$[SU_2 \times U_1] \times S_n$ flavor dynamics and a bound on the number of flavors

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We unify the n quark and n lepton generations within the standard $SU_2 \times U_1$ gauge model by means of the generation symmetry group S_n. We show that no more than five generations of quarks and leptons can be incorporated into the theory. The resultant model always has either one or two exactly conserved multiplicative quantum numbers with eigenvalue ± 1 carried by each generation. These constrain the number of nonzero weak mixing angles between generations so that, even if there exist six or eight quarks, exact Cabibbo universality can hold naturally in the light-four-quark sector. The b quark must then always decay semileptonically with lepton-flavor violation, a key test.

I. INTRODUCTION

In this paper we consider the consequences of unifying n quark generations, and analogously the lepton generations, within the standard $SU_2\times U_1$ gauge model[1] by means of the discrete generation symmetry group S_n. We shall show that, under some simple assumptions to be delineated below, not more than five fermion generations can be incorporated into this generation-symmetric model. Moreover, for all allowed physically interesting n (i.e., $n=3, 4, 5$), there are always either one (π) or two (π and π') multiplicatively conserved quantum numbers carried by all generations in the theory, corresponding to residual unbroken S_2 or $S_2\times S_2$ symmetries of the vacuum of the original S_n-symmetric theory. These multiplicative conservation laws constrain some normally allowed weak mixing angles between generations in the charged-gauge-boson couplings of the quarks to be zero. An attractive consequence of this is that even if there are six or eight quarks exact Cabibbo universality[2] in the light-four-quark sector of the theory can be naturally maintained. This Cabibbo universality results in the b quark decaying purely via Higgs-boson-mediated interactions, *always* semileptonically and with flavor violation as described below. Verification of this unique prediction would strongly point to the existence of real Higgs bosons.

The model we propose for linking the different flavors is $[SU_2\times U_1]_{gauge}\times [S_n]_{generation}$, where S_n is the symmetric (permutation) group of n objects, and n corresponds to the number of quarks (and leptons) of a given charge in the theory—i.e., the number of generations. The logic behind this choice of generation unification group is that, before spontaneous symmetry breaking, the gauge-boson interactions of the n charge $\frac{2}{3}$ quarks, the n charge $-\frac{1}{3}$ quarks, the n charge -1 leptons, and the n neutrinos are already invariant under simultaneous S_n permutations[3] on the n flavor labels of each set of n fermions, since their gauge couplings depend only upon weak isospin and hypercharge, but not on generation number. We regard this S_n generation invariance as the superficial manifestation of a true S_n generation symmetry of the total weak Lagrangian including Higgs-boson and Yukawa interactions. In order to implement S_n invariance in the whole Lagrangian and nevertheless obtain nondegenerate quark and lepton masses after symmetry breaking, one needs Higgs bosons belonging to nonsinglet S_n representations, since singlet Higgs bosons result in an S_n-symmetric fermion mass matrix with degenerate fermions (see Sec. IV below). It is therefore most natural to require the existence of n generations of Higgs-boson doublets in the theory, in direct analogy with the quark and leptons. In this way *all* nongauge particles appear with n flavors, and the model displays a fundamental quark–lepton–Higgs-boson symmetry.

Models based upon this $SU_2\times U_1\times S_n$ picture for $n=2$ and 3 have been recently considered.[4-6] The $n=3$ case with six quarks[6] was shown to lead naturally to rather remarkable and unique predictions for b-quark decay that will soon be tested—namely that b-flavored hadrons *always* decay via Higgs-boson-mediated interactions to final states containing two differently flavored leptons, e.g., $b\to de^+\mu^-$. This absence of gauge-boson-mediated b decay naturally explains exact Cabibbo universality in the light- (u,d,c,s) quark sector. The model for $n=3$ also[6] predicted the existence of rare τ decays such as $\tau\to\mu ee$, $\mu\gamma$ via Higgs bosons of coupling strength $\sqrt{G_F}\, m_\tau$, as well as requiring muonium to antimuonium conversion. This paper is devoted to showing exactly how these features persist if more than six (in general $2n$) quarks exist.

The layout of the paper and a summary of the principal assumptions made are briefly outlined. In Sec. II we define the model $[SU_2\times U_1]_{gauge}\times [S_n]_{generation}$ with its n generations of quarks, leptons, and Higgs bosons. In Sec. III we derive strong constraints on the residual symmetries of the vacuums of the S_n-symmetric Lagrangian after spontaneous symmetry breaking. The main assumption we make in deriving these constraints is to restrict ourselves to minima of the Higgs potential with real vacuum expectation values. This leads to restrictive and highly structured relations between the fermion generations that will be easily verifiable.[7] Section IV derives the mass eigenvalues of the fermions from the Yukawa couplings in the different vacuums allowed by Sec. III, and shows that for $n>5$, some charged fermions must always be exactly degenerate. Since this is not the case for known leptons or quarks, we assume that newly discovered fermions will be similarly nondegenerate, so that the theory cannot incorporate more than five successive generations of quarks and leptons. Section V therefore discusses the physics of the allowed $n=3$, 4, and 5 models, and proves the existence in all cases of at least one and at most two multiplicatively conserved quantum numbers. We then demonstrate how these conservation laws can naturally enforce Cabibbo universality in the light- (u,d,c,s) quark sector even when the theory contains six or eight quarks. Finally, some phenomenological features and tests of the models are briefly sketched. In most cases the crucial test of the model is whether the b quark decays purely semileptonically, and always with lepton flavor violation.[8] Section VI contains the conclusion. Here we also discuss the incorporation of T violation into the theory, and the extension of S_n-generation symmetry to grand unified gauge models.

II. $[SU_2 \times U_1]_{\text{gauge}} \times [S_n]_{\text{generation}}$

The ingredients of the model are as follows:

(A) The usual[1] $SU_2\times U_1$ Yang-Mills gauge bosons $\vec{A}_\mu$ and B_μ which produce γ, $W^\pm$, and Z physical gauge bosons.

(B) The n generations of fermions and Higgs bosons in the theory, each generation containing

(i) one left-handed (LH) lepton isodoublet $l_i=(\nu_i,e_i^-)_L$,

(ii) one right-handed (RH) charged-lepton isosinglet $r_i=(e_i^-)_R$,

(iii) one LH quark isodoublet $\psi_i=(\mathcal{P}_i,\mathcal{N}_i)_L$,

(iv) one RH $\mathcal{P}$-quark isosinglet $(\mathcal{P}_i)_R$,

(v) one RH $\mathcal{N}$-quark isosinglet $(\mathcal{N}_i)_R$,

(vi) one Higgs doublet $\phi_i=(\phi_i^+,\phi_i^0)$.

In all the above categories except (A), the generation index i runs from 1 to n. This means that all particles except the gauge bosons are assigned to n-dimensional reducible representations of S_n. [One such n-dimensional representation can be decomposed into an irreducible S_n singlet and an irreducible $(n-1)$-tuplet of S_n. Most of the proofs below, however, can be understood without making use of formal group theory based upon these representation properties.] Quantum chromodynamics color indices have been suppressed since $(SU_3)_{\text{color}}$ commutes with the flavor group. S_n invariance requires that all interactions be invariant under the $n!$ permutations simultaneously acting on the n generation indices i of all fields.

We do not display the gauge-boson self-interactions here, nor their couplings to the n Higgs bosons; these are similar to the standard $SU_2\times U_1$ model with one Higgs doublet, and lead to the usual[1] W and Z interactions with $M_W/M_Z=\cos\theta_W$ after symmetry breaking.

The coupling of $\vec{A}_\mu$ and B_μ to all leptons and quarks is independent of the generation i, and therefore already S_n invariant. (It was this S_n invariance which we abstracted as the generation symmetry of the whole theory.)

The Yukawa couplings of Higgs bosons to leptons and quarks are given by

$$\mathcal{L}_Y=\sum_{i,j,k=1}^{n}[A_{ijk}\bar{l}_i\phi_j r_k+B_{ijk}\bar{\psi}_i\phi_j(\mathcal{N}_k)_R+C_{ijk}\bar{\psi}_i\tilde{\phi}_j(\mathcal{P}_k)_R+\text{H.c.}], \tag{2.1}$$

where A_{ijk}, B_{ijk}, and C_{ijk} are real if we assume T invariance, and where the whole expression must be symmetrized under the $n!$ S_n permutations on the i,j,k labels. For $n\geq 3$, this results in the n^3 coefficients A_{ijk} depending upon only five independent coupling constants, so that the first term in (2.1) can be written[9] after symmetrization as

$$\mathcal{L}_Y(\text{leptons})=a\bar{l}_i\phi_i r_i+b\bar{l}_i\phi_i r_j+c\bar{l}_i\phi_j r_i+d\bar{l}_j\phi_i r_i+e\bar{l}_i\phi_j r_k, \tag{2.2}$$

where i,j,k in each S_n-symmetric term of (2.2) range from 1 to n subject to the constraint $i\neq j\neq k$. A similar form[9] is valid for the B_{ijk} and C_{ijk} coefficients in the quark Yukawa couplings. [The existence of only five independent coupling constants instead of n^3 for each term in (2.1) can also be seen by decomposing the n fields into the above-mentioned irreducible representations of S_n and using the S_n Clebsch-Gordan coupling coefficients to construct only five possible invariants.]

The Higgs-boson self-interactions are

$$V = -\lambda[\phi_i^\dagger\phi_i] - \gamma[\phi_i^\dagger\phi_j] + A[(\phi_i^\dagger\phi_i)^2] + \tfrac{1}{2}B[(\phi_i^\dagger\phi_i)(\phi_i^\dagger\phi_j + \text{H.c.})]$$
$$+ \tfrac{1}{2}C_1[(\phi_i^\dagger\phi_i)(\phi_j^\dagger\phi_k + \text{H.c.})] + \tfrac{1}{2}C_2[(\phi_i^\dagger\phi_j)(\phi_i^\dagger\phi_k) + \text{H.c.}] + \tfrac{1}{2}C_3[(\phi_i^\dagger\phi_j)(\phi_k^\dagger\phi_i) + \text{H.c.}]$$
$$+ \tfrac{1}{2}D_1[(\phi_i^\dagger\phi_j)^2 + \text{H.c.}] + D_2[|\phi_i^\dagger\phi_j|^2] + D_3[(\phi_i^\dagger\phi_i)(\phi_j^\dagger\phi_j)] + E[(\phi_i^\dagger\phi_j)(\phi_k^\dagger\phi_l)], \tag{2.3}$$

where in each square bracket [ ] one sums over all independent permutations of $i,j,k,l = 1 \cdots n$, subject to the constraint $i \neq j \neq k \neq l$. Equation (2.3) is valid for $n \geq 4$; for $n=3$ (2) the terms with coefficients E (E and C_i) are absent. We take A, B, C_i, D_i, and E real by assuming T invariance.

Note that because three Higgs doublets cannot be combined into an SU_2 invariant, $\mathcal{L}_Y$ and V are accidentally invariant under the discrete symmetry

$$R: \left.\begin{array}{l}\phi_i \to -\phi_i\\ l_i \to -l_i\\ \psi_i \to -\psi_i\end{array}\right\}\text{for all } i. \tag{2.4}$$

III. RESIDUAL SYMMETRIES OF THE VACUUM AFTER SPONTANEOUS SYMMETRY BREAKING

We are now interested in the residual symmetries maintained by the vacuum (i.e., minimum) of the S_n- and R-invariant potential V in (2.3) after spontaneous symmetry breaking. As explained in Sec. I, here we shall restrict ourselves to examining only real (i.e., T-invariant) minima because of the strong and attractive constraints they yield on the fermion generations. Some remarks about complex minima will be made later.[7]

First, we assume the charged fields ϕ_i^+ to have zero vacuum expectation values at the minimum in accord with charge conservation. The neutral fields ϕ_i^0 are assumed to have vacuum expectation values of the same phase (i.e., effectively real and T invariant) at the minimum.[7] The conditions necessary to ensure this have been illustrated for the case $n=3$ in Ref. 6. The potential for the real c-number neutral fields $\phi_i^0 = \rho_i$ then takes the form [analogous to Eq. (2.3)],

$$U(\rho_i) = -\lambda[\rho_i^2] - \gamma[\rho_i\rho_j] + A[\rho_i^4] + B[\rho_i^3\rho_j] + C[\rho_i^2\rho_j\rho_k] + D[\rho_i^2\rho_j^2] + E[\rho_i\rho_j\rho_k\rho_l], \tag{3.1}$$

where $C = C_1 + C_2 + C_3$ and $D = D_1 + D_2 + D_3$. For $n=3$, $E=0$, for $n=2$, both $E=0$ and $C=0$. The following theorem then constrains the symmetries of the T-invariant vacuums of the potential (3.1):

Theorem 1. At any extremum (and therefore any minimum) of $U(\rho_i)$ for $i = 1 \cdots n$, $n \geq 4$, no more than three ρ_i can naturally have different values.

Proof. The extrema of the S_n-invariant $U(\rho_i)$ are given by the n equations

$$\frac{\partial U}{\partial \rho_i} \equiv W[\rho_i;\rho_k] = 0, \tag{3.2}$$

where W is a symmetric third-order polynomial symmetric in ρ_k for $k = 1 \cdots n$, $k \neq i$. Suppose two ρ_i, say ρ_1 and ρ_2, are different. Then

$$W[\rho_1;\rho_k] - W[\rho_2;\rho_k] = 0. \tag{3.3}$$

Since (3.3) is an antisymmetric polynomial under 1 and 2 label exchange, $(\rho_1 - \rho_2)$ must factor out, i.e.,

$$W[\rho_1;\rho_k] - W[\rho_2;\rho_k] \equiv (\rho_1 - \rho_2)Y[\rho_1,\rho_2;\rho_k] = 0, \tag{3.4}$$

where Y is symmetric under both $1 \leftrightarrow 2$ exchange, *and* under the S_{n-2} permutations on the $k = 3 \cdots n$ labels. Assuming $\rho_1 \neq \rho_2$,

$$Y[\rho_1,\rho_2;\rho_k] = 0, \quad k \neq 1,2. \tag{3.5}$$

Similarly if $\rho_2 \neq \rho_3$,

$$Y[\rho_2,\rho_3;\rho_k] = 0, \quad k \neq 2,3. \tag{3.6}$$

Subtracting (3.6) from (3.5) and making use of the antisymmetry under ρ_1 and ρ_3 exchange yields

$$Y[\rho_1,\rho_2;\rho_k] - Y[\rho_2,\rho_3;\rho_k] \equiv (\rho_1 - \rho_3)R[\rho_1,\rho_2,\rho_3;\rho_k] = 0, \quad k \neq 1,2,3 \tag{3.7}$$

where R is symmetric under ρ_k ($k \neq 1,2,3$) permutations. By repeating the above argument with ρ_1, ρ_2, and ρ_3 treated in reverse order, one can see that R is in fact also symmetric under (ρ_1,ρ_2,ρ_3) permutations. Furthermore, the successive factorizations imply that R is a linear function of ρ_i, so that if $\rho_1 \neq \rho_3$, then

$$R = \alpha(\rho_1 + \rho_2 + \rho_3) + \beta \sum_{k=4}^{n} \rho_k = 0, \tag{3.8}$$

where α and β are linearly dependent upon A, B, C, D, E in (3.1).

Similarly if $\rho_2 \neq \rho_3 \neq \rho_4$, (3.8) implies

$$\alpha(\rho_2 + \rho_3 + \rho_4) + \beta\left(\rho_1 + \sum_{k=5}^{n} \rho_k\right) = 0. \tag{3.9}$$

Subtracting (3.8) from (3.9) yields

$$(\alpha - \beta)(\rho_1 - \rho_4) = 0.$$

If $\rho_1 \neq \rho_4$, then $\alpha = \beta$, which is an unnatural constraint on independent quartic coupling constants. Thus no more than three ρ_i can be naturally different and the theorem is proved.

For the potential (3.1) it is easy to check that

$$\alpha = 4A - B + C - 2D\,,\qquad \beta = 3B - 3C + E\,. \tag{3.10}$$

This theorem severely constrains the symmetry of the vacuums of $U(\rho_i)$ in (3.1). For example, for $n=4$, S_4 cannot be broken completely (with real vacuum expectation values) since at the minimum at least two of ρ_1, ρ_2, ρ_3, and ρ_4 must be equal by the theorem, and therefore there is always at least a residual S_2 symmetry in the vacuum.[10] We will shortly categorize the residual symmetries of the real vacuums for low n in this model. First, however, we prove a special result for $n=3$ not covered by the above theorem.

Corollary A. The vacuum of the symmetric potential $U(\rho_i)$ for $n=3$ in (3.1) always has at least a residual S_2 symmetry.

Proof. The only way in which the minimum of $U(\rho_i)$ could possibly have no residual S_n symmetry is if $\rho_1 \neq \rho_2 \neq \rho_3$ at the extremum. We shall show that when this occurs, the three ρ_i are constrained such that there is still an effective S_2 symmetry at the minimum.

Suppose $\rho_1 \neq \rho_2 \neq \rho_3$. Then the arguments leading to (3.8) and (3.10) can be similarly used to deduce

$$(4A - B + C - 2D)(\rho_1 + \rho_2 + \rho_3) = 0\,,$$

so that the only natural condition is

$$\rho_1 + \rho_2 + \rho_3 = 0\,. \tag{3.11}$$

By substituting this into the equation $\partial U/\partial\rho_1 + \partial U/\partial\rho_2 + \partial U/\partial\rho_3 = 0$ (also valid at the minimum), one obtains

$$\rho_1\rho_2\rho_3 = 0\,. \tag{3.12}$$

Thus at least one ρ_i (say ρ_3) must vanish, and then from (3.11) $\rho_1 = -\rho_2 \equiv x$, so that the Higgs-field vacuum expectation values are $(\rho_1, \rho_2, \rho_3) = (x, -x, 0)$.

The original Lagrangian was invariant under S_3 and R [see Eq. (2.4)], and therefore invariant under the subgroup[11] with elements $\{1, P, R, RP\}$, where P is the $1 \leftrightarrow 2$ generation-index permutation operator. The above-mentioned vacuum $(x, -x, 0)$ is still invariant under the subgroup $\{1, RP\}$ which is isomorphic to S_2, and thus even in this case with $\rho_1 \neq \rho_2 \neq \rho_3$, there is still effectively an unbroken S_2.

Corollary B. The vacuums of the S_2 symmetric potential $U(\rho_i)$ for $n=2$ in Eq. (3.1) can have either $\rho_1 = \rho_2$ with a residual S_2 symmetry,[4] or $\rho_1 \neq \rho_2$ with no residual symmetry.

The proof is obtained by straightforward extremization of U.

With these theorems, we can now construct the allowed real minima of S_n-symmetric Higgs potentials after spontaneous symmetry breaking, and examine the residual symmetries of such vacuums. These minima and their symmetries are displayed in Table I. These residual symmetries will yield strong constraints on the fermion masses of the $SU_2 \times U_1 \times S_n$ theory in the next section. Table I covers $n=3$, 4, and 5. For $n \geq 6$, theorem I shows that there is always at least a residual symmetry containing S_3.

The theorems which constrained the vacuums of the theory to those of Table I held only for the real minima we considered. For complex-valued minima (i.e., ρ_i complex), the constraints on the residual symmetry of the vacuum of the S_n theory are much less severe. For example, for S_3 the values $[\langle\phi_1^0\rangle, \langle\phi_2^0\rangle, \langle\phi_3^0\rangle] = \rho[1, e^{2i\pi/3}, e^{4i\pi/3}]$ can extremize the potential and lead to a Z_3-symmetric cyclic model[12] with *all* vacuum expectation values different in their (complex) value, an option not allowed in the real case. Such a vacuum results in a theory[12] in which all particles carry a phase of either 0, $2\pi/3$, or $4\pi/3$ under cyclic Z_3 permutations on the three generation indices, with all interaction vertices in the theory constrained to conserve the phase modulo 2π. Other complex minima which totally break S_3 or S_4 in models with similar Higgs-boson potentials have also been found.[13,14] In view of this greater freedom for minima involving complex-valued $\langle\phi_i^0\rangle$, it is clear that once we allow T-violating complex vacuums in the S_n theory we are no longer so attractively constrained to the restrictive residual symmetries of Table I. If, however, the idea of a multiplicatively conserved generation quantum number that emerges in the remainder of this paper as a consequence of the residual real vacuum symmetries proves correct, these symmetries can be abstracted and maintained even in the case of complex vacuums—i.e., complex minima which allow T violation but still maintain the symmetry structure of Table I and its implications can easily be found—although their emergence in the presence of T violation is no longer so compelling.[7]

IV. FERMION MASSES IN $SU_2 \times U_1 \times S_n$

The mass-generating Yukawa couplings of the n-generation theory were given in Eqs. (2.1) and (2.2). The n Higgs-boson vacuum expectation values determine the residual permutation symmetry of the fermion mass matrix and the whole theory.

Theorem 2. If the residual permutation symmetry of the S_n-symmetric original theory after spontaneous symmetry breaking is S_l, $l \geq 3$, then at least $(l-1)$ fermions of the same charge will

TABLE I. The possible real vacuums and their residual symmetries allowed by theorem I in the $SU_2 \times U_1 \times S_n$ model considered in the text.

No. of Higgs fields n	Vacuum expectation values $[\langle\phi_1^0\rangle, \langle\phi_2^0\rangle \cdots \langle\phi_n^0\rangle]$	Residual S_n symmetry of vacuum and theory
3	$[x,x,x]$	S_3
	$[x,x,y]$	S_2 [b]
	$[x,y,z]=[x,-x,0]$ [a]	S_2
4	$[x,x,x,x]$	S_4
	$[x,y,y,y]$	S_3
	$[x,x,y,y]$	$S_2 \times S_2$ [b]
	$[x,x,y,z]$	S_2 [b]
5	$[x,x,x,x,x]$	S_5
	$[x,y,y,y,y]$	S_4
	$[x,x,y,y,y]$	$S_2 \times S_3$
	$[x,x,x,y,z]$	S_3
	$[x,x,y,y,z]$	$S_2 \times S_2$ [b]

[a] See Corollary A to theorem I.

[b] These vacuums, characterized by a residual symmetry smaller than S_3, are the only ones allowed by theorem II of Sec. IV. They are employed in Sec. V.

be exactly degenerate.

Proof. A residually S_l-symmetric vacuum produces an S_l-symmetric fermion mass matrix M, since S_l is still an unbroken symmetry of the Lagrangian. To examine the degeneracy of eigenvalues of M, it is sufficient to consider the $l\times l$ generation submatrix of the $n\times n$ matrix M. The most general form of $M^\dagger M$ restricted to this l-generation subspace is then

$$M^\dagger M = \begin{pmatrix} A & B & B & B & \cdots \\ B & A & B & B & \cdots \\ B & B & A & B & \cdots \\ B & B & B & A & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{pmatrix}, \qquad (4.1)$$

with l equal diagonal elements, and all off diagonal elements equal. This is easily seen to have one eigenvector $(1,1,\ldots 1)$ with eigenvalue $A+(l-1)B$, and $(l-1)$ degenerate eigenvectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1-l \\ \end{pmatrix},$$

with eigenvalue $A-B$. Thus if the residual vacuum symmetry is S_l, there are $l-1$ degenerate fermions of a given charge.

The large mass differences between known quarks (and leptons) of the same charge suggests that a residual degeneracy is incompatible with experiment. This theory can therefore tolerate only those vacuums whose residual symmetry is smaller than S_3. This restricts the vacuums of Table I in Sec. III further to those marked with a superscript b for $n\leq 5$; for $n>5$ the residual symmetry is always S_3 or greater, with unacceptably degenerate leptons and quarks, so that not more than five quark doublets can be incorporated into the theory.

In an S_n-symmetric theory with only singlet (S_n-invariant) Higgs bosons, no S_n breaking can occur, so that (4.1) is the general form of the mass matrix. As noted in Sec. I, for $n>2$ (more than four quarks) such a theory must have some degenerate fermions. Since six leptons (and probably six quarks) are already known, with no charged-particle degeneracies, we were forced to introduce flavored (nonsinglet) Higgs bosons too.

V. QUARK AND LEPTON PHYSICS FOR $n = 3,4,5$

We have shown that the need to avoid fermion mass degeneracies restricts us to $n\leq 5$,[15] and to vacuums with a residual symmetry smaller than S_3. The only allowed vacuums are then those marked with a superscript b in Table I.

All of these allowed vacuums have either one or two residual S_2 symmetries which are maintained in the full theory. We demonstrate below that these lead respectively to one (π) or two (π and π') multiplicatively conserved generation quantum numbers in the theory. *Only* multiplicative generation number is conserved; additive flavor or generation number is always violated by the Yukawa couplings to Higgs bosons. These couplings to any fermion are generally of order $\sqrt{G_F}m$, where G_F is the Fermi constant and m the mass of the heaviest fermion (quark or lepton) of the same charge as the fermion considered. Since $m=m_\tau$ for the leptons, and $m=m_b$ or m_t for the quarks, these couplings are of order 10^{-2} and therefore not negligible compared to the gauge couplings, unlike the standard model[1] with one Higgs boson, where a light fermion's mass determines its Yukawa coupling. The gauge couplings are similar to the standard case. The Yukawa couplings, however, lead to Higgs-boson-mediated, additive flavor-violating decays that serve to test the model. Finally, we shall show that π conservation for $n=3$, and π and π' conservation for $n=4$ naturally produce exact Cabibbo universality in the light-four-quark sector, even in the presence of two or four extra quarks in the theory. If this mechanism is the correct explanation for the observed universality, its consequence is easily testable: The b quark can then decay only via Higgs-mediated interactions, always semileptonically, and always with additive lepton-flavor violation, as discussed in Ref. 6. This unique prediction will be unambiguously confirmed or invalidated as soon as b-flavored hadrons are produced in e^+-e^- colliding-beam machines. All these results are outlined below for the allowable $n=3$, 4, and 5 models.

$n=3$: $SU_2\times U_1\times S_3$

This model has six quarks and leptons, and is the smallest one consistent with currently known particles. The only allowable vacuums in Table I for $n=3$ are $[x,x,y]$ with an S_2 symmetry under $1\leftrightarrow 2$ flavor exchange, and $[x,-x,0]$ with an effective S_2 symmetry as discussed below (3.12). This latter vacuum can easily be shown to lead to one exactly massless charged lepton, one massless charge $\frac{2}{3}$ (up-type) quark, and one massless charge $-\frac{1}{3}$ (down-type) quark. Only the $[x,x,y]$ vacuum is therefore acceptable.

The theory built upon this vacuum was extensively discussed in Refs. 5 and 6. Because of the $1\leftrightarrow 2$ symmetry in the whole theory, the $1\leftrightarrow 2$ permutation operator commutes with the Hamiltonian. Every state in the theory must therefore be even (multiplicative quantum number $\pi=+1$) or odd ($\pi=-1$) under $1\leftrightarrow 2$ flavor exchange, and all interaction vertices must conserve π multiplicatively. With this information it is easy to construct physical particle eigenstates and count the number of allowed weak mixing angles.

Let the three bare fermion flavors (one to each generation) in any sector (neutrinos, charged leptons, up or down quarks) be denoted $|1\rangle$, $|2\rangle$, and $|3\rangle$. The fermonic π eigenstates are then

$$|\mathrm{I}\rangle=|1-2\rangle\,,\quad \pi=-1\,,$$
$$\left.\begin{array}{l}|\mathrm{II}\rangle=\cos\phi\,|1+2\rangle+\sin\phi\,|3\rangle\\ |\mathrm{III}\rangle=-\sin\phi\,|1+2\rangle+\cos\phi\,|3\rangle\end{array}\right\}\,,\quad \pi=1 \tag{5.1}$$

with ϕ determined by the Yukawa coupling coefficients and the Higgs vacuum expectation values.[6] The gauge bosons $W^\pm$, Z, γ carry no generation number and therefore have $\pi=1$. Since π is conserved at all vertices, the $\pi=+1$ up quarks $|\mathrm{II}\rangle$, and $|\mathrm{III}\rangle$ can decay weakly via $W^\pm$ into any of the $\pi=+1$ down quarks of the type $|\mathrm{II}\rangle$ and $|\mathrm{III}\rangle$, but not into type $|\mathrm{I}\rangle$ quarks with $\pi=-1$. Thus only four quarks are linked to each other via the gauge weak interactions, allowing for only one weak mixing angle. Similarly the $\pi=-1$ up and down quarks couple weakly only to each other. To be consistent with the existence of a nonzero Cabibbo weak mixing angle in the u,d,c,s quark sector, these quarks must be identified with the $\pi=+1$ states. The heavy t and b quarks must therefore carry $\pi=-1$. Since b is lighter than t, and cannot decay weakly to light $\pi=+1$ quarks via the $W^\pm$, b is gauge stable and Cabibbo universality emerges naturally in the light-quark sector.

Neutrinos have been presumed massless, so that leptonic weak mixing angles are by definition zero and cannot be used to assign π values to e, μ, and τ. As in (5.1), two of e, μ, and τ must carry $\pi=+1$, the remainder carrying $\pi=-1$.

Since b is gauge-stable, it can only decay to a light $\pi=+1$ quark (u, d, c, or s) by emitting a virtual $\pi=-1$ Higgs boson. The only on-mass-shell state such a Higgs can decay into consists of two light leptons with opposite π values. (Total lepton number and total quark number are conserved by all interactions.) b quarks therefore always undergo the decay

$$\begin{array}{lcccc} & b\to & q & l & l'\,,\\ \pi: & -1 & +1 & -1 & +1\,,\end{array} \tag{5.2}$$

with manifest lepton-flavor violation 100% of the time. This unique prediction will serve to test the model. Observing the different leptons emitted by decaying b quarks will also allow the determination of leptonic π eigenvalues.

The scale of flavor-violating Yukawa couplings to any lepton is set by $\sqrt{G_F}m$, where m is the largest lepton mass of similar charge. This occurs because of the permutation symmetry in the original Lagrangian. The analogous statement is true in the quark sector. A Yukawa coupling strength of this magnitude $[\sqrt{G_F}\, m_\tau \simeq 6\times 10^{-3}]$ leads to peculiar leptonic decay modes (e.g., radiative τ decay) at appreciable rates, as discussed in Ref. 6.

$n=4$: $SU_2 \times U_1 \times S_4$

This model will contain four physical up quarks (u,c,t,h) and four down quarks (d,s,b,l). From Table I, there are only two allowable vacuums with no fermion degeneracy:

$$\begin{aligned} &\text{vacuum } A\text{: } [x,x,y,z]\text{: } S_2 \text{ symmetric} \\ &\text{vacuum } B\text{: } [x,x,y,y]\text{: } S_2\times S_2 \text{ symmetric}\,. \end{aligned} \tag{5.3}$$

Vacuum A has a residual $1 \leftrightarrow 2$ symmetry, so that all states must be even $(\pi=1)$ or odd $(\pi=-1)$ under $1\leftrightarrow 2$ exchange, with all interactions conserving π multiplicatively. Denoting the bare fermion flavors by $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$, there is one physical eigenstate $|1-2\rangle$ with $\pi=-1$, and three $\pi=+1$ eigenstates which are orthogonal linear combinations of $|1+2\rangle$, $|3\rangle$, and $|4\rangle$. Analogous to the $n=3$ case, the three $\pi=1$ up quarks can decay via W^+ into any of the three $\pi=1$ down quarks, with the usual weak mixing angles for six quarks. Since u, d, c, and s already display experimental Cabibbo mixing, these four quarks must be assigned to the $\pi=1$ states. A possible assignment is therefore

$$\begin{aligned} &\pi=1\text{: } u,d,c,s,t,b\,, \\ &\pi=-1\text{: } h,l\,, \end{aligned} \tag{5.4}$$

where the lighter t and b could be exchanged with h and l as another possibility. The lighter of h and l is now gauge stable, decaying purely via the Yukawa interactions, again semileptonically with lepton-flavor violation in order to conserve π, analogous to the b quark in the $n=3$ case. Exact Cabibbo universality for u, d, c, and s is now unnatural because there are six weakly mixing quarks with three mixing angles and one CP-violating phase possible. We do not discuss lepton assignments here since the analogy with quarks is clear.

Vacuum B has both an independent $1\leftrightarrow 2$ symmetry and a $3\leftrightarrow 4$ symmetry. If this vacuum is chosen for the theory, all physical energy eigenstates must be exactly either even or odd under *both* of these label permutations, with two multiplicatively conserved quantum numbers π and π' corresponding to $1\leftrightarrow 2$ and $3\leftrightarrow 4$ exchange, respectively. The physical eigenstates are therefore

$$\begin{aligned} &|1-2\rangle\,,\quad \pi=-1\,,\pi'=1\,, \\ &|3-4\rangle\,,\quad \pi=1\,,\pi'=-1\,, \\ &\left.\begin{aligned}&\cos\phi\,|1+2\rangle+\sin\phi\,|3+4\rangle \\ &-\sin\phi\,|1+2\rangle+\cos\phi\,|3+4\rangle\end{aligned}\right\}\,,\quad \pi=\pi'=1\,. \end{aligned} \tag{5.5}$$

The $W^\pm$ bosons clearly carry $\pi=\pi'=1$. π and π' conservation forbid the $\pi=\pi'=1$ states from decaying via $W^\pm$ bosons into $\pi=-\pi'$ states; thus the two up quarks with $\pi=\pi'=1$ can decay only into both of the two down quarks with $\pi=\pi'=1$. These four states must therefore be identified with the u, c, d, and s quarks. We thus arrive at the assignments

$$\begin{aligned} &\pi=\pi'=1\text{: } u,d,c,s\,, \\ &\pi=-\pi'=1\text{: } t,b\,, \\ &\pi=-\pi'=-1\text{: } h,l\,. \end{aligned} \tag{5.6}$$

π and π' conservation now naturally ensure exact Cabibbo universality in the light-quark sector, despite the presence of four heavier quarks. Both b and the lighter of h and l are gauge stable, and can only decay to a light quark with $\pi=\pi'=1$ by emitting a virtual $\pi=-\pi'$ Higgs boson; the only light state this virtual Higgs boson can decay into is one consisting of two oppositely flavored leptons, so that in this model the characteristic signature for b decay is a flavor-violating semileptonic final state, as in the $n=3$ model. The π and π' values of the four charged leptons are best discovered by examining the leptons appearing in the final states of heavy-quark decays, in accord with π and π' conservation. Once these are known, rare lepton decays (e.g., radiative decays or decays to three charged leptons[6]) can be calculated and predicted. Finally it is interesting to note that the (t,b) quarks and the (h,l) quarks cannot couple to each other by emission of one gauge or *one* Higgs boson, because π and π' conservation would require emission of a $\pi=\pi'=-1$ boson. It is straightforward to see that no linear function of $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$ can be made simultaneously odd under both $1\leftrightarrow 2$ and $3\leftrightarrow 4$ exchange, so that no such physical Higgs boson exists. Both b and the lighter of (h,l) therefore decay via Higgs bosons predominately to the four light quarks.

$n=5$: $SU_2 \times U_1 \times S_5$

From Table I, the only allowed vacuum is $[x,x,y,y,z]$ with a residual $S_2\times S_2$ symmetry under $1\leftrightarrow 2$ and $3\leftrightarrow 4$ exchange. In this ten-quark and ten-lepton model, the eigenstates in one charge sector are

$$|1-2\rangle:\ \pi=-1,\ \pi'=1\,,\qquad |3-4\rangle:\ \pi=1,\ \pi'=-1\,, \tag{5.7}$$

and three orthogonal mixtures of $|1+2\rangle$, $|3+4\rangle$, and $|5\rangle$ with $\pi=\pi'=1$. Since only states of the same π and π' can mix weakly, u,d,c,s must be assigned to the six $\pi=\pi'=1$ quark eigenstates, together with two other heavier quarks, one of which could be the b. There is therefore room for three weak mixing angles, and one phase, so that there is no exact Cabibbo universality. Two of the four $\pi=-\pi'$ quarks in the states (5.7), however, must be gauge stable, and therefore cannot decay to lighter quarks via $\pi=\pi'=1$ gauge bosons. Leptons can be analogously discussed.

VI. CONCLUSION

We have discussed the implications of using the permutation (symmetry) group S_n as the symmetry group for n fermion generations. In order to avoid fermion degeneracies between different generations we were led to deduce the existence of different generations of Higgs bosons, analogous to the fermion generations. Restricting the theory to have real Higgs-boson vacuum expectation values, we found that only a limited class of vacuums to the theory existed, all with some residual permutation symmetry in the vacuum (Table I). By insisting on no degenerate fermions, we found that $n\leq 5$, i.e., that no more than five generations of fermions could be incorporated into the theory.[15]

For the physically allowed cases $n=3$, 4, or 5, all possible allowed vacuums (the four marked with a superscript b in Table I) have either one (π) or two (π,π') multiplicatively conserved quantum numbers. Given the experimentally observed Cabibbo mixing between the light quarks (u,d,c,s), these must be assigned to $\pi=1$ ($\pi=\pi'=1$) states in the case of one (two) multiplicatively conserved quantum numbers. The b quark can then fall into one of two different classes. Class I vacuums $[(x,x,y)$ and $(x,x,y,y)]$ in Table I can only accomodate the b quark in a $\pi=-1$ or $\pi=-\pi'=1$ state with different π,π' values from the light quarks. In all these cases light-quark Cabibbo universality is natural and the b quark decays purely semileptonically with lepton-flavor violation. In class II vacuums $[(x,x,y,z)$ and $(x,x,y,y,z)]$ in Table I, the b-quark assignment is not unique. If the b quark carries the same π and π' as the light quarks, it can decay via ordinary gauge-boson-mediated weak interactions. Then one of the heavier quarks in the theory must carry a different π or π and π' and decay semileptonically. If, alternatively, the b quark has a $\pi=-\pi'$ assignment, it decays only semileptonically. *An unavoidable prediction of the model is therefore that some heavy quark must decay purely semileptonically with lepton-flavor violation, no matter whether $n=3$, 4, or 5.* If $n=3$ or 4, this may well be the b quark, in which case the prediction and the model will soon be tested in e^+-e^- colliding beams.

Note that even though our discussion has been given for n generations in the $SU_2\times U_1$ gauge group, it is obvious that many of our results on symmetry breaking and the existence of multiplicatively conserved quantum numbers can be trivially extended to different gauge groups or to grand unified models such as SU_5.[16]

The real Higgs-boson vacuum expectation values we have constrained ourselves to in order to obtain these restrictive symmetry schemes after spontaneous symmetry breaking have made the theory naturally T-invariant. Once we allow complex Higgs-boson vacuum expectation values, the vacuums with π and π' conservation in Table I are no longer the only allowed ones. However, complex-valued vacuums with the symmetry structure of Table I abstracted from the real case can still be found, and T violation can be incorporated into the theory in this way while still preserving multiplicative scheme for generations. (The choice of such vacuums in the presence of T violation, however, is no longer as compelling.) T violation can also be incorporated into the theory by explicitly allowing complex Yukawa coupling coefficients *a priori*.

Finally, we note that we have not tackled the problem of predicting quark mass ratios and nonzero weak mixing angles. Some attempts in this direction have been made by the authors listed in Ref. 8. Within the framework of our model here, some solutions to the problem could conceivably be obtained by incorporating the S_n generation symmetry within a larger and more restrictive finite group.

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[7]The assumption of real Higgs vacuum expectation values precludes the possibility of spontaneously broken T invariance. The incorporation of T violation into the model while preserving our results concerning the assignment of a multiplicative quantum number to the generations is sketched at the end of Sec. III and in Sec. VI.

[8]Models using S_n as a discrete symmetry have been proposed also by S. Pakvasa and H. Sugawara, Phys. Lett. 73B, 61 (1978); *ibid.* 82B, 105 (1979); H. Sato, Tokyo University Report No. UT-299, 1978 (unpublished); Nucl. Phys. B148, 433 (1979); D. Wyler, Phys. Rev. D 19, 330 (1979); 19, 3369 (1979); G. Segrè and H. A. Weldon, Phys. Rev. Lett. 42, 1191 (1979). These models differ from the model we consider here in both their choice of representations and/or their vacuum structure. In particular they do not lead to the unique prediction of our model of semileptonic flavor-violating b decay because of their lack of multiplicative quantum number conservation.

[9]See Eqs. (3.3) and (8.1) of Ref. 6 for specific examples for the case $n = 3$.

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