Clouds of strings in general relativity

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A gauge-invariant version of the Stachel string cloud model is presented. Some formal aspects as well as the energy conditions for the model are studied. The general solution to Einstein's equations for a cloud of strings with spherical, plane, and a particular case of cylindrical symmetry are studied. The solution with spherical symmetry is used to construct a double-layer model of a star.

I. INTRODUCTION

In a recent paper[1] an extension of the relativistic "dust cloud" model for a perfect fluid[2] was proposed. In this new model the objects that form the cloud are one-dimensional continua (classical relativistic strings) instead of point particles.

The purpose of this paper is to present a gauge-invariant version of "the incoherent string fluid model" as well as to study some formal aspects and particular solutions of Einstein's equations for the above-mentioned model.

We have two main reasons to study Einstein's equations coupled with a "string cloud." First, the relativistic strings at a classical level can be used to construct good models for many interactions.[3-5] One type of models are the field theories associated with given action-at-a-distance interactions.[4] This formalism has been developed using the same ideas employed in the study of action-at-a-distance interactions between particles.[6,7] Then, as a test of consistency, we must have a reasonable behavior of the gravitational field produced by the main elements of these models[8] (strings). Second, the universe can be represented as a collection of extended (nonpoint) objects.[9] So a "string dust" cosmology should give us a model to investigate properties related with this fact. Also, a quantization of such a model may shed some light on the question of discretization in astronomy.[10]

In the present paper we do not touch upon the cosmological aspects of the string cloud model. In Sec. II we present a summary of the incoherent fluid model; in particular, we stress the role played by the parametrization of the particles' world lines. In Sec. III we present a gauge-invariant version of the string cloud model, following the lines given in Sec. II. The method used to derive these equations is a slight generalization of the one employed in Ref. 1. In Sec. IV we study the different energy conditions for the string cloud model. In Sec. V we present the general solution to Einstein's equations for a cloud of strings with spherical symmetry. The solution is used to construct a double-layer model of a star. In Sec. VI we present the general solution to Einstein's equations for a plane-symmetric cloud of strings. The solution found presents some similarities with the solution to Einstein-Maxwell equations for the same symmetry.[11] In Sec. VII we study the solution to Einstein's equation for a cloud of strings with a particular type of cylindrical symmetry. In Sec. VIII we discuss the problem of the open strings' end points. We also point out some possible applications as well as generalizations of the presented theory.

II. THE DUST CLOUD MODEL

In this section we study the incoherent perfect fluid model, paying special attention to the invariance of the theory under reparametrization of the particles' world lines. The metric of the space-time is $g_{\mu\nu}$ of signature $(+---)$, where the Greek indices run from 0 to 3. The units are chosen so that the speed of light is one. The action of a particle evolving in the space-time is

$$A=\int L d\lambda\,, \tag{2.1}$$

$$L\equiv m(g_{\mu\nu}u^\mu u^\nu)^{1/2}\,, \tag{2.2}$$

$$u^\mu \equiv dz^\mu/d\lambda\,, \tag{2.3}$$

where m is the particle proper mass, $z^\mu=z^\mu(\lambda)$ is the particle world line and λ is an evolution parameter. The energy-momentum tensor of a particle is defined by

$$t_p^{\mu\nu}\equiv 2\frac{\partial L}{\partial g_{\mu\nu}}=mu^\mu u^\nu/(u^\alpha u_\alpha)^{1/2}\,. \tag{2.4}$$

Let us consider a cloud of particles with world lines described[12] by $Z^\mu=Z^\mu(\lambda,\xi,\eta,\zeta)$, where ξ, η, and ζ are variables labeling a particular world line and λ is a parameter describing the evolution of this particular world line (as before). The cloud of particles is also characterized by the proper energy density ρ_p. ρ_p is the energy per unit volume measured by an observer locally at rest with respect to the cloud, i.e., by an observer with the same local mean velocity of the cloud particles. For a discussion of this point see, for instance, Ref. 2, p. 168. Now the energy-momentum tensor of a cloud of particles is

$$T_p^{\mu\nu}=\rho_p u^\mu u^\nu/(u^\alpha u_\alpha)^{1/2}. \quad (2.5)$$

where the velocity vector u^μ is defined by $u^\mu \equiv \partial Z^\mu/\partial\lambda$.

By invariance of the theory under reparametrization we mean invariance under the transformation

$$\lambda \to \lambda^* = \lambda(\lambda, \xi, \eta, \zeta), \quad \frac{\partial\lambda^*}{\partial\lambda} \neq 0. \quad (2.6)$$

We shall also require that the interval $(-\infty, +\infty)$ be left invariant by the transformation (2.6). Note that under the transformation (2.6) Z^μ transforms like a scalar and the velocity as

$$u^{*\mu}(\lambda^*, \xi, \eta, \zeta) = \frac{\partial\lambda}{\partial\lambda^*} u^\mu(\lambda, \xi, \eta, \zeta), \quad (2.7)$$

where

$$u^{*\mu} = \partial Z^\mu/\partial\lambda^*.$$

To have a $T_p^{\mu\nu}$ invariant under reparametrization ρ_p must transform as $(u^\alpha u_\alpha)^{-1/2}$, (for a deeper discussion of this point see Ref. 13). The following expressions with obvious physical meaning are reparametrization invariant,

$$\rho_p(u_\alpha u^\alpha)^{1/2}, \quad (2.8a)$$

$$\rho_p u^\mu, \quad (2.8b)$$

$$u^\mu/(u_\alpha u^\alpha)^{1/2}. \quad (2.8c)$$

Einstein's equations for the cloud are

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = -T_p^{\mu\nu}, \quad (2.9)$$

where we have taken the coupling constant equal to one. The Bianchi identities tell us that

$$\nabla_\mu T_p^{\mu\nu} = 0, \quad (2.10a)$$

i.e.,

$$\nabla_\mu(\rho_p u^\mu) u^\nu/(u^\alpha u_\alpha)^{1/2} + \rho_p u^\mu \nabla_\mu[u^\nu/(u^\alpha u_\alpha)^{1/2}] = 0, \quad (2.10b)$$

where ∇_μ denotes the usual covariant derivative. Upon a multiplication of (2.10b) by $u_\nu/(u^\alpha u_\alpha)^{1/2}$ we get the conservation equation

$$\nabla_\mu(\rho_p u^\mu) = 0, \quad (2.11)$$

and from (2.10) and (2.11) we obtain the motion equation

$$u^\mu \nabla_\mu [u^\nu/(u^\alpha u_\alpha)^{1/2}] = 0. \quad (2.12)$$

We recover the usual expressions for the dust cloud[14] by choosing a parametrization such that $u^\alpha u_\alpha = 1$,

$$T_p^{\mu\nu} = \rho_p u^\mu u^\nu, \quad (2.13)$$

$$\nabla_\mu(\rho_p u^\mu) = 0, \quad (2.14)$$

$$u^\mu \nabla_\mu u^\nu = 0. \quad (2.15)$$

Note that (2.15) implies

$$u^\mu \nabla_\mu (u^\alpha u_\alpha) = 0, \quad (2.16)$$

i.e., the parametrization is propagated by the motion equations. In other words, the condition $u^\alpha u_\alpha = 1$ can be treated as an initial condition. Of course, this is not the only parametrization that enjoys this property; as an example let us choose $(u^\alpha u_\alpha)^{1/2} = \rho_p$. Then the equations are

$$T_p^{\mu\nu} = u^\mu u^\nu, \quad (2.17)$$

$$\nabla_\mu(\rho_p u^\mu) = 0, \quad (2.18)$$

$$u^\mu \nabla_\mu (u^\nu/\rho_p) = 0. \quad (2.19)$$

Note that (2.19) implies

$$u^\mu \nabla_\mu \ln[(u^\alpha u_\alpha)^{1/2}/\rho_p] = 0. \quad (2.20)$$

Hence, as before, the parametrization is propagated by the motion equations.

III. THE STRING CLOUD MODEL

In this section we study the Einstein equations for a cloud of strings using the model discussed in Sec. II as a guideline. The method that we follow to derive the Einstein equations for a cloud of strings is a slight modification of the one presented in Ref. 1. The only difference is that we do not choose a particular parametrization of the strings world sheets at our starting point, in the same sense that for the cloud of particles we did not impose the condition $u^\mu u_\mu = 1$ from the beginning.

The action of a string evolving in the space-time is[15]

$$\mathcal{A} \equiv \int \mathcal{L}\, d\lambda^0\, d\lambda^1, \quad (3.1)$$

$$\mathcal{L} \equiv M\sqrt{-\gamma}, \quad (3.2)$$

where M is a dimensionless constant that characterizes each string[16] and

$$\gamma \equiv \det \gamma_{AB}, \quad (3.3)$$

$$\gamma_{AB} \equiv g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial\lambda^A} \frac{\partial x^\nu}{\partial\lambda^B}. \quad (3.4)$$

$x^\mu = x^\mu(\lambda^A)$ describes the string world sheet. $\lambda^A \equiv (\lambda^0, \lambda^1)$, λ^0, and λ^1 are a timelike and a spacelike parameter.[15]

Associated with the string world sheet we have the bivector

$$\Sigma^{\mu\nu}=\epsilon^{AB}\frac{\partial x^\mu}{\partial\lambda^A}\frac{\partial x^\nu}{\partial\lambda^B}, \tag{3.5}$$

where ϵ^{AB} is the two-dimensional Levi Civita symbol normalized as follows: $\epsilon^{01}=-\epsilon^{10}=1$. Thus, the Lagrangian density can also be written as

$$\mathfrak{L}=M(-\tfrac{1}{2}\Sigma^{\alpha\beta}\Sigma_{\alpha\beta})^{1/2}. \tag{3.2$'$}$$

The energy-momentum tensor for one string is

$$t^{\mu\nu}\equiv 2\frac{\partial}{\partial g_{\mu\nu}}\mathfrak{L}, \tag{3.6}$$

$$t^{\mu\nu}=M\Sigma^{\mu\beta}\Sigma_\beta{}^\nu/(-\gamma)^{1/2}. \tag{3.7}$$

Let us consider a cloud of strings with world sheets described by $X^\mu=X^\mu(\lambda^A,\xi,\eta)$, where ξ and η are variables labeling a particular world sheet and λ^A are parameters describing the evolution of this particular world sheet. The cloud of strings is also characterized by a proper density ρ. Now the energy-momentum tensor for a cloud of strings is[17]

$$T^{\mu\nu}=\rho\Sigma^{\mu\beta}\Sigma_\beta^\nu/(-\gamma)^{1/2}. \tag{3.8}$$

By invariance of the theory under reparametrization we mean invariance under the transformation

$$\lambda^0\rightarrow\lambda^{0*}=\lambda^{0*}(\lambda^A,\xi,\eta), \tag{3.9a}$$

$$\lambda^1\rightarrow\lambda^{1*}=\lambda^{1*}(\lambda^A,\xi,\eta). \tag{3.9b}$$

We shall require that the Jacobian of the above transformation be different from zero and that the interval $(-\infty,+\infty)$ be left invariant by the transformation (3.9a). We shall also refer to the above-mentioned invariance as "gauge invariance." Note that under the transformation (3.9) the world sheet X^μ transforms as a scalar and the bivector $\Sigma^{\mu\nu}$ as

$$\Sigma^{\mu\nu*}(\lambda^{A*},\xi,\eta)=J\left(\frac{\lambda^C}{\lambda^{B*}}\right)\Sigma^{\mu\nu}(\lambda^A,\xi,\eta), \tag{3.10}$$

where

$$\Sigma^{\mu\nu*}=\epsilon^{A*B*}\frac{\partial}{\partial\lambda^{A*}}X^\mu\frac{\partial}{\partial\lambda^{B*}}X^\nu. \tag{3.11}$$

To have a $T^{\mu\nu}$ invariant under a reparametrization of the string's world sheets ρ must transform as $(-\gamma)^{-1/2}$ (for a deeper discussion of this point see Ref. 13). The following quantities are manifestly gauge invariant:

$$\rho(-\gamma)^{1/2}, \tag{3.12a}$$

$$\rho\Sigma^{\mu\nu}, \tag{3.12b}$$

$$\Sigma^{\mu\nu}/(-\gamma)^{1/2}. \tag{3.12c}$$

Note the similarity of these expressions with the corresponding expression for particles, (2.8).

We shall refer to the gauge-invariant quantity $(-\gamma)^{1/2}\rho$ as the gauge-invariant density.

We have two ways to characterize the strings: first, by its world sheet, i.e., $x^\mu=x^\mu(\lambda^A)$ and second, by a surface forming bivector $\Sigma^{\mu\nu}$. This last characterization, called the intrinsic characterization,[1] is more useful for our purpose. The conditions for $\Sigma^{\mu\nu}$ to be a surface forming bivector[18] are

$$\Sigma^{\mu[\alpha}\Sigma^{\beta\gamma]}=0, \tag{3.13}$$

$$\nabla_\mu\Sigma^{\mu[\alpha}\Sigma^{\beta\gamma]}=0, \tag{3.14}$$

where the square brackets denote antisymmetrization in the enclosed indices. The first condition tells us that the bivector is simple, i.e., that it can be written as $\epsilon^{AB}X_A^\mu X_B^\nu$, and the second tells us that the two vectors X_0 and X_1 fit together to form a surface, i.e., $X_A^\mu=\partial_A X^\mu$. Two useful identities are

$$\Sigma^{\alpha\beta}\nabla_\alpha\Sigma_{\beta\nu}=\tfrac{3}{2}\Sigma^{\alpha\beta}\partial_{[\alpha}\Sigma_{\beta\nu]}-\tfrac{1}{4}\nabla_\nu(\Sigma^{\alpha\beta}\Sigma_{\alpha\beta}), \tag{3.15}$$

$$\Sigma^{\mu\alpha}\Sigma_{\alpha\beta}\Sigma^{\beta\nu}=\gamma\Sigma^{\nu\mu}. \tag{3.16}$$

The identity (3.15) is valid for any antisymmetric tensor and (3.16) is a consequence of (3.13), (3.3), and (3.5).

The Bianchi identity and (3.8) give us

$$\nabla_\mu(\rho\Sigma^{\mu\beta})\Sigma_\beta{}^\nu/(-\gamma)^{1/2}+\rho\Sigma^{\mu\beta}\nabla_\mu[\Sigma_\beta{}^\nu/(-\gamma)^{1/2}]=0. \tag{3.17}$$

Upon multiplication of Eq. (3.17) by $\Sigma_{\nu\alpha}/(-\gamma)^{1/2}$ we get

$$[\nabla_\mu(\rho\Sigma^{\mu\beta})\Sigma_\beta{}^\nu\Sigma_{\nu\alpha}/\gamma]=0. \tag{3.18}$$

In finding this result we have made use of (3.15) and (3.13). From (3.17) and (3.18) we get

$$\nabla_\mu(\rho\Sigma^{\mu\beta})\Sigma_\beta{}^\nu=0. \tag{3.19}$$

Using the special representation of $\Sigma^{\mu\nu}$ given by Eq. (3.5) and adapting the coordinates to the parametrization we get the "conservation law"

$$\nabla_\mu(\rho\Sigma^{\mu\nu})=0. \tag{3.20}$$

Note that Eqs. (3.20) and (3.13) imply Eq. (3.14). From (3.19) and (3.17) we obtain

$$\Sigma^{\mu\beta}\nabla_\mu[\Sigma_\beta^\nu/(-\gamma)^{1/2}]=0. \tag{3.21}$$

Taking the particular representation of $\Sigma^{\mu\beta}$ given by (3.5) one can cast (3.21) into

$$\nabla^{(\gamma)A}\nabla_A^{(\gamma)}X^\mu+\left\{{\mu \atop \alpha\beta}\right\}\gamma^{AB}\partial_A X^\alpha\partial_B X^\beta=0, \tag{3.21$'$}$$

where $\nabla_A^{(\gamma)}$ denotes the covariant derivative on the surface world sheet. Equation (3.21′) is the equation for a minimal two-dimensional surface embedded in a Riemannian space.[19] The fact that the string equation of motion (3.21) can be derived from the Bianchi identity is well known.[20]

Equations (3.21) and (3.19) are manifestly gauge invariant. Furthermore, if we choose a particular parametrization, let us say

$$\Sigma^{\mu\nu}\Sigma_{\mu\nu} = -2\sigma^2 , \tag{3.22}$$

Eq. (3.21) will propagate it. From (3.21), (3.22), (3.15), (3.16), and (3.13) we get

$$\Sigma^{\mu\nu}\nabla_\mu \ln(-\sigma^2/\Sigma_{\alpha\beta}\Sigma^{\alpha\beta}) = 0 . \tag{3.23}$$

Thus, the parametrization (3.22) is propagated by the field equations.

Hence, in summary we have that the Einstein equations plus their integrability conditions for a cloud of strings are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\rho \Sigma_\mu{}^\alpha \Sigma_{\alpha\nu}/(-\gamma)^{1/2} \tag{3.24}$$

and Eqs. (3.13), (3.20), and (3.21). Equation (3.24) can also be cast as

$$R_{\mu\nu} = \rho[(-\gamma)^{1/2} g_{\mu\nu} - \Sigma_\mu{}^\alpha \Sigma_{\alpha\nu}/(-\gamma)^{1/2}] . \tag{3.25}$$

$$R_{\mu\nu} = -[\rho/(-\gamma)^{1/2}](\Sigma_\mu{}^\alpha \Sigma_{\alpha\nu} + \tfrac{1}{2} g_{\mu\nu} \Sigma^{\alpha\beta}\Sigma_{\alpha\beta}) . \tag{3.26}$$

Equation (3.25) will be useful when dealing with particular symmetries and Eq. (3.26) shows a remarkable resemblance with Einstein-Maxwell equations. Equation (3.20) is equivalent to

$$\partial_\mu[\rho(-g)^{1/2}\Sigma^{\mu\nu})] = 0 . \tag{3.27}$$

For *some particular cases we* can have

$$\Sigma^{\alpha\beta}\partial_{[\alpha}\Sigma_{\beta\nu]} = 0 , \tag{3.28}$$

e.g., if we have only $\Sigma_{01} = -\Sigma_{10} \neq 0$. When (3.28) holds, Eq. (3.21) is equivalent to

$$(\Sigma^{\mu\beta}\Sigma_{\beta\nu}/\gamma + \delta^\mu{}_\nu)\partial_\mu(-\gamma)^{1/2} = 0 , \tag{3.29}$$

as a consequence of (3.15).

The closest analog to the "usual" gauge employed when dealing with particles $u^\mu u_\mu = 1$ is $\gamma = -1$, i.e., $\Sigma_{\mu\nu}\Sigma^{\mu\nu} = -2$. This is the gauge used by Stachel[1,21] to study the string cloud model. Note that this gauge restricts the model because, in general, γ can take the value zero. In particular, for open strings we have that the usual boundary conditions imply $\gamma = 0$ at the open strings' end points.[17] Also closed strings can have $\gamma = 0$ at some singular points.[22]

To end this section let us study the behavior of $\Sigma^{\mu\nu}$ and ρ when the space-time has a symmetry described by a Killing vector[23] ξ^μ, i.e.,

$$\mathcal{L}_\xi g_{\mu\nu} = 0 , \tag{3.30}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative. It is defined as[23]

$$\mathcal{L}_\xi A_{\mu\nu} \equiv \xi^\lambda \partial_\lambda A_{\mu\nu} + A_{\lambda\nu}\partial_\mu \xi^\lambda + A_{\mu\lambda}\partial_\nu \xi^\lambda , \tag{3.31}$$

where $A_{\mu\nu}$ represents an arbitrary second-rank covariant tensor. For scalars the Lie derivative reduces to $\mathcal{L}_\xi \phi = \xi^\lambda \partial_\lambda \phi$. As a consequence of (3.30) we have

$$\mathcal{L}_\xi R_{\mu\nu} = 0 . \tag{3.32}$$

Equations (3.24) imply that

$$\mathcal{L}_\xi \Sigma_{\mu\nu} = 0 , \tag{3.33}$$

$$\mathcal{L}_\xi \rho = 0 , \tag{3.34}$$

i.e., that a space-time symmetry must also be a symmetry of $\Sigma^{\mu\nu}$ and ρ.

IV. ENERGY CONDITIONS

In this section we study the restrictions that different energy conditions impose on the bivector $\Sigma^{\mu\nu}$. We have that the most natural energy conditions[24] are the following:

(a) The weak energy condition, i.e.,

$$T^{\mu\nu}U_\mu U_\nu \geq 0 , \tag{4.1}$$

where U_μ is an arbitrary timelike vector:

$$U_\mu U^\mu > 0 . \tag{4.2}$$

From (3.8) and (4.1) we find

$$\rho U_\mu \Sigma^{\mu\alpha}\Sigma_\alpha{}^\nu U_\nu/(-\gamma)^{1/2} \geq 0 . \tag{4.3}$$

(b) The dominant energy condition, i.e.,

$$T^{\mu\nu}U_\mu U_\nu > 0 , \tag{4.4}$$

$$U_\mu T^{\mu\alpha} T_\alpha{}^\nu U_\nu \geq 0 . \tag{4.5}$$

From the identity (3.16) we find that this condition gives

$$\rho^2 U_\mu \Sigma^{\mu\alpha}\Sigma_\alpha{}^\nu U_\nu \geq 0 . \tag{4.6}$$

(c) The strong energy condition,[25] i.e.,

$$(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) U^\mu U^\nu \geq 0 . \tag{4.7}$$

From (4.2) and (3.8) we find that (4.7) is equivalent to

$$\rho \hat{U}^\mu \Sigma_\mu{}^\alpha \Sigma_{\alpha\nu} \hat{U}^\nu/(-\gamma)^{1/2} > \rho(-\gamma)^{1/2} , \tag{4.8}$$

where

$$\hat{U}^\mu = U^\mu/(U^\alpha U_\alpha)^{1/2} . \tag{4.9}$$

Note that the expressions (4.3), (4.6), and (4.8) are gauge invariant.

V. STRING CLOUD WITH SPHERICAL SYMMETRY

In this section we study the general solution to Einstein's equations for a cloud of strings with spherical symmetry. The method that we follow to solve the above-mentioned equations is as follows: First, we solve the Einstein equations for the general static spherical symmetric metric[26]

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{5.1}$$

where ν and λ are functions of r. Next we prove that the static solution is the general one.

The symmetries of the space-time under consideration restrict the density ρ and the bivector $\Sigma_{\mu\nu}$ to be functions of r alone. Furthermore, these symmetries restrict $\Sigma_{\mu\nu}$ to have only two independent components different from zero, Σ_{01} and Σ_{23}. The previous statements can be proved directly from (3.33), (3.31) and the explicit form of the three ξ^{μ} that characterize the spherical symmetry.[27] Also, it is a rather intuitive fact that a spherically symmetric bivector $\Sigma^{\mu\nu}$ must have its "electric vector" (Σ^{0i}, $i=1,2,3$) as well as its "magnetic vector" ($\Sigma^k \equiv \Sigma^{ij}$, cyclic permutation in i,j,k) along the radial direction. Equations (3.13) and the condition $\gamma<0$ tell us that out of Σ_{01} and Σ_{23} only Σ_{01} survives. Thus, the general solution to Eq. (3.20) is

$$\Sigma^{01} = \frac{a}{\rho r^2} e^{-(\lambda+\nu)/2}, \tag{5.2}$$

where a is an integration constant. Note that the guage-invariant density $(-\gamma)^{1/2}\rho$ has the value

$$(-\gamma)^{1/2}\rho = a/r^2; \tag{5.3}$$

thus, a is a positive constant.

Equation (3.29) is trivially satisfied by (5.2). From (5.1)–(5.3) we find that the Einstein equations (3.21) reduce to

$$2\nu'' - \lambda'\nu' + 4\nu'/r + \nu'^2 = 0, \tag{5.4}$$

$$2\nu'' - \lambda'\nu' - 4\lambda'/r + \nu'^2 = 0, \tag{5.5}$$

$$e^{-\lambda}[1 + \tfrac{1}{2} r(\nu' - \lambda')] - 1 = -a. \tag{5.6}$$

From (5.4) and (5.5) we find

$$\nu' + \lambda' = 0, \tag{5.7}$$

so

$$\nu = -\lambda, \tag{5.8}$$

where we have omitted the constant of integration, because it can always be set equal to zero by a suitable redefinition of the time coordinate.

Equations (5.6) and (5.7) give us

$$(e^{-\lambda} r)' = 1 - a. \tag{5.9}$$

Thus,

$$e^{\nu} = e^{-\lambda} = 1 - a - 2m/r, \tag{5.10}$$

where m is an integration constant.

Now let us consider the general case, i.e., $\Sigma^{\mu\nu} = \Sigma^{\mu\nu}(t,r)$, $\rho=\rho(t,r)$, $\lambda=\lambda(t,r)$, and $\nu=\nu(t,r)$. As before, we find that the only surviving components of $\Sigma_{\mu\nu}$ are $\Sigma_{01} = -\Sigma_{10}$. Thus, we have

$$T_{01} = \rho \Sigma_0{}^{\mu} \Sigma_{\mu 1}/(-\gamma)^{1/2} = 0. \tag{5.11}$$

But $T_{01}=0$ and the Einstein equations (3.21) imply that λ is time independent,[28] and from this fact it follows that Σ_{01}, ρ, and ν are also time independent. In other words, we have a "Birkhoff theorem" for the cloud of strings. Thus, the solution exhibited before is the general solution for the symmetry under consideration.

The strong energy condition for the present solution reduces to

$$r^2[(\hat{U}^2)^2 + \sin^2\theta(\hat{U}^3)^2] > 0. \tag{5.12a}$$

The weak and the dominant energy conditions give us

$$r^2[(U^2)^2 + \sin^2\theta(U^3)^2] \geq 0. \tag{5.12b}$$

Note that these conditions do not impose any restriction on the value of the constant a.

The constants a and m are not related, thus, the metric

$$ds^2 = \left(1 - a - \frac{2m}{r}\right)dt^2 - \left(1 - a - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{5.13}$$

represents the space-time associated with a particle of mass m centered at the origin of the system of coordinates surrounded by a spherical cloud of strings of gauge-invariant density $(-\gamma)^{1/2}\rho = a/r^2$. The solution (5.13) has a horizon of radius

$$r_s = \frac{2m}{1-a}, \quad a \neq 1. \tag{5.14}$$

If a is less than one we have that the cloud of strings enlarge the Schwarzschild radius of the particle by the amount $(1-a)^{-1}$. When $a>1$, (5.13) represents a homogenous space-time. The cloud of strings alone ($m=0$) does not have horizons; it only has a naked singularity at $r=0$. This can be seen from the fact that

$$R = 2(-\gamma)^{1/2}\rho = 2a/r^2, \tag{5.15}$$

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} + \frac{16ma}{r^5} + \frac{4a^2}{r^4}. \tag{5.16}$$

Note that if $m=0$ and $a=1$, we still have a well-behaved metric as (5.15) and (5.16) indicate. This fact can also be seen by making a simple change of coordinates in (5.13).

As an application of the previous results, let us construct a simple model of a star. Because of the singular character of $(-\gamma)^{1/2}\rho$ at $r=0$, (5.13) cannot be used alone to construct a star, but it can be used as an outer layer of a multilayer star. A usual multilayer model of a star[29] is a model where each layer is formed by a fluid with a different state equation. Hence, the solution to Einstein's equations for a fluid sphere can be used to describe physical properties of this system by a suitable choice of the state equation for each layer. The main mathematical problem presented by these models is that the metrics associated with each layer must match continuously, one to each other. And the metric that represents the outermost layer must match continuously to the Schwarzschild metric. As an example we shall consider a two-layer star, the core formed by a perfect fluid of constant proper density ρ_0 and pressure p, and the outer layer formed by a cloud of strings. Thus the metric and the pressure for the "inner layer" is[30]

$$ds^2 = B^2[3(1-\tfrac{1}{3}\rho_0 r_0^2)^{1/2} - (1-\tfrac{1}{3}\rho_0 r^2)^{1/2}]\,dt^2 - \frac{dr^2}{1-\frac{1}{3}\rho_0 r^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{5.17}$$

$$P = \rho_0 \frac{(1-\frac{1}{3}\rho_0 r^2)^{1/2} - (1-\frac{1}{3}\rho_0 r_0^2)^{1/2}}{3(1-\frac{1}{3}\rho_0 r_0^2)^{1/2} - (1-\frac{1}{3}\rho_0 r^2)^{1/2}}\,, \tag{5.18}$$

where the range r is $0<r<r_0$, r_0 is the "radius" of the core, and B is a constant to be determined. Note that the pressure vanishes at $r=r_0$. The metric for the outer layer is taken as (5.13), but now the coordinate r is restricted to $r_0<r<r_1$, where r_1 is the star "radius." Outside the star $(r>r_1)$ the metric is taken as

$$ds^2 = (1-2M/r)dt^2 - (1-2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{5.19}$$

i.e., the Schwarzschild metric.

The condition that (5.17) and (5.13), and (5.13) and (5.19) match continuously at $r=r_0$ and $r=r_1$, respectively, gives us

$$B^2 = \tfrac{1}{4}\,, \tag{5.20}$$

$$2m = \tfrac{1}{3}\rho_0 r_0^3 - a r_0\,, \tag{5.21}$$

$$2M = a(r_1 - r_0) + \tfrac{1}{3}\rho_0 r_0^3\,. \tag{5.22}$$

The model under consideration has a clear meaning only if the three line elements of the different regions of the space-time are static. This condition is implemented by

$$\rho_0 r_0^3 < 1\,, \tag{5.23}$$

$$r_0 > \frac{2m}{1-a}\,,\quad a<1 \tag{5.24}$$

$$r_1 > 2M\,. \tag{5.25}$$

The condition (5.23) and Eqs. (5.21) and (5.22) imply (5.24) and (5.25). Thus, the model is consistent.

The model that we have considered presents a novelty in the fact that the different layers that form the star are made of different kinds of matter. We have presented the most simple model to show with an example that the matching of a perfect fluid with a string dust can be performed. This fact can be used to construct more elaborated models of star, e.g., the classical analog of a star formed by quarks attached by strings.[13] (This point is under active consideration by the author.)

VI. STRING CLOUD WITH PLANE SYMMETRY

In this section we study the general solution to Einstein's equations for a cloud of strings with plane symmetry. The general plane-symmetric metric in null coordinates is[31]

$$ds^2 = 2e^{\omega}du\,dv - e^{\mu}(dx^2 + dy^2)\,, \tag{6.1}$$

where ω and μ are functions of u and v. The symmetries of (6.1) restrict $\Sigma^{\mu\nu}$ and ρ to be only functions of u and v. Furthermore, these symmetries restrict $\Sigma_{\mu\nu}$ to have only two independent components Σ_{+-} and Σ_{23}, where by "+" and "−" we refer to the coordinates u and v, respectively. Equation (3.9) and the condition $\gamma<0$ tell us that only Σ_{+-} and Σ_{-+} are different from zero. Thus, the general solution to Eq. (3.20) is

$$\Sigma^{+-} = (a/\rho)e^{-(\omega+\mu)}\,, \tag{6.2}$$

where a is an integration constant. From (6.2) we find that the gauge-invariant density $(-\gamma)^{1/2}\rho$ has the value

$$(-\gamma)^{1/2}\rho = a e^{-\mu}\,, \tag{6.3}$$

thus, a must be such that $(-\gamma)^{1/2}\rho>0$.

Equation (3.29) is trivially satisfied by (6.2). Now the Einstein equations (3.25) reduce to

$$\mu_{++} + \tfrac{1}{2}\mu_+^2 - \mu_+\omega_+ = 0\,, \tag{6.4}$$

$$\mu_{+-} + \omega_{+-} + \tfrac{1}{2}\mu_+\mu_- = 0\,, \tag{6.5}$$

$$\mu_{--} + \tfrac{1}{2}\mu_-^2 - \mu_-\omega_- = 0\,, \tag{6.6}$$

$$(\mu_{++} + \mu_+\mu_-)e^{\mu} = a e^{\omega}\,, \tag{6.7}$$

where we have introduced the notation $\mu_+ \equiv \partial_+\mu$, $\omega_- \equiv \partial_-\omega$, etc.

From (6.4) and (6.5) we get

$$\omega - \tfrac{1}{2}\mu = \ln[\mu_+ G_-(v)]\,, \tag{6.8a}$$

$$\omega - \tfrac{1}{2}\mu = \ln[\mu_- F_+(u)]\,, \tag{6.8b}$$

where F and G are arbitrary functions of their arguments. Thus, Eqs. (6.8) give us

$$\mu_+ G_- = \mu_- F_+\,. \tag{6.9}$$

This condition tells us that μ is a function of u and

v of the form

$$\mu=\mu[F(u)+G(v)]. \qquad (6.10)$$

So from (6.10) and (6.8) we have

$$e^{\omega}=\mu' e^{\mu/2}F_{+}G_{-}, \qquad (6.11)$$

where a prime denotes differentiation with respect to $t=F(u)+G(v)$. When we put (6.10) and (6.11) into (6.5) and (6.7) we get

$$\mu''+\omega''+\tfrac{1}{2}\mu'^2=0, \qquad (6.12)$$

$$\mu''+\mu'^2=a\mu' e^{-\mu/2}. \qquad (6.13)$$

From (6.11) and (6.13) we have

$$\omega''=-\tfrac{1}{2}(\mu''+a\mu' e^{-\mu/2}), \qquad (6.14)$$

and from (6.14) and (6.13) we recover (6.12). Thus, we only need to solve (6.13) that can be cast into the form

$$(e^{\mu})''=2a(e^{\mu/2})'. \qquad (6.15)$$

This ordinary differential equation is solved by

$$e^{\mu}-C_1\ln(e^{\mu/2}+C_1)=at+C_2, \qquad (6.16)$$

where C_i are arbitrary constants. When $C_i=0$, solution (6.16) must be changed to

$$e^{\mu}=(at+C_3)^2. \qquad (6.17)$$

From (6.1) and (6.11) we get

$$ds^2=4\Omega dF dG-e^{\mu}(dx^2+dy^2), \qquad (6.18)$$

where

$$\Omega\equiv(e^{\mu/2})'=(e^{\mu})''/2a. \qquad (6.19)$$

Letting $z=G-F$ and $t=G+F$ (as before), we have

$$ds^2=\Omega(t)(dt^2-dz^2)-e^{\mu(t)}(dx^2+dy^2), \qquad (6.20)$$

where $\mu(t)$ is the solution to (6.16) or (6.17).

The strong energy condition for the present case reduces to

$$e^{\mu}[(\hat{U}^2)^2+(\hat{U}^3)^2]>0. \qquad (6.21a)$$

The weak and dominant energy conditions give us

$$e^{\mu}[(U^2)^2+(U^3)^2]\geq 0. \qquad (6.21b)$$

Thus, Eqs. (6.21) restrict e^{μ} to be positive and Eq. (6.3) implies $a>0$.

We have for this solution a Killing vector $(\xi_{\alpha}=\delta_{\alpha}{}^1)$ that is a consequence of the field equations (Birkhoff theorem). The norm of this Killing vector is $\xi^{\alpha}\xi_{\alpha}=-\Omega$, so its character will depend on the sign of Ω; in particular, when $C_1=0$, we have $\Omega=a>0$. In this late case the space-time is homogeneous. Note that this solution is quite similar to the general solution to Einstein-Maxwell equations for the same symmetry.[11] This fact is not too surprising if one considers Eqs. (3.26).

VII. STRING CLOUD WITH CYLINDRICAL SYMMETRY

In this section we study the solution of Einstein for a cloud of strings when the metric is given by[32]

$$ds^2=e^{2(\nu-\lambda)}(dt^2-dr^2)-e^{-2\lambda}r^2d\phi^2-e^{2\lambda}dz^2, \qquad (7.1)$$

where ν and λ are functions of t and r.

As in the previous cases we find that $\Sigma_{01}=-\Sigma_{10}\neq 0$ and that the integrability conditions are fulfilled by

$$\Sigma^{01}=\frac{a}{\rho r}e^{2(\lambda-\nu)}, \qquad (7.2)$$

$$(-\gamma)^{1/2}\rho=a/r, \qquad (7.3)$$

where a is a positive constant.

The Einstein equations for this case reduce to

$$\nu_{00}-\nu_{11}-\nu_1/r-\lambda_{00}+\lambda_{11}+\lambda_1/r+2\lambda_0{}^2=0, \qquad (7.4)$$

$$\nu_{11}-\nu_1/r-\nu_{00}+\lambda_{00}-\lambda_{11}-\lambda_1/r+2\lambda_1{}^2=0, \qquad (7.5)$$

$$2\lambda_0\lambda_1-\nu_0/r=0 \qquad (7.6)$$

$$r(\lambda_{00}-\lambda_{11}-\lambda_1/r)=-ae^{2(\nu-\lambda)}, \qquad (7.7)$$

$$r(\lambda_{00}-\lambda_{11}-\lambda_1/r)=ae^{2(\nu-\lambda)}, \qquad (7.8)$$

where we have introduced the notation $\partial_0\nu\equiv\partial_t\nu$, $\lambda_1\equiv\partial_r\lambda$, etc.

From (7.7) and (7.8) we find that $a=0$, and thus $\rho=0$. The same result can be achieved as follows. From (7.4) and (7.5) we get

$$\lambda_0{}^2+\lambda_1{}^2-\nu_1/r=0, \qquad (7.9)$$

and (7.6) and (7.9) tell us that the integrability condition for ν is

$$\lambda_{00}-\lambda_{11}-\lambda_1/r=0. \qquad (7.10)$$

Thus, from either (7.7) or (7.8) we conclude that $a=0$. In other words, the symmetries of (7.1) are too restrictive to allow a solution of Einstein's equations in this case. It is a well-known fact that there are particular metrics that are too restrictive to be solutions of Einstein's equations for some models of matter. For instance, a metric whose Ricci scalar is identically zero, when coupled via Einstein's equations to a cloud of particles, will produce $\rho_p\equiv 0$. This will also be the case when coupled with a cloud of strings. An example of a metric with $R=0$ is

$$ds^2=2H(u,x,y)du^2+2dudv-dx^2-dy^2. \qquad (7.11)$$

Finally, we want to point out that the metric (7.1) is not the most general metric with cylindrical symmetry.[32]

VIII. DISCUSSION

In the study of the string cloud model it has not been considered the behavior of the open strings' end points.[15] If we do not impose the usual[15] boundary conditions at the strings' end points we have that each string in the cloud is not a closed system, i.e., we have energy going out of the string by their end points. But the cloud as a whole is a closed system as a consequence of the Bianchi identities ($\nabla_\mu T^{\mu\nu}=0$). So the model studied in this paper forms a well-defined physical system, even in the case of open strings without the usual boundary conditions. The problem indicated above does not appear when we consider that each string that forms the cloud is closed.[33]

If we impose the condition that no energy escapes from the strings' end points, we find that $\gamma=0$ at the strings' end points.[4] And $\gamma=0$ implies that the end points move with the speed of light.[4] Thus, when $\gamma=0$ the concepts of proper energy density ρ and gauge-invariant density $(-\gamma)^{1/2}\rho$ are meaningless. In this case we must replace in (3.8) $\rho/(-\gamma)^{1/2}$ by w and $\Sigma^{\mu\nu}$ by $K^{\mu\nu}$, where w is an energy density and $K^{\mu\nu}$ a null bivector ($K^{\mu\nu}K_{\mu\nu}=0$). Completely null strings have been studied by Schild.[34] The behavior of the gravitational field in the weak-field approximation for a particular open string is discussed in Ref. 8.

We have a similar situation for particles that move with the speed of light ($u^\alpha u_\alpha=0$); ρ_p as well as $\rho_p(u^\alpha u_\alpha)^{1/2}$ are meaningless. We must replace in (2.5) $\rho_p/(u^\alpha u_\alpha)^{1/2}$ by w_p and u^μ by k^μ, where w_p is the particles' energy density and k^μ is a null vector. Particular solutions to the Einstein equations for beams of photons are known.[35] It is interesting to point out that the general solution to Einstein's equations in the weak-field approximation for a beam of photons is unknown.[36] Also, the general solution to Maxwell's equations for a null current is unknown.[37]

To use the string cloud model to construct more complete models of stars than the one presented in Sec. IV as well as to study some of its cosmoogical consequences we need to introduce the concept of pressure, in other words, to transform the "cloud" in a "fluid." Part of this program has been already completed.[38]

The motivation that we had to study the string cloud model is also valid to the study of a "membrane cloud model," i.e., now the constituents of the cloud are two-dimensional objects.[39] Work done along this line will be soon reported.

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