations of motion (3.11)–(3.12) as

$$[h, X] \equiv q , \tag{3.43}$$

$$[h, q] = 2q + X(4h - 1) , \tag{3.44}$$

$$q \equiv \sin z - 2\cos z \frac{d}{dz} = -2iP , \tag{3.45}$$

the iterated commutators in (3.42) close in the following sense:

$$X_0 = X , \tag{3.46}$$

$$X_1 = q , \tag{3.47}$$

$$X_2 = 2q + X(4h - 1) , \tag{3.48a}$$

$$\ldots$$

$$X_n = q g_n(h) + X f_n(h) . \tag{3.48b}$$

One can then use the procedure described in Sec. IV of paper I to evaluate the infinite sum in (3.42), and obtain

$$X(t) = X e^{-i\omega_0 t}\left[\cos\omega_H t - i\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t\right]$$
$$+ P e^{i\omega_0 t} 2\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t , \tag{3.49}$$

where

$$\omega_0 = \mathcal{E}_0/\hbar , \tag{3.50}$$

and ω_H is the operator analog of the classical frequency ω_c,

$$\omega_H = \left[\frac{2a^2}{m}(U_0 + H)\right]^{1/2} = 2[\mathcal{E}_0(U_0 + H)]^{1/2}/\hbar . \tag{3.51}$$

Similarly,

$$P(t) = P e^{-i\omega_0 t}\left[\cos\omega_H t + i\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t\right]$$
$$- X e^{i\omega_0 t}(h - \tfrac{1}{4}) 2\left(\frac{\omega_0}{\omega_H}\right)\sin\omega_H t . \tag{3.52}$$

The above formulas for $X(t)$ and $P(t)$ are not manifestly Hermitian, even though they follow from (3.42). The operators X and P are written to the left of operators that are functionals of the Hamiltonian. These expressions could be made manifestly Hermitian by writing them as one-half the sum of the given forms plus the adjoints of the given forms. [The adjoint forms emerge if one chooses to write the factors in the iterated commutators (3.42) in the opposite order.] However, it is more useful to write them in the forms above because the functionals on the right produce simple results when acting on the eigenstate decomposition of the coherent states.

Specifically, taking the expectation values of $X(t)$ and $P(t)$ between coherent states by using the decomposition (3.27) and the other results of this section one obtains closed-form results. They are listed in Appendix A of the following paper III,[16] for conciseness. Both $\langle X(t)\rangle$ and $\langle P(t)\rangle$ given in Eqs. (III-A2) and (III-A3) are real, as they should be if $X(t)$ and $P(t)$ are Hermitian operators. Further, the phase factor $\exp(i\omega_0 t)$ in the expressions for $X(t)$ and $P(t)$ is required if (III-A2) and (III-A3) are to be real.

Finally, one can calculate $\langle X^2(t)\rangle$ and $\langle P^2(t)\rangle$ by, for example, calculating $\langle n|X(t)|\mathrm{CS}\rangle$, taking the absolute square, and summing over all n. The results are also given in Appendix A of the following paper.[16]

E. Harmonic-oscillator limit

Defining the harmonic-oscillator (HO) limit as

$$\lim_{\mathrm{HO}} \equiv \lim \begin{cases} \lambda \to \infty \\ a \to 0 \\ \lambda a^2 \to m\omega/\hbar , \end{cases} \tag{3.53}$$

all our results go over to the harmonic-oscillator case discussed in I. The eigenenergies (3.19) trivially become those of the harmonic oscillator (I-2.28). The eigenstate wave functions (3.18) also become the HO eigenstate wave functions (I-2.9), complete with the normalization constants. To see this, write the associated Legendre functions of (2.18) in terms of Gegenbauer polynomials and use the known limiting relationship between these polynomials and Hermite polynomials. Also use

$$\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = e^a . \tag{3.54}$$

Finally, the Pöschl-Teller MUCS (3.20) become the harmonic-oscillator coherent states (I-2.8) in the limit (3.53), as can be shown by the use of (3.54) and Stirling's approximation.

Because of the above properties and the similarity of the operator formulas for $X(t)$ and $P(t)$ when compared to the harmonic-oscillator operator formulas for $x(t)$ and $p(t)$, a first-order analytic approximation to $X(t)$ and $P(t)$ in the HO limit (3.53) shows that they will follow the same classical motion. This holds similarly for $X^2(t)$ and $P^2(t)$. However, higher-order terms show that this classical motion cannot proceed forever. As shall be discussed in the numerical results of paper V, a rough measure of the coherence time is the time during which the wave packet remains localized about $x_c(t)$. This time can be increased by decreasing the parameter $\langle H\rangle/U_0$ or by increasing the number of eigenstates that have a significant overlap with the coherent state.

F. n-Independent raising and lowering operators, and other coherent states

As we have emphasized, for systems whose energy levels are not equally spaced one must further generalize the known AOCS and DOCS concepts. Applied to the PT system, the procedure we gave in Sec. V of paper I takes the n-dependent raising and lowering operators (3.29) and makes the transformation

$$(\sin z)(n+\lambda) \to (\sin z)h^{1/2}, \tag{3.55}$$

so that

$$A_n^{\pm} \to A^{\pm} = (\sin z)h^{1/2} \mp \cos z \frac{d}{dz}. \tag{3.56}$$

Observe that one can then write the Hamiltonian as

$$H \to \mathcal{H} = \tfrac{1}{2}\mathcal{E}_0[A^+A^- + A^-A^+], \tag{3.57}$$

because

$$\mathcal{H}|n\rangle = E_n|n\rangle. \tag{3.58}$$

Defining the AOCS, ψ_α, as eigenstates of the annihilation operator A^- yields

$$A^-\psi_\alpha = \sum_{n=0}^{\infty} a_n(n+\lambda)D(n-1,\lambda)\psi_{n-1} \tag{3.59}$$

$$\equiv \alpha\psi_\alpha = \alpha\sum_{n=0}^{\infty} a_n\psi_n. \tag{3.60}$$

Therefore,

$$\frac{a_{n+1}}{a_n} = \alpha\left(\frac{(\lambda+n)}{(n+1)(2\lambda+n)(\lambda+n+1)}\right)^{1/2}, \tag{3.61}$$

and ψ_α can be written as

$$\psi_\alpha = N_\alpha\sum_{n=0}^{\infty}\alpha^n\left[\frac{1}{n!}\frac{\Gamma(2\lambda)}{\Gamma(2\lambda+n)}\frac{\lambda}{\lambda+n}\right]^{1/2}\psi_n, \tag{3.62}$$

$$N_\alpha = \frac{1}{[\lambda\Gamma(2\lambda)S(|\alpha|)]^{1/2}}, \tag{3.63}$$

where

$$S(|\alpha|) = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!(n+\nu)\Gamma(2\lambda+n)} \tag{3.64}$$

$$= \frac{1}{|\alpha|^{2\lambda}}\int_0^{2|\alpha|} dx\, I_{2\lambda-1}(x), \tag{3.65}$$

and I is the modified Bessel function defined by[16]

$$I_\rho(y) = \sum_{n=0}^{\infty}\frac{(y/2)^{\rho+2n}}{n!\,\Gamma(\rho+n+1)}. \tag{3.66}$$

The ψ_α obtained above are *not* equivalent to the MUCS as can be seen by direct comparison with Eq. (3.27). Further, these AOCS *are* equivalent to a set of DOCS, which are not given by an exponential displacement operator acting on the ground state. Specifically, using (3.30) one has

$$\psi_n = \left(\frac{(\lambda+n)}{(\lambda+n-1)(2\lambda+n-1)n}\right)^{1/2} A^+\psi_{n-1}$$

$$= \left[\left(\frac{\lambda+n}{\lambda}\right)\frac{\Gamma(2\lambda)}{\Gamma(2\lambda+n)n!}\right]^{1/2}(A^+)^n\psi_0. \tag{3.67}$$

Putting (3.67) into (3.62) and using (3.66) one has

$$\psi_\alpha = N_\alpha\Gamma(2\lambda)[(\alpha A^+)^{-\lambda+1/2}I_{2\lambda-1}(2[\alpha A^+]^{1/2})]\psi_0. \tag{3.68}$$

Thus, by example we have shown that in general DOCS need not be defined in terms of exponential displacement operators, and furthermore that the DOCS as well as the AOCS need not be equivalent to the MUCS. Also observe that, at least in the form of (3.62), the AOCS-DOCS do not have the relatively simple analytic forms of our MUCS of Eq. (3.15). In addition to this reason, we have chosen to emphasize the MUCS in our study because of their intuitive relationship to the classical motion.

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