PHYSICAL REVIEW D VOLUME 20, NUMBER 5 1 SEPTEMBER 1979

η^0-π^0 mixing effects and $\Delta S = 1$ weak decays

Barry R. Holstein
Physics Division, National Science Foundation, Washington, D.C. 20550
(Received 6 April 1979)

η^0-π^0 mixing induces calculable $\Delta I = 3/2$ effects in $\Delta S = 1$ nonleptonic weak decay amplitudes. These effects are evaluated and compared to experimentally measured quantities.

A recent article by Gross, Treiman, and Wilczek (GTW)[1] treated the effects of the u,d mass difference on isospin-violating processes. They pointed out that $m_d - m_u \neq 0$ induces η^0-π^0 and Λ^0-Σ^0 mixing, which leads to (generally small) corrections to familiar results—e.g., the vanishing of the vector coupling constant $g_V(0)$ in

$$\Sigma^\pm \to \Lambda^0 + e^\pm + \nu_e$$

is no longer required.

We wish here to note that similar considerations lead to small but calculable violations of the $\Delta I = \frac{1}{2}$ rule in $\Delta S = 1$ weak decays. In order to estimate the size of these effects, we note that in the diagonalized mass matrix the physical η^0-π^0 states have the form

$$\pi^0 = \cos\lambda\, P^3 + \sin\lambda\, P^8, \quad \eta^0 = -\sin\lambda\, P^3 + \cos\lambda\, P^8 \tag{1a}$$

where the mixing angle is given by

$$\sin\lambda = \frac{\sqrt{3}}{4}\,\frac{m_d - m_u}{m_s} \tag{1b}$$

in the "tadpole" approximation.[2] There is presumably also an electromagnetic contribution, but this vanishes in the soft-meson limits and will be assumed negligible here.[3]

Because of this mixing the amplitudes for emission of charged and neutral pions are no longer simply related by a Clebsch-Gordan coefficient. Thus, for π^0 emission

$$\langle \pi_q^0 \beta | H_w | \alpha \rangle \approx \langle \pi_q{}^3 \beta | H_w | \alpha \rangle + \sin\lambda \langle \eta_q \beta | H_w | \alpha \rangle , \tag{2}$$

where π^3 is the isotopic-spin partner of the charged pions.

Consider, for example, the nonleptonic decays $K \to 2\pi$. We can parametrize them as

$$\begin{aligned} A(K^0 \to \pi^3\pi^3) &= -(\tfrac{1}{3})^{1/2} f_1 + \frac{2}{\sqrt{15}} f_3 , \\ A(K^0 \to \pi^+\pi^-) &= (\tfrac{1}{3})^{1/2} f_1 + \frac{1}{\sqrt{15}} f_3 , \\ A(K^+ \to \pi^+\pi^3) &= (\tfrac{3}{10})^{1/2} f_3 , \end{aligned} \tag{3}$$

where f_1, f_3 measure the intrinsic $\Delta I = \frac{1}{2}, \frac{3}{2}$ decay amplitudes. Experimentally we find[4]

$$\begin{aligned} y &= \frac{\sqrt{2}\, A(K^+ \to \pi^+\pi^0)}{2A(K^0 \to \pi^+\pi^-) - A(K^0 \to \pi^0\pi^0)} \\ &= 0.032 \pm 0.001 . \end{aligned} \tag{4}$$

Usually this is interpreted as a 3% $\Delta I = \frac{1}{2}$ rule violating component

$$(\tfrac{1}{5})^{1/2} \frac{f_3}{f_1} = 0.032 . \tag{5}$$

However, including the effects of η^0-π^0 mixing we have

$$\begin{aligned} A(K^+ \to \pi^+\pi^0) &= (\tfrac{3}{10})^{1/2} f_3 + \sin\lambda\, A(K^+ \to \pi^+\eta) , \\ A(K^0 \to \pi^0\pi^0) &= -(\tfrac{1}{3})^{1/2} f_1 + \frac{2}{\sqrt{15}} f_3 + 2\sin\lambda\, A(K^0 \to \pi^0\eta) , \\ A(K^0 \to \pi^+\pi^-) &= (\tfrac{1}{3})^{1/2} f_1 + \frac{1}{\sqrt{15}} f_3 . \end{aligned} \tag{6}$$

Hence

$$y \cong \frac{(\frac{1}{5})^{1/2} f_3 + (\frac{2}{3})^{1/2} \sin\lambda\, A(K^+ \to \pi^+\eta)}{f_1} . \tag{7}$$

Since such terms are already suppressed by $\sin\lambda$ we assume $\Delta I = \frac{1}{2}$ dominance for $A(K \to \pi\eta)$. Then

$$A(K^+ \to \pi^+\eta) = \sqrt{2}\, A(K^0 \to \pi^0\eta) , \tag{8}$$

and since the η is soft—$q_\eta{}^2 = m_\pi{}^2$—we can reliably estimate these matrix elements using current-algebra and PCAC (partially conserved axial-vector current) techniques[5]

$$\begin{aligned} A(K^0 \to \pi^0\pi^0) &\underset{q_\pi \to 0}{\longrightarrow} -\frac{i}{2F_\pi} \langle \pi^0 | H_w | K^0 \rangle , \\ A(K^0 \to \pi^0\eta) &\underset{q_\eta \to 0}{\longrightarrow} \frac{i}{2F_\pi} \sqrt{3} \langle \pi^0 | H_w | K^0 \rangle . \end{aligned} \tag{9}$$

Thus we have

$$A(K^0 \to \pi^0\eta) \cong -\sqrt{3}\, A(K^0 \to \pi^0\pi^0) \approx f_1 . \tag{10}$$

Then

$$y = (\tfrac{1}{5})^{1/2} \frac{f_3}{f_1} + \frac{2}{\sqrt{3}} \sin\lambda . \tag{11}$$

Since

$$\frac{2}{\sqrt{3}}\sin\lambda \cong \frac{1}{2}\;\frac{m_d - m_u}{m_s} \approx 0.012\,, \tag{12}$$

we see that about forty percent of the experimental $\Delta I=\frac{1}{2}$ violating amplitude is accounted for by η^0-π^0 mixing, so that only sixty percent need be a result of intrinsic $\Delta I=\frac{3}{2}$ terms—thus bona fide $\Delta I=\frac{3}{2}$ terms are even smaller than usually believed

$$(\tfrac{1}{5})^{1/2}\frac{f_3}{f_1} = 0.020\,. \tag{13}$$

Now consider $K\to 3\pi$ decays. Here we assume an amplitude which is linear in the pion energy

$$\begin{aligned}
A(K^0\to\pi^0\pi^0\pi^0) &= -\tfrac{1}{3}\sqrt{2}\,[3(\alpha_1+\sqrt{2}\,\alpha_3)+M_K(\beta_1+\sqrt{2}\,\beta_3)]\,,\\
A(K^0\to\pi^+\pi^-\pi^0) &= \tfrac{1}{3}\sqrt{2}\,[\alpha_1+\sqrt{2}\,\alpha_3+E_0(\beta_1+\sqrt{2}\,\beta_3)+(3/\sqrt{10})(E_- - E_+)\gamma_3]\,,\\
A(K^+\to\pi^+\pi^+\pi^-) &= \tfrac{1}{3}\sqrt{2}\,\{2(\alpha_1-\sqrt{\tfrac{1}{2}}\,\alpha_3)+(M_K-E_-)[\beta_1-(\tfrac{1}{2})^{1/2}\beta_3]+(9/2\sqrt{10})(E_- - \tfrac{1}{3}M_K)\gamma_3\}\,,\\
A(K^+\to\pi^+\pi^0\pi^0) &= -\tfrac{1}{3}\sqrt{2}\,\{\alpha_1-(\tfrac{1}{2})^{1/2}\alpha_3+E_+[\beta_1-(\tfrac{1}{2})^{1/2}\beta_3]+(9/2\sqrt{10})(E_+ - \tfrac{1}{3}M_K)\gamma_3\}\,,
\end{aligned} \tag{14}$$

where the subscripts 1, 3 represent the $\Delta I=\frac{1}{2}, \frac{3}{2}$ contributions to the decay and E_i represents the energy of the ith pion. Now write the $K\to 3\pi$ amplitudes via

$$A(K\to\pi^a\pi^a\pi^b) = A_0\left[1-\lambda\,\frac{2m_K}{m_\pi^{\,2}}\left(E_b-\frac{m_K}{3}\right)\right]\,, \tag{15}$$

where A_0 is the mean decay amplitude and λ is the slope. We construct the $\Delta I=\frac{1}{2}$ rule violating quantities[4]

$$\begin{aligned}
v_1 &= \frac{1}{4}\left(\frac{A_0^{++-}}{A_0^{+-0}}\right)^2 - 1 = 0.216\pm 0.020\,,\\
v_2 &= -\frac{1}{2}\;\frac{\lambda^{+00}}{\lambda^{++-}} - 1 = 0.308\pm 0.051\,.
\end{aligned} \tag{16}$$

Using straightforward current-algebra PCAC techniques[5] one can—neglecting η^0-π^0 mixing—calculate v_1, v_2 in terms of the parameter y which measures $\Delta I=\frac{1}{2}$ violation in the $K\to 2\pi$ system. We find in this way[6]

$$\begin{aligned}
v_1 &= 6y \cong 0.19\,,\\
v_2 &= \tfrac{27}{2}\,y \cong 0.43\,.
\end{aligned} \tag{17}$$

Of course, here v_1, v_2 are considered to arise entirely from intrinsic $\Delta I=\frac{3}{2}$ effects. Including η^0-π^0 mixing we must write

$$\begin{aligned}
A(K^0\to 3\pi^0) &\cong A(K^0\to 3\pi^3) + 3\sin\lambda\, A(K^0\to\pi^0\pi^0\eta)\,,\\
A(K^0\to\pi^+\pi^-\pi^0) &\cong A(K^0\to\pi^+\pi^-\pi^3)\\
&\quad + \sin\lambda\, A(K^0\to\pi^+\pi^-\eta)\,,\\
A(K^+\to\pi^+\pi^+\pi^-) &= A(K^+\to\pi^+\pi^+\pi^-)\,,\\
A(K^+\to\pi^+\pi^0\pi^0) &\cong A(K^+\to\pi^+\pi^3\pi^3)\\
&\quad + 2\sin\lambda\, A(K^+\to\pi^+\pi^0\eta)\,.
\end{aligned} \tag{18}$$

Requiring consistency with soft-pion and soft-η limits in a linear approximation (again we emphasize that $q_\eta^{\,2}=m_\pi^{\,2}$ so that this limit should be as good as that expected for soft pions alone) we find (using $\Delta I=\frac{1}{2}$ dominance)

$$\begin{aligned}
A(K^0\to\pi^0\pi^0\eta) &= -\frac{i}{2F_\pi}f_1\,,\\
A(K^0\to\pi^+\pi^-\eta) &= \frac{i}{2F_\pi}f_1\left(1+\frac{2}{M_K}(E_- - E_+)\right)\,,\\
A(K^+\to\pi^+\pi^0\eta) &= \frac{i}{\sqrt{2}F_\pi}f_1\,\frac{2}{M_K}(E_0-E_+)\,.
\end{aligned} \tag{19}$$

The final result is then

$$\begin{aligned}
\alpha_1 &= -i\,\frac{\sqrt{3}}{2F_\pi}f_1\left(1+\frac{1}{\sqrt{3}}\sin\lambda\right)\,,\\
M_K\beta_1 &= i\,\frac{\sqrt{3}}{F_\pi}f_1(1+\sqrt{3}\sin\lambda)\,,\\
\alpha_3 &= -\frac{i}{2F_\pi}(\tfrac{3}{10})^{1/2}f_3 + i\,\frac{\sqrt{2}}{F_\pi}f_1\sin\lambda\,,\\
M_K\beta_3 &= i\,\frac{5}{2F_\pi}(\tfrac{3}{10})^{1/2}f_3 - i\,\frac{3}{\sqrt{2}\,F_\pi}f_1\sin\lambda\,,\\
M_K\gamma_3 &= i\,\frac{3}{\sqrt{2}\,F_\pi}\sqrt{3}\,f_3 + i\,\frac{\sqrt{10}}{F_\pi}f_1\sin\lambda\,.
\end{aligned} \tag{20}$$

One can now calculate the $\Delta I=\frac{1}{2}$ rule violating parameters v_1, v_2

$$\begin{aligned}
v_1 &= 6\sqrt{3}\sin\lambda + 6(\tfrac{1}{5})^{1/2}\frac{f_3}{f_1} = 6y + 2\sqrt{3}\sin\lambda\,,\\
v_2 &= 3\sqrt{3}\sin\lambda + \tfrac{27}{2}(\tfrac{1}{5})^{1/2}\frac{f_3}{f_1} = \tfrac{27}{2}y - 6\sqrt{3}\sin\lambda\,.
\end{aligned} \tag{21}$$

Using the canonical value for $\sin\lambda$ [Eq. (1b)], we find

$$\begin{aligned}
v_1 &= 0.036 + 6y \cong 0.228\,,\\
v_2 &= -0.108 + \tfrac{27}{2}y \cong 0.32\,,
\end{aligned} \tag{22}$$

which are in reasonable agreement with the experimental numbers. Again we see that a substantial fraction of the $\Delta I=\frac{1}{2}$ rule violating amplitude (~30%) arises from simple η^0-π^0 mixing.

In the case of nonleptonic hyperon decay we must also account for the effects of Λ^0-Σ^0 mixing,

$$\begin{aligned}\Lambda^0 &= \cos\rho B^8 + \sin\rho B^3\,,\\ \Sigma^0 &= -\sin\rho B^8 + \cos\rho B^3\,,\end{aligned} \tag{23a}$$

where

$$\sin\rho \approx 0.4\,\frac{m_d - m_u}{m_s} \approx 0.0096\,. \tag{23b}$$

GTW have given arguments that again this should be the dominant effect.

We find then

$$\begin{aligned}A(\Lambda^0\to p\pi^-) &\cong A(B^8\to p\pi^-) + \sin\rho A(B^3\to p\pi^-)\,,\\ A(\Lambda^0\to n\pi^0) &\cong A(B^8\to n\pi^3) + \sin\rho A(B^3\to n\pi^0)\\ &\quad + \sin\lambda A(B^8\to n\eta)\,.\end{aligned} \tag{24}$$

Using soft-π-η limits with PCAC, we can exploit the property of the weak Hamiltonian that[7]

$$[F_a^5, H_w] = [F_a, H_w]\,, \tag{25}$$

where

$$F_a^5 = \int d^3x\, A_a^0(\vec{x},t), \quad F_a = \int d^3x\, V_a^0(\vec{x},t) \tag{26}$$

are the axial-vector and vector charges. Since

$$\begin{aligned}F_a &= I_a, \quad a=1,2,3\,,\\ F_8 &= \tfrac{1}{2}\sqrt{3}\,Y,\end{aligned} \tag{27}$$

where I and Y are the isotopic-spin and hypercharge operators, respectively, we can write (for S-wave amplitudes)[8]

$$\begin{aligned}A(\Lambda^0\to p\pi^-) &\cong \frac{-i}{\sqrt{2}\,F_\pi}[\langle n|H_w|B^8\rangle\\ &\quad + \sin\rho(\langle n|H_w|B^3\rangle\\ &\quad - \sqrt{2}\,\langle p|H_w|\Sigma^+\rangle)]\,,\\ A(\Lambda^0\to n\pi^0) &\cong \frac{i}{2F_\pi}[\langle n|H_w|B^8\rangle(1-\sqrt{3}\sin\lambda)\\ &\quad + \sin\rho\,\langle n|H_w|B^3\rangle]\,.\end{aligned} \tag{28}$$

Now use the observation that the SU(3)-octet contribution to $\bar{B}B$ matrix elements is the dominant one to write

$$\begin{aligned}A(\Lambda^0\to p\pi^-) &\cong \frac{-i}{\sqrt{2}\,F_\pi}[D+3F-\sqrt{3}\sin\rho(D-F)+\sigma_3]\,,\\ A(\Lambda^0\to n\pi^0) &\cong \frac{i}{2F_\pi}\,[(D+3F)(1-\sqrt{3}\sin\lambda)\\ &\quad + \sqrt{3}\sin\rho(D-F)-2\sigma_3]\,,\end{aligned} \tag{29}$$

where D, F are the octet $\langle B'|H_w|B\rangle$ coupling constants, and σ_3 is a bona fide $\Delta I=\frac{3}{2}$ amplitude contributing to the decay. For an estimate, we use the empirical value $F\approx -2D$.[9] Then

$$\begin{aligned}R_\Lambda &= \frac{A(\Lambda^0\to p\pi^-)+\sqrt{2}\,A(\Lambda^0\to n\pi^0)}{\sqrt{2}\,A(\Lambda^0\to p\pi^-)-A(\Lambda^0\to n\pi^0)}\\ &= (\tfrac{2}{3})^{1/2}\sin\lambda - 2(\tfrac{2}{3})^{1/2}\sin\rho\,\frac{D-F}{D+3F}+z\\ &\approx 0.018+z\,,\end{aligned} \tag{30}$$

where

$$z = \frac{\sqrt{2}\,\sigma_3}{D+3F}\,, \tag{31}$$

to be compared to the experimental value

$$R_\Lambda^{\text{expt}} = -0.027\pm 0.008\,. \tag{32}$$

Thus the η^0-π^0 mixing effect in this case is opposite in sign to the experimental $\Delta I=\frac{1}{2}$ violating effect, so that the intrinsic $\Delta I=\frac{3}{2}$ amplitude is not the simple value

$$z\approx -0.027\,, \tag{33}$$

derived neglecting η^0-π^0 mixing but rather the larger value

$$z\approx -0.043\,. \tag{34}$$

In a similar fashion we find

$$\begin{aligned}A(\Xi^-\to\Lambda^0\pi^-) &\cong \frac{-i}{\sqrt{2}\,F_\pi}[D-3F-\sqrt{3}\sin\rho(D+F)+\sigma_3']\,,\\ A(\Xi^0\to\Lambda^0\pi^0) &\cong \frac{i}{2F_\pi}[(D-3F)(1-\sqrt{3}\sin\lambda)\\ &\quad + \sqrt{3}\sin\rho(D+F)-2\sigma_3']\,.\end{aligned} \tag{35}$$

Then

$$\begin{aligned}R_\Xi &= \frac{A(\Xi^-\to\Lambda\pi^-)+\sqrt{2}\,A(\Xi^0\to\Lambda\pi^0)}{\sqrt{2}\,A(\Xi^-\to\Lambda\pi^-)-A(\Xi^0\to\Lambda\pi^0)}\\ &= (\tfrac{2}{3})^{1/2}\sin\lambda - 2(\tfrac{2}{3})^{1/2}\sin\rho\,\frac{D+F}{D-3F}+z'\\ &\approx 0.011+z'\,,\end{aligned} \tag{36}$$

where

$$z' = \frac{\sqrt{2}\,\sigma_3'}{D-3F}\,. \tag{37}$$

Here again comparison with the experimental value

$$R_\Xi^{\text{expt}} = -0.030\pm 0.011 \tag{38}$$

implies that the η^0-π^0 effect goes in the opposite direction from the experimental number, so that the bona fide $\Delta I=\frac{3}{2}$ amplitude z' is not given by its naive value

$$z'\approx -0.030 \tag{39}$$

calculated omitting η^0-π^0 mixing, but rather is given by

$$z' \approx -0.041\,. \qquad (40)$$

Finally, for Σ-hyperon decays we have

$$A(\Sigma^+ \to n\pi^+) \cong \frac{-i}{2\sqrt{2}\,F_\pi}\,\sigma_3'' ,$$

$$A(\Sigma^- \to n\pi^-) \cong \frac{i}{F_\pi}\left[\sqrt{3}\,(D-F) + \frac{3}{4\sqrt{2}}\,\sigma_3''\right], \qquad (41)$$

$$A(\Sigma^+ \to p\pi^0) \cong \frac{i}{2F_\pi}\,[\sqrt{6}\,(D-F)(1-\sqrt{3}\,\sin\lambda) - \sigma_3''] ,$$

so that

$$R_\Sigma = \frac{A(\Sigma^+ \to n\pi^+) - A(\Sigma^- \to n\pi^-) + \sqrt{2}\,A(\Sigma^+ \to p\pi^0)}{A(\Sigma^- \to n\pi^-)}$$

$$\cong -\sqrt{3}\,\sin\lambda + z''$$

$$\approx -0.018 + z'' , \qquad (42)$$

where

$$z'' = -\frac{3\sqrt{3}}{4\sqrt{2}}\,\frac{\sigma_3''}{D-F} , \qquad (43)$$

to be compared to the experimental value

$$R_\Sigma^{\text{expt}} = 0.12 \pm 0.05\,. \qquad (44)$$

Here also then the bona fide $\Delta I = \frac{3}{2}$ term is modified by mixing from its naive value

$$z'' \approx 0.12 \qquad (45)$$

to

$$z'' \approx 0.14\,. \qquad (46)$$

We conclude that η^0-π^0 mixing effects, although small, have a significant effect on the "measured" size of intrinsic $\Delta I = \frac{3}{2}$ weak amplitudes. For kaon (hyperon) decays the required $\Delta I = \frac{3}{2}$ amplitude is smaller (larger) than usually assumed. These changes do not necessarily pose particular theoretical difficulties, inasmuch as there exists at present no reliable means of calculating nonleptonic weak amplitudes. Nevertheless, they could prove troublesome in that the "enhancement factor" for the purely weak $\Delta I = \frac{1}{2}$ amplitude relative to the $\Delta I = \frac{3}{2}$ amplitude in $K \to 2\pi$ is now roughly 30 instead of 20. Renormalization-group quantum-chromodynamic enhancement calculations,[10] which have had difficulty in generating this factor of 20, may be hard pressed to attain a factor of 30.

---

[1]D. J. Gross, S. B. Treiman, and F. Wilczek, Phys. Rev. D 19, 2188 (1979).

[2]That is, we are using

$$M^2_{\alpha\beta} = -\frac{1}{F_\pi^{\,2}}\,\langle 0|\,[F_\alpha^5,\,[F_\beta^5,\,m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s]]\,|0\rangle\,.$$

[3]We note

$$A(0,p) = A(q,0) = 0\,,$$

$$A(q,p) = \int d^4x\,e^{ik\cdot x}D_{\mu\nu}(x)\,\langle \eta_q|\,T[J^\mu_{\text{em}}(x)J^\nu_{\text{em}}(0)]\,|\pi_p\rangle\,.$$

[4]C. Bricman *et al.*, Phys. Lett. 75B, 1 (1978).

[5]S. Adler and R. Dashen, *Current Algebra* (Benjamin, New York, 1966); B. Renner, *Current Algebras and Their Applications* (Pergamon, Oxford, 1968); Y. Nambu, Phys. Rev. Lett. 4, 380 (1960); M. Gell-Mann and M. Levy, Nuovo Cimento 17, 705 (1960).

[6]B. Holstein, Phys. Rev. 183, 1228 (1969); Y. Nambu and Y. Hara, Phys. Rev. Lett. 16, 875 (1966); C. Bouchiat and Ph. Meyer, Phys. Lett. 25B, 282 (1967). $\eta - \pi$ mixing effects and nonleptonic K decay have been discussed in a different way by M. Bace, Phys. Rev. D 4, 2838 (1971); A. Goyal and L. F. Li, *ibid.* 4, 2012 (1971); B. Holstein, *ibid.* 6, 3266 (1972).

[7]E. Golowich and B. R. Holstein, Phys. Rev. Lett. 35, 831 (1975).

[8]We deal here only with the parity-*violating* weak hyperon amplitudes, avoiding the traditional difficulties in understanding the parity-conserving piece of the decay.

[9]B. W. Lee, Phys. Rev. 170, 1359 (1969); B. Holstein, Nuovo Cimento A2, 561 (1971); M. Gronau, Phys. Rev. Lett. 28, 188 (1972); Phys. Rev. D 5, 118 (1972).

[10]M. K. Gaillard and B. W. Lee, Phys. Rev. Lett. 33, 108 (1974); G. Altarelli and L. Maiani, Phys. Lett. 52B, 351 (1974).