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# Final state of the evolution of the interior of a charged black hole

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We describe the dynamical evolution of scalar, electromagnetic, and gravitational test fields on the interior of a Reissner-Nordström (spherically symmetric and electrically charged) black hole. The instability of the hole's Cauchy horizon is discussed in detail in terms of the divergences of the energy densities of the test fields as measured by a freely falling observer approaching the Cauchy horizon. The late-time development of the fields is discussed and a picture of the final state for the interior (in terms of classical fields) is developed. We conclude that the Cauchy horizon of the analytically extended Reissner-Nordström solution is highly unstable and not a physical feature of a realistic gravitational collapse.

## I. INTRODUCTION

The Reissner-Nordström black hole represents the unique static exterior of a collapsed spherically symmetric distribution of charge and mass. However, the interior of the analytically extended solution possesses a Cauchy horizon prohibiting any deterministic future based on the hyperbolic Einstein field equations. In a previous paper[1] (hereafter referred to as I) we discussed the behavior of a test scalar field near the hole's Cauchy horizon where the field's energy density was shown to develop singularities and suggested a disruption of the horizon (e.g., through the back reaction of the singular energy density on the curvature). In this paper we continue the scalar field analysis, extend the development to include electromagnetic and gravitational perturbations, and investigate the problem of the final state of the evolution of the interior of a charged black hole for late times.

We assume that a nearly spherical star with a net electric charge has undergone a gravitational collapse with small deviations from spherical symmetry in the matter density and charge density at the moment when the surface of the star crossed the event horizon. All the perturbations from spherical symmetry are assumed to be weak enough so that we can neglect their back reaction on the spacetime metric at the moment of crossing. Mashhoon[2] has recently investigated the spherical charged collapse of a perfect fluid and found that while the exterior geometry was necessarily Reissner-Nordström type, the interior geometry collapsed behind an apparent horizon to a spacelike curvature singularity. Doroshkevich and Novikov[3] have investigated the final-state problem for perturbations inside a Schwarzschild black hole, where the evolution in time of a perturbation ceases at the spacelike curvature singularity. The geometry at the Cauchy horizon inside the Reissner-Nordström black hole is smooth and regular and gives rise to dramatically different features not found in the Schwarzschild interior (cf. paper I).

In Sec. II we solve the scalar wave equation for small wave number and use this solution to discuss the behavior of the scalar field in a "neighborhood" of the intersection of future timelike infinity, the event horizon, the Cauchy horizon, and the curvature singularity on the Carter-Penrose diagram of an analytically extended Reissner-Nordström black hole (point B in Fig. 1 of paper I). As will be shown approaching B along a spacelike hypersurface $r=\text{const}$ (for $r_-<r<r_+$), i.e., as $t\to\infty$ for $r$ held fixed, a perturbation of multiple index $l$ will decay as $t^{-2l-2}$ and the field between the $r_+$ and $r_-$ horizons (i.e., the interior as we have called it) does not develop any pathologies. However, if we approach B or the $r_-$ horizon along the world line of a freely falling observer then the energy density as measured by the observer is blue-shifted and diverges exponentially in $r^*$ [or as $(r-r_-)^{-1}$] near the $r_-$ horizon. This suggests that a curvature singularity develops, topologically similar to that of the Schwarzschild black-hole interior.

 

In Sec. III we formulate and discuss the electromagnetic and gravitational perturbation problems. Using a Regge-Wheeler-type formalism[4–7] combined with the techniques developed previously to analyze the scalar case, we extend the above conclusions on instability.

In Sec. IV we consider the features of perturbations arising from stationary sources in the exterior region. As an example, we present the astrophysically interesting case of a hole in a uniform magnetic field and demonstrate that the "stationary" field which threads through the interior does not disrupt the $r_-$ horizon (as also was the case with the stationary scalar field discussed in paper I).

In Sec. V we discuss our conclusions reached by treating the perturbations as classical fields. By a classical field we mean that the smallest values of our field amplitudes are still much larger than the corresponding quantum amplitudes. We do not consider quantum-mechanical processes here. While these processes undoubtedly influence the structure of the evolving singularity (see, e.g., Ref. 8), they do not alter our main conclusions.

## II. SCALAR FIELD

In this section we conclude the scalar field analysis begun in paper I and compute the asymptotic behavior of the scalar test field in a neighborhood of the point B (cf. Fig. 1 of paper I) for the final-state analysis. We match the field to a power law on the $r_+$ horizon which had developed from the late-time field in the exterior due to the backscatter of radiation. Using a small-wave-number approximate solution, we evolve this field through the interior up to the $r_-$ horizon. With this solution we investigate the asymptotic form of the field as point B is approached in a spacelike direction ($r=\text{const}$, $t\to\infty$) and in a null direction along the $r_-$ horizon ($u=\infty$, $v\to\infty$).

Using the notation of paper I, we write the scalar field solution as

$$\phi(t,r,\theta,\varphi)=\sum_{l,m} Y_{lm}(\theta,\varphi)\int_{-\infty}^{\infty} dk\, e^{-ikt}\frac{1}{r}\psi_{lmk}(r)\,, \tag{1}$$

where $\psi_{lmk}(r)$ satisfies the evolution equation

$$\frac{d^2\psi_{lmk}}{dr^{*2}}+\left\{k^2+\frac{(r_+-r)(r-r_-)}{r^2}\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}-\frac{2Q^2}{r^4}\right]\right\}\psi_{lmk}=0\,, \tag{2}$$

and $r^*$ and $r$ are related (on the interior) by the equation

$$r^*=-r-\frac{1}{\kappa_+}\ln\left(1-\frac{r}{r_+}\right)+\frac{1}{\kappa_-}\ln\left(\frac{r}{r_-}-1\right)\,, \tag{3}$$

with $\kappa_\pm=(r_+-r_-)/r_\pm^{\,2}$. We will also use the null coordinates $u=-r^*-t$ and $v=-r^*+t$.

Let $\psi_k(r)$ be the left-going solution to Eq. (2) defined by

$$\psi_k(r)\sim e^{ikr^*} \text{ as } r^*\to\infty \text{ (i.e., as } r\to r_+)\,.$$

Then $\psi_k^*(r)$ is a second linearly independent solution. A general solution (for each set of multipole indices $l,m$ which we suppress for brevity) may be written in the form

$$\psi(r,t)=\int_{-\infty}^{\infty} dk\, e^{-ikt}[a(k)\psi_k(r)+b(k)\psi_k^*(r)]\,. \tag{4}$$

The coefficients $a(k)$ and $b(k)$ are to be determined from the initial conditions on the $r_+$ horizon and the conditions on the collapsing star.

A comparison of the asymptotic form of Eq. (4) near the $r_+$ horizon with the late-time field in the exterior suggests (cf. Ref. 3) that we choose

$$a(k)=b(k)[-1+B(k)]\,, \tag{5}$$

where $[B(k)-1]$ is the reflection coefficient of a wave scattering off the static potential in the exterior of the black hole.[9] [The Fourier coefficients based on the $(t,r)$ coordinates in the interior and exterior regions can be equated across the $r_+$ horizon for infalling waves only. This may be seen by comparing the $(t,r)$-based transforms with a transform based on, e.g., Eddington-Finkelstein coordinates $(v,r)$ which are nonsingular at the $r_+$ horizon.] The form of $B(k)$ is such as to represent a power-law-type trail on the $r_+$ horizon for large values of $t$. The function $b(k)$ is determined by the details of the collapse as the star crosses the $r_+$ horizon (which we take to be at $v=0$). In what follows, all we will need are a few details on the analytic properties of $B(k)$ and $b(k)$. (For an analysis in the exterior, see Sibgatullin and Alekseev.[9])

The existence of power-law tails in the exterior region implies that $B(k)$ has, at least, a branch point at $k=0$. (This follows from a Weiner-Hopf-type analysis of the asymptotic behavior of Fourier transforms. Sibgatullin and Alekseev[9] have further demonstrated that the branch point has the character of a logarithmic branch point.) Using time dilation arguments, Price has shown that in the exterior the field on the surface of the star as the star crosses the $r_+$ horizon has the asymptotic form

$$\phi \sim \text{const} \times \theta(v_{\text{ext}}) + \text{const} \times e^{-u_{\text{ext}}/2r_+},$$

which when Fourier transformed implies that $b(k)$ has a simple pole at $k=0$.

As $t\to\infty$ (i.e., a neighborhood of point B), the dominant contribution to $\psi(r,t)$ in Eq. (4) comes from the modes in a neighborhood of $k=0$ and we may expect the solution for $|kM| \ll 1$ to Eq. (2) to apply. We use the now common technique[10–15] of matching the leading terms in $k$ of the wavelike solutions at $r=r_+$ with the low-$k$ solutions on the interior. This technique matches only the dominant terms and ignores, e.g., the logarithmic singularities which are of higher order in $k$. The $k=0$ solutions can be written in closed form and are given by

$$\psi_{k=0}(r) = \alpha r P_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right) + \beta r Q_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right), \quad (6)$$

where $P_l$ is the Legendre polynomial, $Q_l$ is the Legendre function of the second kind, and $\alpha$ and $\beta$ are constants.

Near the $r_+$ horizon, $\psi_k(r)$ has the form of an ingoing wave of unit amplitude, viz., $\psi_k(r)=e^{ikr*}$. For $|krr^*| \ll 1$ we may neglect the $k^2$ term in Eq. (2) and match the leading terms in $\psi_k(r)$ in the region $-1/k \ll r^* \ll -M$ with the $\psi_{k=0}$ solution and we find to $O(k^2)$

$$\psi_k(r) = \frac{r}{r_+}\left[P_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right) + 2ik\frac{1}{\kappa_+}Q_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right)\right] \approx \left[1 - ik\frac{1}{\kappa_+}\ln\left(\frac{r-r_-}{r_+-r_-}\right)\right] \text{ as } r\to r_+ . \quad (7)$$

Near the $r_-$ horizon, $\psi_k(r)$ evolves to a solution of Eq. (2) with the asymptotic form given by

$$\psi_k(r) \approx \tilde{A}(k)\, e^{ikr^*} + \tilde{B}(k)\, e^{-ikr^*} \text{ as } r\to r_- . \quad (8)$$

The Wronskian for two solutions of Eq. (2) is independent of $r$ and for $\psi_k(r)$ and $\psi_k^*(r)$ it has the value given by

$$W_{r^*}[\psi_k, \psi_k^*] = -2ik . \quad (9)$$

This in turn implies that $|\tilde{A}(k)|^2 - |\tilde{B}(k)|^2 = 1$ for all $k$. Comparing Eqs. (7) and (8) in the region $M \ll r^* \ll 1/k$, we find for $|kM| \ll 1$ the scattering amplitudes $A$ and $B$ are given by

$$\tilde{A}(k) = \frac{(-1)^l}{2}\left(\frac{r_+}{r_-}+\frac{r_-}{r_+}\right) + O(k), \quad (10)$$

$$\tilde{B}(k) = -\frac{(-1)^l}{2}\left(\frac{r_+}{r_-}-\frac{r_-}{r_+}\right) + O(k). \quad (11)$$

[Owing to the exponential decay of the potential in Eq. (2) as $|r^*| \to \infty$ (see paper I), the coefficients $\tilde{A}$ and $\tilde{B}$ are, in fact, analytic at $k=0$.[15]]

We now consider Eq. (4) with the Fourier coefficients related by Eq. (5), viz.,

$$\psi(r,t) = \int_{-\infty}^{\infty} e^{-ikt} b(k)[(B(k)-1)\psi_k(r) + \psi_k^*(r)]dk . \quad (12)$$

We take the branch cut [to define $B(k)$] to run from $k=0$ to $k=-i\infty$ along the negative imaginary axis and deform the contour of integration into the lower half of the complex $k$ plane. The pole of $b(k)$ at $k=0$ produces a cancellation between the second and third terms $(-\psi_k+\psi_k^*)$ [cf. Eqs. (10) and (11)]. Any additional poles of $b(k)$ below the real axis will give terms that exponentially damp as $t\to\infty$. There remains the integral along the cut from $k=0$ to $k=-i\infty$. As previously discussed, the reflection coefficient $B(k)$ is chosen to reproduce the power-law tail on the $r_+$ horizon (expected from the late-time development of a collapse) with the asymptotic form given by

$$\psi(r,t) \sim \frac{\text{const}}{v^{2l+2}} = \frac{\text{const}}{(t-r^*)^{2l+2}} \text{ for } u\to-\infty,\ v \gg M . \quad (13)$$

At intermediate times ($r$ = constant) using the $|kM| \ll 1$ solutions given in Eq. (7), we find

$$\psi(r=\text{const},t) = \text{const} \times \int_0^{\infty} dk\, e^{-kt}(k^{2l+1})\psi_k(r) = \frac{\text{const}}{t^{2l+2}} \times rP_l\left(\frac{2r-r_+-r_-}{r_+-r_-}\right) \times \left[1 + O\left(\left|\frac{r^*}{t}\right|\right)\right] \text{ as } t\to\infty . \quad (14)$$

Near the $r_-$ horizon approaching the point B (i.e., for $v\to\infty$ with $u<0$) using Eq. (8), the Fourier synthesis gives the result

$$\psi(r,t) \sim \int_0^{\infty} dk\, k^{2l+1}[A(0)\,e^{-kv} + B(0)\,e^{ku}] = \text{const} \times \left[\frac{A(0)}{v^{2l+2}} + \frac{B(0)}{u^{2l+2}}\right]. \quad (15)$$

Recalling that near the $r_+$ horizon

$$\psi = \text{const} \times v^{-(2l+2)}, \quad (16)$$

we see that Eqs. (14)–(16) present the complete structure of the scalar field near the point B. (N.B. The regions in which each of these expressions are applicable overlap.)

In paper I we considered what a freely falling observer (with a four-velocity $U^\alpha$) crossing the $r_-$ horizon would observe for the energy density in the scalar field and demonstrated that the energy density was a function of $\phi_{,\alpha}U^\alpha$. For an observer crossing $v=\infty$ this is given by

$$\phi_{,\alpha}U^\alpha \sim \frac{\partial\phi}{\partial v}e^{(\kappa_-/2)(u+v)} + \text{const}\times\frac{\partial\phi}{\partial u} \text{ as } v\to\infty\,. \quad (17)$$

If the point B is approached along a surface of constant $r$, then since $u+v=-2r^*$ we find using Eqs. (14) and (17) that

$$\phi_{,\alpha}U^\alpha \sim \text{const}/t^{2l+3} \text{ as } t\to\infty\,.$$

Hence, there is no pathological behavior of $\phi$ approaching the point B in this direction. However, if the point B is approached along the $r_-$ horizon, then using Eqs. (15) and (17) we find that

$$\phi_{,\alpha}u^\alpha \sim (\text{const}/v^{2l+3})\,e^{(\kappa_-/2)v} \text{ as } v\to\infty\,.$$

Hence, approaching the point B by running along the $r_-$ horizon (i.e., by taking the order of the limits as $v\to\infty$, then $u\to-\infty$ in the previous expression) indicates that a singularity may develop near the Cauchy horizon in an arbitrarily small neighborhood of the point B. These conclusions when combined with the analysis in paper I indicate that the entire Cauchy horizon is unstable to perturbation produced by a scalar test field.

## III. ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATIONS

In this section we augment the previous analysis and extend our considerations to include electromagnetic and gravitational perturbations of the interior. Owing to the presence of the background electric field, the situation is mathematically more difficult. Electromagnetic and gravitational perturbations are described by coupled sets of wave equations which correspond physically to the conversion of electromagnetic perturbations into gravitational perturbations and vice versa by the catalytic action of the background fields. However, owing to the efforts of Zerilli,[5] Moncrief,[6,7] Sibgatullin and Alekseev,[9] and Chandrasekhar,[16,17] it is possible to obtain decoupled equations which upon separation satisfy potential-like equations similar to Eq. (2) (see Fig. 1).

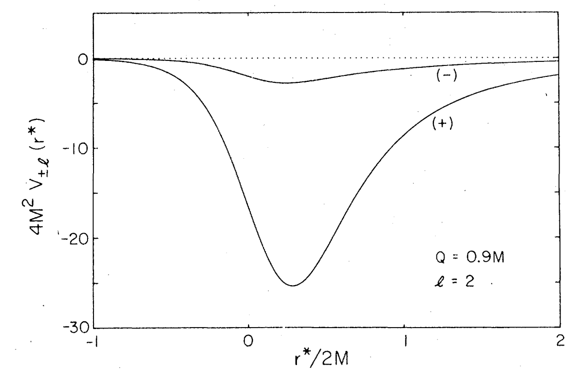


FIG. 1. The potential of the separated wave equation for electromagnetic and gravitational perturbations. The particular case $Q=0.9$M and $l=2$ is shown. The potentials have the same overall features for other values of the parameters $Q$ and $l$, as in the scalar case.

Following Moncrief and Zerilli, we expand the electromagnetic vector potential, the electromagnetic field tensor, and the metric perturbations of the Reissner-Nordström background in terms of the Regge-Wheeler spherical harmonics.[4,18] The Einstein-Maxwell equations for the perturbed quantities have been decoupled by Moncrief[6,7] into two second-order wave equations for the even-parity and odd-parity cases. Chandrasekhar,[16] using an approach based on the Newman-Penrose formalism,[12,19] has shown how to derive solutions of the even-parity Moncrief equations from the solutions of the much simpler odd-parity Moncrief equations. Hence it is sufficient to consider only the odd-parity Moncrief equations given by Ref. 20:

$$\left\{\frac{d^2}{dr^{*2}}+k^2-\frac{\Delta}{r^2}\left[l(l+1)-\frac{3M}{r}+\frac{4Q^2}{r^2}\right]\right\}\begin{pmatrix}\hat\pi_g\\ \hat\pi_f\end{pmatrix}=\frac{\Delta}{r^3}\begin{pmatrix}3M & 2Q[(l-1)(l+2)]^{1/2}\\ 2Q[(l-1)(l+2)]^{1/2} & -3M\end{pmatrix}\begin{pmatrix}\hat\pi_g\\ \hat\pi_f\end{pmatrix}, \quad (18)$$

where $\Delta=(r-r_+)(r-r_-)$ and where Moncrief's variables $\pi_g$ and $\pi_f$ are related to Zerilli's electromagnetic field $f_{23}$ and metric perturbation $h_1$ (in the Regge-Wheeler odd-parity gauge although the variables $\pi_g$ and $\pi_f$ have gauge-invariant significance) by

$$\hat\pi_f=-\frac{2[(l-1)(l+2)]^{1/2}}{l(l+1)}f_{23}=[(l-1)(l+2)]^{1/2}\pi_f\,,$$

$$\hat\pi_g=(l-1)(l+2)\frac{\Delta}{ikr}h_1=\frac{\pi_g}{l(l+1)}\,.$$

This system is decoupled by the linear transformation that diagonalizes the right-hand side of Eq. (18) (see Matzner[20] for details) and the resulting combinations here called $R_\pm$ satisfy the equations

$$\frac{d^2R_\pm}{dr^{*2}}+\left\{k^2-\frac{(r-r_+)(r-r_-)}{r^2}\left[\frac{l(l+1)}{r^2}-\frac{3M}{r^3}\pm\frac{C}{r^3}+\frac{4Q^2}{r^2}\right]\right\}R_\pm=0\,, \qquad (19)$$

where

$$C=[9M^2+4Q^2(l-1)(l+2)]^{1/2}\,. \qquad (20)$$

The electromagnetic and gravitational perturbations are then extracted from the $R_\pm$ solutions by the relations

$$\hat{\pi}_f=\cos\psi R_+-\sin\psi R_-\,, \qquad (21)$$

$$\hat{\pi}_g=\sin\psi R_++\cos\psi R_-\,, \qquad (22)$$

where

$$\sin(2\psi)=+2\frac{Q[(l-1)(l+2)]^{1/2}}{C}\,. \qquad (23)$$

The decoupled perturbation equations for $R_\pm$ [Eq. (19)] are very similar to Eq. (2) for the scalar field and the general qualitative features of the evolution of the scalar field also hold for $R_\pm$. For brevity we outline only the essential details since the scalar case was carried out in such detail. Near either horizon Eq. (19) has solutions of the form $R\sim e^{\pm ikr*}$. For the case $|kM|\ll 1$ we can match the leading terms of these wave solutions from the $r_+$ horizon to the $r_-$ horizon through the use of the $k=0$ solution as was done with the scalar case, and obtain analytical expressions for the low-frequency fields between the horizon. The $k=0$ solutions are discussed in detail in the Appendix. Here we note that the solutions are generalizations of Eq. (6), one being a polynomial in $r$ and one being a polynomial times a logarithm that diverges on either horizon [cf. Eq. (A7) (in the Appendix)].

We start with initially infalling waves ($R_\pm\sim e^{+ikr*}$) on the $r_+$ horizon. According to Sibgatullin and Alekseev's[19] analysis in the exterior region ($r>r_+$), the power-law tails, reflection coefficients, and general analytical features for the Fourier coefficients defined (as in the scalar field case) by expansions of the $R_\pm$ fields on the $r_+$ horizon are qualitatively similar to the scalar field case. Following a similar analysis that leads from Eq. (12) to Eqs. (14) and (15), we obtain the following picture for the development of gravitational and electromagnetic perturbations inside the black hole: On the spacelike surface $r=$const for $t\gg M$ we find

$$R_\pm(r=\text{const},t\gg M)\sim\frac{D_\pm}{t^{2l+2}}\left[1+O\left(\frac{|r^*|}{t}\right)\right]\,, \qquad (24)$$

and near the $r_-$ horizon approaching the point B we find

$$R_\pm(v\to\infty,\ u<0)\sim D_\pm\left[\frac{A_\pm(0)}{v^{2l+2}}+\frac{B_\pm(0)}{u^{2l+2}}\right]\,, \qquad (25)$$

where the $D_\pm$ are constants. From Eqs. (21) and (22) it follows that the electromagnetic and metric perturbations have similar power-law developments near the $r_-$ horizon. [The complete perturbed Maxwell field and the perturbed metric follow from a knowledge of $\pi_f$ and $\pi_g$ (see Moncrief and Zerilli for details).] Consequently, the energy density in the electromagnetic field and the energy density in the Landau-Lifshitz pseudotensor describing the energy density in the perturbed gravitational field will have similar power-law falloff relative to the frame stationary in $(r,t)$ coordinates. Therefore, exactly as in the scalar field case, when the fields and the energy tensors are referred to a frame carried by a freely falling observer (cf. Gürsel *et al.*, paper I) the power-law falloff of the fields and energy densities are overcome by the exponential blue-shift factor [cf. Eq. (17)] of the observer's frame as she approaches the horizon.

## IV. STATIONARY EXTERNAL SOURCES

For the special case of perturbations that are independent of $t$ (i.e., stationary in the exterior and homogeneous in the interior), the Moncrief variable $\pi_g$ is not well defined and the derivation that leads to Eq. (18) breaks down. However, it is possible to still obtain an appropriate set of decoupled equations. Using Zerilli's[5] notation in the odd-parity Regge-Wheeler gauge (i.e., $h_2=0$) Maxwell's equations for the $t$-independent case are given by

$$l(l+1)f_{12}=\frac{d}{dr}f_{23}\,, \qquad (26)$$

$$\frac{d}{dr}\left(\frac{\Delta}{r^2}f_{12}\right)-\frac{1}{r^2}f_{23}=\frac{d}{dr}\left(\frac{Q}{r^2}h_0\right)\,, \qquad (27)$$

$$f_{02}=0\,, \qquad (28)$$

and the relevant Einstein equations are given by

$$\frac{d^2h_0}{dr^2}-\left[\frac{2}{r^2}+\frac{l(l+1)-2}{r^2}\left(\frac{r^2}{\Delta}\right)\right]h_0=\frac{4Q}{r^2}f_{12}$$

$$=\frac{4Q}{r^2l(l+1)}\frac{d}{dr}f_{23}\,, \qquad (29)$$

$$h_1=0\,, \qquad (30)$$

where

$$\frac{\Delta}{r^2}=1-\frac{2M}{r}+\frac{Q^2}{r^2}.$$

Solving Eqs. (4.1) and (4.2) for $f_{23}$, we find

$$\frac{d}{dr}\left(\frac{\Delta}{r^2}\frac{df_{23}}{dr}\right)-\frac{l(l+1)}{r^2}f_{23}=l(l+1)\frac{d}{dr}\left(\frac{Q}{r^2}h_0\right). \tag{31}$$

The problem is how to deal with $h_0$ and $f_{23}$ in Eqs. (29) and (31). As Moncrief has pointed out,[6] Zerilli's odd-parity (what he calls "magnetic") equations for the functions

$$R^{(m)}_{\mathrm{LM}}=\frac{\Delta}{r^3}h_1 \tag{32}$$

and

$$f^{(m)}_{\mathrm{LM}}=\frac{1}{l(l+1)}f_{23} \tag{33}$$

may be decoupled by a linear transformation; however, for the $t$-independent case (i.e., $k\equiv\omega=0$) the variable $h_1$ vanishes. This difficulty may be avoided and the decoupling of Eqs. (29) and (31) accomplished by considering Zerilli's Eqs. (14) and (16):

$$l(l+1)f_{02}=-i\omega f_{23}, \tag{34}$$

$$\left(i\omega-\frac{2i\lambda}{r^2}\frac{\Delta}{r^2}\frac{1}{\omega}\right)h_1=\frac{4Q}{r^2}\frac{f_{23}}{l(l+1)}-r^2\frac{d}{dr}\left(\frac{h_0}{r^2}\right). \tag{35}$$

Taking the $\omega\to0$ limit, we find

$$\lim_{\omega\to0}\left[2i\lambda\left(\frac{\Delta}{r^3}\frac{h_1}{\omega}\right)\right]=\lim_{\omega\to0}2i\lambda\left(\frac{R^{(m)}_{lM}}{\omega}\right)=r^3\frac{d}{dr}\left(\frac{h_0}{r^2}\right)-\frac{4Q}{r}f^{(m)}_{lM}. \tag{36}$$

Upon substitution of the new variable $\pi_0$ defined by

$$\pi_0=r^3\frac{d}{dr}\left(\frac{h_0}{r^2}\right)-\frac{4Q}{r}f^{(m)}_{lM} \tag{37}$$

in Eqs. (29) and (31), we find a pair of suitable equations to describe the $t$-independent case, viz.,

$$\left\{r^3\frac{d}{dr}\left(\frac{\Delta}{r^2}\frac{d}{dr}\right)-\left[l(l+1)r+\frac{4Q^2}{r}\right]\right\}\begin{pmatrix}\pi_0\\ f^{(m)}_{lM}\end{pmatrix}=\begin{pmatrix}-6M & 8\lambda Q\\ Q & 0\end{pmatrix}\begin{pmatrix}\pi_0\\ f^{(m)}_{lM}\end{pmatrix}. \tag{38}$$

It is interesting to note that when Eq. (38) is decoupled, the resulting equations are identical to Eq. (19) with $k=0$.

We shall illustrate the case of an electromagnetic perturbation arising from a source current in the exterior that is at rest with respect to the black hole by the example of a charged black hole in a uniform static magnetic field. This situation is mathematically similar to the scalar case treated in paper I. The problem is to extend the field to the interior and solve for the behavior of the field near the $r_-$ horizon.

To match onto a uniform field at $r=\infty$ we keep only the $l=1$ dipole term in the multipole expansion and align the external magnetic field along the $z$ axis. For this case the Maxwell equations decouple from the gravitational perturbation and we find the solution that matches onto a uniform field for large $r$ is given by

$$f_{23}=A\left(1-\frac{r^2}{Q^2}\right), \tag{39}$$

where $A$ is a constant determined by the magnitude of the external field. From Eq. (26) we deduce for $f_{12}$ the value

$$f_{12}=-Ar/Q^2, \tag{40}$$

and with these values the field two-form is given by

$$F=-B\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{1/2}\sin\theta\,\omega^{\hat r}\wedge\omega^{\hat\varphi}+B\frac{Q^2}{r^2}\left(-3+\frac{r^2}{Q^2}\right)\cos\theta\,\omega^{\hat\theta}\wedge\omega^{\hat\varphi}, \tag{41}$$

where $B=(3/4\pi)^{1/2}A/Q^2$ and $[\omega^{\hat i}]$ is the orthonormal frame given by

$$\omega^{\hat t}=\frac{(|\Delta|)^{1/2}}{r}dt,\quad \omega^{\hat\theta}=r\,d\theta,$$

$$\omega^{\hat r}=\frac{r}{(|\Delta|)^{1/2}}dr,\quad \omega^{\hat\varphi}=r\sin\theta\,d\varphi.$$

Notice that the field between $r_-$ and $r_+$ contains an electric part ($F_{\hat r\hat\varphi}$) due to the fact that the orthonormal frame we have chosen is not stationary in this region.

This solution is finite at either horizon and like the scalar case (cf. paper I) does not disrupt the $r_-$ horizon. The field does not depend on the time of an external observer, hence any observer falling into the hole from the outside at any time will always see the same electromagnetic field.

## V. CONCLUSION

We have calculated the evolution of scalar, electromagnetic, and gravitational test fields in the interior of the Reissner-Nordström geometry near the "intersection" of the event horizon

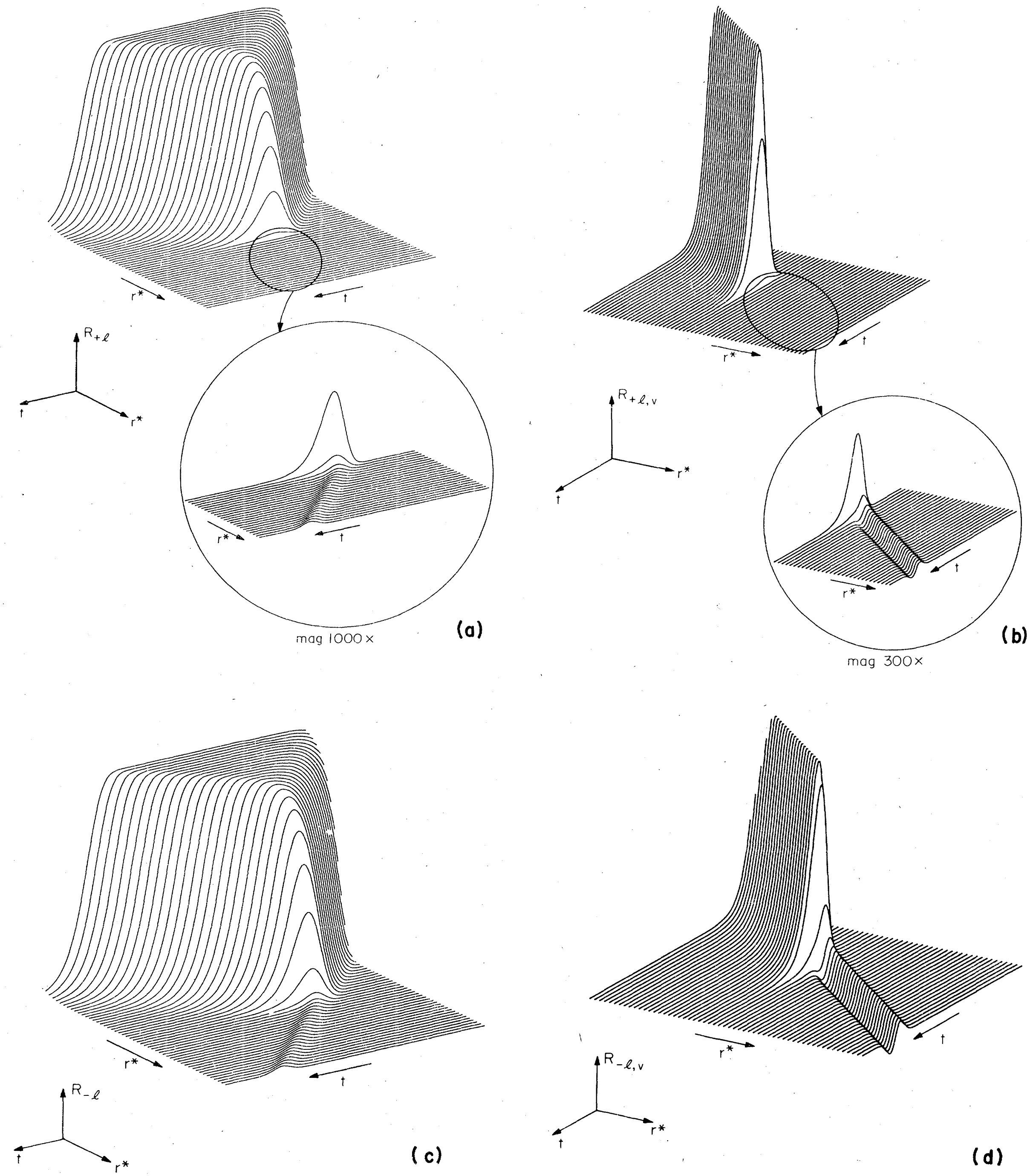


FIG. 2. These figures summarize the results of a numerical integration of the equation $R_{\pm l, r^* r^*} - R_{\pm l, tt} = V_{\pm l}(r^*) R_{\pm l}$. The results presented here are for the particular case $Q = 0.9$M and $l = 2$; however, additional integrations with alternative parameters support the qualitative features illustrated here. (a)–(d) show $R_{+l}$, $R_{+l,v}$ and $R_{-l}$, $R_{-l,v}$, respectively, in relief as functions of $r^*$ and $t$. The $r^*$ coordinate ranges from 50M (near event horizon) to $-48$M (near Cauchy horizon) and $t$ ranges from $-50$M to 50M. The separation between lines of constant $r^*$ is 2M for (a) and (c) and 1.6 M for (b) and (d). The initial wave form is taken to be a Gaussian of unit amplitude. These are shown on or below the figures. The asymptotic exponential falloff of the derivatives of $R_+$ and $R_-$ near the Cauchy horizon is shown in (e)–(h). (e) and (f) show the exponential behavior of $R_{+,u}$ and $R_{-,u}$, respectively, at constant $v$ in the neighborhood of the right-hand side of the $r_-$ horizon. (g) and (h) show the analogous behavior of $R_{+,v}$ and $R_{-,v}$, respectively, at constant $u$ near the left-hand side of the $r_-$ horizon.

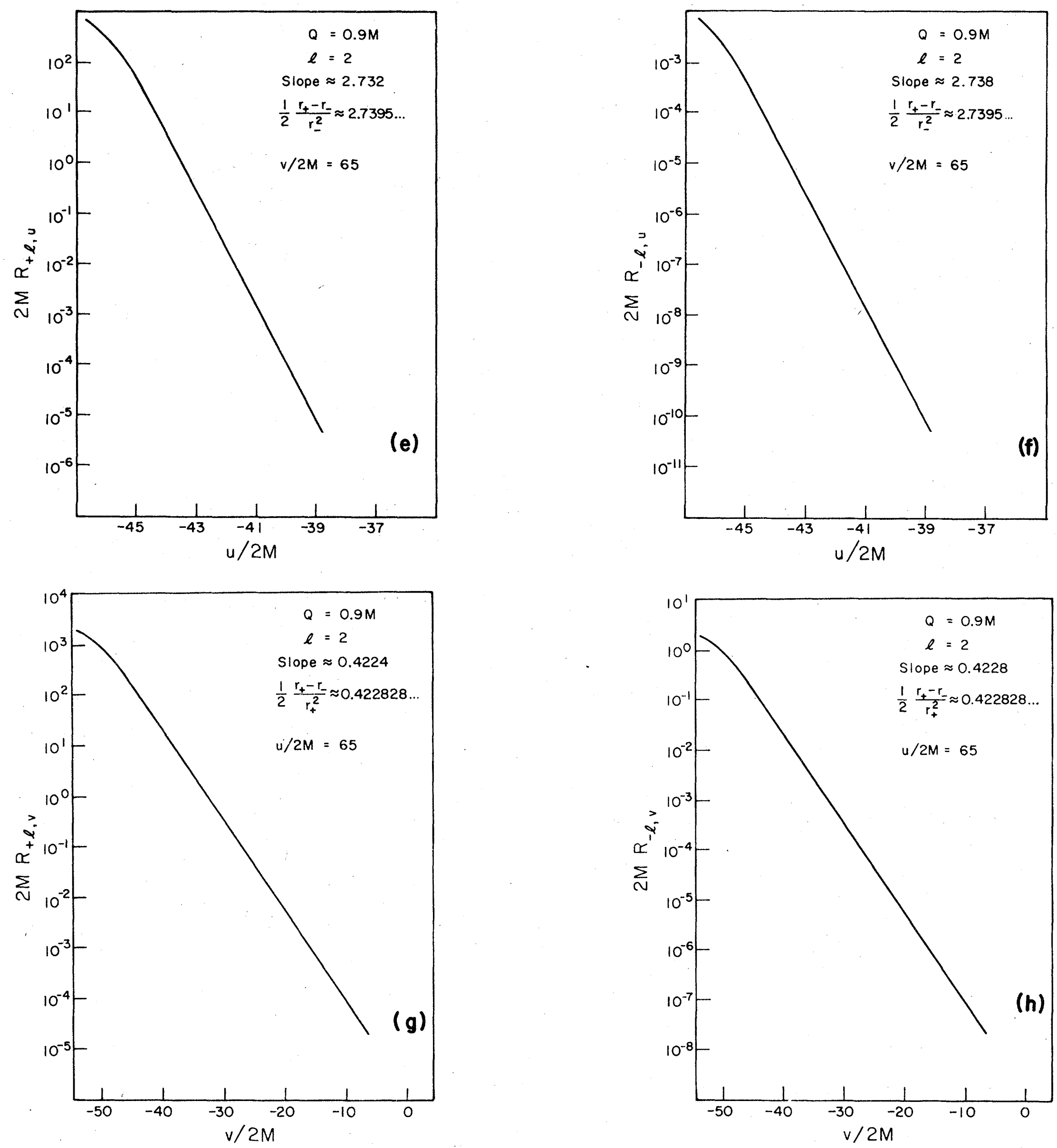


FIG. 2. *(Continued)*

($r=r_+$) and the Cauchy horizon ($r=r_-$) from initial data appropriate for long times after the formation of the charged black hole. The behavior of the fields in this region is qualitatively independent of details of the collapse process. In particular, the results are independent of whether the surface of the collapsing body approaches the left or right part of the $r_-$ horizon (Fig. 2).

In this sense these results are universal and may be extended to the final stage of the evolution of a charged black hole, not just a collapsing body. The general picture is that at fixed times ($r=$constant) from the event horizon all perturbations damp according to a power-law-type behavior as we move away from the surface of a collapsing body (i.e., as $t\to\infty$). The metric becomes increasingly spherically symmetric, similar to the behavior of perturbations inside a Schwarzschild black hole (Ref. 3). However, freely falling observers will see an infinitely blue-shifted energy density or an infinite tidal shear as they approach the $r_-$ horizon. This in turn suggests an instability in the geometry developing within the $r_-$ horizon and its possible transformation into a spacelike curvature singularity.

We have not taken into account the quantum-mechanical process of pair creation by the classical electromagnetic and gravitational fields. In the exterior region ($r>r_+$) the Hawking process takes place on a characteristic time scale $G^2M^3/\hbar c^4$. The background electric charge $Q$ may discharge itself by means of $e^+-e^-$ pair creation in the outer region via electromagnetic interactions in a characteristic time $GM/c^3(e^2/\hbar c)^{-5/2}$ if $eQ \gg G^2 m_e{}^2 M^2/\hbar c$, but for smaller values of $Q$ the process proceeds much more slowly (and even stops completely if $eQ<4Gm_eM$). The Hawking effect will then dominate only long after the formation of the tails. Preliminary considerations[8] indicate that these effects also lead to a disruption of the $r_-$ horizon. We hope to return to this question elsewhere.

## ACKNOWLEDGMENTS

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## APPENDIX: ZERO-FREQUENCY, ODD-PARITY SOLUTIONS OF THE ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATION EQUATIONS

The electromagnetic-gravitational perturbations in the small-wave-number limit satisfy Eq. (19) with $k=0$. [Also, see Sec. IV Eq. (38).] Setting $k=0$ and changing the dependent variable by

$$G_\pm = rR_\pm$$

and using a dimensionless independent variable $x=r/2M$, Eq. (19) becomes

$$\frac{d^2G_\pm}{dx^2}+\left(\frac{-4}{x}+\frac{1}{x-r'_+}+\frac{1}{x-r'_-}\right)\frac{dG_\pm}{dx}+\left[\frac{-(l-1)(l+2)x+C_{\pm l}-3}{x(x-r'_+)(x-r'_-)}\right]G_\pm=0\,, \tag{A1}$$

where $r'_+=r_+/2M$, $r'_-=r_-/2M$, and

$$C_{\pm l}=\frac{1}{2M}\left\{3M-(\pm)[9M^2+4Q^2(l-1)(l+2)]^{1/2}\right\}.$$

Notice that

$$C_{\pm l}(C_{\pm l}-3)=-4\alpha^2[2-l(l+1)]\,, \tag{A2}$$

where $\alpha=Q/2M$.

This equation is of the form

$$\frac{d^2y}{dq^2}+\left(\frac{\gamma}{q}+\frac{\delta}{q-1}+\frac{\epsilon}{q-a}\right)\frac{dy}{dq}+\left[\frac{\eta\beta q-p}{q(q-1)(q-a)}\right]y=0\,, \tag{A3}$$

if we set $q=x/r'_-$, $a=r'_+/r'_-$, $p=-(C_{\pm l}-3)/r'_-)$, $\gamma=-4$, $\delta=1$, $\epsilon=1$, $\eta=-(l+2)$, and $\beta=(l-1)$. $\eta$, $\beta$, $\gamma$, $\delta$, and $\epsilon$ satisfy the relation

$$\eta+\beta-\gamma-\delta-\epsilon+1=0\,.$$

Such an equation is called Heun's equation[21] and represents a Fuchsian equation[22] of the second order with four singularities at $x=0$, 1, $a$, and $\infty$. The constant $p$ is called the accessory parameter whose presence is due to the fact that the solution to a Fuchsian equation of second order with four or more singularities is not completely determined by the positions of the singularities and the exponents at those singularities.[23]

Equation (A3) [or its equivalent Eq. (A1)] is known to admit polynomial solutions if both of the following conditions hold:[24]

(i) $\eta$ or $\beta$ should be a negative integer.

(ii) The accessory parameter should have one of its special values called the eigenvalues.

Condition (i) is clearly satisfied, but at this point one is not sure about (ii). In what follows, we will examine the solutions and the nature of the singularities of Eq. (A1). We will drop the $\pm$ subscript on $G_\pm$ and $C_{\pm l}$ with the understanding that one takes the appropriate root of (A2) for the case one has. Set

$$G=\sum_n a_n x^{n+s}\,. \tag{A4}$$

One then obtains the indicial equation

$$s(s-5)=0 \text{ or } s_1=5 \text{ and } s_2=0\,.$$

We will try to obtain the solutions of (A1) by the method of Frobenius.[25]

In this method, one chooses the smaller root of the indicial equation, namely $s_2=0$. Taking $a_0$ a constant different from zero, we find $a_1 = ((C_l - 3)/4\alpha^2)\, a_0$ and the remaining coefficients are given by the recursion relation

$$\alpha^2(n+2)(n-3)a_{n+2} + [n(2-n) + C_l]a_{n+1} + (n-l-2)(n+l-1)a_n = 0\,. \quad \text{(A5)}$$

When $n=3$, Eq. (A5) implies a consistency relation between $a_4$ and $a_3$ which were calculated before. This is a fifth-degree equation in $C_l$ and is of the form

$$\{C_l(C_l-3) + 4\alpha^2[2 - l(l+1)]\}\{\text{a cubic in } C_l\} = 0\,. \quad \text{(A6)}$$

Equation (A2) implies that this is always satisfied. [Incidentally, the cubic shares no common roots with Eq. (A2).] Since Eq. (A6) is always satisfied, the coefficient $a_5$ is left totally arbitrary. Two linearly independent solutions of Eq. (A1) can be obtained by setting $a_0=1$, $a_5=0$ and $a_0=0$, $a_5=1$. Call these solutions $G_1$ and $G_2$. From Eq. (A4) it follows that both $G_1$ and $G_2$ are regular at $x=0$, i.e., the point $x=0$ is an "apparent" singular point of (A1). [An apparent singular point of the differential equation $y'' + P(x)y' + Q(x)y = 0$ is the one at which the coefficients $P(x)$ and $Q(x)$ blow up but the solutions do not.]

The following tests[25] can be made to show that $x=0$ is an apparent singularity:

(i) Write the equation as $y'' + P(x)y' + Q(x)y = 0$.

(ii) Set $P(x) = p(x)/x$; $p(0)$ should be a negative integer (−4 in our case).

(iii) Calculate $s_1 - s_2$. It should be a nonzero integer (5 in our case).

(iv) Finally, the case with two arbitrary constants (as in the case of $G_1$ and $G_2$ above) should result.

If any of these fail, the singularity is real.

We will use these tests to show that the singularities at $x = r'_+$ and $x = r'_-$ are real [i.e., one of the solutions at $x=r'_+$ and $x=r'_-$ contains $\ln(r'_+ - x)$ and $\ln(x - r'_-)$, respectively]. Consider now the recursion relation (A5). Notice that when $n = l+2$, the coefficient multiplying $a_n$ vanishes. This fact can be used to generate a polynomial solution to Eq. (A1). The prescription is the following:

(i) Set $a_0=1$, leave $a_5$ arbitrary.

(ii) Calculate $a_{l+3}$. Choose $a_5$ so that $a_{l+3}=0$. (This is always possible since $a_{l+3}=0$ is a linear equation in $a_5$.)

Then $a_{l+4}$ will be zero because of Eq. (A5) and so will $a_{l+5}, \ldots$ . Hence the series will terminate and one will obtain a polynomial of order $l+2$ as a solution. This procedure corresponds to taking the right linear combination of $G_1$ and $G_2$ above to get a polynomial.

The first few of the solutions are given below:

$l=1$: (special case)

$$G = 1 - \frac{3}{4\alpha^2}x + \frac{6}{24\alpha^4}x^3\,,$$

$l=2$:

$$G = 1 + \frac{C_l - 3}{4\alpha^2}x - \frac{C_l - 3}{4\alpha^4}x^3 - \frac{1}{\alpha^4}x^4\,,$$

$l=3$:

$$G = 1 + \frac{C_l - 3}{4\alpha^2}x - \frac{C_l - 3}{2\alpha 4}x^3 - \frac{5}{\alpha 4}x^4 - \frac{30}{(C_l - 8)\alpha 4}x^5\,,$$

$l=4$:

$$G = 1 + \frac{C_l - 3}{4\alpha^2}x - \frac{5(C_l - 3)}{6\alpha 4}x^3 - \frac{15}{\alpha 4}x^4 + \frac{21(C_l - 15)}{2\alpha^4(C_l - 6 - 6\alpha^2)}x^5 + \frac{84}{\alpha 4}\frac{1}{(C_l - 6 - 6\alpha^2)}x^6\,.$$

The region of interest in our case is $r'_- \le x \le r'_+$. But since the solutions above are polynomials, they will serve as well as any other regular solution of Eq. (A1) between $r'_-$ and $r'_+$. (Indeed, these are the solutions that match to the waves outside properly.) We should now determine the solution of Eq. (A1) which is not regular at $r'_+$ or $r'_-$. The singularities at these points are real. This can be seen as follows. (We will take $r'_-$ as an example; the results are identical for $r'_+$.)

Try an expansion of the form

$$G = \sum_n b_n x'^{n+s}\,,$$

where $x' = x - r'_-$. One obtains the indicial equation $s^2 = 0$, i.e., $s_1 = s_2 = 0$. Write the equation as $\ddot{y} + P'(x')\dot{y} + Q'(x')y = 0$, where the single overdot denotes differentiation with respect to $x'$.

Now set $P'(x') = p'(x')/x'$, then $p'(0) = 1$. Using the method of Frobenius[25] does not lead to the case with two arbitrary constants. It is clear that all of the tests mentioned before give negative results. Hence the solution at $r'_-$ will contain $\ln(x') = \ln(x - r'_-)$. If $G'$ is the polynomial found above, then the irregular solution at $x = r'_-$ is of the form

$$G'' = AG' \ln(x - r'_-) + \sum_{n=0}^{\infty} b_n x'^n\,, \quad \text{(A7)}$$

where $A$ is a constant and the $b_n$'s can be determined by the method of undetermined coefficients. The precise form of the $b_n$'s is not important in the calculations we do in the main portion of the paper.

From the expressions written above, it follows that for the regular solution

$$R'_\pm = \frac{G'_\pm}{r}\,, \quad \frac{R'_\pm(r_+)}{R'_\pm(r_-)} = (-1)^{l+1}\frac{r_+}{r_-}\,.$$

Let us normalize this solution so that $R'_\pm(r_+)=1$, then

$$R'_\pm(r_-)=(-1)^{l+1}\frac{r_-}{r_+}.$$

If $R''_\pm$ denotes the irregular solution normalized in such a manner that

$$R''_\pm(\text{as } r\to r_+)=\ln\left(\frac{r_+-r}{r_+}\right)+\text{const}$$

then

$$R''_\pm(\text{as } r\to r_-)=A\ln\left(\frac{r-r_-}{r_-}\right)+\text{const}$$

The constant $A$ can be determined by the Wronskian of Eq. (A1),

$$W=\frac{dR'_\pm}{dr}R''_\pm-\frac{dR''_\pm}{dr}R'_\pm=\text{const},$$

near the points $r^*\to-\infty$ and $r^*\to+\infty$. We obtain

$$A=-R'_\pm(r_-)=(-1)^l\frac{r_-}{r_+}.$$

From this one can determine the zero-$k$ behavior of the coefficients $\tilde{A}(k)$ and $\tilde{B}(k)$ defined in the main portion of the paper. The answer has the same form as in the scalar case.


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