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Coherent states for general potentials. I. Formalism

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We first review the properties of the harmonic-oscillator coherent states which can be equivalently defined as (a) a specific subset of the x-p minimum-uncertainty states, (b) eigenstates of the annihilation operator, or (c) states created by a certain unitary exponential displacement operator. Then we present a new method for finding coherent states for particles in general potentials. Its basis is the desire to find those states which most nearly follow the classical motion, but it is most nearly a generalization of the minimum-uncertainty method. The properties of these states are discussed in detail. Next we show that the annihilation operator and displacement operator methods, as heretofore defined, cannot be applied to general potentials (whose eigenvalues are not equally spaced). We define a generalization of these methods but show that the states so defined are not, in general, equivalent to the minimum-uncertainty coherent states. We discuss a number of properties of our coherent states and the procedures we have used.

I. INTRODUCTION

In 1926 Schrödinger[1] discovered what have come to be called the coherent states of the harmonic oscillator.[2-5] In his paper,[1] the third[6] of the series of articles which dealt with his discovery of wave mechanics, Schrödinger was interested in finding quantum-mechanical states which followed the motion of a classical particle in a given potential. Studying the harmonic oscillator, Schrödinger found such states, commenting at the end,[6] "We can definitely foresee that, in a similar way, wave groups can be constructed which move around highly quantized Kepler ellipses and are the representation by wave mechanics of the hydrogen electron. But the technical difficulties in the calculation are greater than in the especially simple case which we have treated here."

Schrödinger's states, which are a special set of Gaussians, became popular during the 1960's for their usefulness in describing the radiation field[2-5] and by now have become known to a wide segment of the physics community. Except for a few studies,[7,8] which we will discuss in paper IV, Schrödinger's prediction that classical states could be found for other systems has, until recently,[9,10] mainly been implemented for systems whose eigenspectra are equally spaced.[11-18] We will see below that it is precisely because the harmonic-oscillator eigenvalues are equally spaced that the harmonic-oscillator coherent states can follow the classical motion and cohere forever.

The coherent states of the *harmonic oscillator* are defined in one of three equivalent ways: (a) Minimum-uncertainty coherent states (MUCS) are that two-parameter set of states which minimize the position-momentum uncertainty relation,[19] subject to the restriction that the ground state be in the set. (b) Annihilation-operator coherent states (AOCS) are the eigenstates of the annihilation operator and are parametrized by a complex eigenvalue. (c) Displacement-operator coherent states (DOCS) are those states which are created from the ground state by a particular unitary displacement operator. Generalizations of the AOCS[15] and the DOCS[16] have been proposed from the group-theory point of view, but only rigorously applied to systems with equally spaced eigenvalues. Recently we proposed[9,10] a method that is most closely a generalization of the MUCS, but which has as its physical basis the original motivation of Schrödinger: to find states which follow the classical motion. This method applies to more general systems having unequally spaced levels and a continuum.

The present series of papers describes in detail and extends the results of our first paper[9] and gives a more complete discussion. The motivation for this work is to find coherent states for systems with unequally spaced levels. One can use these states to describe more general coherent quantum-mechanical systems. Ultimately one can hope to generalize this work for use in quantum field theory. In both of the above types of physical applications, past discussions have essentially been limited to equally spaced level systems. This has been emphasized by Shelepin,[20] "If the coherent states of oscillators have been discussed extensively and found a wide range of applications, then the attempt to generalize them to nonequidistant systems has encountered a number of difficulties." It is the primary object of this series to focus on how these difficulties can be overcome. The new coherent states we obtain can be used directly with coherent quantum systems. Further, the raising and lowering operator techniques which we discuss point the way towards the use of our techniques in quantum field theory.

In Sec. II of this paper we review the harmonic-oscillator coherent states, showing how they are derived from the MUCS, AOCS, and DOCS points of view mentioned above. Section III contains a description of our method for obtaining general coherent states and some properties of these states. Our method is to seek those states which most closely approximate the classical motion of a particle. Therefore, we require that the states be well localized in position and momentum. To obtain our coherent states we use a generalization of the minimum-uncertainty method for more complicated uncertainty relations. Therefore, for simplicity we will call our states MUCS. (Appendix A describes how to find states which minimize generalized uncertainty relations.)

The time evolution of our coherent states and of the operators associated with them is discussed in Sec. IV. In Sec. V we point out that, to be useful for general systems, the annihilation operator[15] and displacement operator[16] methods must be generalized further. We do this, showing that the resulting AOCS and DOCS are not of necessity equivalent to the MUCS. In Sec. VI we discuss our coherent states and the procedures we have used, closing in Sec. VII with a summation of the results and goals of this series.

Papers II and III of this series will describe in detail the results for four exactly solvable one-dimensional potentials: two confining potentials, the harmonic oscillator with centripetal barrier and the symmetric Pöschl-Teller potentials, in paper II and two nonconfining examples, the symmetric Rosen-Morse[21] and the Morse potentials, in paper III. One advantage of the MUCS here and elsewhere is that, for solvable examples, the MUCS can readily be put into analytic closed forms. Paper IV will describe a generalization of our method to spherically symmetric potentials, concentrating on the radial portion of the problem. In paper V numerical results will be shown for the time evolution[22] of our analytic MUCS, and paper VI will give a set of conclusions.

II. HARMONIC-OSCILLATOR COHERENT STATES

A. Minimum-uncertainty coherent states

For the harmonic oscillator, with

$$V(x)=\tfrac{1}{2}kx^2\equiv\tfrac{1}{2}m\omega^2x^2\,, \tag{2.1}$$

the solutions to the classical problem

$$E=\tfrac{1}{2}p^2/m+V(x) \tag{2.2}$$

are

$$x(t)=[2E/(m\omega^2)]^{1/2}\sin(\omega t+\phi)\,, \tag{2.3}$$

$$p(t)=[2mE]^{1/2}\cos(\omega t+\phi)\,. \tag{2.4}$$

The quantum operators whose uncertainty relation is to be minimized are x and

$$p=\frac{\hbar}{i}\frac{d}{dx}\,. \tag{2.5}$$

One has

$$[x,p]=i\hbar\,, \tag{2.6}$$

implying that

$$(\Delta x)^2(\Delta p)^2\geq\hbar^2/4\,. \tag{2.7}$$

The normalized states which satisfy the equality in (2.7) are, from (A4) and (A5),

$$\psi_{\text{CS}}(x)=[2\pi(\Delta x)^2]^{-1/4}\times\exp\left\{-\left[\frac{x-\langle x\rangle}{2(\Delta x)}\right]^2+\frac{i}{\hbar}\langle p\rangle x\right\}. \tag{2.8}$$

If this set of coherent states (CS) is to be capable of representing a classical particle with the minimum allowable (quantum) energy, the set must contain the ground state. The normalized eigenfunctions of the quantum harmonic oscillator are[23]

$$\psi_n=\left(\frac{a_0}{\pi^{1/2}2^n n!}\right)^{1/2}\exp(-\tfrac{1}{2}a_0{}^2x^2)H_n(a_0x)\,, \tag{2.9}$$

$$a_0\equiv(m\omega/\hbar)^{1/2}\equiv1/(2^{1/2}x_0)\,. \tag{2.10}$$

Demanding that the ground-state ($n=0$) wave function be a special case of (2.8) for $\langle x\rangle=\langle p\rangle=0$ yields the restriction

$$(\Delta x/\Delta p)^2=1/(m\omega)^2\,, \tag{2.11}$$

or

$$(\Delta x^2)=(2a_0{}^2)^{-1}=x_0{}^2\,. \tag{2.12}$$

We emphasize this last restriction. With it the coherent-state wave packets will follow the motion of a classical particle and retain their shape. If a different value of $(\Delta x/\Delta p)$ were to be chosen, the packet would not keep its shape.[24]

We will discuss the properties and motion of these states after making the connection to the equivalent AOCS and DOCS definitions.

B. Annihilation-operator coherent states

Write x and p in terms of the usual annihilation and creation operators,

$$x=[\hbar/(2m\omega)]^{1/2}(a^-+a^+)\,, \tag{2.13}$$

$$p=[m\hbar\omega/2]^{1/2}(a^--a^+)/i\,, \tag{2.14}$$

$$a^{\pm}=(2m\hbar\omega)^{-1/2}(m\omega x\mp ip)\,, \tag{2.15}$$

$$[a^-,a^+]=1\,. \tag{2.16}$$

Consider the eigenstates of the destruction operator:

$$a^- |\alpha\rangle = \alpha |\alpha\rangle , \quad (2.17)$$

where $\alpha = u + iv$ is any complex number. The solution can be written in terms of number states:

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle . \quad (2.18)$$

However, by using the fact[23] that the Gaussian is the generating function of the Hermite polynomials $H_n(y)$,

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\exp(-\tfrac{1}{2}y^2)H_n(y) = \exp(-\xi^2 + 2\xi y - \tfrac{1}{2}y^2) , \quad (2.19)$$

Eqs. (2.9)–(2.12), (2.18), and (2.19) can be combined to give

$$|\alpha\rangle = [2\pi(\Delta x)^2]^{-1/4} \times \exp\left[-\frac{x^2}{4(\Delta x)^2} + \frac{x\alpha}{\Delta x} - \frac{1}{2}(\alpha^2 + |\alpha|^2)\right] . \quad (2.20)$$

Keeping the restrictions (2.11) and (2.12), and taking the parameter α to be

$$\alpha = \frac{\langle x\rangle}{2\Delta x} + \frac{i}{\hbar}\langle p\rangle \Delta x = \frac{1}{2}\left[\frac{\langle x\rangle}{\Delta x} + i\frac{\langle p\rangle}{\Delta p}\right], \quad (2.21)$$

one obtains

$$|\alpha\rangle = \exp[i\langle p\rangle\langle x\rangle/(2\hbar)]\psi_{\rm CS} . \quad (2.22)$$

Thus, the AOCS $|\alpha\rangle$ are exactly the MUCS $\psi_{\rm CS}$, up to the irrelevant phase factor in (2.22). The connection can be seen another way. Equations (2.17) and (2.21) are equivalent to the x, p minimum uncertainty defining Eq. (A4) subject to (2.11) and (2.12).

C. Displacement-operator coherent states

These states are defined as those states which are created by the unitary displacement operator

$$D(\alpha) = \exp(\alpha a^+ - \alpha^* a^-) . \quad (2.23)$$

That $D(\alpha)$ is unitary follows since $(\alpha a^+ - \alpha^* a^-)$ is anti-Hermitian.

The equivalence of the displacement-operator coherent states to the annihilation-operator coherent states can be shown with the aid of the Baker-Campbell-Hausdorff identity,[4]

$$\exp(A)\exp(B) = \exp(A + B + \tfrac{1}{2}[A,B]) \quad \text{if} \quad 0 = [A,[A,B]] = [B,[A,B]] . \quad (2.24)$$

Using (2.24) and

$$a^{\pm}|n\rangle = (n + \tfrac{1}{2} \pm \tfrac{1}{2})^{1/2}|n \pm 1\rangle , \quad (2.25)$$

one has

$$D(\alpha)|0\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^+)\exp(-\alpha a)|0\rangle \quad (2.26a)$$

$$= \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^+)|0\rangle \quad (2.26b)$$

$$= \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n(a^+)^n}{n!}|0\rangle \quad (2.26c)$$

$$= |\alpha\rangle . \quad (2.26d)$$

D. Properties of the coherent states

The harmonic oscillator has several related features that are crucial to the special properties of its coherent states: The classical frequency is independent of the energy, the eigenenergies are equally spaced

$$H|n\rangle = \hbar\omega(n + \tfrac{1}{2})|n\rangle , \quad (2.27)$$

and the raising and lowering operators are independent of n, allowing a unique factorization of the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^+a^- + \tfrac{1}{2}) . \quad (2.28)$$

Using (2.18) and (2.27),

$$|\alpha_t\rangle = \exp(-iHt/\hbar)|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{(\alpha)^n\exp[-i\omega t(n+\frac{1}{2})]}{(n!)^{1/2}}|n\rangle = \exp(-i\omega t/2)|\alpha(t)\rangle , \quad (2.29a)$$

$$\alpha(t) = \alpha e^{-i\omega t} . \quad (2.29b)$$

Thus, the coherent-state wave packets $\psi_\alpha^*\psi_\alpha$ have the same shape, with the center evolving in time from $\mathrm{Re}[\alpha]$ to $\mathrm{Re}[\alpha\exp(-i\omega t)]$. Trivially, of course, they retain the same energy:

$$\langle\alpha|H|\alpha\rangle = \hbar\omega(|\alpha|^2 + \tfrac{1}{2}) = \langle\alpha_t|H|\alpha_t\rangle . \quad (2.30)$$

Using the operator equations of motion,

$$i\hbar\dot{x} = [x,H] = i\hbar p/m , \quad (2.31)$$

$$i\hbar\dot{p} = [p,H] = -i\hbar m\omega^2 x , \quad (2.32)$$

one can calculate

$$x(t) \equiv e^{iHt/\hbar}xe^{-iHt/\hbar} \quad (2.33)$$

$$= \sum_{n=0}^{\infty}\left(\frac{it}{\hbar}\right)^n\frac{1}{n!}([H,)^n x(])^n \quad (2.34)$$

$$= x\cos\omega t + \frac{p}{m\omega}\sin\omega t . \quad (2.35)$$

The notation in Eq. (2.34) is that[10] $(\cdot)^n$ means the quantity in the parentheses is to be written out n times, yielding an n-fold iterated commutator which is easily evaluated to give the well-known results (2.35) for the harmonic oscillator. In Sec.

IV we shall explain these nested commutators more fully for use with general potentials. Similarly,

$$p(t)=p\cos\omega t-m\omega x\sin\omega t\,. \tag{2.36}$$

$x^2(t)$ and $p^2(t)$ can be calculated in the same manner, or simply by squaring (2.35) and (2.36). The results are

$$x^2(t)=[x^2-H/(m\omega^2)]\cos(2\omega t)+H/(m\omega^2)+\frac{\{x,p\}}{2m\omega}\sin(2\omega t) \tag{2.37a}$$

$$=x^2\cos^2\omega t+\frac{p^2}{m^2\omega^2}\sin^2\omega t+\frac{\{x,p\}}{m\omega}\cos\omega t\sin\omega t\,, \tag{2.37b}$$

$$p^2(t)=(p^2-mH)\cos(2\omega t)+mH-\tfrac{1}{2}m\omega\{x,p\}\sin(2\omega t) \tag{2.38a}$$

$$=p^2\cos^2\omega t+m^2\omega^2x^2\sin^2\omega t-m\omega\{x,p\}\cos\omega t\sin\omega t\,. \tag{2.38b}$$

Then, by using (2.13)–(2.17) in the above, one finds

$$\langle\alpha|x(t)|\alpha\rangle=\left(\frac{2\hbar}{m\omega}\right)^{1/2}(u\cos\omega t+v\sin\omega t)\,, \tag{2.39}$$

$$\langle\alpha|x^2(t)|\alpha\rangle=\left(\frac{\hbar}{m\omega}\right)(2u^2\cos^2\omega t+2v^2\sin^2\omega t+4uv\sin\omega t\cos\omega t+\tfrac{1}{2})\,, \tag{2.40}$$

$$\langle\alpha|p(t)|\alpha\rangle=(2m\hbar\omega)^{1/2}(v\cos\omega t-u\sin\omega t)\,, \tag{2.41}$$

$$\langle\alpha|p^2(t)|\alpha\rangle=(m\omega\hbar)(2v^2\cos^2\omega t+2u^2\sin^2\omega t-4uv\sin\omega t\cos\omega t+\tfrac{1}{2})\,, \tag{2.42}$$

and

$$[\Delta x(t)]^2=\tfrac{1}{2}\hbar/(m\omega)\,, \tag{2.43}$$

$$[\Delta p(t)]^2=\tfrac{1}{2}m\omega\hbar\,. \tag{2.44}$$

Therefore, the shape and the minimum-uncertainty property of the wave packet are preserved in time.

Finally, from (2.39) we can also see that the center of the coherent-state wave packet follows the motion of a classical particle. Defining ϕ by

$$\tan\phi=u/v\,, \tag{2.45}$$

Eq. (2.39) becomes

$$\langle\alpha|x(t)|\alpha\rangle=\left[\frac{2(\hbar\omega|\alpha|^2)}{m\omega^2}\right]^{1/2}\sin(\omega t+\phi)\,. \tag{2.46}$$

This is exactly the Eq. (2.3) for the classical motion, except that the classical energy E has been replaced by

$$\hbar\omega|\alpha|^2=\langle\alpha|H|\alpha\rangle-\langle 0|H|0\rangle\,. \tag{2.47}$$

That is, one subtracts the ground-state energy from E in the expression for the amplitude of $\langle\alpha|x(t)|\alpha\rangle$.

III. GENERALIZED COHERENT STATES

In obtaining generalized coherent states,[9,10,21] our goal was to find states which are localized, follow the classical motion, and disperse as little as possible with time. To do this we first considered the bound-motion problem of a conserved one-dimensional classical Hamiltonian

$$E=\frac{1}{2m}p^2+V(x)=\frac{m}{2}\dot{x}^2+V(x)\,, \tag{3.1}$$

where $V(x)$ has only one minimum. Because the problem is one dimensional, the bound motion is a simple closed orbit in a p-x phase-space plot. Therefore, there exists a one-to-one mapping of this orbit onto general coordinates, X_c and P_c, such that the orbit is an ellipse. In particular, with these transformations X_c and P_c will vary sinusoidally with time.

To be explicit, for potentials with one confining region, there exist variables $X_c(x)$ and $P_c=m\dot{X}_c=(m[X'(x)]\dot{x})$ which are solutions of (3.1) and whose time variations are given by

$$X_c=A(E)\sin[\omega_c(E)t]\,, \tag{3.2}$$

$$P_c=mA(E)\omega_c(E)\cos[\omega_c(E)t]\,. \tag{3.3}$$

Because

$$\dot{x}^2=2(E-V)/m\,, \tag{3.4a}$$

$X_c(x)$ is the solution of the equation

$$X_c'=\frac{d}{dx}X_c(x)=\omega_c\left(\frac{m(A^2-X_c^2)}{2[E-V(x)]}\right)^{1/2}\,. \tag{3.4b}$$

Equations (3.2) and (3.3) imply that the classical equations of motion are

$$\dot{X}_c=P_c/m\,, \tag{3.5}$$

$$\ddot{X}_c=\frac{\dot{P}_c}{m}=-\omega_c^2(E)X_c\,. \tag{3.6}$$

Note that we are not employing a Hamiltonian formalism. In particular, P_c is a generalized velocity times mass, *not* the momentum canonically conjugate to X_c (see Appendix B).

Thus, Eq. (3.1) is replaced by a form which is similar to the harmonic-oscillator equation for a given energy. That is, this transformation is equivalent to rewriting (3.1) as

$$\frac{1}{2}m\omega_c^2(E)A^2(E)=\frac{1}{2m}P_c^2+\frac{1}{2}m\omega_c^2X_c^2\,. \tag{3.7}$$

Now define the quantum operators

$$X\equiv X_c(x)\,, \tag{3.8}$$

$$P\equiv\tfrac{1}{2}(X_c'p+pX_c')\,, \tag{3.9}$$

whose commutator is

$$[X,P]\equiv iG\,. \tag{3.10}$$

The generalized uncertainty relation for X and P is[5,25,26]

$$(\Delta X)^2(\Delta P)^2/\langle G\rangle^2\geq \tfrac{1}{4}\,. \tag{3.11}$$

Next, as described in Appendix A, obtain those states which minimize this uncertainty relation.

Our physical ansatz is that for the generalized coherent states one should take that two-parameter subset of the above set of minimum-uncertainty states specified by a certain value of $\Delta X/\Delta P$. This value is chosen such that the ground state belongs to the set of coherent states. The rationale is that if the coherent states are to approximate the motion of a classical particle, the states should be localized. Therefore, we start with minimum-uncertainty states and demand that one be able to represent a particle of any energy, including the minimum or ground-state energy.

In the one-dimensional examples treated in Ref. 9, X_c was independent of the classical constant of the motion E. As Eq. (3.4) demonstrates, this is not generally true. If $X_c=X_c(x,E)$, to obtain the appropriate quantum operator X one must make the replacement $E\to H$, possibly allowing for a zero-point energy contribution, and symmetrize. (The Morse potential[10] provides an example of this situation.)

Just as for the harmonic oscillator, for our examples it turns out that one can *exactly* express the generalized operators X and P in terms of the raising and lowering operators of the discrete energy eigenstates. However, because energy levels are not in general equally spaced, the raising and lowering operators can be n dependent and $(A_n^+)^\dagger \neq (A_n^-)$. Thus, the general representations of the operators X and P are of the more complicated forms

$$X=K_1(n)\{[A_n^-+(A_n^+)^\dagger]+[A_n^++(A_n^-)^\dagger]\}\,, \tag{3.12}$$

$$P=\frac{1}{i}K_2(n)\{[A_n^-+(A_n^+)^\dagger]-[A_n^++(A_n^-)^\dagger]\}\,, \tag{3.13}$$

where $K_1(n)$ and $K_2(n)$ are n-dependent c numbers. The above is exactly true even for an energy-dependent example,[10] if the appropriate zero-point energy is inserted along with the Hamiltonian in the substitution for the classical energy.

The forms of (3.12) and (3.13) imply that, just as the classical variables X_c and P_c are the "natural" ones in that they vary sinusoidally as $(\omega_c t)$, the quantum generalizations X and P are the "natural" operators because they have nonzero matrix elements *only* between adjacent eigenstates:

$$\langle n|X|m\rangle=\langle n|P|m\rangle=0 \quad \text{if } m\neq n\pm 1\,. \tag{3.14}$$

IV. TIME EVOLUTION OF THE COHERENT STATES

After obtaining these minimum-uncertainty coherent states for a given system, one wants to understand their properties, especially their coherence with time. One way to do this is to decompose the coherent states into energy eigenstates. In particular, write

$$\psi_{CS}(x)=\sum_n a_n\psi_n(x)+\text{continuum}\,, \tag{4.1}$$

$$\begin{aligned}\psi_{CS}(x,t)&=\exp(-iHt/\hbar)\psi_{CS}(x)\\&=\sum_n a_n\exp(-iE_nt/\hbar)\psi_n(x)\\&\quad+\text{continuum}\,.\end{aligned} \tag{4.2}$$

The continuum contributions, if they exist, will become plane waves in the far continuum region. They will, therefore, not contribute to the oscillatory part of ψ_{CS}. (We will give an example in paper III.) For our examples,[9,10] the bound-state decompositions (4.2) can always be done analytically.

Similarly, one can derive the time-dependent operators $X(t)$, $P(t)$, and their squares, and calculate their expectation values in the coherent states. This is a generalized version of the calculations described in (2.33)–(2.38). Again, for our examples, this calculation can be done exactly.

Start with the quantum analogs of the classical equations of motion (3.5)–(3.6). These turn out to be of the form

$$\dot{X}=(-i/\hbar)[X,H]=P/m\,, \tag{4.3}$$

$$\dot{P}=(-i/\hbar)[P,H]=XB_1(H)+iPB_0\,, \tag{4.4}$$

where $B_1(H)$ is a simple polynomial in H and B_0 is a constant.

With the aid of (4.3) and (4.4) one can calculate

$$\begin{aligned}X(t)&\equiv e^{iHt/\hbar}Xe^{-iHt/\hbar}\\&=Xe^{i\phi(t)}[\cos\omega_H t+ij_1(H)\sin\omega_H t]\\&\quad+Pe^{i\phi(t)}j_2(H)\sin\omega_H t\,,\end{aligned} \tag{4.5}$$

$$\begin{aligned}P(t)&=Pe^{i\phi(t)}[\cos\omega_H t-ij_3(H)\sin\omega_H t]\\&\quad+Xe^{i\phi(t)}j_4(H)\sin\omega_H t\,,\end{aligned} \tag{4.6}$$

where ω_H is an operator that has the functional form of the classical frequency with E replaced by H, $\phi(t)$ is a real function of t, and $j_1(H)$, $j_2(H)$, $j_3(H)$, and $j_4(H)$ are explicit operator functions of H. These expressions are obtained as follows.

Consider the general relation[25]

$$e^{+\lambda V}Qe^{-\lambda V}=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}([V,)^nQ(])^n\equiv\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}Q_n\,, \tag{4.7}$$

where, in the iterated commutator Q_n, the symbol

$(\cdot)^n$ means that the quantity enclosed in the parenthesis is to be repeated n times. Specifically,

$$Q_0 = Q\ ,\quad Q_1 = [V,Q]\ ,\quad Q_2 = [V,[V,Q]]\ ,\quad \ldots \tag{4.8a}$$

and generally

$$Q_{n+1} = [V, Q_n]\ . \tag{4.8b}$$

Now observe that one can simplify the equations by reducing them to dimensionless form. Let $1/a$ be a characteristic length specified by the potential and define

$$\mathcal{E}_0 \equiv \hbar^2 a^2/2m\ . \tag{4.9}$$

Measuring distance and energy in these units, the equations can be rewritten easily in terms of dimensionless variables and operators:

$$\tau = \mathcal{E}_0 t/\hbar\ , \tag{4.10}$$

$$z \equiv ax\ , \tag{4.11}$$

$$h \equiv (H + \text{const})/\mathcal{E}_0\ , \tag{4.12}$$

$$q \equiv \text{const}(\hbar a^2)^{-1} P\ . \tag{4.13}$$

(Specific use of this will be made in the examples treated in papers II and III.)

With these definitions, one can write

$$\begin{aligned} X(t) &= e^{ih\tau} X e^{-ih\tau} \\ &= \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} ([h,)^n X(])^n \\ &\equiv \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} X_n\ . \end{aligned} \tag{4.14}$$

Using the dimensionless versions of the equations of motion (4.3) and (4.4),

$$[h, X] = q\ , \tag{4.15}$$

$$[h, q] = qb_0 + Xb_1(h)\ , \tag{4.16}$$

one has

$$X_0 = X\ , \tag{4.17a}$$

$$X_1 = q\ , \tag{4.17b}$$

$$X_2 = Xb(h) + qb_0\ . \tag{4.17c}$$

We find for all our examples that b_0 is a constant and $b_1(h)$ is a first-order polynomial in h,

$$b_1(h) = (\text{const})h + \text{const}\ . \tag{4.18}$$

Because of (4.17c), the iterated commutators in (4.14) close on themselves in the following sense:

$$X_n = Xf_n(h) + qg_n(h)\ . \tag{4.19}$$

Then

$$X_{n+1} = Xf_{n+1}(h) + qg_{n+1}(h) \tag{4.20a}$$

$$= X[g_n(h)b_1(h)] + q[f_n(h) + b_0 g_n(h)]\ . \tag{4.20b}$$

Comparing the two lines of (4.20),

$$f_{n+1}(h) = g_n(h) b_1(h)\ , \tag{4.21}$$

and hence

$$g_{n+1}(h) = b_0 g_n(h) + g_{n-1} b_1(h)\ . \tag{4.22}$$

The recurrence relation (4.22) can be solved with a trial solution

$$g_n \propto r^n\ . \tag{4.23}$$

From (4.22) one finds two roots

$$r_\pm = \tfrac{1}{2} b_0 \pm \tfrac{1}{2}[b_0^2 - 4b_1(h)]^{1/2}\ , \tag{4.24}$$

so g_n has the form

$$g_n = Ar_+^n + Br_-^n\ . \tag{4.25}$$

However, $g_0 = 0$ implies $A = -B$, and $g_1 = b_0$ implies

$$-B = A = (b_0^2 + 4h)^{-1/2}\ . \tag{4.26}$$

Thus,

$$X(t) = \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} [Xf_n(h) + qg_n(h)]\ , \tag{4.27}$$

where $g_n(h)$ and $f_n(h)$ are given by (4.24)–(4.26) and (4.21). Straightforward algebra then allows the final form (4.5) to be obtained. A similar calculation can be done for $P(t)$.

V. OTHER GENERALIZED COHERENT STATES

A. Annihilation-operator coherent states

It is possible to generalize the annihilation-operator definition[15] to systems without equally spaced levels. For our exactly solvable examples,[9,10] the raising and lowering operators can be written in the form

$$A_n^\pm = Xc_1(n) \mp c_2 X' \frac{d}{dx}\ . \tag{5.1}$$

[Equations (3.12) and (3.13) follow from this form.] These operators are n dependent, and raise or lower the nth eigenstate only. $A_n^\pm$ applied to $|m \neq n\rangle$ does not produce a pure state. E_n is an invertible function of n,

$$E_n = g(n)\ , \tag{5.2}$$

$$n = g^{-1}(E_n)\ . \tag{5.3}$$

By substituting $g^{-1}(H)$ for n in Eq. (5.1), one obtains operators which raise or lower any eigenstate:

$$A^\pm \equiv Xc_1[g^{-1}(H)] \mp c_2 X' \frac{d}{dx}\ , \tag{5.4}$$

$$A^{\pm}|n\rangle = C_n^{\pm}|n \pm 1\rangle . \tag{5.5}$$

Thus, a possible definition for generalized AOCS is those states which are eigenfunctions of the generalized annihilation operator A^-:

$$A^-|\alpha\rangle = \alpha|\alpha\rangle . \tag{5.6}$$

In papers II and III we will investigate examples of such AOCS and will find for unequally spaced systems that they are more difficult to deal with analytically than are the minimum-uncertainty coherent states (For example, the normalization constant or the continuum contribution is not obtained in analytic closed form.) They are not equivalent to our minimum-uncertainty coherent states. Moreover, in a numerical example discussed in paper V, they are found to be slightly more spread out.

The use of the operators $A^{\pm}$ also allows a connection to the harmonic-oscillator result

$$H = \hbar\omega(a^+a^- + \tfrac{1}{2}) = \hbar\omega\tfrac{1}{2}(a^+a^- + a^-a^+) . \tag{5.7}$$

One might think that it is the first equality, giving a number operator form to H, that is the property to be generalized. However, we will observe in papers II, III, and IV that for our solvable examples there is an operator $\mathcal{H}$ such that for the discrete portion of the spectrum

$$\mathcal{H}|n\rangle = \bar{E}_n|n\rangle , \tag{5.8}$$

$$\mathcal{H} \equiv \text{const} \times (\mathcal{A}^+\mathcal{A}^- + \mathcal{A}^-\mathcal{A}^+) , \tag{5.9}$$

where $\bar{E}_n$ is either the eigenvalue E_n or differs from E_n by a constant, and $\mathcal{A}^{\pm}$ is either $A^{\pm}$ or simply related to it.

B. Displacement-operator coherent states

In an attempt to generalize the definition[16] of coherent states from systems with equally spaced levels to systems with unequally spaced levels, one might first attempt to modify (2.23) and (2.26) by replacing $a^{\pm}$ by the operator $A^{\pm}$ defined in (5.4):

$$\tilde{D}(\alpha) = \exp(\alpha A^+ - \alpha^* A^-) \overset{?}{=} \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha A^+) . \tag{5.10}$$

However, in general $(A^+)^\dagger \neq A^-$, so in general $\tilde{D}(\alpha)$ will not be unitary and the last equality will not hold.

Further, by specific example, one can show that the appropriate displacement operator is *not* always an exponential operator. In papers II and III we will discuss examples in which annihilation-operator coherent states defined by (5.6) can also be created by displacement operators which are not exponential operators.

We briefly mention one of these examples here, the symmetric Pöschl-Teller potential

$$V(x) = \mathcal{E}_0\lambda(\lambda - 1)\tan^2 z \equiv U_0 \tan^2 z , \tag{5.11}$$

$$z = ax, \quad \mathcal{E}_0 = \left(\frac{\hbar^2 a^2}{2m}\right), \tag{5.12}$$

with

$$X = \sin z , \tag{5.13}$$

$$P = \frac{\hbar a^2}{2i}\left(\cos z \frac{d}{dz} + \frac{d}{dz}\cos z\right). \tag{5.14}$$

We will demonstrate in paper II that the states defined by

$$A^-|\alpha\rangle = \alpha|\alpha\rangle , \tag{5.15}$$

with

$$A^{\pm} = \sin z[(H + U_0)/\mathcal{E}_0]^{1/2} \mp \cos z \frac{d}{dz} , \tag{5.16}$$

are

$$|\alpha\rangle = N a^{1/2}(\cos z)^{1/2} e^{\alpha \sin z}(\alpha)^{-\lambda+1/2} \times J_{\lambda-1/2}(\alpha \cos z) , \tag{5.17}$$

$$N = \left(\tfrac{1}{2}\Gamma(2\lambda + 1)|\alpha|^{-2\lambda}\int_0^{|\alpha|} dy I_{2\lambda-1}(2y)\right)^{-1/2} . \tag{5.18}$$

These are equivalent to the states defined by

$$D(\alpha)|0\rangle = |\alpha\rangle , \tag{5.19}$$

$$D(\alpha) = N\Gamma(2\lambda)(\alpha A^+)^{-\lambda+1/2} I_{2\lambda-1}(2(\alpha A^+)^{1/2}) , \tag{5.20}$$

where J and I are the standard Bessel functions.

This example demonstrates that in general (a) the DOCS are not the same as the minimum-uncertainty coherent states, and (b) the appropriate displacement operator $D(\alpha)$ is not in general an exponential, but can be a more complicated functional.

VI. DISCUSSION

Before proceeding in the following two papers to investigate four systems in detail, we make a number of comments here on our coherent states and on the procedures we have used.

First, we point out that the construction of the minimum-uncertainty coherent states can be carried through approximately, if necessary. Equation (3.4) is amenable to analytic or numerical perturbation methods, as are the defining equations (A3) and (A4) for the uncertainty relation (3.11). In fact, given the operators X and P for the Morse oscillator, we obtained approximate analytic solutions, as described in Ref. 10 and in paper III.

Second, observe that the classical transformation from x and p to the variables X_c and P_c is *not* a canonical transformation. $P_c = pX_c'$ is not the canonically conjugate momentum.

$$P_{c(\mathrm{canon})}=p/X' . \quad (6.1)$$

If (6.1) is symmetrized to obtain an appropriate quantum operator, the set of states which minimizes the uncertainty product defined by the commutator

$$\left[X,\frac{1}{2}\left(p\frac{1}{X'}+\frac{1}{X'}p\right)\right]=i\hat{G}=i \quad (6.2)$$

in general will not include the ground state as a special case. (We discuss this in Appendix B.) That is, this set of minimum-uncertainty states does not provide a satisfactory quantum approximation to the classical problem because it does not have a state corresponding to a particle at rest at the minimum of the potential.

Next, notice that there is an ambiguity in the definitions of X_c and P_c. If X_c and P_c vary as $\sin\omega_c t$ and $\cos\omega_c t$, then $F_1(E)X_c$ and $F_2(E)P_c$, where F_1 and F_2 are arbitrary functions of energy, will also vary as $\sin\omega_c t$ and $\cos\omega_c t$. [This can be seen from (3.2) since A depends upon E to begin with.] Therefore, in effect, we make the further physical ansatz that one should consider an X_c which has the "simplest" possible E dependence.

We choose the "simplest" X and P operators as the basis of our MUCS, instead of more complicated operators. This is entirely consistent with the harmonic-oscillator solution. There is nothing in that system which requires one to choose x and p as the natural variables over, say, $E^{1/2}x$ and $E^{1/2}p$. On esthetic and physical grounds one chooses x and p and these variables yield the most classical coherent states.

Grounds of "simplicity" are also important in choosing the $A^{\pm}$, and hence the AOCS and DOCS. The $A^{\pm}$ are obtained from the $A_n^{\pm}$ which are appropriate for the eigenfunctions of the potential at hand. For instance, the $A_n^{\pm}$ for the symmetric Pöschl-Teller potential are, from Eq. (5.16),

$$A_n^{\pm}=(n+\lambda)\sin z \mp \cos z\frac{d}{dz} . \quad (6.3)$$

One could multiply the above operators by any random function of n, obtain new $\hat{A}_n^{\pm}$, and hence entirely different $\hat{A}^{\pm}$. The eigenfunctions of the new operators $\hat{A}^{-}$ would be different possible definitions of annihilation-operator coherent states and similarly one could use these operators to obtain different displacement-operator coherent states. Clearly, some criterion must be used to decide among the various possibilities.

We will see that for the AOCS-DOCS, "simplicity" amounts to using those $A_n^{\pm}$ which in their n-independent forms can describe the Hamiltonian $\mathcal{H}$ as in Eq. (5.9).

Finally, for potentials with more than one minimum, it is in principle possible to obtain minimum-uncertainty coherent states, although in general it is mathematically intractable. For explicitness, consider the confining potential

$$V(x)=U-2(UR)^{1/2}x^2+Rx^4 , \quad (6.4)$$

which has two minima

$$V(x=\pm(UR)^{1/4})=0 , \quad (6.5)$$

and a maximum

$$V(x=0)=U . \quad (6.6)$$

Classically, there are three possibilities. For $E<U$, the particle is confined either to the left or right of $x=0$. For $E>U$, the particle will travel through both regions. Thus, for the classical problem, there are three different types of solutions (in terms of elliptic integrals), each with its corresponding X_c and P_c. Each of these will produce, by the techniques described above, its own set of coherent states. But we know that one of those packets, say for $E<U$ and $x>0$, should eventually tunnel to the region $x<0$.

By anticipating the numerical time-evolution solutions that will be discussed in paper V, one can understand this process. In general, at $t=0$ the coherent states start off as minimum-uncertainty wave packets and eventually disperse. The coherent states are a superposition of eigenstates carefully chosen to cancel each other out except in a limited region. However, as the superposition evolves in time the cancellation gradually gets worse (because the eigenvalues are not equally spaced), until finally the wave packet is spread out.

For the quartic potential tunneling can simply be viewed as part of the process by which the coherent superposition of eigenstates slowly evolves out of the proper phase relationships.

Thus, this double-well dispersion is not fundamentally different from a single-well dispersion. Quantum mechanically there is no difference between an originally coherent packet dispersing with time to a position away from the classical position that is in the same well or in a different well it has "tunneled" to. It still has dispersed.

VII. CONCLUSIONS

This paper has described our minimum-uncertainty method for obtaining coherent states for general potentials. The reader should be aware that we have made two physical *Ansätze*. They have not been rigorously proven to be correct. But in the numerous examples we describe in this series they always work.

Our first ansatz came from our search for the classical motion. We began by looking for variables X_c and P_c which vary sinusoidally as the classical $(\omega_c t)$. The classical motion is an ellipse in the X_c-P_c plane, and in terms of X_c and P_c the classical problem is similar to the harmonic oscillator. This is why we made the first physical ansatz of taking the variables X_c and P_c to be the "natural classical variables" of the problem, and the ones on which to concentrate.

Our second physical ansatz is to take these variables and form "natural quantum operators" out of them. Just as the natural classical variables describe the classical motion most simply, the natural quantum operators are chosen as those operators which can best define a classical motion for the quantum system. Hence, the associated uncertainty relation should best define classical motion in an expectation value sense. This second ansatz can be further justified by the facts that the MUCS always turn out to include the ground state as a special case, and further that the natural quantum operators can always be obtained from Hermitian sums and differences of the quantum raising and lowering operators.

Thus, one has a connected chain of steps: (a) the classical problem; (b) the "natural classical variables," (c) the "natural quantum operators," (d) the raising and lowering operators for the quantum eigenvalue problem. *A priori* one has no rigorous reason to believe that such a sequence exists. Nonetheless, it does. In paper VI we will give an explanation of how this chain can be understood.

We have also pointed out how the annihilation operator and displacement operator methods can be generalized to systems without equal level spacing. One changes n-dependent raising and lowering operators into n-independent raising and lowering operators with the aid of functionals of the Hamiltonian. In our examples we find that given a set of AOCS, defined by operators which satisfy Eq. (5.9) for the Hamiltonian, one can always find a set of DOCS equivalent to them,[26] but these AOCS-DOCS will not necessarily be defined by a unitary exponential displacement operator, nor will they necessarily be equivalent to our MUCS.

The methods we have used are analytic and serve as a complement to the group-theoretic point of view.[15-18,27,28] Note that the operators $A_n^{\pm}$ and $(A_n^{\pm})^{\dagger}$, which arise naturally in our approach, do not have simple commutation relations. They do not appear to belong to an elementary Lie algebra of the sort heretofore employed in the construction of generalized coherent states. Nevertheless, we assume that our method and the Lie-group methods are related simply because of the connections of Lie-group theory to the special functions of mathematical physics.[28] It would be interesting to see our special case results obtained from the group-theory point of view.

As mentioned in the Introduction, these results have immediate interesting applications to coherent quantum-mechanical systems.[20,29] Moreover, with the results of this series, one might be able to extend studies[20,26,29-34] of field-theoretic coherent systems to general (nonharmonic-oscillator) backgrounds. We have discussed this possibility with many of our colleagues and hope such a study can be implemented.

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APPENDIX A: MINIMUM-UNCERTAINTY STATES

As is well known,[5,35,36] given two Hermitian operators A and B, whose commutation relation is given by

$$[A, B] = iG\,, \tag{A1}$$

where G may also be an operator, there is an associated uncertainty relation

$$(\Delta A)^2(\Delta B)^2/\langle G\rangle^2 \geq \tfrac{1}{4}\,, \tag{A2}$$

$$(\Delta A)^2 \equiv \langle A^2\rangle - \langle A\rangle^2\,. \tag{A3}$$

The three parameter set of states which minimize Eq. (A2) is composed of the solutions to the eigenvalue equation

$$\left(A + \frac{i\langle G\rangle}{2(\Delta B)^2}B\right)\psi = \left(\langle A\rangle + \frac{i\langle G\rangle}{2(\Delta B)^2}\langle B\rangle\right)\psi\,. \tag{A4}$$

If $\langle G\rangle$ is positive definite, Eq. (A4) can also be written

$$\left(\frac{A}{\Delta A} + i\frac{B}{\Delta B}\right)\psi = \left(\frac{\langle A\rangle}{\Delta A} + i\frac{\langle B\rangle}{\Delta B}\right)\psi\,. \tag{A5}$$

Note that the four parameters $\langle A\rangle$, $\langle A^2\rangle$, $\langle B\rangle$, and $\langle B^2\rangle$ are not independent because they satisfy the equality in Eq. (A2). The remaining three independent parameters may be reduced to two by imposing another condition: that the set include the ground state as a special case. This can be stated as a restriction on $(\Delta A)^2/(\Delta B)^2$, leading to the minimum-uncertainty coherent states (MUCS).

APPENDIX B: CANONICAL TRANSFORMATION

The classical canonical transformation from the variables x, p to the variables X_c and $P_{c(\mathrm{canon})}$, is defined by the Poisson bracket relation

$$[X_c, P_{c(\text{canon})}]_{x,p} = 1\,. \tag{B1}$$

If X_c is only a function of x,

$$P_{c(\text{canon})} = p/X' \tag{B2}$$

(plus a constant which can be dropped). Then the canonical quantum operator $P_{(\text{canon})}$ will have the form

$$P_{(\text{canon})} = \frac{1}{2i}\left(\frac{1}{X'}\frac{d}{dx} + \frac{d}{dx}\frac{1}{X'}\right), \tag{B3}$$

giving

$$[X, P_{(\text{canon})}] = i\,, \tag{B4}$$

$$(\Delta X)^2 (\Delta P_{(\text{canon})})^2 \geq \tfrac{1}{4}\,. \tag{B5}$$

Compare these expressions to the results in Sec. III: $P_c = pX_c'$ and see Eqs. (3.9), (3.10), and (3.11).

A specific example shows that in general the minimum-uncertainty states for the relation (B5) do not contain the ground state as a special case. For the symmetric Rosen-Morse potential

$$V(x) = \mathcal{E}_0 s(s+1)\tanh^2 z\,, \quad z \equiv ax\,, \tag{B6}$$

and X_c is

$$X_c = \sinh z\,. \tag{B7}$$

Therefore,

$$X = \sinh z\,, \tag{B8}$$

$$P_{(\text{canon})} = \frac{1}{2i}\left(\frac{1}{\cosh z}\frac{d}{dz} + \frac{d}{dz}\frac{1}{\cosh z}\right) = \frac{1}{i}\left(\frac{1}{\cosh z}\frac{d}{dz} - \frac{1}{2}\frac{\sinh z}{\cosh^2 z}\right), \tag{B9}$$

and

$$[X, P_{(\text{canon})}] = i\,. \tag{B10}$$

The defining equation for the minimum-uncertainty states reduces to

$$\left[\sinh z\left(1 - \frac{B}{2\cosh^2 z}\right) + \frac{B}{\cosh z}\frac{d}{dz}\right]\psi_{\text{MUS}} = C\psi_{\text{MUS}}\,, \tag{B11}$$

$$B = (\Delta P_{(\text{canon})})^{-2}\,, \tag{B12}$$

$$C \equiv u + iv = \langle \sinh z\rangle + iB\langle P_{(\text{canon})}\rangle\,. \tag{B13}$$

The solution is

$$\psi_{\text{MUS}} = N(\cosh^{1/2} z)\exp\left(-\frac{1}{2B}\sinh^2 z + \frac{C}{B}\sinh z\right). \tag{B14}$$

Clearly no choice of the three parameters B, u, and v will reduce ψ_{MUS} to the ground-state wave function which is proportional to $(\cosh z)^{-s}$ (see paper III).

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