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High-energy predictions in quantum chromodynamics

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In this paper the methods of cut vertices are applied to deeply inelastic electron scattering, single-particle hadron production in e^+e^- annihilation, multiparticle inclusive production in e^+e^- annihilation, μ-pair production in hadron-hadron collisions, wide-angle inclusive hadron production in hadron-hadron collisions, and single-particle production in deeply inelastic electron scattering. A generalized Wilson expansion is discussed in detail for two-particle inclusive production in e^+e^- annihilation. The question of soft and wee partons is discussed and the cancellation of wee partons demonstrated. The relation of cut vertices to the parton model is briefly discussed.

I. INTRODUCTION

In this paper we shall discuss some of the predictions which quantum chromodynamics (QCD) gives at high energies. We shall use the method of cut vertices which was developed in Ref. 1. Suitable moments of certain high-energy processes factorize into several parts, one of which is a dimensionless factor which obeys a Callan-Symanzik equation. This factor is the analog of the Wilson singular function which appears in the operator product expansion. The other factors are the cut vertices, V_σ. The cut vertices may refer to incoming particles, spacelike cut vertices, in which case they correspond to matrix elements of composite operators for positive integral σ. The cut vertices may also refer to outgoing particles, timelike cut vertices, in which case they have no known connection with local operators.

Section II describes the QCD predictions in detail. After listing the Feynman rules for cut vertices in Sec. II A we go on to discuss deeply inelastic electron scattering in B, single-particle production in e^+e^- collisions in C, multiparticle inclusive production in e^+e^- collisions in D, two-particle production in e^+e^- collisions in E, μ-pair production in hadron-hadron collisions in F, wide-angle hadron production in hadron-hadron collisions in G, and single-particle production in deeply inelastic electron scattering in H. In each case we discuss only those predictions which firmly follow from QCD at high energies. Some of these sections have considerable overlap with the work of other authors.[2-6] Many of the results have been anticipated by the practitioners of the parton model.[7-11] However, since we have a precise formalism we are able to see which of the parton-model results are unambiguous QCD predictions and which are more model dependent. Let us briefly summarize Sec. II.

A. Cut vertices are given for QCD. Both spacelike and timelike vertices are listed.

B. Deeply inelastic electron scattering is discussed from the point of view of cut vertices. This section contains no new results, but may serve to show the reader how cut vertices act in a familiar setting.

C. Single-particle inclusive hadron production in e^+e^- annihilation is discussed. Moments of the cross section obey a Callan-Symanzik equation. It is observed that by taking differences of cross sections the singlet cut vertices are eliminated. Thus it is relatively easy to obtain a cross section whose moments behave as a single unique power of $\ln Q^2$. In particular, certain differences of multiplicities approach a constant at large Q^2. If it were not for the complication of heavy-quark thresholds and the necessity[12] of removing those hadrons coming from the weak decays of charmed hadrons and τ's, this process would serve as an exceptionally easy test of QCD. Whether in fact the tests of QCD are easily made here will depend on how easily the above complications can be overcome.

D. This section, multiparticle inclusive production in e^+e^- annihilation, is presumably of formal interest only.

E. In two-particle inclusive production in e^+e^- annihilation there are three separate kinematic regions. In terms of the angle, θ, between the outgoing observed particles in the center of mass of the e^+e^- these regions are (i) $\theta \neq 0, \pi$, (ii) $\theta = \pi$, and (iii) $\theta = 0$. In region (i) the moments

$$\int \omega_1^{\sigma_1 - 2}\omega_2^{\sigma_2 - 2} d\omega_1 d\omega_2 2E_1 2E_2 \frac{d\sigma}{d^3p_1 d^3p_2}$$

obey a Callan-Symanzik equation at large Q^2. In the above $\omega_1 = 2p_1 \cdot q/Q^2$ and $\omega_2 = 2p_2 \cdot q/Q^2$. An order-$g^2$ QCD calculation along with the Callan-Symanzik equation gives the asymptotic form of the moments in terms of timelike cut vertices and calculable Q^2 and θ dependences. Taking appropriate differences of cross sections allows one to obtain a set of moments having a single and calculable power of $\ln Q^2$.

In region (ii), the two-jet-dominated region, a result analogous to the Drell-Yan formula results. However, in order to obtain such a result it appears important to integrate over a region of θ including zero. In this case a better coordinate system is $q=((Q^2+\underline{q}^2)^{1/2}, \underline{q}, 0)$, $p_1=(E_{p_1}, 0, 0, -p_1)$, $p_2=(E_{p_2}, 0, 0, p_2)$. Then in this coordinate system the moments

$$\int d^2\underline{q}\,\omega_1^{\sigma_1-2}\omega_2^{\sigma_2-2}d\omega_1 d\omega_2 2E_1 2E_2\frac{d\sigma}{d^3p_1d^3p_2}$$

factor into cut vertices times singular functions which obey the Callan-Symanzik equation and which can be calculated to dominant order from a zeroth-order QCD graph. There appears to be no fixed-$\underline{q}^2$ result for small $\underline{q}^2$.

Region (iii) is technically virtually identical to the single-particle production process.

F. In massive-μ-pair production consider the coordinate system $p_1=(E_1, 0, 0, p_1)$, $p_2=(E_2, 0, 0, p_2)$, $q=((Q^2+\underline{q}^2)^{1/2}, \underline{q}, 0)$. Then predictions can be given simply for moments of the form (i)

$$\int d^2\underline{q}\,\omega_1^{-\sigma_1}\omega_2^{-\sigma_2}d\omega_1 d\omega_2\frac{d\sigma}{d^4q}$$

or of the form (ii)

$$\int \omega_1^{-\sigma_1}\omega_2^{-\sigma_2}d\omega_1 d\omega_2\frac{d\sigma}{d^4q}$$

where in (ii) $\underline{q}^2/Q^2=\lambda$ is fixed as $Q^2\to\infty$; $\omega_1=2p_1\cdot q/Q^2$, and $\omega_2=2p_2\cdot q/Q^2$.

In case (i) the moments factorize into products of spacelike cut vertices and singular functions. The singular functions are determined from a zeroth-order QCD calculation. In case (ii) an order-g^2 calculation augmented by the renormalization group gives definite predictions in terms of cut vertices and the calculable singular functions. The cut vertices appearing here are identical to those appearing in deeply inelastic electron and neutrino scattering. These results are already widely believed to hold in QCD though the necessity of integrating over $\underline{q}$ has perhaps not been widely appreciated. The question of wee partons is discussed for the technically similar process of two-particle inclusive hadron production in e^+e^- annihilation.

G. Inclusive hadron production in hadron-hadron collisions appears to be the one case where the renormalization-group predictions cannot be reduced to moments. In this process there are three invariants. One may take moments in any two of the relevant scaling variables, but the other invariant must remain as the large momentum in the Callan-Symanzik equation. An integrodifferential Callan-Symanzik equation is given and solved for this process. Our conclusions are in agreement with the asymptotic predictions of Feynman, Field, and Fox, except that we find no room for the idea of a transverse-momentum smearing due to the hadron wave function.

H. Single-particle inclusive production in deeply inelastic electron scattering is technically similar to μ-pair production and two-particle production in e^+e^- annihilation. In the coordinate system $p=(E_p, 0, 0, -p)$, $p_1=(E_{p_1}, 0, 0, p_1)$, $q=(0, \underline{q}, (Q^2-\underline{q}^2)^{1/2})$ one has predictions for moments in $\omega=2p\cdot q/Q^2$ and $\omega_1=-2p_1 q/Q^2$ either when, (i) $\underline{q}^2$ is integrated over or when, (ii) $\underline{q}^2/Q^2=\lambda$ is held fixed. (p refers to the target hadron, and p_1 the observed final-state hadron.) At large Q^2 moments of the structure functions factor into a spacelike vertex for p, a timelike vertex for p_1, and a singular function which is calculable in QCD. Region (i) corresponds to the decay of the struck parton, and region (ii) is a hard-scattering process.

In Sec. III the details of how a generalized Wilson expansion emerges in two-particle inclusive production in e^+e^- annihilation are given. Technically this process is almost identical to μ-pair production in hadron-hadron collisions and to single-particle production in deeply inelastic electron scattering.

In Sec. III A we discuss in detail how the oversubtractions are done for the relevant amplitudes. A generalization of Zimmermann's forest formula is given.

In Sec. III B we discuss some examples which illustrate the subtraction procedure. Illustrations of soft and wee partons are also given.

In Sec. III C we discuss the question of the wee partons in detail. Although this is not necessary for deriving the generalized Wilson expansion it is necessary in order to obtain a Callan-Symanzik equation. A general argument, using gauge invariance, is given for the cancellation of the wee-parton contributions between various graphs. A really complete discussion of why the oversubtraction procedure works cannot be given without a generalization of Zimmerman's complete forests. These are forests that are chosen to correspond to a particular momentum flow in a graph. Such a discussion seems too technical for the purpose of this paper and so a heuristic discussion of the generalized Bogoliubov-Parasiuk-Hepp-Zimmerman (BPHZ) method as applied to the process at hand is given.

In Sec. IV the relation of cut vertices to the parton model is considered. Many of the intuitive ideas of parton distributions, such as positivity of the number density for a particular type of quark, do not follow in any obvious manner from renormalized QCD.

Finally, only an Abelian gauge theory has been dealt with in detail in this paper, as far as questions of renormalization are concerned. We presume, but we have not examined this in detail, that the arguments can be carried through for non-Abelian theories with some slight modification. All the phenomenological results, however, are stated for a color gauge theory with three flavors.

II. PHYSICAL PREDICTIONS FROM QCD

In this section we shall discuss what types of physical predictions one gets from QCD. The following list is not meant to be complete but includes those processes which we feel will be of most experimental and theoretical interest in the near future. We shall reduce all of the predictions to those of moments, except in the case of wide-angle particle production in hadron-hadron collisions where this is impossible. We believe that the predictions for moments are the most certain tests of QCD.

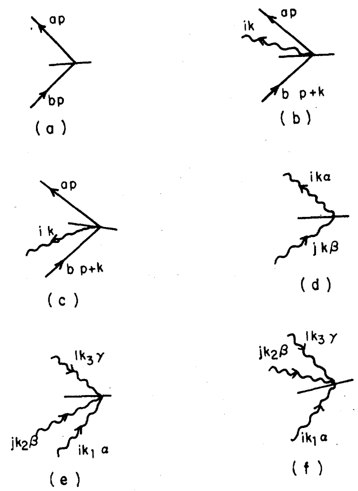

FIG. 1. Spacelike cut vertices.

A. Cut vertices

We begin by listing the elementary (bare) cut vertices[1] which are needed in later sections of this paper. For spacelike vertices one has the terms shown in Fig. 1. The arrows in Fig. 1 refer to the momentum labels and not to flavor or color charge. The bare vertices are

Fig. 1(a): $\Gamma_\sigma^{a,b}(p)=\gamma_-p_-^{\sigma-1}\delta_{ab}\theta(p_-)$,

Fig. 1(b): $\Gamma_\sigma^{ai,b}(p,k)=N_a g(\gamma_-/k_-)(p+k)_-^{\sigma-1}T^i_{ab}\theta(p_-+k_-)$,

Fig. 1(c): $\Gamma_\sigma^{a,bi}(p,k)=-N_a g(\gamma_-/k_-)p_-^{\sigma-1}T^i_{ab}\theta(p_-)$,

Fig. 1(d): $\Gamma^{ij}_{\sigma\alpha\beta}(k)=4\delta_{ij}[g_{\alpha\beta}k_-^2-k_-(g_{\alpha-}k_\beta+g_{\beta-}k_\alpha)+g_{\alpha-}g_{\beta-}k^2]k_-^{\sigma-2}\theta(k_-)$,

Fig. 1(e):
$$\begin{aligned}\Gamma^{ij,l}_{\sigma\alpha\beta,\gamma}(k_1,k_2,k_3)=&(2igC_{ijl}/k_{2-})\theta(k_{3-})g_{\beta-}[g_{\alpha\gamma}k_{1-}k_{3-}-k_{1-}g_{\gamma-}k_{3\alpha}\\&-k_{3-}g_{\alpha-}k_{1\gamma}+g_{\alpha-}g_{\gamma-}k_1\cdot k_3]k_{3-}^{\sigma-2}\\&+2igC_{ijl}\theta(k_{3-})[k_{3-}(g_{\beta-}g_{\alpha\gamma}-g_{\alpha-}g_{\beta\gamma})+g_{\gamma-}(k_{3\beta}g_{\alpha-}-k_{3\alpha}g_{\beta-})]k_{3-}^{\sigma-2}\\&+\text{terms where }(i,\alpha,k_1)\leftrightarrow(j,\beta,k_2).\end{aligned}$$

$\Gamma^{i,jl}_{\sigma\alpha,\beta\gamma}(k_1,k_2,k_3)$ is identical to $\Gamma^{ij,l}_{\sigma\alpha\beta,\gamma}(k_1,k_2,k_3)$ except for a factor of (−1) along with the change $\theta(k_{3-})k_{3-}^{\sigma-2}$ to $\theta(-k_{1-})(-k_{1-})^{\sigma-2}$. In the above $N_a=+1$ if a is a fermion of momentum p and $N_a=-1$ if a is an antifermion of momentum p. The C_{ijl} are the structure functions of the color group, G. The indices a,b refer to a representation, R, for fermions and a representation $\bar{R}$ for antifermions. The vector index for the gluons is understood to be + in Fig. 1(c) and Fig. 1(d). Timelike vertices are shown in Fig. 2. The formulas are identical to the spacelike case with the replacement of σ by $-\sigma+1$ and the omission of the $\theta(k_-)$ factors.

So far we have explicitly indicated color indices and suppressed flavor indices. In the following parts of this section it will be more convenient to suppress color indices, since we shall deal with physical states which are color singlets, and exhibit flavor indices. We suppose that the flavor group is SU(3), though the generalization to SU(4) is trivial.

B. Deeply inelastic electron scattering

In deeply inelastic scattering off a proton one is concerned with the matrix element

$$W_{\mu\nu}=\frac{4\pi^2E_p}{m}\int d^4x\,e^{iqx}\langle p|j_\nu(x)j_\mu(0)|p\rangle,$$

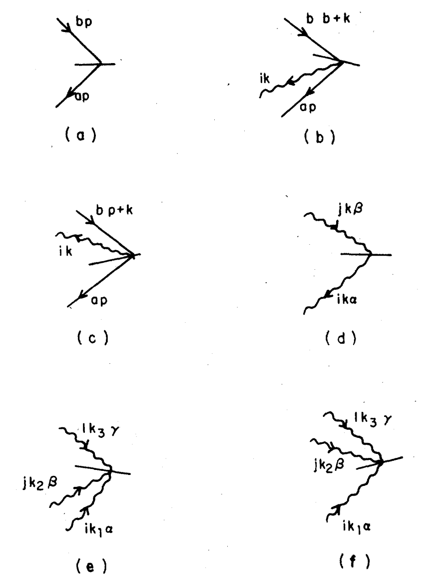

FIG. 2. Timelike cut vertices.

where a spin average over the proton is assumed. One can write $W_{\mu\nu}$ as

$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_L + \frac{1}{m^2}\left[p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + g_{\mu\nu}\frac{(p\cdot q)^2}{q^2}\right] W_2 .$$

The moments of W_L and W_2 obey

$$\int_1^\infty d\omega\, \omega^{-\sigma-2} \nu W_2 = \int_0^1 dx\, x^\sigma \nu W_2 = \sum_{i=1}^{4} C^{(i)}_{\sigma+2} E^{(i)}_{\sigma+2}(Q^2) \quad (1)$$

and

$$m\int_1^\infty d\omega\, \omega^{-\sigma-1} W_L = m\int_0^1 dx\, x^{\sigma+1} W_L = \sum_{i=1}^{4} C^{(i)}_\sigma F^{(i)}_\sigma(Q^2) , \quad (2)$$

where,[13] defining $v_\sigma = p_-^{-\sigma}\Gamma_\sigma$,

$$C^{(1)}_\sigma = p_-^{-\sigma}\Gamma_\sigma = v_\sigma , \quad (3)$$

$$C^{(2)}_\sigma = p_-^{-\sigma}\sum_a \Gamma^a_\sigma(\tfrac{1}{2}\lambda_0)_{aa} = \sum_a v^a_\sigma(\tfrac{1}{2}\lambda_0)_{aa} = v^s_\sigma + v^{\bar{s}}_\sigma , \quad (4)$$

$$C^{(3)}_\sigma = p_-^{-\sigma}\sum_a \Gamma^a_\sigma(\tfrac{1}{2}\lambda_3)_{aa} = \sum_a v^a_\sigma(\tfrac{1}{2}\lambda_3)_{aa} = v^{O_3}_\sigma + v^{\bar{O}_3}_\sigma , \quad (5)$$

$$C^{(4)}_\sigma = p_-^{-\sigma}\sum_a \Gamma^a_\sigma(\tfrac{1}{2}\lambda_8)_{aa} = \sum_a v^a_\sigma(\tfrac{1}{2}\lambda_8)_{aa} = v^{O_8}_\sigma + v^{\bar{O}_8}_\sigma . \quad (6)$$

$C^{(1)}$ and $C^{(2)}$ are singlet contributions, while $C^{(3)}$ and $C^{(4)}$ are flavor octets. $E^{(3)}$ and $E^{(4)}$ obey the same Callan-Symanzik equation. The a sum goes over $u, d, s, \bar{u}, \bar{d}, \bar{s}$.

Neglecting renormalization

$$\Gamma^a_\sigma = \int d^4k\, T^{aa}_{rs}(p,k)(\gamma_-)_{rs} k_-^{\sigma-1}\theta(k_-) , \quad (7)$$

with

$$T^{aa}_{rs}(p,k) = \frac{(2\pi)^3 E_p}{m}\int d^4k\, e^{-ikx}\langle p|\bar{\psi}^a_r(x)\psi^a_s(0)|p\rangle \quad (8)$$

for $a = u, d, s$ and

$$T^{aa}_{rs}(p,k) = \frac{(2\pi)^3 E_p}{m}\int d^4k\, e^{-ikx}\langle p|\psi^a_s(x)\bar{\psi}^a_r(0)|p\rangle \quad (9)$$

for $a = \bar{u}, \bar{d}, \bar{s}$. Γ_σ is given by

$$\Gamma_\sigma = \int d^4k\, T^{ii}_{\alpha\beta}\, 4[g_{\alpha\beta}k_-^2 - k_-(g_{\alpha-}k_\beta + g_{\beta-}k_\alpha) + g_{\alpha-}g_{\beta-}k^2]k_-^{\sigma-2}\theta(k_-) , \quad (10)$$

with

$$T^{ii}_{\alpha\beta}(p,k) = \frac{(2\pi)^3 E_p}{m}\int d^4x\, e^{-ikx}\langle p|A^i_\alpha(x)A^i_\beta(0)|p\rangle . \quad (11)$$

There is a sum over i in (10) and (11) and a sum over fermion colors in (8) and (9).

The renormalization necessary to give (7) and (10) a real meaning is discussed in Ref. 1. The renormalization prescription forces one to introduce the bare vertices shown in Figs. 1(c), 1(d), 1(e), and 1(f) along with higher vertices not shown there. For $\sigma = 2, 4, 6, \ldots$ the vertices Γ_σ, $\sum_a \Gamma^a_\sigma(\frac{1}{2}\lambda_0)_{aa}$, and $\sum_a \Gamma^a_\sigma(\frac{1}{2}\lambda_i)_{aa}$ correspond to matrix elements of the usual local operators occuring in an analysis of inelastic electron scattering. We emphasize, though, that the moment relations (1) and (2) and the corresponding Callan-Symanzik equations for the E's are true for any complex values of σ so long as the integrals converge.

C. $e^+ + e^- \to$ hadron(p) + anything

The formalism for the process $e^+ + e^- \to h(p)$ + anything has been discussed in detail in Ref. 1. We here list the results only for completeness. The cross section for the process shown in Fig. 3 is

$$E_p\frac{d\sigma}{d^3p} = \frac{2m\alpha^2}{(q^2)^2}\left(2\overline{W}_L - \frac{\omega}{m}\frac{1+\cos^2\theta}{2}\nu\overline{W}_2\right) , \quad (12)$$

where

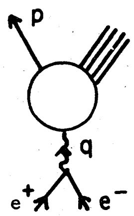

FIG. 3. Single-particle inclusive hadron production in e^+e^- annihilation.

$$\overline{W}_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\overline{W}_L + \frac{1}{m^2}\left(p_\mu p_\nu - \frac{p\cdot q}{q^2}(p_\mu q_\nu + p_\nu q_\mu) + g_{\mu\nu}\frac{(p\cdot q)^2}{q^2}\right)\overline{W}_2 .$$

For large q^2 the moment equations are

$$\int_0^1 \nu \overline{W}_2 \omega^{\sigma-1} d\omega = \sum_{i=1}^4 \overline{C}_\sigma^{(i)} \overline{E}_\sigma^{(i)}(q^2), \quad (13)$$

$$m\int_0^1 \overline{W}_L \omega^{\sigma-2} d\omega = \sum_{i=1}^4 \overline{C}_\sigma^{(i)} \overline{F}_\sigma^{(i)}(q^2), \quad (14)$$

where the $\overline{E}$ and $\overline{F}$ obey

$$\left[\left(-q^2\frac{\partial}{\partial q^2} + \beta\frac{\partial}{\partial g}\right)\delta_{ij} - \overline{\gamma}_{ij}^\sigma\right] E_\sigma^{(j)}(q^2) = 0 . \quad (15)$$

Equation (15) includes all factors of $(\ln q^2)^n$, but neglects powers of m^2/q^2. In the above the $\overline{C}$'s are defined by equations identical to (3)–(5), but now with timelike cut vertices. The cut vertices are given by equations identical to (7) and (10) except that in (7) $\gamma_- k_-^{\sigma-1}\theta(k_-)$ is replaced by $\gamma_- k_-^{\sigma}$, and in (10) $k_-^{\sigma-2}\theta(k_-)$ is replaced by $k_-^{-\sigma-1}$ and where now, for spin summed external fermions

$$T_{rs}^{aa}(p,k) = -\left(\frac{\gamma\cdot p + m}{2m}\right)_{tu} \int d^4x d^4y d^4z\, e^{+ip(z-y) - ikx} \times \langle \overline{T}(\overline{\Psi}_t(y)\psi_s^a(0)) T(\Psi_u(z)\overline{\psi}_r^a(x))\rangle . \quad (16)$$

In (16) the field Ψ represents a composite particle, and the propagator of the Ψ is understood to be amputated.

Probably the cleanest test of QCD available in single-particle inclusive annihilation is in the difference of the cross sections for, say, K^+ and K^0 production,

$$E_p\frac{d\sigma^{I=1}}{d^3p} = \frac{1}{2}\left(E_p\frac{d\sigma^{K^+}}{d^3p} - E_p\frac{d\sigma^{K^0}}{d^3p}\right).$$

In this cross section only the octet piece survives in (13) and (14). If one defines invariant amplitudes for the process so that

$$E_p\frac{d\sigma^{I=1}}{d^3p} = \frac{2m\alpha^2}{(q^2)^2}\left(2\overline{W}_L^{I=1} - \frac{\omega}{m}\frac{1+\cos^2\theta}{2}\nu\overline{W}_2^{I=1}\right), \quad (17)$$

then

$$\int_0^1 \omega^{\sigma-1} d\omega\, \nu \overline{W}_2^{I=1} \to d_{2\sigma}(\ln q^2)^{-A_\sigma},$$
$$m\int_0^1 \omega^{\sigma-2} d\omega \overline{W}_L \to d_{L\sigma}(\ln q^2)^{-A_\sigma}. \quad (18)$$

Here

$$A_\sigma = \frac{3C_2(R)}{22C_2(G) - 8T(R)} \times\left(1 - \frac{2}{(\sigma-1)(\sigma-2)} + 4\sum_{l=1}^\infty \frac{\sigma-2}{(l+1)(l+\sigma-1)} - \frac{4}{\sigma-1}\right). \quad (19)$$

In case $\sigma - 2$ is an integer ≥ 2,

$$A_\sigma = \frac{3C_2(R)}{22C_2(G) - 8T(R)}\left(1 - \frac{2}{(\sigma-1)(\sigma-2)} + 4\sum_{l=2}^{\sigma-2}\frac{1}{l}\right).$$

For color and flavor SU(3), $3C_2(R)/[22C_2(G) - 8T(R)] = \frac{2}{27}$ and $A_3 = 0$, $A_4 = \frac{16}{81}$, $A_5 = \frac{25}{81}$, $A_6 = \frac{157}{405}$, etc. The same factor of $(\ln q^2)^{-A_\sigma}$ occurs in other processes, for example, in the difference, $E_p d\sigma^p/d^3p - E_p d\sigma^N/d^3p$, of the proton and neutron production cross sections. Thus

$$\frac{\int_0^1 \omega^{\sigma-2} d\omega\left(E_p\frac{d\sigma^p}{d^3p} - E_p\frac{d\sigma^n}{d^3p}\right)}{\int_0^1 \omega^{\sigma-2} d\omega\left(E_p\frac{d\sigma^{K^+}}{d^3p} - E_p\frac{d\sigma^{K^0}}{d^3p}\right)} = r$$

should be independent of q^2 and independent of θ at large values of q^2. Corrections to constancy in r are down by a power of Q^2 by factorization.

In some cases it may be more convenient to integrate over electron angles in the center-of-mass system of the e^+e^-. Defining this cross section to be $d\tilde{\sigma}/d^3p$ (see Appendix A) one can write

$$\int (2\pi)^3 2E\frac{d\tilde{\sigma}}{d^3p}\omega^{\sigma-2}d\omega = \frac{8\pi^2\alpha^2}{3(Q^2)^2}\sum_{ij}\tilde{v}_\sigma^i R_{ji}(Q_1^2, Q^2) E_\sigma^i(Q_0^2, g(Q^2, Q_0^2)), \quad (20)$$

where the i,j indices take on values $g, S, \overline{S}, O_3, \overline{O}_3, O_8, \overline{O}_8$. In particular, the average multiplicity is given by a moment like the above. Unfortunately, the moment, $\sigma = 3$, is divergent. However,[14] if one takes differences of multiplicities to eliminate the singlet contributions one finds, for example,

$$\bar{n}_{\pi^+} - \bar{n}_{\pi^0} = \frac{1}{\sigma}\int d^3p\left(\frac{d\bar{\sigma}^{\pi^+}}{d^3p} - \frac{d\bar{\sigma}^{\pi^0}}{d^3p}\right)$$
$$= \frac{1}{\sigma}\int (2\pi)^3 2E\left(\frac{d\bar{\sigma}^{\pi^+}}{d^3p} - \frac{d\bar{\sigma}^{\pi^0}}{d^3p}\right)\omega d\omega\frac{Q^2}{16\pi^2}$$
$$= \frac{\alpha^2}{6Q^2\sigma}(\ln Q^2/Q_1^2)^{-A_3}(\bar{v}_3^{O_3} + \bar{v}_3^{\bar{O}_3})$$
$$\times E_3^{O_3}(Q_0^2, g(Q^2, Q_0^2)).$$

Using $A_3 = 0$ and $E_3^{O_3} = 4\pi/3$ from perturbation theory, we obtain

$$\bar{n}_{\pi^+} - \bar{n}_{\pi^0} = \frac{2\pi\alpha^2}{9Q^2\sigma}(\bar{v}_3^{O_3} + \bar{v}_3^{\bar{O}_3}) \tag{21}$$

with the v's referring to the π^+. Similar results hold for $\bar{n}_{K^+} - \bar{n}_{K^0}$, $\bar{n}_p - \bar{n}_N$, etc. The lack of any $\ln Q^2$ dependence comes from $A_3 = 0$ and can be understood in the following way. A_3 is equal to the A_1^{GW} of Gross and Wilczek[15] if their A_n^{GW} are continued in n in the natural way. $n = 1$ would indicate that a vector current should be occurring in their Wilson expansion. However, their Wilson expansion does not have odd n, and the continued $n = 1$ term does not refer to a local operator. In fact, the $n = 1$ continued term acts much like a vector current except that quarks and antiquark contribute with the same sign. To lowest order in g^2 this does not bother the conservation of that vector current, and so the resulting anomalous dimension is zero.

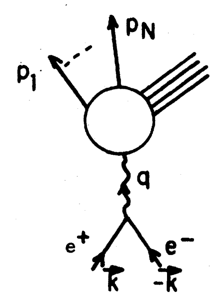

FIG. 4. N-particle inclusive hadron production in e^+e^- annihilation.

Finally, we comment (see Appendix A) that (20) can also be written as[8]

$$(2\pi)^3 2E\frac{d\bar{\sigma}}{d^3p} = \frac{8\pi^2\alpha^2}{3(Q^2)^2}\sum_i \omega v^i(\omega, Q^2)E^i,$$

where the E^i are $E^G = 0$, $E^S = E^{\bar{S}} = \frac{8}{3}\pi(\frac{2}{3})^{1/2}$, $E^{O_3} = E^{\bar{O}_3} = \frac{4}{3}\pi$, and $E^{O_8} = E^{\bar{O}_8} = 4\pi/3\sqrt{3}$. This form is exactly as in a naive parton model except that v^i has a well-determined Q^2 dependence. (20) is general while the above form uses asymptotic freedom explicitly.

D. $e^+ + e^- \to h_1(p_1) + h_2(p_2) + \cdots + h_N(p_N)$ + anything

Consider the process shown in Fig. 4. The cross section is

$$[(2\pi)^3]^N 2E_1\cdots 2E_N\frac{d\sigma(\vec{k})}{d^3p_1\cdots d^3p_N} = \frac{(e^2)^2}{2(q^2)^3}F_{N\nu\mu}(k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu} - k_1\cdot k_2 g_{\mu\nu}), \tag{22}$$

where

$$F_{N\nu\mu} = \int d^4x_1\cdots d^4x_N d^4y_1\cdots d^4y_N d^4x\,\exp\left(iqx - i\sum_i p_i(x_i - y_i)\right)$$
$$\times\langle\bar{T}(\phi_1(x_1)\cdots\phi_N(x_N)j_\nu(x))T(\phi_1(y_1)\cdots\phi_N(y_N)j_\mu(0))\rangle. \tag{23}$$

Equations (22) and (23) are written for the production of scalar particles and an amputation of the general ϕ propagators is understood. If the angles of $\vec{k}$, the e^+ momentum, are integrated over in the center-of-mass system of the e^+e^-, one obtains

$$\int\frac{d\Omega(\vec{k})}{4\pi}[(2\pi)^3]^N 2E_1\cdots 2E_N\frac{d\sigma(\vec{k})}{d^3p_1\cdots d^3p_N} = [(2\pi)^3]^N 2E_1\cdots 2E_N\frac{d\sigma}{d^3p_1\cdots 1^3p_N}$$
$$= -\frac{8\pi^2\alpha^2}{3(q^2)^2}F_N, \tag{24}$$

where $F_N = \sum_\mu F_{N\mu\mu}$.

For $N \geq 2$, F_N depends on $3N - 2$ variables. We shall find it convenient to use the coordinate system where $q = (Q, 0, 0, 0)$ and to label the other $3N - 3$ variables by the magnitudes of the momenta and the angles of the outgoing particles. Thus

$$p_{1\mu} = (E_1, 0, 0, -p_1),$$
$$p_{2\mu} = (E_2, -p_2\sin\theta_2, 0, -p_2\cos\theta_2), \tag{25}$$
$$p_{i\mu} = (E_i, -p_i\sin\theta_i\cos\phi_i, -p_i\sin\theta_i\sin\phi_i, -p_i\cos\theta_i)$$

The independent variables are taken to be Q, p_i, θ_i, and ϕ_i. Define $\omega_i = 2p_i \cdot q/q^2$. Then for large q^2

$$\int_0^1 \omega_1{}^{\sigma_1-2} \cdots \omega_N{}^{\sigma_N-2} d\omega_1 \cdots d\omega_N F_N = F_{\sigma_1\cdots\sigma_N}(Q^2, \theta, \phi), \tag{26}$$

with

$$\begin{aligned} F_{\sigma_1\cdots\sigma_N}(Q^2,\theta,\phi) &= p_{1-}{}^{\sigma_1-1}\cdots p_{i-}{}^{\sigma_i-1}(\theta_i,\phi_i)\cdots p_{N-}{}^{\sigma_N-1}(\theta_N,\phi_N)(Q^2)^{-N+1} \\ &\quad\times \sum_{\alpha_1\cdots\alpha_N} \Gamma_{\sigma_1}^{\alpha_1}(p_1)\cdots\Gamma_{\sigma_N}^{\alpha_N}(p_N)\, E_{\sigma_1\cdots\sigma_N}^{\alpha_1\cdots\alpha_N}(Q^2,\theta,\phi) \\ &= (Q^2)^{-N+1} \sum_{\alpha_1\cdots\alpha_N} v_{\sigma_1}^{\alpha_1}\cdots v_{\sigma_N}^{\alpha_N} E_{\sigma_1\cdots\sigma_N}^{\alpha_1\cdots\alpha_N}(Q^2,\theta,\phi). \end{aligned} \tag{27}$$

In (27) $p_-(\theta,\phi) = E_p + |\vec{p}|$ and the Γ_σ's are the cut vertices discussed in part C of this section. α takes on values 1 to 7. $\alpha=1$ refers to the gluon vertex, while $\alpha=2,3,4$ refer to u,d,s vertices, and $\alpha=5,6,7$ refer to $\bar{u},\bar{d},\bar{s}$ vertices. The Callan-Symanzik equation is

$$\sum_{i=1}^N \left[\frac{1}{N}\left(-Q^2\frac{\partial}{\partial Q^2}+\beta\frac{\partial}{\partial g}\right)\delta_{\alpha_i\alpha_i'} - \bar{\gamma}_\sigma^{\alpha_i\alpha_i'}\right] \times E_{\sigma_1\cdots\sigma_N}^{\alpha_1\cdots\alpha_i'\cdots\alpha_N} = 0. \tag{28}$$

Equation (28) is very complicated for all but small N. If one were content to take only the truly dominant part at large $\ln q^2$, the term given by the maximum eigenvalue of the γ's, the result would be

$$E_{\sigma_1\cdots\sigma_N}^{\alpha_1\cdots\alpha_N}(Q^2,\theta\phi) \to C_{\sigma_1\cdots\sigma_N}^{\alpha_1\cdots\alpha_N}(Q^2,\theta,\phi)(\ln Q^2)^{\mathfrak{a}}, \tag{29}$$

where

$$\mathfrak{a} = -\sum_{i=1}^N A_{\sigma_i}$$

and A_σ is the maximum eigenvalue of $\bar{\gamma}_\sigma^{\alpha\beta}$. If one restricts the θ_i to lie, say, in the hemisphere $0 \le \theta_i \le \pi/2$, then

$$C_{\sigma_1\cdots\sigma_N}^{\alpha_1\cdots\alpha_N}(Q^2,\theta,\phi) = C_{\sigma_1\cdots\alpha_N}^{\alpha_1\cdots\alpha_N}(\theta,\phi)(\ln Q^2)^{-N+1}, \tag{30}$$

and C is determined from the tree graph structure of QCD. If the angles are not restricted, then several of the particles may belong to the same jet and the behavior of $C_{\sigma_1\cdots\sigma_N}^{\alpha_1\cdots\alpha_N}(Q^2,\theta,\phi)$ is not so simple. Since the case of $N>2$ appears to be only of formal interest for the near future, we shall now go on to a detailed discussion of the situation when $N=2$.

E. $e^+ + e^- \to h(p_1) + h(p_2)$ + anything

The reaction $e^+ + e^- \to h(p_1)+h(p_2)+X$ separates naturally into three different kinematic regions. In the coordinate system where $q=(Q,0,0,0)$, $p_{1\mu}=(E_{p_1},0,0,-p_1)$, $p_{2\mu}=(E_{p_2},-p_2\sin\theta,0,-p_2\cos\theta)$, these regions are (i) θ fixed and $\theta\ne 0,\pi$, (ii) $\theta=\pi$, and (iii) $\theta=0$. In each case timelike vertices rather than spacelike vertices enter since the particles p_1 and p_2 are outgoing. We begin with (i).

(i) When θ is fixed away from 0 or π we are in effect requiring that three jets[17] form the dominant production mechanism. Equation (27) can now be written as

$$\begin{aligned} F_{\sigma_1\sigma_2}(q^2,\theta) &= \frac{1}{Q^2} p_{1-}{}^{\sigma_1-1} p_{2-}{}^{\sigma_2-1} \\ &\quad\times \sum_{\alpha_1,\alpha_2} \Gamma_{\sigma_1}^{\alpha_1}(p_1)\Gamma_{\sigma_2}^{\alpha_2}(p_2) E_{\sigma_1\sigma_2}^{\alpha_1\alpha_2}(Q^2,\theta) \\ &= \frac{1}{Q^2}\sum_{\alpha_1\alpha_1} v_{\sigma_1}^{\alpha_1} v_{\sigma_2}^{\alpha_2} E_{\sigma_1\sigma_2}^{\alpha_1\alpha_2}(Q^2,\theta). \end{aligned} \tag{31}$$

It is simplest to change to a basis where the Γ's are written as

$$\begin{aligned} \Gamma_\sigma^G &= \Gamma_\sigma^1, \\ \Gamma_\sigma^S &= \sum_{\alpha=2}^4 \Gamma_\sigma^\alpha(\tfrac{1}{2}\lambda_0)_{\alpha\alpha}, \\ \Gamma_\sigma^{\bar{S}} &= \sum_{\alpha=5}^7 \Gamma_\sigma^\alpha(\tfrac{1}{2}\lambda_0)_{\alpha\alpha}, \\ \Gamma_\sigma^{O,i} &= \sum_{\alpha=2}^4 \Gamma_\sigma^\alpha(\tfrac{1}{2}\lambda_i)_{\alpha\alpha}, \\ \Gamma_\sigma^{\bar{O},i} &= \sum_{\alpha=5}^7 \Gamma_\sigma^\alpha(\tfrac{1}{2}\lambda_i)_{\alpha\alpha}. \end{aligned} \tag{32}$$

There is an analogous formula for the v's. There are five possible independent vertices for each outgoing particle and thus 25 possible terms in (31). In fact there are only 13 independent terms. These are S_1S_2, $S_1\bar{S}_2$, S_1G_2, S_1O_2, $S_1\bar{O}_2$, G_1S_2,

G_1G_2, G_1O_2, O_1S_2, $O_1\bar{S}_2$, O_1G_2, O_1O_2, $O_1\bar{O}_2$. Here, for example, G_1O_2 means that the term $\Gamma^G_{\sigma_1}(p_1)\Gamma^O_{\sigma_2}(p_2)$ occurs. Clearly 13 terms is too large a number to comfortably consider. Fortunately, by forming differences of cross sections it is possible to get down to a simple result. To this end form the difference of $\pi_1^+\pi_2^+ + \pi_1^-\pi_2^- - \pi_1^+\pi_2^- - \pi_1^-\pi_2^+$ cross sections. One finds

$$\int \omega_1^{\sigma_1-2}\omega_2^{\sigma_2-2}d\omega_1 d\omega_2 2E_1 2E_2\left(\frac{d\sigma^{\pi^+\pi^+}}{d^3p d^3p_2}-\frac{d\sigma^{\pi^+\pi^-}}{d^3p_1 d^3p_2}\right)$$
$$=\frac{640\alpha^2}{243\pi(11-\frac{2}{3}n_f)}\frac{[\ln(Q^2/Q_1^2)]^{-A_{\sigma_1}-A_{\sigma_2}}}{(Q^2)^3\ln Q^2}(\tilde{v}_{\sigma_1}^{O_3}\tilde{v}_{\sigma_2}^{\bar{O}_3}+\tilde{v}_{\sigma_1}^{\bar{O}_3}\tilde{v}_{\sigma_2}^{O_3})$$
$$\times\left[\frac{\lambda}{1+\lambda}F(\sigma_2,\sigma_1,\sigma_1+\sigma_2-2,1/(1+\lambda))[B(\sigma_2-2,\sigma_1)+B(\sigma_1-2,\sigma_2)]\right.$$
$$\left.+(2/\lambda)B(\sigma_2-1,\sigma_1-1)F(\sigma_2-1,\sigma_1-1,\sigma_1+\sigma_2-2,1/(1+\lambda))\right], \quad (33)$$

where $\lambda=(1+\cos\theta)/(1-\cos\theta)$, B is the beta function, and F is the hypergeometric function. Of course, it is not necessary to fix θ in (33). Regions of θ can be integrated so long as one stops away from $\theta=0$ and $\theta=\pi$. A similar relation holds for moments of

$$2E_1 2E_2\left(\frac{d\sigma^{K^+K^+}}{d^3p_1 d^3p_2}+\frac{d\sigma^{K^0K^0}}{d^3p_1 d^3p_2}-2\frac{d\sigma^{K^+K^0}}{d^3p_1 d^3p_2}\right). \quad (34)$$

In each case the $C_{\sigma_1\sigma_2}(\theta)$ is determined from an order-g^2 perturbation calculation, and the Γ^O_σ are the same as occur in single-particle inclusive annihilation.

(ii) When $\theta=\pi$ the analysis outlined above breaks down as discussed in Sec. III. What must be done is to integrate over a region around $\theta=\pi$. To do this it is convenient to consider, temporarily, a prime coordinate system where

$$q'=((Q^2+\underline{q}^2)^{1/2},\underline{q},0),\quad p_1'=(E_{p_1'},0,0,-p_1'),\quad p_2'=(E_{p_2'},0,0,p_2').$$

The relation between $\underline{q}$ and θ is given by $\underline{q}^2=Q^2(1+\cos\theta)/(1-\cos\theta)$ when $E_p, E_{p'}\gg m$. The amplitude

$$F_2(Q^2,\omega_1,\omega_2)=\int d^2\underline{q}F_2(Q^2,\omega_1,\omega_2,\underline{q}^2)=2\pi Q^2\int\frac{d\cos\theta}{(1-\cos\theta)^2}F_2(Q^2,\omega_1\omega_2,\theta) \quad (35)$$

is appropriate for taking moments. The region of θ integration may be any region including $\pi=\theta$ but excluding $\theta=0$. We find

$$2E_1 2E_2\frac{d\sigma}{d^3p_1 d^3p_2}\omega_1^{\sigma_1-2}\omega_2^{\sigma_2-2}d^2\underline{q}d\omega_1 d\omega_2=\frac{16\alpha^2}{3(Q^2)^2N_c}\sum(\tilde{v}^i_{\sigma_1}\tilde{v}^j_{\sigma_2})Q_{i'j'}R^{\sigma_1}_{ii'}(Q_1^2,Q^2)R^{\sigma_2}_{jj'}(Q_1^2,Q^2). \quad (36)$$

This simplifies considerably if singlets are avoided by taking differences. For example,

$$\int 2E_1 2E_2\left(\frac{d\sigma^{\pi^+\pi^+}}{d^3p_1 d^3p_2}-\frac{d\sigma^{\pi^+\pi^-}}{d^3p_1 d^3p_2}\right)\omega_1^{\sigma_1-2}\omega_2^{\sigma_2-2}d^2\underline{q}d\omega_1 d\omega_2=\frac{320\alpha^2}{81(Q^2)^2}\tilde{v}^{O_3}_{\sigma_1}\tilde{v}^{\bar{O}_3}_{\sigma_2}[\ln(Q^2/Q_1^2)]^{-A_{\sigma_1}-A_{\sigma_2}}. \quad (37)$$

The Q^2 dependence of (37) is characteristic of a two-jet process. The matrix Q_{ij} is

$$Q=\frac{1}{9}\begin{array}{c|ccccccc|} & G & S & O_3 & O_8 & \bar{S} & \bar{O}_3 & \bar{O}_8\\ \hline G & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ S & 0 & 0 & 0 & 0 & 4 & \sqrt{6} & \sqrt{2}\\ O_3 & 0 & 0 & 0 & 0 & \sqrt{6} & 5 & \sqrt{3}\\ O_8 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{3} & 3\\ \bar{S} & 0 & 4 & \sqrt{6} & \sqrt{2} & 0 & 0 & 0\\ \bar{O}_3 & 0 & \sqrt{6} & 5 & \sqrt{3} & 0 & 0 & 0\\ \bar{O}_8 & 0 & \sqrt{2} & \sqrt{3} & 3 & 0 & 0 & 0 \end{array}.$$

Equation (36) can be written in the alternative form

$$2E_1 2E_2\frac{d\sigma}{d^3p_1 d^3p_2}=\frac{16\alpha^2}{3(Q^2)^2N_c}\sum_{i,j}\omega_1 v^i(\omega_1,Q^2)\omega_2 v^j(\omega_2,Q^2)Q_{ij}.$$

(iii) When $\theta=0$ it is convenient to parametrize the momenta by $q_\mu=(Q,0,0,0)$, $p_{1\mu}=(E_p,0,0,-p)$, $p_{2\mu}=(E_{p_2},\underline{p}_2,-xp)$. For fixed $\omega_2=2p_2\cdot q/Q^2$ and fixed $\underline{p}_2$, $\theta\to 0$ as $Q^2\to\infty$. The relevant amplitude [see Eq. (24)] is $F_2(Q^2,\omega_1,x,\underline{p}_2)$. Define

$$F_\sigma(Q^2,x,\underline{p}_2^2)=\int\omega_1^{\sigma-2}d\omega_1 F_2. \quad (38)$$

Then

$$F_\sigma(Q^2, x, \underline{p}_2{}^2) = \sum_\alpha v_\sigma^\alpha(x, \underline{p}_2{}^2) E_\sigma^\alpha(Q^2), \qquad (39)$$

where E_σ^α is the same E as appears in single-particle inclusive annihilation. In this region of fixed x and fixed $\underline{p}_2{}^2$ two-particle inclusive annihilation is not essentially different from a single-particle inclusive process.

F. $h(p_1) + h(p_2) \to \mu^+\mu^-(q)$ + anything

The Drell-Yan[18] formula for massive μ-pair production states that

$$\frac{d\sigma}{d^4q} = \frac{4\pi\alpha^2}{3N_c(Q^2)^2} \sum_a Q_a{}^2 \int dx_1 dx_2 v^a(x_1) v^{\bar{a}}(x_2) \times \delta(x_1 x_2 - Q^2/s). \qquad (40)$$

$s = (p_1 + p_2)^2$ and p_1, p_2 are the momenta of the incoming hadrons. The a sum goes over $u, d, s, \bar{u}, \bar{d}, \bar{s}$. Q_a is the charge of the a quark. $v^a(x)$ is the parton number density, discussed in more detail in Sec. IV. N_c is the number of colors, which is always taken to be 3. When the radiative corrections of QCD[7,9,10] are taken seriously, Eq. (40) does not remain exact but the resulting structure bears a recognizable resemblance to the Drell-Yan formula. Define

$$\frac{d\sigma}{d^4q} = \frac{4\pi\alpha^2}{3Q^2} \frac{W(s, Q^2)}{s}, \qquad (41)$$

with

$$W = -16\pi^2 E_1 E_2 \sum_n (2\pi)^4 \delta(p_1 + p_2 - q - p_n) \times \langle p_1 p_2(+) | j_\mu(0) | n \rangle \times \langle n | j_\mu(0) | p_1 p_2(+) \rangle. \qquad (42)$$

We suppose that a coordinate has been chosen so that $(p_1)_\mu = (E_1, 0, 0, -p_1)$, $(p_2)_\mu = (E_2, 0, 0, p_2)$, $q_\mu = ((Q^2 + \underline{q}^2)^{1/2}, \underline{q}, 0)$. Then there are two situations for which exact equations can be given at large Q^2. The first case, (i), is when an integral over $d^2\underline{q}$ is performed. The second case, (ii), is when $\underline{q}^2 = \lambda Q^2$ and λ is fixed.

(i) We begin by considering the case where the variable $\underline{q}^2$ has been integrated over. Consider then

$$\tilde{F}_{\sigma_1\sigma_2}(Q^2) = \int \frac{d\sigma}{d^4q} \omega_1{}^{-\sigma_1} \omega_2{}^{-\sigma_2} d\omega_1 d\omega_2 d^2\underline{q}, \qquad (43)$$

where $\omega_1 = 2p_1 \cdot q/Q^2$ and $\omega_2 = 2p_2 \cdot q/Q^2$. As discussed for two-particle inclusive production in e^+e^- collisions, it is crucial that a $d^2\underline{q}$ integral be taken. There appears to be no simple result if $\underline{q}$ is held fixed on the order of hadronic masses. For large Q^2 one can write

$$\tilde{F}_{\sigma_1\sigma_2}(Q^2) = \frac{8\pi\alpha^2}{3(Q^2)^2} \sum_{i,j} v_{\sigma_1}^i v_{\sigma_2}^j E_{\sigma_1\sigma_2}^{ij}(Q^2), \qquad (44)$$

where the i,j sum goes over $G, S, O_3, O_8, \bar{S}, \bar{O}_3, \bar{O}_8$. The v_σ^i are the same spacelike cut vertices as appear in deeply inelastic electron scattering. The $E_{\sigma_1\sigma_2}^{ij}$ obey a Callan-Symanzik equation with usual spacelike γ_σ^{ij}'s. Using asymptotic freedom we can write

$$\tilde{F}_{\sigma_1\sigma_2}(Q^2) = \frac{8\pi\alpha^2}{3(Q^2)^2 N_c} \sum_{i,j} \tilde{v}_{\sigma_1}^i \tilde{v}_{\sigma_2}^i \times \sum_{i',j'} R_{ii'}^{\sigma_1}(Q_1{}^2, Q^2) R_{jj'}^{\sigma_2}(Q_1{}^2, Q^2) Q_{i'j'}. \qquad (45)$$

As discussed in Appendix A we suppose $Q_1{}^2$ is large enough that the γ's can be taken from lowest-order perturbation theory for the evaluation of the R_{ij}'s. The i, j, i', j' sum goes over $G, S, O_3, O_8, \bar{O}_3, \bar{O}_8, \bar{S}$. It may be convenient to write (45) as

$$\tilde{F}_{\sigma_1\sigma_2}(Q^2) = \frac{8\pi\alpha^2}{3(Q^2)^2 N_c} \sum_{i,j} v_{\sigma_1}^i(Q^2) v_{\sigma_2}^j(Q^2) Q_{ij}, \qquad (45')$$

where

$$v_\sigma^i(Q^2) = \sum_j \tilde{v}_\sigma^j R_{ji}^\sigma(Q_1{}^2, Q^2) = \sum_j v_\sigma^j R_{ji}^\sigma(Q_0{}^2, Q^2).$$

Another alternative form to (45) is

$$\tilde{F}(x_1, x_2, Q^2) = \frac{8\pi\alpha^2}{3(Q^2)^2 N_c} \sum_{i,j} x_1 v^i(x_1, Q^2) \times x_2 v^j(x_2, Q^2) Q_{ij},$$

where

$$\tilde{F}(x_1, x_2, Q^2) = \int \frac{d\sigma}{d^4q} d^2\underline{q}$$

and $x_i = Q^2/2p_i \cdot q$. The $v^i(x, Q^2)$ are defined like the corresponding timelike vertices with ω replaced by x. The Q_{ij} are as before.

(ii) When $\underline{q}^2 = \lambda Q^2$,[19] we write

$$\tilde{F}_{\sigma_1\sigma_2}(Q^2, \lambda) = \int \frac{d\sigma}{d^4q} \omega_1{}^{-\sigma_1} \omega_2{}^{-\sigma_2} d\omega_1 d\omega_2. \qquad (46)$$

Then for large Q^2

$$\tilde{F}_{\sigma_1\sigma_2}(Q^2, \lambda) = \frac{\alpha^2}{6\pi^2(Q^2)^3} \sum_{i,j} v_{\sigma_1}^i v_{\sigma_2}^j E_{\sigma_1\sigma_2}^{ij}(Q^2, \lambda), \qquad (47)$$

where the i,j sum goes over $G, S, O_3, O_8, \bar{S}, \bar{O}_3, \bar{O}_3, \bar{O}_8$ and the E^{ij} obey a Callan-Symanzik equation

$$\left(-Q^2 \frac{\partial}{\partial Q^2} + \beta \frac{\partial}{\partial g}\right) E_{\sigma_1\sigma_2}^{ij}(Q^2, \lambda) = \gamma_{\sigma_1}^{ii'} E_{\sigma_1\sigma_2}^{i'j}(Q^2\lambda) + \gamma_{\sigma_2}^{jj'} E_{\sigma_1\sigma_2}^{ij'}(Q^2, \lambda). \qquad (48)$$

Using the graphs shown in Fig. 5 and the Callan-Symanzik equation we find .

$$\tilde{F}_{\sigma_1\sigma_2}(Q^2,\lambda)=\frac{8\alpha^2}{33-2n_f}\frac{1}{(Q^2)^3\ln Q^2}\sum_{i,j}(\tilde{v}^i_{\sigma_1}\tilde{v}^j_{\sigma_2})\sum_{i',j'}R^{\sigma_1}_{ii'}(Q_1{}^2,Q^2)R^{\sigma_2}_{jj'}(Q_1{}^2,Q^2)Q'_{i'j'}v^{\sigma_1\sigma_2}_{i'j'}(\lambda)\,. \tag{49}$$

Q'_{ij} is given by

$$Q'=\frac{1}{9}\begin{array}{c|ccccccc} & G & S & O_3 & O_8 & \bar{S} & \bar{O}_3 & \bar{O}_8 \\ \hline G & 0 & 6\sqrt{2/3} & 3 & \sqrt{3} & 6\sqrt{2/3} & 3 & \sqrt{3} \\ S & 6\sqrt{2/3} & 0 & 0 & 0 & 4 & \sqrt{6} & \sqrt{2} \\ O_3 & 3 & 0 & 0 & 0 & \sqrt{6} & 5 & \sqrt{3} \\ O_8 & \sqrt{3} & 0 & 0 & 0 & \sqrt{2} & \sqrt{3} & 3 \\ \bar{S} & 6\sqrt{2/3} & 4 & \sqrt{6} & \sqrt{2} & 0 & 0 & 0 \\ \bar{O}_3 & 3 & \sqrt{6} & 5 & \sqrt{3} & 0 & 0 & 0 \\ \bar{O}_8 & \sqrt{3} & \sqrt{2} & \sqrt{3} & 3 & 0 & 0 & 0 \end{array}\,.$$

For quark-antiquark terms

$$V^{\sigma_1\sigma_2}=\frac{8(1+\lambda)^{-\sigma_1-\sigma_2}}{9\lambda}\left[(1+\lambda)F\left(\sigma_1,\sigma_2,\sigma_1+\sigma_2+2,\frac{1}{1+\lambda}\right)[B(\sigma_2,\sigma_1+2)+B(\sigma_1,\sigma_2+2)]\right.$$
$$\left.+2B(\sigma_1+1,\sigma_2+1)F\left(\sigma_1+1,\sigma_2+1,\sigma_1+\sigma_2+2,\frac{1}{1+\lambda}\right)\right]\,. \tag{50}$$

For quark-gluon and antiquark-gluon terms

$$V^{\sigma_1\sigma_2}=-\frac{(1+\lambda)^{-\sigma_1-\sigma_2}}{6\lambda}\left[(1+\lambda)B(\sigma_2,\sigma_1+1)F\left(\sigma_1,\sigma_2,\sigma_1+\sigma_2+1,\frac{1}{1+\lambda}\right)\right.$$
$$+\frac{\lambda^2}{1+\lambda}B(\sigma_2+2,\sigma_1+1)F\left(\sigma_1+2,\sigma_2+2,\sigma_1+\sigma_2+3,\frac{1}{1+\lambda}\right)$$
$$\left.-2B(\sigma_2+1,\sigma_1+3)F\left(\sigma_1+1,\sigma_2+1,\sigma_1+\sigma_2+4,\frac{1}{1+\lambda}\right)\right]\,. \tag{51}$$

G. $h(p_1)+h(p_2)\to h(p_3)$ + anything

In this section we shall discuss large-transverse-momentum inclusive hadron production in hadron-hadron collisions.[20] We shall see a difference in this process from all the others we have so far discussed. In the above process two spacelike vertices (parton distribution functions) and one timelike cut vertex (parton decay function) appear. However, it is not possible to perform a diagonalization so that a simple Callan-Symanzik equation emerges.

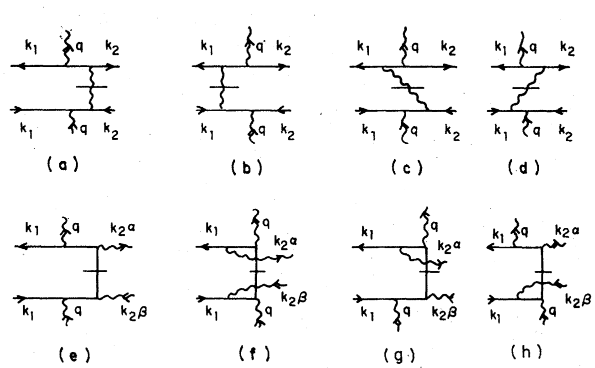

FIG. 5. The hard-scattering graphs used as input to the Callan-Symanzik equation for large-q^2 μ-pair production. With a reversal of various arrows of the k_1, k_2 lines, these graphs are also the relevant graphs for a similar region in $e^+ + e^- \to h(p_1)+h(p_2)+X$ and in $e+h(p)\to e+h(p_1)+X$.

Since we shall not be able to completely diagonalize the high momentum equation let us begin by reminding the reader, for notational purposes, how a diagonalized and undiagonalized equation are related. Consider, for example, Eq. (20)

$$\int(2\pi)^3 2E\frac{d\bar{\sigma}}{d^3p}\omega^{\sigma-2}d\omega=\frac{8\pi^2\alpha^2}{3(Q^2)^2}\sum_{i,j}\tilde{v}^i_\sigma R^\sigma_{ij}(Q_1{}^2,Q^2)\times E^j_\sigma(Q_0{}^2,g(Q^2,Q_0{}^2))\,. \tag{20}$$

$d\bar{\sigma}/d^3p$ is the single-particle inclusive cross section in e^+e^- collisions with an average being taken over electron directions in the center of mass of the e^+e^- system. Define

$$\tilde{v}^i(\omega)=\frac{1}{2\pi i}\int_{L-i\infty}^{L+i\infty} d\sigma\omega^{-\sigma}\tilde{v}^i_\sigma, \tag{52}$$

then

$$\tilde{v}^i_\sigma=\int_0^1 d\omega\omega^{\sigma-1}\tilde{v}^i(\omega), \tag{53}$$

where L is taken to the right of all singularities in the σ plane. Then

$$(2\pi)^3 2E\frac{d\tilde{\sigma}}{d^3p} = \frac{8\pi^2\alpha^2}{3(Q^2)^2}\sum_i \int_\omega^1 \frac{d\omega_1}{\omega_1}\tilde{v}^i(\omega_1)E^i(\omega/\omega_1,Q^2), \tag{54}$$

where

$$E^i(\omega,Q^2)=\frac{1}{2\pi i}\int d\sigma\omega^{-\sigma}E^i_\sigma(Q_1{}^2,Q^2)$$

and

$$E^i_\sigma(Q^2,Q_1{}^2)=\sum_j R^\sigma_{ij}(Q_1{}^2,Q^2)E^j(Q_0{}^2,g(Q^2,Q_0{}^2)).$$

$E^i(\omega,Q^2)$ obeys the Callan-Symanzik equation

$$\left(-Q^2\frac{\partial}{\partial Q^2}+\beta\frac{\partial}{\partial g}\right)E^i(\omega,Q^2) = \sum_j \int \frac{d\omega_1}{\omega_1}\gamma^{ij}(\omega_1)E^j(\omega/\omega_1,Q^2)$$

with

$$\gamma^{ij}(\omega)=\frac{1}{2\pi i}\int d\sigma\omega^{-\sigma}\gamma^{ij}_\sigma .$$

The advantage of (20) over (54) is that (20) has a definite Q^2 behavior, while (54) has an extremely complicated Q^2 behavior depending on ω. These two equations have, of course, equivalent content.

Consider now the process $h_1(p_1)+h_2(p_2)\to h_3(p_3)$ +anything. The inclusive cross section $2Ed\sigma/d^3p_3$ depends on three variables. We shall use a coordinate system where $(p_1)_\mu=(E_1,0,0,-p_1)$, $(p_2)_\mu=(E_2,0,0,p_2)$, $(p_3)_\mu=((\underline{p}_3{}^2+m^2)^{1/2},\underline{p}_3,0)$. We choose the three independent variables to be $y_1=\underline{p}_3{}^2/p_1\cdot p_3$, $y_2=\underline{p}_3{}^2/p_2\cdot p_3$, $\underline{p}_3{}^2$. We presume $\underline{p}_3{}^2$ is large compared to all relevant hadronic masses. Then a discussion much like that given in Sec. III leads to the equation

$$2E_{p_3}\frac{d\sigma}{d^3p_3}=\int_0^1\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\sum_{i,j,k}v_1^i(x_1)v_2^j(x_2)v_3^k(x_3)V^{ijk}(x_1\hat{p}_1,x_2\hat{p}_2,x_3\hat{p}_3). \tag{55}$$

The i,j,k sum takes values $G,S,O_3,O_8,\overline{S},\overline{O}_3,\overline{O}_8$. $v_1^i(x_1)$ and $v_2^j(x_2)$ are spacelike vertices where the subscripts label particle types rather than moments. $v_3^k(x_3)$ is a timelike cut vertex referring to the outgoing particle p_3. $(\hat{p}_1)_\mu=(p_1,0,0,-p_1)$, $(\hat{p}_2)_\mu=(p_2,0,0,p_2)$, $(\hat{p}_3)_\mu=(|\underline{p}_3|,\underline{p}_3,0)$. If we call $V^{ikj}(\hat{p}_1,\hat{p}_2,\hat{p}_3)$ $=V^{ijk}(y_1,y_2,\underline{p}_3{}^2)$, then $V^{ijk}(x_1\hat{p}_1,x_2\hat{p}_2,x_3\hat{p}_3)=V^{ijk}(y_1x_3/x_1,y_2x_3/x_2,x_3^2\underline{p}_3{}^2)$ and (55) becomes

$$2E_{p_3}\frac{d\sigma}{d^3p_3}=\int_0^1\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\sum_{i,j,k}v_1^i(x_1)v_2^j(x_2)v_3^k(x_3)V^{ijk}(y_1x_3/x_1,y_2x_3/x_2,x_3^2\underline{p}_3{}^2). \tag{56}$$

Equation (56) is the completely undiagonalized form of a generalized Wilson light-cone expansion for this process.

Now define

$$R(x,\underline{p}^2/\underline{p}_0{}^2)=\frac{1}{2\pi i}\int d\sigma x^{-\sigma}R^\sigma(\underline{p}_0{}^2,p^2),$$

where, as before,

$$R^\sigma(\underline{p}_0{}^2,\underline{p}^2)=O\exp\left(-\int_{\underline{p}_0{}^2}^{\underline{p}^2}\frac{d\underline{p}'^2}{(\underline{p}')^2}\gamma_\sigma[g(\underline{p}'^2,\underline{p}_0{}^2)]\right).$$

Then $R(x,\underline{p}^2/\underline{p}_0{}^2)$ obeys the matrix Callan-Symanzik equation

$$\left(-\underline{p}^2\frac{\partial}{\partial\underline{p}^2}+\beta\frac{\partial}{\partial g}\right)R(x,\underline{p}^2/\underline{p}_0{}^2)=\int\frac{dx'}{x'}\gamma(x')R(x/x',\underline{p}^2/\underline{p}_0{}^2).$$

The solution for V is

$$V^{ijk}(y_1,y_2,\underline{p}_3{}^2)=\int\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\sum_{i',j',k'}R_{ii'}\left(x_1,\frac{\underline{p}_3{}^2}{\underline{p}_0{}^2}\right)R_{jj'}\left(x_2,\frac{\underline{p}_3{}^2}{\underline{p}_0{}^2}\right)R_{kk'}(x_3,\underline{p}_3{}^2/\underline{p}_0{}^2) \times V^{i'j'k'}(y_1x_3/x_1,y_2x_3/x_2,x_3^2\underline{p}_0{}^2,g(\underline{p}_3{}^2,\underline{p}_0{}^2)). \tag{57}$$

One can easily check that V obeys the Callan-Symanzik equation.

$$\left(-\underline{p}_3{}^2\frac{\partial}{\partial \underline{p}_3{}^2}+\beta\frac{\partial}{\partial g}\right)V^{ijk}(y_1,y_2,\underline{p}_3{}^2)=\int\frac{dx_1}{x_1}\sum_{i'}\gamma^{ii'}(x_1)V^{i'jk}(y_1/x_1,y_2,\underline{p}_3{}^2)$$
$$+\int\frac{dx_2}{x_2}\sum_{j'}\gamma^{jj'}(x_2)V^{ij'k}(y_1,y_2/x_2,\underline{p}_3{}^2)$$
$$+\int\frac{dx_3}{x_3}\sum_{k'}\gamma^{kk'}(x_3)V^{ijk'}(y_1x_3,y_2x_3,x_3{}^2\underline{p}_3{}^2)\,. \tag{58}$$

Equations (56) and (57) exhibit the information one obtains from the renormalization group in wide-angle scattering. It is possible, and straightforward, to diagonalize the y_1 and y_2 dependences of (56) leaving only a dx_3/x_3 integration. In that case the first two terms on the right-hand side of (58) become matrix multiplications rather than convolutions. It appears impossible to do any further diagonalization, so that the Callan-Symanzik equation must always remain a convolution equation.

Equations (56) and (57) can also be written in the form

$$2E_{p_3}\frac{d\sigma}{d^3p_3}=\int_0^1\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\sum_{i,j,k}v_1^i(x_1,Q^2)v_2^j(x_2,Q^2)v_3^k(x_3;Q^2)V^{ijk}\left(\frac{y_1x_3}{x_2},\frac{y_2x_3}{x_2},x_3{}^2\underline{p}_0{}^2,g(\underline{p}_3{}^2,\underline{p}_0{}^2)\right), \tag{59}$$

where the V^{ijk} can be determined from lowest-order perturbation theory. An equation like (59) is the starting point of an analysis of the type done by Feynman, Field, and Fox.[20]

Finally, it has been a common practice[20] dealing with wide-angle scattering as well as in μ-pair production to distinguish between a transverse momentum due to the hard scattering and one due to the wave function of the hadron.[21,22,23] We are unable to find any place for such a separation in our formalism, and we suspect that the concept of the "intrinsic transverse momentum in a hadron" is a difficult one to talk about sensibly. We think that any analysis that depends crucially on the existence of a wave-function distribution of transverse momentum is probably not a tight test of QCD.

H. $e+h(p)\to e+h(p_1)+X$

In single-particle inclusive hadron production in deeply inelastic electron scattering one has[24]

$$\frac{d\sigma}{dQ^2d\nu d\kappa_1 d\nu_1}=\frac{4\pi\alpha^2}{(Q^2)^2}\left(\frac{\epsilon'}{\epsilon}\right)[\mathcal{W}_2\cos^2(\tfrac{1}{2}\theta)+2\mathcal{W}_1\sin^2(\tfrac{1}{2}\theta)], \tag{60}$$

where

$$\mathcal{W}_{\mu\nu}=\frac{4\pi^2E_p}{m}\sum_n d^3p_1\delta\left(\kappa_1-\frac{p\cdot p_1}{m}\right)\delta\left(\nu_1-\frac{p_1\cdot q}{m_1}\right)\times\langle p|j_\mu(0)|p_1n\rangle\langle np_1|j_\nu(0)|p\rangle\times(2\pi)^4\delta^4(p+q-p_1-p_n)\,. \tag{61}$$

In the above ϵ is the energy of the initial electron, ϵ' the energy of the final electron, and θ the electron angle of scattering in the laboratory system. In order to obtain a somewhat simpler amplitude to deal with, consider the coordinate system

$$q_\mu=(0,0,0,Q)\,,$$
$$p_\mu=(\overline{E},0,0,-\overline{p})\,,$$
$$k_\mu=\overline{\epsilon}(1,\sin\overline{\theta}\cos\overline{\phi},\sin\overline{\theta}\sin\overline{\phi},\cos\overline{\theta})\,,$$
$$p_{1\mu}=(\overline{E}_1,\overline{p}_1\sin\overline{\theta}_1\cos\overline{\phi}_1,\overline{p}_1\sin\overline{\theta}_1\sin\overline{\phi}_1,\overline{p}_1\cos\overline{\theta}_1)\,,$$
$$k'=k-q\,.$$

Then one can easily verify that

$$F(\omega,\omega_1,Q^2\kappa_1)=\frac{1}{m}\int(k\cdot p)^2(15\cos^2\overline{\theta}-7)\cos^2\overline{\theta}\times d\cos^2\overline{\theta}\frac{2E_1d\sigma}{dQ^2d\nu d^3p_1}d\overline{\phi}_1\,,$$

$$F=\frac{2\nu}{mm_1}\int(k\cdot p)^2(15\cos^2\overline{\theta}-7)\cos^2\overline{\theta}\times d\cos^2\overline{\theta}\frac{d\sigma}{dQ^2d\nu d\nu_1 d\kappa_1}\,, \tag{62}$$

$$F=-\frac{2\nu m\pi\alpha^2}{m_1Q^2}\mathcal{W}_{\mu\mu}\,. \tag{63}$$

F is given by

$$F=-\frac{\alpha^2(2\pi)^3}{(2\pi)^2(Q^2)}E_p(2\pi)^3E_{p_1}\times\sum_n\langle p|j_\mu(0)|p_1n\rangle\langle p_1n|j_\mu(0)|p\rangle\times(2\pi)^4\delta^4(p+q-p_1-p_n)\,. \tag{64}$$

In the following we shall understand a spin sum over p_1 and a spin average over p. $\omega=2p\cdot q/Q^2$, $\omega_1=-2p_1\cdot q/Q^2$, $\kappa_1=p\cdot p_1/m$.

The predictions of QCD are most easily obtained in a somewhat different coordinate system. Consider the system where

$$p_\mu = (E_p, 0, 0, -p),$$
$$p_{1\mu} = (E_{p_1}, 0, 0, p_1),$$
$$q_\mu = (0, \underline{q}, (Q^2 - \underline{q}^2)^{1/2}).$$

In this system we view F as a function of $\omega, \omega_1, Q^2, \underline{q}^2$.

(i) Suppose $\underline{q}^2$ is not necessarily large. Then we must include an integral over $\underline{q}$ in order to get definite predictions from the renormalization group. Define

$$F_{\sigma\sigma_1}(Q^2) = \int d^2\underline{q}\, d\omega\, d\omega_1 F(\omega, \omega_1, Q^2, \underline{q}^2)\omega^{-\sigma-1}\omega_1^{\sigma_1-2}; \qquad (65)$$

then for large Q^2

$$F_{\sigma\sigma_1}(Q^2) = \frac{8\pi^2\alpha^2}{Q^2}\sum_{ij} v_\sigma^i v_{\sigma_1}^j E_{\sigma\sigma_1}^{ij}(Q^2), \qquad (66)$$

where v_σ^i is a spacelike cut vertex referring to the target particle and $v_{\sigma_1}^j$ is a timelike cut vertex referring to the produced particle. $E_{\sigma\sigma_1}^{ij}$ obeys a Callan-Symanzik equation. Using asymptotic freedom we may write

$$F_{\sigma\sigma_1}(Q^2) = \frac{16\pi^2\alpha^2}{3Q^2}\sum_{ij} Q_{ij} v_\sigma^i(Q^2) v_{\sigma_1}^j(Q^2) \qquad (67)$$

or

$$F(x, \omega_1, Q^2) = \frac{16\pi^2\alpha^2}{3Q^2}\sum_{ij} Q_{ij}\, x v^i(x, Q^2)\omega_1 v^j(\omega_1, Q^2),$$

where $x = 1/\omega$ and

$$F(x, \omega_1, Q^2) = \int d^2\underline{q}\, F\left(\frac{1}{x}, \omega_1, Q^2, \underline{q}^2\right).$$

(ii) Suppose $\underline{q}^2/Q^2 = \lambda$ is fixed. Now one must calculate graphs like those shown in Fig. 5, but with k_2 an outgoing line. Using asymptotic freedom we find

$$F_{\sigma\sigma_1}(Q^2, \lambda) = \frac{\alpha^2 g^2(Q^2)}{\pi(Q^2)^2}\left(\frac{\lambda}{1-\lambda}\right)^{-\sigma_1} \times \sum_{ij} Q_{ij}' v_\sigma^i(Q^2) v_{\sigma_1}^j(Q^2) V_{\sigma\sigma_1}^{ij}(\lambda). \qquad (68)$$

For quark-antiquark terms

$$V_{\sigma\sigma_1}(\lambda) = \frac{4}{9}\left[\frac{\lambda}{1-\lambda}B(\sigma_1 - 2, \sigma + 2)F\left(\sigma_1, \sigma_1 - 2, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right) + B(\sigma_1, \sigma)F\left(\sigma_1 - 2, \sigma_1, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right) + \frac{2}{1-\lambda}B(\sigma_1 - 1, \sigma + 1)F\left(\sigma_1 - 1, \sigma_1 - 1, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right)\right]. \qquad (69)$$

For i = quark or antiquark and j = gluon

$$V_{\sigma\sigma_1}(\lambda) = -\frac{1}{12}\frac{1}{1-\lambda}\left[B(\sigma_1 - 1, \sigma + 1)F\left(\sigma_1, \sigma_1 - 1, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right) + \lambda^2 B(\sigma_1, \sigma)F\left(\sigma_1 - 2, \sigma_1, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right) + 2\lambda^2 B(\sigma_1 - 3, \sigma + 3)F\left(\sigma_1 - 1, \sigma_1 - 3, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right)\right]. \qquad (70)$$

For i = gluon and j = quark or antiquark

$$V_{\sigma\sigma_1}(\lambda) = -\frac{1}{12}\frac{1}{1-\lambda}\left[B(\sigma_1, \sigma)F\left(\sigma_1 - 1, \sigma_1, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right) + \lambda^2 B(\sigma_1 - 2, \sigma + 2)F\left(\sigma_1 - 1, \sigma_1 - 2, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right) - 2\lambda^2 B(\sigma_1 - 1, \sigma + 1)F\left(\sigma_1 - 3, \sigma_1 - 1, \sigma + \sigma_1, 1 - \frac{1}{\lambda}\right)\right]. \qquad (71)$$

III. A GENERALIZED WILSON EXPANSION FOR MULTIPARTICLE PROCESS

In this section a discussion will be given as to how a generalized Wilson expansion emerges for multiparticle process. Rather than deal with all the processes discussed in Sec. II, we shall limit ourselves to a detailed discussion of $e^+ + e^- \to h(p_1) + h(p_2) + X$. The derivations of the other results listed in Sec. II take a similar form though the details differ considerably from case to case. We shall only discuss the case of an Abelian gauge theory where both the fermion and gauge field have mass m. The equality of masses is taken only for convenience. We shall first state the result of the generalized Wilson expansion. Then some examples illustrating this result will be given. Finally, arguments supporting the conclusions will be given.

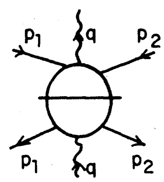

FIG. 6. The Feynman graphs contributing to F.

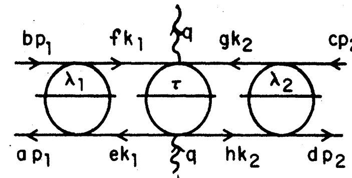

FIG. 7. A particular decomposition of a given graph of F.

A. The form of the expansion

Let

$$2E_1 2E_2 \frac{d\sigma}{d^3p_1 d^3p_2} = -\frac{8\pi^2\alpha^2}{(2\pi)^6 3(Q^2)^2} F(\omega_1, \omega_2, Q^2, \underline{q}^2), \tag{72}$$

where F is the set of all cut Feynman diagrams with contracted current indices shown in Fig. 6 for outgoing fermion lines. We choose the coordinate system $q_\mu = ((Q^2+\underline{q}^2)^{1/2}, \underline{q}, 0)$, $p_{1\mu} = (E_{p_1}, 0, 0, 0, -p_1)$, $p_{2\mu} = (E_{p_2}, 0, 0, p_2)$.

We begin by reminding the reader of some notational conventions.[25] A renormalization part, γ, is a proper subgraph which has superficial degree of divergence greater than or equal to zero, with one exception. The exception is the four-photon vertex which has superficial degree of divergence equal to zero, but does not diverge when a gauge-invariant combination of graphs is taken. Thus a subgraph, γ, having only four external photon lines need not be considered a renormalization part. A forest, U, of a graph G is a set of nonoverlapping renormalization parts of G. U may include the null set and the graph G.

Define t_2^γ to be the usual subtraction operator[26] for a massive vector-meson theory. That is, if γ has degree of divergence equal to zero, then $t_2^\gamma \gamma(p_1, p_2, \ldots) = \gamma(0, 0, \ldots)$. If γ has degree of divergence equal to one,

$$t_2^\gamma \gamma(p_1, p_2, \ldots) = \gamma(0, 0, \ldots) + p_{1\mu}\frac{\partial}{\partial k_\mu}\gamma(k, 0, \ldots)|_{k=0} + p_{2\mu}\frac{\partial}{\partial k_\mu}\gamma(0, k, \ldots)|_{k=0} + \cdots .$$

If γ has degree of divergence equal to two, t_2^γ gives all terms to second order in the Taylor series.

The fully renormalized F is given in terms of the renormalized F^u by the BPHZ[26-28] formula

$$F = \sum_U \prod_{\gamma\in U} (-t_2^\gamma) F^u . \tag{73}$$

In order to ensure gauge invariance, we suppose that a gauge-invariant regularization is performed.

The usual way of obtaining a Wilson expansion is to do additional subtractions on the amplitude F. Consider a particular graph G contributing to F. Suppose G can be decomposed topologically as shown in Fig. 7. All external propagators are contained in λ_1 and λ_2 rather than in τ. We further suppose that p_1 is a fermion, p_2 an antifermion, k_1 a fermion, and k_2 an antifermion. (We shall, of course, later add in the contributions having k_1 an antifermion and k_2 a fermion.) Then

$$F(p_1, p_2, q) = (\gamma\cdot p_1 + m)_{ba}(\gamma\cdot p - m)_{dc}\int d^4k_1 d^4k_2 T^{\lambda_1}_{abfe}(p_1, k_1) T^{\lambda_2}_{dcgh}(p_2, k_2) V^\tau_{efgh}(k_1, k_2, q), \tag{74}$$

where the k_1 and k_2 propagators are contained in T^{λ_1} and T^{λ_2}, respectively. We define t^τ by the rule

$$t^\tau V^\tau_{efgh}(k_1, k_2, q) = \hat{V}^\tau_{efgh}(\hat{k}_1, \hat{k}_2, q),$$

where $\hat{k}_{1+} = \hat{\underline{k}}_1 = \hat{k}_{2-} = \hat{\underline{k}}_2 = 0$, $\hat{k}_{1-} = k_{1-}$, $\hat{k}_{2+} = k_{2+}$, and $\hat{V}^\tau$ means that only the $(\gamma_-)_{ef}(\gamma_+)_{gh}$ term of V^τ is kept. Another way of writing this is

$$\begin{aligned} t^\tau V^\tau_{efgh}(k_1, k_2, q) &= (\gamma_-)_{ef}(\gamma_+)_{gh}\frac{1}{16}\sum_{\substack{e'f'\\ g'h'}} (\gamma_+)_{f'e'}(\gamma_-)_{h'g'} V^\tau_{e'f'g'h'}(\hat{k}_1, \hat{k}_2, q) \\ &= (\gamma_-)_{ef}(\gamma_+)_{gh} v^\tau(\hat{k}_1, \hat{k}_2, q). \end{aligned} \tag{75}$$

Call

$$F^\tau(p_1, p_2, q) = (\gamma\cdot p_1 + m)_{ba}(\gamma\cdot p_2 - m)_{dc}\int d^4k_1 d^4k_2 T^{\lambda_1}_{abfe}(p_1, k_1) T^{\lambda_2}_{dcgh}(p_2, k_2) t^\tau V^\tau_{efgh}(k_1, k_2, q).$$

Then the coupling of T^{λ_1} to V is only through k_{1-} and the coupling of T^{λ_2} to V is only through k_{2+}. We can also write F^τ in the form

$$F^\tau(p_1,p_2,q)=(\gamma\cdot p_1+m)_{ba}(\gamma\cdot p_2-m)_{dc}\int d^4k_1\sum_{ef}T^{\lambda_1}_{abef}(p_1,k_1)(\gamma_-)_{ef}$$
$$\times\int d^4k_2\sum_{gh}T^{\lambda_2}_{dcgh}(p_2,k_2)(\gamma_+)_{gh}v^\tau(\hat{k}_1,\hat{k}_2,q)\,. \tag{76}$$

Define

$$F^\tau_{\sigma_1\sigma_2}(Q^2)=\int d\omega_1 d\omega_2 d^2\underline{q}\,\omega_1{}^{\sigma_1-2}\omega_2{}^{\sigma_2-2}F^\tau(\omega_1,\omega_2,Q^2,\underline{q}^2)\,. \tag{77}$$

Then

$$F^\tau_{\sigma_1\sigma_2}(Q^2)=\frac{Q^2}{2}p_{1-}{}^{\sigma_1-1}(\gamma\cdot p_1)_{ba}\int d^4k_1\sum_{ef}T^{\lambda_1}_{abfe}(p_1,k_1)(\gamma_-)_{ef}k_{1-}{}^{-\sigma_1}$$
$$\times p_{2+}{}^{\sigma_2-1}(\gamma\cdot p_2)_{dc}\int d^4k_2\sum_{gh}T^{\lambda_2}_{dcgh}(p_2,k_2)(\gamma_+)_{gh}k_{2+}{}^{-\sigma_2}$$
$$\times\int d^2\underline{q}\frac{Q^2}{Q^2+\underline{q}^2}d\omega_1 d\omega_2\,\omega_1{}^{\sigma_1-1}\omega_2{}^{\sigma_2-1}v^\tau(\omega_1,\omega_2,Q^2,\underline{q}^2)\,. \tag{78}$$

Thus

$$F^\tau_{\sigma_1\sigma_2}(Q^2)=8Q^2v^S_{\sigma_1}v^{\bar{S}}_{\sigma_2}\int d^2\underline{q}\,d\omega_1 d\omega_2\frac{Q^2}{Q^2+\underline{q}^2}v^\tau(\omega_1,\omega_2,Q^2,\underline{q}^2)\omega_1{}^{\sigma_1-1}\omega_2{}^{\sigma_2-1}$$
$$=v_{\sigma_1}v_{\sigma_2}\frac{1}{Q^2}E^{\tau,S\bar{S}}_{\sigma_1\sigma_2}(Q^2)\,, \tag{79}$$

where

$$v^S_{\sigma_1}=p_{1-}{}^{\sigma_1-1}\tfrac{1}{4}\,\mathrm{tr}\gamma\cdot p_1\Gamma^S_{\sigma_1}(p_1)\,,\quad v^{\bar{S}}_{\sigma_2}=p_{2+}{}^{\sigma_2-1}\tfrac{1}{4}\,\mathrm{tr}\gamma\cdot p_2\Gamma^{\bar{S}}_{\sigma_2}(p_2)\,.$$

Remember, Γ_{σ_1} involves a bare fermion cut vertex and Γ_{σ_2} a bare antifermion cut vertex. At this point the v_σ's are unrenormalized cut vertices. If we renormalize the cut vertices by applying the subtraction operator t_1^γ to renormalization parts of λ_1 and λ_2, considered now as graphs of cut vertices, we get an equation just like (79) except that v_{σ_1} and v_{σ_2} are now renormalized. The subtraction operator, t_1^γ, for cut vertices is described in detail in Ref. 1.

If we now sum over τ and over all graphs we obtain

$$v^S_{\sigma_1}v^{\bar{S}}_{\sigma_2}\frac{1}{Q^2}E^{S\bar{S}}_{\sigma_1\sigma_2}(Q^2)=\sum_\tau\sum_{U_1\in\mathfrak{U}(\lambda_1)}\sum_{U_2\in\mathfrak{U}(\lambda_2)}\sum_{U_3\in\mathfrak{U}(\tau)}\prod_{\gamma_1\in U_1}(-t_1^{\gamma_1})\prod_{\gamma_2\in U_2}(-t_1^{\gamma_2})t^\tau\prod_{\gamma_3\in U_3}(-t_2^{\gamma_3})$$
$$\times\int\omega_1{}^{\sigma_1-2}\omega_2{}^{\sigma_2-2}d\omega_1 d\omega_2 d^2\underline{q}F^u(p_1,p_2,q)\,. \tag{80}$$

In (80) γ_1 is a renormalization part of λ_1, considered as a graph contributing to a cut vertex. γ_2 is a renormalization part of λ_2, considered as a graph contributing to a cut vertex, and γ_3 is a renormalization part of τ. U_1 is a forest of λ_1, U_2 is a forest of λ_2, and U_3 is a forest of τ. $\mathfrak{U}(\lambda_1)$ is the set of all forests of λ_1, $\mathfrak{U}(\lambda_2)$ is the set of all forests of λ_2, and $\mathfrak{U}(\tau)$ is the set of all forests of τ.

Of course, so far we have only taken the contribution to v_{σ_1} which consists of a bare cut vertex having a fermion below the cut. The contributions involving an antifermion below the cut along with the fermion-antifermion many-gluon bare vertices and the two-gluon bare vertex will be added in shortly. However, before doing this let us indicate the relationship between the $F_{\sigma_1\sigma_2}$ given by

$$F_{\sigma_1\sigma_2}(Q^2)=\int\omega_1{}^{\sigma_1-2}\omega_2{}^{\sigma_2-2}d\omega_1 d\omega_2 d^2\underline{q}$$
$$\times\sum_U\prod_{\gamma\in U}(-t_2^\gamma)F^u(\omega_1,\omega_2,Q^2,\underline{q}^2) \tag{81}$$

and the expression (80). This relationship is

$$F_{\sigma_1\sigma_2}(Q^2)=v^S_{\sigma_1}v^{\bar S}_{\sigma_2}\frac{1}{Q^2}E^{S\bar S}_{\sigma_1\sigma_2}(Q^2)$$
$$+\sum_U \prod_{\gamma\in U}(-t_1^\gamma)F^u_{\sigma_1\sigma_2}(Q^2). \quad (82)$$

In (82) the "renormalization parts," γ, consist of three types of subgroups. (i) γ may be any subgraph having superficial degree of divergence greater than or equal to zero. (ii) γ may consist of any amputated graph of the type shown in Fig. 6, with p_i replaced by k_i, where k_1 is a fermion and k_2 an antifermion line. (iii) γ may be of a type to be specified below.

In (82) the forests U consist of sets of nonoverlapping renormalization parts with the restriction that each forest contains at most one element of class (ii) and, further, that a forest contains elements of class (iii) only if it contains an element of class (ii). We can now define an element, γ, of class (iii) in the following manner. Let τ be an element of (ii). Then the full graph is topologically broken up as in Fig. 7. The elements of class (iii) are renormalization parts of λ_1 and λ_2 where λ_1 and λ_2 are considered as graphs of cut vertices where the e,f lines are

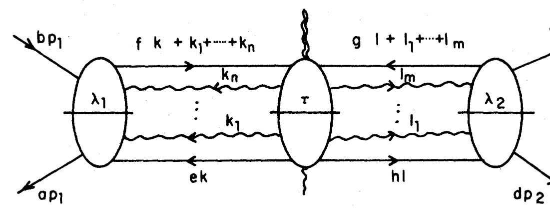

FIG. 8. A decomposition of F showing many photons.

multiplied by a $(\gamma_-)_{ef}$ and the g,h lines are multiplied by a $(\gamma_+)_{gh}$. Then if γ is of class (ii), t_1^γ is the same as the t^τ discussed above. If γ is of class (iii), t_1^γ is the same as the subtraction operator for cut vertices discussed in Ref. 1.

Our next step is to consider decompositions different than those shown in Fig. 7. Consider a decomposition of the form shown in Fig. 8. There are n photon lines of momentum $k_1 \cdots k_n$ connecting λ_1 to τ, the first i of which lie below the cut in τ. There are m photon lines of momentum $l_1 \cdots l_m$ connecting λ_2 to τ, the first j of which lie below the cut in τ. All propagators are in λ_1 and λ_2 rather than in τ. m or n but not both may be zero.

This contribution to F is given by

$$F(p_1,p_2,q)=(\gamma\cdot p_1+m)_{ba}(\gamma\cdot p_2-m)_{dc}\int d^4k_1\cdots d^4k_n d^4l_1\cdots d^4l_m d^4k d^4l\, T^{\lambda_1,i}_{alfe}(p_1,k,k_1,\ldots k_n)$$
$$\times T^{\lambda_2,j}_{dcgh}(p_2,l,l_1,\ldots l_m)V^{\tau,ij}_{efgh}(k,k_1,\ldots l_1,l,q). \quad (83)$$

The photons with momentum $k_1,\ldots,k_n$ end on γ_+'s in τ, while the photons with momentum $l_1,\ldots,l_m$ end on γ_-'s in τ. τ may be disconnected, but the photon lines must end on some part of τ. No sum of photon lines may form all the external lines of a disconnected component of τ.

We define

$$t^\tau V^{\tau,ij}_{efgh}(k,l,q)=\hat V^{\tau,ij}_{efgh}(\hat k,\hat l,q),$$

where the ^ on V indicates that only the $(\gamma_-)_{ef}(\gamma_+)_{gh}$ part of V is taken. $\hat k_+=\underline{\hat k}=\hat l_-=\underline{\hat l}=0$, $k_-=\hat k_-$, $\hat l_+=l_+$. $\hat V$ obeys the Ward identity

$$\prod_{r=1}^n k_{r-}\prod_{s=1}^m l_{s+}\hat V^{\tau,ij}_{efgh}(\hat k,\hat k_1,\ldots,\hat l_1,\hat l,q)=(-1)^{n+m-i-j}g^{n+m}V^\tau_{efgh}(\hat k+\hat k_1+\cdots+\hat k_i,\hat l+\hat l_1+\cdots+\hat l_j,q). \quad (84)$$

The Ward identity clearly determines $\hat V^{\tau,ij}$. The further discussion for the decomposition shown in Fig. 8 now follows parallel to the discussion for the decomposition shown in Fig. 7. (The Ward identity makes the two discussions almost identical.) We arrive at a formula identical to (82) where the v's stand for the bare two-fermion many-photon contributions to the cut vertices.

Next consider a decomposition of the sort shown in Fig. 9. Again τ is completely amputated. The contribution to F is

$$F(p_1,p_2,q)=(\gamma\cdot p_1+m)_{ba}(\gamma\cdot p_2-m)_{dc}\int d^4k_1 d^4k_2 T^{\lambda_1}_{abfe}(p_1,k_1)T^{\lambda_2}_{dc\alpha\beta}(p_2,k_2)V^\tau_{ef\alpha\beta}(k_1,k_2,q). \quad (85)$$

Now $V^\tau_{ef\alpha\beta}$ can always be written as (see Appendix B)

$$V^\tau_{ef\alpha\beta}(k_1,k_2,q)=\sum_{\alpha'\beta'}\tilde V^\tau_{ef\alpha'\beta'}(k_1,k_2,q)[g_{\alpha\beta}k_{2\alpha'}k_{2\beta'}+g_{\alpha\alpha'}g_{\beta\beta'}k_2{}^2-g_{\alpha\alpha'}k_{2\beta}k_{2\beta'}-g_{\beta\beta'}k_{2\alpha}k_{2\alpha'}]. \quad (86)$$

We define t^τ by the rule

$$t^\tau \tilde{V}^\tau_{ef\alpha'\beta'}(k_1,k_2,q) = 4(\gamma_-)_{ef}\sum_{e'f'}\tfrac{1}{16}(\gamma_+)_{f'e'}\tilde{V}^\tau_{e'f'}(\hat{k}_1,\hat{k}_2,q)g_{\alpha'+}g_{\beta'+}$$
$$= 4g_{\alpha'+}g_{\beta'+}(\gamma_-)_{ef}v^\tau(\hat{k}_1,\hat{k}_2,q). \qquad (87)$$

Then

$$t^\tau V^\tau_{ef\alpha\beta}(k_1,k_2,q) = (\gamma_-)_{ef}4[g_{\alpha\beta}k_{2+}{}^2 - k_{2+}(g_{\alpha+}k_{2\beta}+g_{\beta+}k_{2\alpha}) + g_{\alpha+}g_{\beta+}k_2{}^2]v^\tau(\hat{k}_1,\hat{k}_2,q). \qquad (88)$$

Further, define

$$F^\tau(p_1,p_2,q) = (\gamma\cdot p_1+m)_{ba}(\gamma\cdot p_2-m)_{dc}\int d^4k_1\sum_{ef}T^{\lambda_1}_{abfe}(p_1,k_1)(\gamma_-)_{ef}$$
$$\times\int d^4k_2\sum_{\alpha\beta}T^{\lambda_2}_{dc\alpha\beta}(p_2,k_2)4[g_{\alpha\beta}k_{2+}{}^2-k_{2+}(g_{\alpha+}k_{2\beta}+g_{\beta+}k_{2\alpha})$$
$$+g_{\alpha+}g_{\beta+}k_2{}^2]v^\tau(\hat{k}_1,\hat{k}_2,q). \qquad (89)$$

Taking moments we obtain

$$F^\tau_{\sigma_1\sigma_2}(Q^2) = \tfrac{1}{2}Q^2 p_{1-}{}^{\sigma_1-1}(\gamma\cdot p_1)_{ba}\int d^4k_1 T^{\lambda_1}_{abfe}(p_1,k_1)(\gamma_-)_{ef}(k_{1-})^{-\sigma_1}$$
$$\times(\gamma\cdot p_2)_{dc}\int d^4k_2 T^{\lambda_2}_{dc\alpha\beta}(p_2,k_2)4[g_{\alpha\beta}k_{2+}{}^2-k_{2+}(g_{\alpha+}k_{2\beta}+g_{\beta+}k_{2\alpha})+g_{\alpha+}g_{\beta+}k_2{}^2](k_{2+})^{-\sigma_2-1}$$
$$\times\int d^2\underline{q}\frac{Q^2}{\underline{q}^2+Q^2}\int d\omega_1 d\omega_2\,\omega_1{}^{\sigma_2-1}\omega_2{}^{\sigma_2-1}v^\tau(\omega_1,\omega_2,Q^2,\underline{q}^2). \qquad (90)$$

We can write this as

$$F^\tau_{\sigma_1\sigma_2}(Q^2) = v^S_{\sigma_1}v^G_{\sigma_2}\frac{1}{Q^2}E^{\tau,SG}_{\sigma_1\sigma_2}(Q^2), \qquad (91)$$

where

$$v^S_{\sigma_1} = p_{1-}{}^{\sigma_1-1}\tfrac{1}{4}\,\mathrm{tr}\gamma\cdot p_1\Gamma^S_{\sigma_1}(p_1),$$
$$v^G_{\sigma_2} = p_{2+}{}^{\sigma_2-1}\tfrac{1}{4}\,\mathrm{tr}\gamma\cdot p_2\Gamma^G_{\sigma_2}(p_2),$$

and with the vertex Γ^G defined by

$$(\Gamma^G_{\sigma_2})_{cd} = \int d^4k_2 T^{\lambda_2}_{dc\alpha\beta}(p_2,k_2)$$
$$\times 4[g_{\alpha\beta}k_{2+}{}^2-k_{2+}(g_{\alpha+}k_{2\beta}+g_{\beta+}k_{2\alpha})$$
$$+g_{\alpha+}g_{\beta+}k_2{}^2]k_{2+}{}^{-\sigma_2-1}. \qquad (92)$$

As in our previous discussion we now use the t_1^γ operation to renormalize $v^S_{\sigma_1}$ and $v^G_{\sigma_2}$. We then arrive at a formula exactly like (80) with $v^{\overline{S}}_\sigma$ replaced by $v^G_{\sigma_2}$ and $E^{S\overline{S}}_{\sigma_1\sigma_2}$ replaced by $E^{SG}_{\sigma_1\sigma_2}$. We immediately arrive at

$$F_{\sigma_1\sigma_2}(Q^2) = v^S_{\sigma_1}v^G_{\sigma_2}\frac{1}{Q^2}E^{SG}_{\sigma_1\sigma_2}(Q^2)$$
$$+\sum_U\prod_{\gamma\in U}(-t_1)^\gamma F^u_{\sigma_1\sigma_2}(Q^2), \qquad (93)$$

where now, in (93), the renormalization parts, γ, consist of three classes of subgraphs. (i) γ may be any subgraph having superficial degree of divergence greater than or equal to zero. (ii) γ may consist of any amputated graph of the form τ shown in Fig. 9. (iii) γ may be of a type to be specified below.

In (93) the forests U consist of sets of nonoverlapping renormalization parts with the restriction that each forest contain at most one element of class (ii) and, further, that a forest contains elements of class (iii) only if it contains an element of class (ii). We can now define an element γ, of class (iii) in the following manner. Let τ be an element of (ii). Then the full graph is topologically broken up as in Fig. 9. The elements of class (iii) are renormalization parts of λ_1 and λ_2 with λ_1 and λ_2 considered as graphs of cut vertices. The ef lines are multiplied by a $(\gamma_-)_{ef}$ factor, and the $\alpha\beta$ lines by a

$$4[g_{\alpha\beta}k_{2+}{}^2-k_{2+}(g_{\alpha+}k_{2\beta}+g_{\beta+}k_{2\alpha})+g_{\alpha+}g_{\beta+}k_2{}^2]$$

factor. If γ is of class (ii) t_1^γ is as t^τ in (87). If

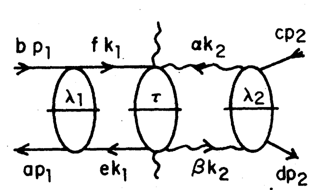

FIG. 9. A decomposition of F involving two fermions and two photons.

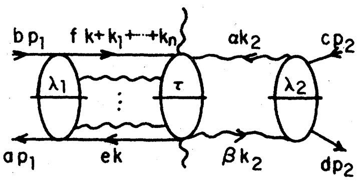

FIG. 10. A decomposition of F involving two photons and two fermions along with many photons.

γ is of class (iii) the t_1^γ is the same as the subtraction operator for cut vertices discussed in Ref. 1.

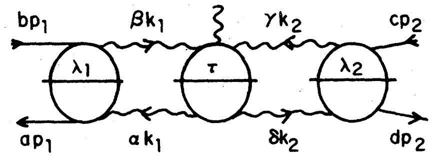

FIG. 11. A photonic decomposition of F.

Decompositions of the type shown in Fig. 10 are simply related to those of Fig. 9 by the Ward identity and so will not be discussed in detail. We must finally consider decompositions of the type shown in Fig. 11. Then

$$F(p_1,p_2,q)=(\gamma\cdot p_1+m)_{ba}(\gamma\cdot p_2-m)_{dc}\int d^4k_1d^4k_2T^{\lambda_1}_{ab\beta\alpha}(p_1,k_1)T^{\lambda_2}_{dc\gamma\delta}(p_2,k_2)V^{\tau}_{\alpha\beta\gamma\delta}(k_1,k_2,q)\,. \tag{94}$$

Now V^τ can be written as (see Appendix B)

$$V^{\tau}_{\alpha\beta\gamma\delta}=\tilde V^{\tau}_{\alpha'\beta'\gamma'\delta'}[g_{\alpha\beta}k_{1\alpha'}k_{1\beta'}+g_{\alpha\alpha'}g_{\beta\beta'}k_1^{\,2}-g_{\alpha\alpha'}k_{1\beta}k_{1\beta'}-g_{\beta\beta'}k_{1\alpha}k_{1\alpha'}]$$
$$\times[g_{\gamma\delta}k_{2\gamma'}k_{2\delta'}+g_{\gamma\gamma'}g_{\delta\delta'}k_2^{\,2}-g_{\gamma\gamma'}k_{2\delta}k_{2\delta'}-g_{\delta\delta'}k_{2\gamma}k_{2\gamma'}]\,. \tag{95}$$

We define t^τ by the rule

$$t^\tau\tilde V^{\tau}_{\alpha\beta\gamma\delta}(k_1,k_2,q)=\tilde V^{\tau}_{++--}(\hat k_1,\hat k_2,q)g_{\alpha-}g_{\beta-}g_{\gamma+}g_{\delta+}=16g_{\alpha-}g_{\beta-}g_{\gamma+}g_{\delta+}v^\tau(\hat k_1,\hat k_2,q)\,. \tag{96}$$

Then

$$t^\tau V^{\tau}_{\alpha\beta\gamma\delta}(k_1,k_2,q)=4[g_{\alpha\beta}k_{1-}^{\,2}-k_{1-}(g_{\alpha-}k_{1\beta}+g_{\beta-}k_{1\alpha})+g_{\alpha-}g_{\beta-}k_1^{\,2}]v^\tau(\hat k_1,\hat k_2,q)$$
$$\times4[g_{\gamma\delta}k_{2+}^{\,2}-k_{2+}(g_{\gamma+}k_{2\delta}+g_{\delta+}k_{2\gamma})+g_{\gamma+}g_{\delta+}k_2^{\,2}]\,. \tag{97}$$

Defining F^τ as usual and taking moments we obtain

$$F^\tau(Q^2)=v^G_{\sigma_1}v^G_{\sigma_2}\frac{1}{Q^2}E^{\tau,GG}_{\sigma_1\sigma_2}(Q^2)\,. \tag{98}$$

where

$$v^G_\sigma=p_-^{\;\sigma-1}\tfrac{1}{4}\,\mathrm{tr}\gamma\cdot p\Gamma^G_\sigma(p)\,,$$

with Γ^G_σ as given in (92) and

$$E^{\tau,GG}_{\sigma_1\sigma_2}=\int 8d^2\underline{q}\frac{(Q^2)^3}{Q^2+\underline{q}^2}d\omega_1d\omega_2\omega_1^{\;\sigma_1-1}\omega_2^{\;\sigma_2-1}$$
$$\times v^\tau(\omega_1,\omega_2,Q^2,\underline{q}^2)\,. \tag{99}$$

Following the procedure discussed several times already, we obtain

$$F_{\sigma_1\sigma_2}(Q^2)=v^G_{\sigma_1}v^G_{\sigma_2}\frac{1}{Q^2}E^{GG}_{\sigma_1\sigma_2}(Q^2)$$
$$+\sum_U\prod_{\gamma\in U}(-t_1)^\gamma F^u_{\sigma_1\sigma_2}(Q^2)\,. \tag{100}$$

Finally, we should add in antiparticle contributions for particle p_1 and particle contributions for particle p_2. When this is done we may write

$$F_{\sigma_1\sigma_2}(Q^2)=\sum_{\alpha,\beta}v^\alpha_{\sigma_1}v^\beta_{\sigma_2}\frac{1}{Q^2}E^{\alpha\beta}_{\sigma_1\sigma_2}(Q^2)$$
$$+\sum_U\prod_{\gamma\in U}(-t_1)^\gamma F^u_{\sigma_1\sigma_2}(Q^2)\,, \tag{101}$$

where α,β equals $G,S,\bar S$. U is any forest of the type previously discussed in this section, and γ is any renormalization part previously discussed in this section. The first term on the right-hand side of (101) is the generalization of the Wilson expansion. We have yet to argue that the second term on the right-hand side of (101) is small compared to the first term by a power of Q^2 when Q^2 is large. We shall give this argument after some examples are discussed.

B. Some examples

In this section we shall exhibit some examples of Eq. (101). That is, we shall show at what rate the second term on the right-hand side of (101), $F^{\mathrm{reg}}_{\sigma_1\sigma_2}(Q^2)$, vanishes as Q^2 becomes large. In particular, we shall see that $F^{\mathrm{reg}}_{\sigma_1\sigma_2}(Q^2)$ is of the order of $(1/Q^2)F_{\sigma_1\sigma_2}(Q^2)$.

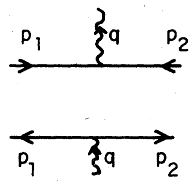

FIG. 12. The lowest-order graph contributing to the Drell-Yan process.

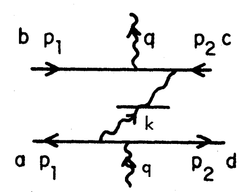

FIG. 13. The second-order radiative correction to the Drell-Yan process.

We begin with the trivial example of the graph shown in Fig. 12. Take p_1 a fermion and p_2 an antifermion. Then

$$F^{(u)}(\omega_1, \omega_2, Q^2, \underline{q}^2) = e^2(2\pi)^4\delta^4(q-p_1-p_2) \times \mathrm{tr}(\gamma\cdot p_1+m)\gamma_\mu(\gamma\cdot p_2-m)\gamma_\mu$$
$$= -128\pi^4e^2\omega_1\omega_2\delta(1-\omega_1) \times\delta(1-\omega_2)\delta^2(\underline{q})+O\left(\frac{1}{Q^2}\right).$$

Thus

$$F^{(u)}_{\sigma_1\sigma_2}(Q^2) = -128\pi^4e^2+O\left(\frac{1}{Q^2}\right).$$

In this case $F_{\sigma_1\sigma_2}(Q^2)$ is given by

$$F^{\mathrm{reg}}_{\sigma_1\sigma_2}(Q^2) = F^{(u)}_{\sigma_1\sigma_2}(Q^2) - t_1F^{(u)}_{\sigma_1\sigma_2}(Q^2).$$

Now the dominant term in $F^{(u)}_{\sigma_1\sigma_2}(Q^2)$ does not depend on external masses so it is clear from (75) that $F^{\mathrm{reg}}_{\sigma_1\sigma_2}$ is down by $1/Q^2$ from $F_{\sigma_1\sigma_2}$.

Now consider the somewhat more complicated example shown in Fig. 13. In this case

$$F^{(u)} = g^2\frac{-2\pi\delta((q-p_1-p_2)^2-m^2)}{[(p_1-q)^2-m^2][(p_2-q)^2-m^2]}\mathrm{tr}\{(\gamma\cdot p_1+m)\gamma_\alpha[\gamma\cdot(q-p_2)+m]\gamma_\mu(\gamma\cdot p_2-m)\gamma_\alpha[\gamma\cdot(p_1-q)+m]\gamma_\mu\}. \tag{102}$$

For convenience in writing we define some new amplitudes. Let

$$\Gamma_1(p_1,k) = \frac{-(2\pi)g\delta(k^2-m^2)\gamma_\alpha[\gamma\cdot(p_1+k)+m]}{(2\pi)^4[(p_1+k)^2-m^2]}, \tag{103a}$$

$$\Gamma_2(p_2,k) = \frac{(2\pi)g\delta(k^2-m^2)\gamma_\alpha[\gamma\cdot(p_2+k)-m]}{(2\pi)^4[(p_2+k)^2-m^2]}, \tag{103b}$$

$$V_2(p_1,p_2,k) = -g\frac{\gamma_\mu\otimes\gamma_\alpha[\gamma\cdot(p_2+k)-m]\gamma_\mu(2\pi)^4\delta(q-p_1-p_2-k)}{(p_2+k)^2-m^2}, \tag{103c}$$

$$V_1(p_1,p_2,k) = g\frac{\gamma_\alpha[\gamma\cdot(p_1+k)+m]\gamma_\mu\otimes\gamma_\mu(2\pi)^4\delta(q-p_1-p_2-k)}{(p_1+k)^2-m^2} \tag{103d}$$

The graphs corresponding to the Γ's and V's are shown in Fig. 14. The ⊗ in the above formulas indicates the Dirac-index structure to be found in Fig. 14. That is for the graph shown in Fig. 13 with Dirac indices a, b, c, d

$$[M_1\otimes M_2]_{abcd} = (M_1)_{ad}(M_2)_{cb}.$$

We can write (102) as

$$F^u = (\gamma\cdot p_1+m)_{bc}(\gamma\cdot p_2-m)_{dc}\int d^4k[\Gamma_1(p_1,k)V_2(p_1,p_2,k)]_{abcd}. \tag{104}$$

The oversubtracted amplitude, $F^{\mathrm{reg}}_{\sigma_1\sigma_2}(Q^2)$, is given by applying the formula (101). We obtain

$$F^{\mathrm{reg}}_{\sigma_1\sigma_2}(Q^2) = \int d\omega_1 d\omega_2 d^2\underline{q}\,\omega_1^{\sigma_1-2}\omega_2^{\sigma_2-2}(\gamma\cdot p_1+m)_{ba}(\gamma\cdot p_2-m)_{dc}$$
$$\times\int d^4k[\Gamma_1(p_1,k)V_2(p_1,p_2,k)-\Gamma_1(p_1,k)V_2(\hat{p}_1,\hat{p}_2,\hat{k})-V_1(\hat{p}_1,\hat{p}_2,\hat{k})\Gamma_2(p_2,k)$$
$$-\Gamma_1(\hat{p}_1,k)V_2(p_1,p_2,k)+\Gamma_1(\hat{p}_1,k)V_2(\hat{p}_1,\hat{p}_2,\hat{k})+V_1(\hat{p}_1,\hat{p}_2,\hat{k})\Gamma_2(\hat{p}_2,k)]_{abcd}.$$

In (105) we have not explicitly done the γ-matrix projections which (75) and (84) require. However, in this simple case the appropriate projections are guaranteed by the $(\gamma \cdot p_1 + m)_{ba}$ and $(\gamma \cdot p_2 - m)_{dc}$ terms. In the above $\hat{p}_{1+} = \underline{\hat{p}}_1 = \hat{p}_{2-} = \underline{\hat{p}}_2 = 0$, $\hat{p}_{1-} = p_{1-}$, $\hat{p}_{2+} = p_{2+}$. When evaluated in V_2, $\hat{k}_+ = \underline{\hat{k}} = 0$, $\hat{k}_- = k_-$. When evaluated in V_1, $\hat{k}_- = \underline{\hat{k}} = 0$, $\hat{k}_+ = k_+$.

Let us consider four separate regions of k integration in (105). (i) $(k+p_1)^2 = O(m^2)$, $(k+p_2)^2 = O(Q^2)$; (ii) $(k+p_2)^2 = O(m^2)$, $(k+p_1)^2 = O(Q^2)$; (iii) $(k+p_1)^2 = O(Q^2)$, $(k+p_2)^2 = O(Q^2)$; and (iv) $(k+p_1)^2(k+p_2)^2$ both of order $(Q^2)^{1/2}$. This does not exhaust all of the k integration volume, of course, but the other regions are easily examined. The reader will note in our massive theory that it is impossible to make both $(p+k_1)^2$ and $(p+k_2)^2$ the order of m^2 simultaneously. Begin with (i). When $(k+p_1)^2$ is finite and $(k+p_2)^2 = O(Q^2)$, $V_2(p_1,p_2,k) \approx V_2(\hat{p}_1,\hat{p}_2,\hat{k})$ and $\Gamma_2(p_2,k) \approx \Gamma_2(\hat{p}_2,k)$. This is true only after an integration over $d^2\underline{q}$ has been performed. Without the $d^2\underline{q}$ integral in (81) we are unable to derive a generalization of the Wilson expansion. Corrections to the above equalities are always manifestly down by a factor of $1/Q^2$. In this case we write (105) as

$$F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2) = \int d\omega_1 d\omega_2 d^2\underline{q}\,\omega_1{}^{\sigma_1-2}\omega_2{}^{\sigma_2-2}(\gamma \cdot p_1 + m)_{ba}(\gamma \cdot p_2 - m)_{dc}$$
$$\times \int d^4k \{\Gamma_1(p_1,k)[V_2(p_1,p_2,k) - V_2(\hat{p}_1,\hat{p}_2,\hat{k})] - \Gamma_1(\hat{p}_1,k)[V_2(p_1,p_2,k) - V_2(\hat{p}_1,\hat{p}_2,\hat{k})]$$
$$- V_1(\hat{p}_1,\hat{p}_2,\hat{k})[\Gamma_2(p_2,k) - \Gamma_2(\hat{p}_2,k)]\}_{abcd}\,. \tag{106}$$

Thus, there is a cancellation of the dominant terms in region (i). Region (ii) is completely equivalent to region (i) by the symmetry of the graph. In region (iii) we still have $V_2(p_1,p_2,k) \approx V_2(\hat{p}_1,\hat{p}_2,\hat{k})$ and $\Gamma_2(p_1,k_2) \approx \Gamma_2(\hat{p}_2,k)$ along with a similar relation for $V_2 \to V_1$ and $\Gamma_2 \to \Gamma_1$. Thus in this region the dominant contribution also cancels in evaluating $F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2)$ for large Q^2. Region (iv) is completely analogous to region (iii).

Finally, we shall discuss the graph shown in Fig. 15. As in the previous example we define some new amplitudes. Let

$$\Gamma_1(p_1,k) = \frac{g\gamma_\alpha[\gamma \cdot (p_1+k) + m]}{(2\pi)^4[(p_1+k)^2 - m^2]}\,\frac{-i}{k^2 - m^2}\,, \tag{107a}$$

$$\Gamma_2(p_2,k) = \frac{g\gamma_\alpha[\gamma \cdot (p_2-k) - m]}{(2\pi)^4[(p_2-k)^2 - m^2]}\,\frac{i}{k^2 - m^2}\,, \tag{107b}$$

$$V_2(p_1,p_2,k) = -\frac{g\gamma_\mu[\gamma \cdot (p_2-k) - m]\gamma_\alpha \otimes \gamma_\mu}{(p_2-k)^2 - m^2}(2\pi)^4\delta^4(q - p_1 - p_2)\,, \tag{107c}$$

$$V_1(p_1,p_2,k) = \frac{g\gamma_\alpha[\gamma \cdot (p_1+k) + m]\gamma_\mu \otimes \gamma_\mu}{(p_1+k)^2 - m^2}(2\pi)^4\delta^4(q - p_1 - p_2)\,. \tag{107d}$$

There amplitudes are illustrated in Fig. 16. The graph in Fig. 15 can then be written as

$$F^u = (\gamma \cdot p_1 + m)_{ba}(\gamma \cdot p_2 - m)_{dc}\int d^4k[\Gamma_1(p_1,k)V_2(p_1,p_2k)]_{abcd}\,. \tag{108}$$

The oversubtracted amplitude, $F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2)$, is given by applying the formula (101). We obtain an equation identical to (105) in structure. Let us consider three regions of k integration. (i) $(p_1+k)^2 = O(Q^2)$, $(p_2+k)^2$ and k^2 arbitrary; (ii) $(p_2-k)^2 = O(Q^2)$, $(p_1+k)^2$ and k^2 arbitrary; and (iii) $(p_1+k)^2$, $(p_2+k)^2$, k^2 all on the order of m^2.

For region (i) we write

$$F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2) = \int d\omega_1 d\omega_2 d^2\underline{q}\,\omega_1{}^{\sigma_1-2}\omega_2{}^{\sigma_2-2}(\gamma \cdot p_1 + m)_{ba}(\gamma \cdot p_2 - m)_{dc}$$
$$\times \int d^4k\{[V_1(p_1,p_2,k) - V_1(\hat{p}_1,\hat{p}_2,\hat{k})]\Gamma_2(p_2,k) + [V_1(\hat{p}_1,\hat{p}_2,k) - V_1(\hat{p}_1,\hat{p}_2,\hat{k})]\Gamma_2(\hat{p}_2,k)$$
$$- [\Gamma_1(p_1,k) - \Gamma_1(\hat{p}_1,k)]V_2(\hat{p}_1,\hat{p}_2,\hat{k})\}_{abcd} \tag{109}$$

for $k^2 < O((Q^2)^{1/2})$, while we write

$$F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2) = \int d\omega_1 d\omega_2 d^2\underline{q}\,\omega_1^{\sigma_1-2}\omega_2^{\sigma_2-2}(\gamma\cdot p_1+m)_{ba}(\gamma\cdot p_2-m)_{dc}$$
$$\times\int d^4k\{[\Gamma_1(p_1,k)V_2(p_1,p_2,k)-\Gamma_1(\hat{p}_1,k)V_2(\hat{p}_1,\hat{p}_2,k)]$$
$$-[\Gamma_1(p_1,k)-\Gamma_1(\hat{p}_1,k)]V_2(\hat{p}_1,\hat{p}_2,k)-V_1(\hat{p}_1,\hat{p}_2,\hat{k})[\Gamma_2(p_2,k)-\Gamma_2(\hat{p}_2,k)]\}_{abcd} \qquad (110)$$

for $k^2>O((Q^2)^{1/2})$. In each case the terms in the square brackets cancel out, and the result is at least of order $1/(Q^2)^{1/2}$ down from the unsubtracted amplitude. Region (ii) works exactly as region (i).

Region (iii) is the region where $k_+=O(1/Q)$, $k_-=O(1/Q)$ with $\underline{k}^2=O(m^2)$. We handle this region simply by observing that the k contour can always be distorted out of the entire region. For example, the term of the form

$$-\int d^4k\,\Gamma_1(p_1,k)V_2(\hat{p}_1,\hat{p}_2,\hat{k})$$

can be written as

$$g^2\int\frac{dk_+dk_-d^2\underline{k}\,\mathfrak{N}}{[2k_+k_- -\underline{k}^2-m^2+i\epsilon][2k_+(p_{1-}+k_-)-\underline{k}^2+p_1^2-m^2+i\epsilon][-2k_-(p_{2+}+k_+)-\underline{k}^2-m^2+i\epsilon]},$$

where $\mathfrak{N}$ is a polynomial in p_1,p_2,k. The poles in the $1/(k^2+i\epsilon)$ term are not located near the origin in the k_+,k_- plane for k_+,k_- small. Thus we may distort k_+ into the upper half-plane and k_- into the lower half-plane and move completely out of region (iii). Again we have found $F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2)$ down by at least a power of $1/Q$ from $F_{\sigma_1\sigma_2}(Q^2)$.

Finally, let us comment on these three examples, the graphs of Figs. 12, 13, and 15, respectively. For the graphs of Figs. 13 and 15 the region where $(p_1+k)^2=O((Q^2)^{1/2})$, $(p_2+k)^2=O((Q^2)^{1/2})$, $k^2=O(m^2)$ is the wee-parton region[18] for gluon exchange. We note that the oversubtraction scheme works graph by graph, though for purposes of gauge invariance it is useful to consider Figs. 13 and 15 together. For the generalized Wilson expansion, the first term on the right-hand side of Eq. (101), it is necessary to keep the graphs corresponding to Figs. 13 and 15 together, otherwise the cut vertices do not appear in gauge-covariant combinations. For the graph of Fig. 15 the region where $(p_1+k)^2$, $(p_2+k)^2$, and k^2 are all of order m^2 is a softer region than the wee-parton region, and it does not contribute at all, graph by graph. The wee-parton region also does not contribute a net result when all graphs are summed to a given order. This will be discussed further in the next part of this section.

C. The correctness of the expansion

In this section we shall expand on and try to elucidate a number of the arguments and rules given in parts A and B of this section. We shall also give a heuristic argument as to why the second term on the right-hand side of Eq. (101) is small by a power of $1/Q$ compared to the first term. (In order to present this argument in its most precise form, both for single-particle inclusive production in e^+e^- collisions and in the various multiparticle generalizations, it is necessary to extend the idea of a complete forest given by Zimmermann.[26] This has been done but the complexity of the technical details make it more appropriate for a separate paper.)

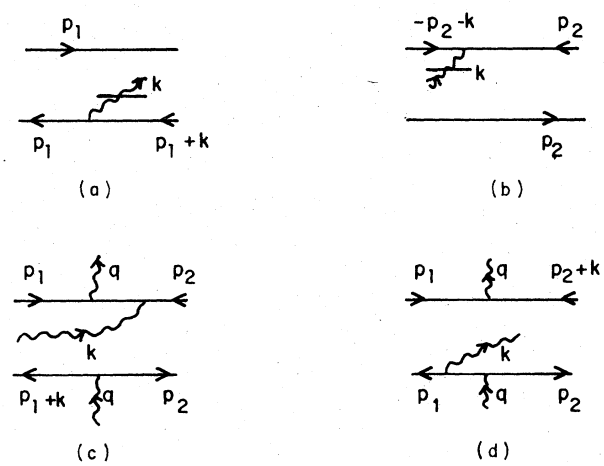

FIG. 14. (a) $\Gamma_1(p_1,k)$, (b) $\Gamma_2(p_2,k)$, (c) $V_2(p_1,p_2,k)$, (d) $V_1(p_1,p_2,k)$.

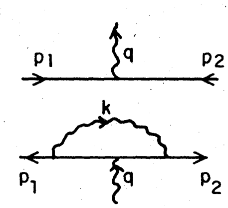

FIG. 15. A second-order radiative correction to the Drell-Yan process.

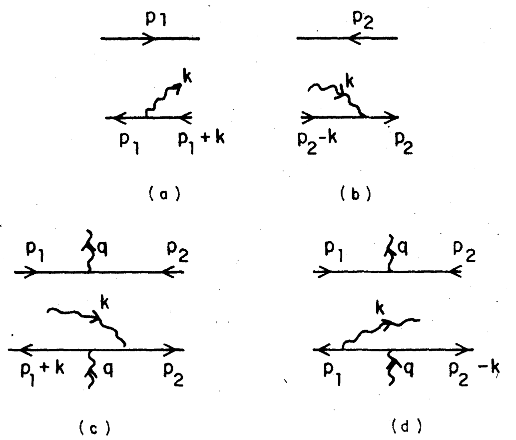

FIG. 16. (a) $\Gamma_1(p_1,k)$, (b) $\Gamma_2(p_2,k)$, (c) $V_2(p_1,p_2,k)$, (d) $V_1(p_1,p_2,k)$.

1. The problem of the wee partons

The question of the generality of the Drell-Yan formula, even in a cutoff or soft field theory version of the parton models, has been a topic of much discussion in the past.[18,29,30,31] The process $e^+ + e^- \to h(p_1)+h(p_2)+X$ is in many respects similar to μ-pair production in hadron-hadron collisions though the interpretation of the cancellation of wee gluons is more direct in the $e^+ - e^-$ case. Firstly, we shall call gluons having $k_+ \cdot k_- \ll m^2$ soft gluons rather than wee gluons. As we have already seen in the example of part B of this section, this region, though potentially dangerous, does not contribute graph by graph in perturbation theory. The contour distortion argument given in the last section is almost trivially generalized to an arbitrary graph as it only makes use of the $i\epsilon$ prescription. (It should be noted that such soft photons can never cross a cut in a graph and so all relevant propagators have either a $+i\epsilon$ or all relevant propagators have a $-i\epsilon$.)

Now, as we have seen in part B of this section and as we shall see further in Sec. III C 2, Eq. (101), does not require any special cancellation among wee partons between different graphs. However, the singular function $E^{\alpha\beta}_{\sigma_1\sigma_2}(Q^2)$ does not obey a Callan-Symanzik equation until such a cancellation occurs. This peculiar situation comes about because the renormalized cut vertex shown in Figs. 17(a) and 17(b) is finite for each graph separately. However, the unrenormalized cut vertex has cancelling dk_-/k_- integrals between the graph in Fig. 17(a) and the one in Fig. 17(b). [See Eq. (30) and (31) of Ref. 1.] This means that the soft-mass insertion does not lower the dimension appropriately unless the two graphs are considered together. The cancellation of the k_- divergence between the graphs 17(a) and 17(b) is in fact an example of the cancellation of wee gluons between different graphs.

Let us examine the above cancellation a little more closely. The graphs corresponding to Figs. 17(a) and 17(b) have the expression,

$$\Gamma^u_\sigma(p) = -\frac{g^2}{(2\pi)^4}\int d^4k \frac{\gamma_-[\gamma\cdot(p+k)+m]\gamma_-}{(p+k)^2-m^2+i\epsilon}\left[-2\pi\delta(k^2-m^2)\frac{(p+k)_-{}^{-\sigma}}{k_-}+\frac{i}{k^2-m^2+i\epsilon}\frac{p_-{}^{-\sigma}}{k_-}\right].$$

It is straightforward to see that $\Gamma^u_\sigma(p)-\Gamma^u_\sigma(\hat{p})$ has no difficulties in the integration region around $k_-=0$. The same is true for a soft-mass insertion on the fermion line. To do a soft-mass insertion on the photon line it is convenient to call the photon mass μ and to consider $\partial/\partial_{\mu^2}\Gamma^u_\sigma(p)$, where now we would like not to have to perform any subtractions. The graph with a photon mass insertion is, however, ambiguous at present. We can make it precisely by adopting the following definition for the unrenormalized integral:

$$\Gamma^u_\sigma(p) = -\frac{g^4}{(2\pi)^4}\int d^4k \frac{\gamma_-[\gamma\cdot(p+k)+m]\gamma_-}{(p+k)^2-m^2+i\epsilon}\,\frac{i}{k^2-\mu^2+i\epsilon}\,\frac{p_-{}^{-\sigma}}{k_- - i\epsilon}$$

$$-\frac{g^2}{(2\pi)^4}\int_C d^4k \frac{\gamma_-[\gamma\cdot(p+k)+m]\gamma_-}{(p+k)^2-m^2}\,\frac{i}{k^2-\mu^2}\,\frac{(p+k)_-{}^{-\sigma}}{k_-}\Theta(k_+),$$

where C indicates a small counterclockwise contour in the k_- plane enclosing only the $k_- = (\underline{k}^2+\mu^2)/2k_+$ pole. The above definition does not change the renormalized vertex functions and gives results for mass-inserted graphs equivalent to dimensional regularization in transverse variables. We may write the above as

$$\Gamma^u_\sigma(p) = -\frac{g^2}{(2\pi)^4}\int_{C'} d^4k \frac{\gamma_-[\gamma\cdot(p+k)+m]\gamma_-}{(p+k)^2-m^2+i\epsilon}\,\frac{i}{k^2-\mu^2}\,\frac{p_-{}^{-\sigma}}{k_-}$$

$$-\frac{g^2}{(2\pi)^4}\int_C \frac{d^4k\,\gamma_-[\gamma\cdot(p+k)+m]\gamma_-}{[(p+k)^2-m^2]}\,\frac{i}{k^2-\mu^2}\,\frac{1}{k_-}[(p+k)_-{}^{-\sigma}-p_-{}^{-\sigma}]\Theta(k_+),$$

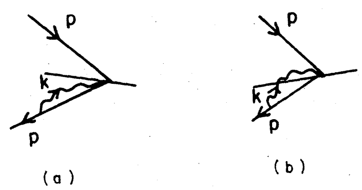

FIG. 17. Particular cuts of a cut vertex.

where the contour C' is on the real axis except for small detours *below* the singularities in k_- at $k_-=0$ and at $k_-=(\underline{k}^2+\mu^2)/2k_+$. Now mass insertions in either fermion lines or photon lines render Γ^u_σ, convergent while the renormalized vertex function is unchanged. The contour C' makes the cancellation of wee partons more apparent. The above argument is easily extended to higher-order graphs.

Let us now consider the question of wee partons in the process $e^+ + e^- \to h(p_1)+h(p_2)+X$. Consider the two terms shown in Figs. 18(a) and 18(b). We suppose $k_-/p_{1-} \ll 1$ and $k_+/p_{2+} \ll 1$. The minus in the figures means that one end of the photon ends on a γ_- in a region of the graph where the minus component of the momentum is on the order of p_{1-}. At the other end of the photon line there is a γ_+ ending on a line having plus component of the momentum on the order of p_{2+}. We suppose that the coordinate system is $p_1=(E_1,0,0,-p_1)$, $p_2=(E_2,0,0,p_2)$, and $q=((Q^2+\underline{q}^2)^{1/2},\underline{q},0)$. We further suppose that an integral over $d^2\underline{q}$ has been done. (If $\underline{q}^2$ is of order Q^2, this integral is not necessary.) ω_1 and ω_2 can either be fixed away from 1 or moments can be taken. Call the contribution due to Fig. 18(a)

$$F_a(Q^2,\omega_1,\omega_2) = -\int \frac{d^4k}{k_- - i\epsilon}\,\frac{-i}{k^2-\mu^2+i\epsilon}\,\mathfrak{M}_a(p_1,p_2,q,k)\,d^2\underline{q}\,, \tag{111}$$

and the contribution from the term in Fig. 18(b)

$$F_b(Q^2,\omega_1,\omega_2) = \int \frac{d^4k}{k_-}\, - 2\pi\delta(k^2-\mu^2)\,\mathfrak{M}_b(p_1,p_2,q,k)\,d^2\underline{q}\,. \tag{112}$$

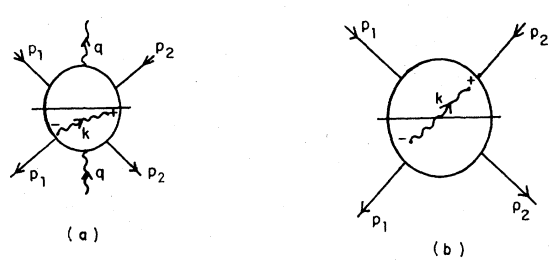

FIG. 18. (a) An uncut wee-photon exchange. (b) A cut wee-photon exchange.

Then, so long as ω_1, ω_2 are not near 1

$$\int d^2\underline{q}\,\mathfrak{M}_a(p_1,p_2,q,k) = \int d^2\underline{q}\,\mathfrak{M}_b(p_1,p_2,q,k) \tag{113}$$

Let us outline how (113) can be seen. Consider the terms shown in Fig. 19. Using the Ward identity the terms shown in Fig. 19 can be written as shown in Fig. 20. Now the terms in Figs. 20(a) and 20(b) do not immediately give the result (113) because the momentum labellings are not the same on the graphs. Consider the term shown in Fig. 20(b). We may change $\underline{q}$ into $\underline{q}+\underline{k}$ since there is a $d^2\underline{q}$ in (113). Then the transverse-momentum labellings are the same in both 20(a) and 20(b). We may let $l_- \to l_- - k_-$ in the right-hand part of the graph in Fig. 20(b) without changing the left-hand part of the graph because p_- is large and ω_1 is not near 1. Now the momentum labellings are the same in Figs. 20(a) and 20(b) except for the + component differing by a k_+ on the right-hand side of the graph. However, so long as ω_2 is not near 1, the k_+ momentum is negligible in the right-hand part of the graph and can be dropped. Thus for the decomposition shown in Fig. 19 we obtain (113). (113) can similarly be obtained for the other classes of graphs. We can now combine (111) and (112) into a single term with the k_- contour C' as described in our previous discussion of the cut vertex.

For our purposes the precise meaning of the cancellation of wee partons is that soft-mass insertions lower the dimensions of graphs appropriately. The dangerous situation is a mass insertion in a photon line connecting opposite-moving fermion lines which are not too far off mass shell. The contour distortion just exhibited is what allows this photon to act much as if it were in the Euclidean region.

2. *The smallness of* $F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2)$

We shall now present a heuristic argument as to why $F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2)$ is small compared to $F_{\sigma_1\sigma_2}(Q^2)$. For a really proper treatment of this problem one should choose a particular momentum flow, define a set of complete forests with respect to that

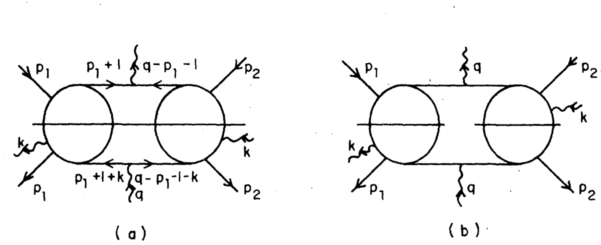

FIG. 19. (a) A particular decomposition of Fig. 18(a). (b) A particular decomposition of Fig. 18(b).

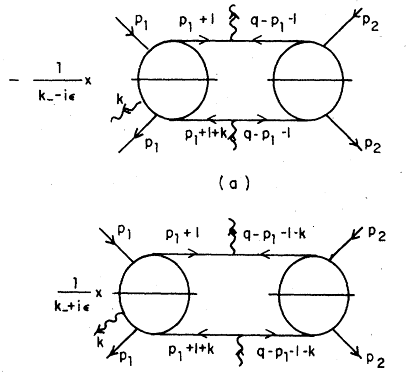

FIG. 20. (a) An evaluation of Fig. 19(a) from the Ward identity. (b) An evaluation of Fig. 19(b) from the Ward identity.

flow, and show that the subtractions take away the dominant term. However, it is possible to understand directly from (101) that the subtractions do indeed remove the dominant power.

We begin by considering the particular breakup, of a graph contributing to $F_{\sigma_1\sigma_2}(Q^2)$, shown in Fig. 7. We now interpret this graph as a particular momentum flow in which the large momentum q flows through τ, but not through λ_1 and λ_2. This means that k_1 is parallel to p_1, and k_2 is parallel to p_2. If the large momentum flows through τ then τ depends on $k_1 \cdot q$, $k_2 \cdot q$, and $k_1 \cdot k_2$, but not on $k_1{}^2$ and $k_2{}^2$. Thus the operator t^τ does not alter the values of the invariant amplitudes when the large momentum flows through τ. The subtraction indicated in Eq. (75) further picks out only the $\gamma_- \otimes \gamma_+$ part of the tensor structure of V^τ. However, this is the dominant tensor structure after the traces are taken with $\gamma \cdot p_1$ and $\gamma \cdot p_2$. In Eq. (101) there occurs in $F^{\text{reg}}_{\sigma_1\sigma_2}(Q^2)$ a term of the form

$$-8Q^2 \int d^2q\, d\omega_1 d\omega_2 \frac{Q^2}{Q^2 + \underline{q}^2} \times v^\tau(\omega_1, \omega_2, Q^2, \underline{q}^2)\omega_1{}^{\sigma_1 - 1}\omega_2{}^{\sigma_2 - 1}$$

multiplied by renormalized cut vertices, where v^τ is given by (75) as a part of $t^\tau V^\tau$. It is this term which exactly cancels the term where the momentum flows through τ but not through λ_1 and λ_2.

The breakups shown in Figs. 9–11 correspond to the momentum q flowing through the τ portions of these graphs but not through the λ_1 and λ_2 parts. The subtractions indicated in the $F^{\text{reg}}_{\sigma_1\sigma_2}$ part of Eq. (101) subtract these momentum flows off also. Thus no matter what route the large momentum q chooses to flow, the subtraction in (101) deletes that particular route.

The above description is, of course, the interpretation one usually has in a BPHZ subtraction scheme. The addition of the ideas of complete forests and a power-counting theorem make the above heuristic arguments rigorous. Complete forests can be defined for the processes of this paper, but a rigorous power-counting theorem has not yet been developed.

IV. RELATION OF CUT VERTICES TO THE PARTON MODEL

In this section we shall discuss the relationship between cut vertices and the parton model. We shall do this only for deeply inelastic electron scattering. The discussion would go pretty much the same for all spacelike vertices. The interpretation of timelike cut vertices in terms of parton decay distributions will not be discussed, since even in the naive parton model the physical meaning of a parton decay distribution is not so clear and appealing as for parton distributions inside a hadron.

A parton-model description of deeply inelastic electron scattering gives, for example,

$$\nu W_2(x, Q^2) = F_2(x) = x\{\tfrac{4}{9}[u(x) + \bar{u}(x)] + \tfrac{1}{9}[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)]\},$$

where $x = Q^2/2m\nu$ and Q^2 is assumed large. $u(x)$ is the number density for *up* quarks having z component of the momentum equal to xp, where the momentum of the proton is $p^\mu = (E_p, 0, 0, p)$. In the context of the naive parton model this interpretation is correct so long as $xp \gg 1$.

In terms of cut vertices

$$\int_0^1 \nu W_2 x^\sigma dx = \sum_{i=1}^{4} C^{(i)}_{\sigma+2} E^{(i)}_{\sigma+2}(Q^2),$$

where the $C^{(i)}_\sigma$ are given by Eqs. (3)–(6). In the naive parton model the $i = 1$ term, the gluon contribution, does not exist and the $E^{(i)}$ are independent of Q^2 since scaling is exact. In this model the values of the $E^{(i)}$ are

$$E^{(2)}_\sigma = \tfrac{2}{9},\quad E^{(3)}_\sigma = \tfrac{1}{3},\quad \text{and } E^{(4)}_\sigma = 1/(3\sqrt{3}).$$

Thus

$$\int_0^1 \nu W_2 x^\sigma dx = \sum_a v^a_{\sigma+2}\left[\frac{2}{9} + \frac{1}{3}\left(\frac{\lambda_3}{2}\right)_{aa} + \frac{1}{3\sqrt{3}}\left(\frac{\lambda_8}{2}\right)_{aa}\right] = \tfrac{4}{9}(v^u_{\sigma+2} + v^{\bar{u}}_{\sigma+2}) + \tfrac{1}{9}(v^d_{\sigma+2} + v^{\bar{d}}_{\sigma+2} + v^s_{\sigma+2} + v^{\bar{s}}_{\sigma+2}).$$

This means that $u(x) = v^u(x)$ etc., where

$$v^a(x) = \frac{1}{2\pi}\int d\sigma x^{-\sigma} v^a_\sigma .$$

The cut vertices are then moments of the quark number densities. In a theory with no ultraviolet divergences other than in the self-mass, the number density is also given in terms of the Fock-space wave function of the proton, $|\psi_p\rangle$, by

$$v(x) = \int d^3\vec{k}\,\delta(x - k_3/p_3)(\psi_{\vec{p}}|a^\dagger_{\vec{k}} a_{\vec{k}}|\psi_{\vec{p}}),$$

where $\vec{p}$ is along the z direction and xp_z is large. Flavor, color, and spin indices have been suppressed in the above formula.

The naive parton model cannot be a realistic description since all $\ln q^2$ terms have been neglected. Let us go back and write our previous equation as

$$\int_0^1 \nu W_2 x^\sigma dx = \sum_{\alpha=1}^{7} v^\alpha_{\sigma+2} E^\alpha_{\sigma+2}(Q^2),$$

where the notation is such that $\alpha = 1, 2, 3, 4, 5, 6, 7$ means gluons, $u, d, s, \bar{u}, \bar{d}, \bar{s}$, respectively. The E^α_σ obey

$$\left(-Q^2\frac{\partial}{\partial Q^2} + \beta\frac{\partial}{\partial g}\right)E^\alpha_\sigma(Q^2) = \sum_{\alpha'}\gamma^{\alpha\alpha'}_\sigma E^{\alpha'}_\sigma(Q^2),$$

which can be solved as

$$E^\alpha_\sigma(Q^2) = R^\sigma_{\alpha\alpha'}(Q^2, Q_0^2)E^{\alpha'}_\sigma(Q_0^2, g(Q^2, Q_0^2)),$$

where

$$R^\sigma(Q^2, Q_0^2) = O\exp\left(-\int_{Q_0^2}^{Q^2}\frac{dQ'^2}{Q'^2}\gamma_\sigma[g(Q'^2, Q_0^2)]\right).$$

In the above O is a Q^2-ordered integral and $g(Q^2, Q_0^2)$ is the usual running coupling constant normalized so that $g(Q_0^2, Q_0^2) = g$. Then

$$\int_0^1 \nu W_2 x^\sigma dx = \sum_{\alpha\alpha'} v^\alpha_{\sigma+2} R^{\sigma+2}_{\alpha\alpha'}(Q^2, Q_0^2) \times E^{\alpha'}_{\sigma+2}(Q_0^2, g(Q^2, Q_0^2)).$$

Often in the literature

$$\frac{1}{2\pi i}\int x^{-\sigma}d\sigma \sum_{\alpha'} v^{\alpha'}_\sigma R_{\alpha'\alpha}(Q^2, Q_0^2) = v^\alpha(x, Q^2)$$

is taken to be a Q^2-dependent parton number density. We shall continue to discuss $v^\alpha(x) = (1/2\pi i)\int x^{-\sigma}d\sigma v^\alpha_\sigma$, though our comments would apply equally to $v^\alpha(x, Q^2)$.

The question is whether, after renormalization, it still makes sense to view $v^\alpha(x)$ as a number density. It is true that the electric charge, the baryon number, and momentum sum rules still can be maintained in the simple form

$$1 = \sum_a Q_a \int_0^1 \frac{dx}{x} v^a(x),$$

$$1 = \sum_a Q^B_a \int_0^1 \frac{dx}{x} v^a(x),$$

$$1 = \sum_\alpha \int_0^1 dx\, v^\alpha(x).$$

Here $Q_u = \frac{2}{3}$, $Q_d = -\frac{1}{3}$, $\cdots$, and $Q^B_u = Q^B_d = \frac{1}{3}$, $\cdots$. However, we can see no reason, for example, for insisting that $v^s(x)$ is a number density for strange quarks inside a hadron. $v^s(x)$ is a renormalization-dependent quantity, and we cannot even see any reason why it must be positive. Thus in QCD the parton number density distribution may very well not have all the attributes one usually assumes for a number density.

ACKNOWLEDGMENT

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APPENDIX A

In this appendix we shall write the solution of the Callan-Symanzik equation in a convenient SU(3) basis for comparison with experimentally measured quantities. If $d\sigma/d^3p$ is the cross section for $e^+ + e^- \to \text{hadron}(p) + \text{anything}$, define

$$(2\pi)^3 2E\frac{d\bar{\sigma}}{d^3p} = \int\frac{d\Omega(\vec{k})}{4\pi}(2\pi)^3 2E\frac{d\sigma}{d^3p}, \tag{A1}$$

where the $d\Omega(\vec{k})/4\pi$ integral averages the cross section over electron angles in the center-of-mass system of the e^+e^-.

Write

$$(2\pi)^3 2E\frac{d\bar{\sigma}}{d^3p} = -\frac{8\pi^2\alpha^2}{3(Q^2)^2}F, \tag{A2}$$

where

$$F = \int d^4x d^4y d^4z\, e^{iqy + ip(z-x)} \times \sum_\mu \langle \bar{T}(\phi(x)j_\mu(y))T(\phi(z)j_\mu(0))\rangle$$

and the external propagators have been amputated. Then

$$\int (2\pi)^3 2E\frac{d\bar{\sigma}}{d^3p}\omega^{\sigma-2}d\omega = \frac{8\pi^2\alpha^2}{3(Q^2)^2}\sum_\alpha v^\alpha_\sigma E^\alpha_\sigma, \tag{A3}$$

where the α sum goes over $g, u, d, s, \bar{u}, \bar{d}, \bar{s}$. It is conveneint to change to an SU(3) basis. Define

$$v_\sigma^S = \sum_{a=u,d,s} v_\sigma^a(\tfrac{1}{2}\lambda_0)_{aa}\,,$$

$$v_\sigma^{\bar S} = \sum_{a=\bar u,\bar d,\bar s} v_\sigma^a(\tfrac{1}{2}\lambda_0)_{aa}\,,$$

$$v_\sigma^{O_3} = \sum_{u\cdots} v_\sigma^a(\tfrac{1}{2}\lambda_3)_{aa}\,,$$

$$v_\sigma^{O_8} = \sum_{a=u\cdots} v_\sigma^a(\tfrac{1}{2}\lambda_8)_{aa}\,, \tag{A4}$$

$$v_\sigma^{\bar O_3} = \sum_{a=\bar u\cdots} v_\sigma^a(\tfrac{1}{2}\lambda_3)_{aa}\,,$$

$$v^{\bar O_8} = \sum_{a=\bar u\cdots} v_\sigma^a(\tfrac{1}{2}\lambda_8)_{aa}\,.$$

The relationship between the bases is

$$v^\alpha = C^{\alpha i} v^i\,, \tag{A5}$$

$$v^i = \tfrac{1}{2} v^\alpha C^{\alpha i}\,, \tag{A6}$$

for $i = g, S, O_3, O_8, \bar S, \bar O_3, \bar O_8$. $C^T C = 2$ and

$$C^{\alpha i} = \begin{array}{c} \\ g \\ u \\ d \\ s \\ \bar u \\ \bar d \\ \bar s \end{array}\begin{array}{c} \begin{array}{ccccccc} g & S & O_3 & O_8 & \bar S & \bar O_3 & \bar O_8 \end{array} \\ \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 1 & 1/\sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & -1 & -1/\sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & -2/\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2/3} & 1 & 1/\sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{2/3} & -1 & 1/\sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{2/3} & 0 & -2/\sqrt{3} \end{pmatrix} \end{array}.$$

The pecular normalization of C is due to the requirement that $\mathrm{tr}\lambda_i\lambda_j = 2\delta_{ij}$.

(A3) can be written as

$$\int (2\pi)^3 2E\frac{d\bar\sigma}{d^3p}\,\omega^{\sigma-2}d\omega = \frac{8\pi^2\alpha^2}{3(Q^2)^2}\sum_i v_\sigma^i E_\sigma^i\,, \tag{A7}$$

where $E_\sigma^i = \frac{1}{2}E_\sigma^\alpha C^{\alpha i}$. The Callan-Symanzik equation is

$$DE_\sigma^\alpha = \gamma_\sigma^{\alpha\alpha'} E_\sigma^{\alpha'}$$

or, in the new basis,

$$DE_\sigma^i = \gamma_\sigma^{ij} E_\sigma^j\,, \tag{A8}$$

where

$$\gamma_\sigma^{ij} = \gamma_\sigma^{\alpha\alpha'} C^{\alpha i} C^{\alpha' j}/2\,.$$

Then

$$E_\sigma^i(Q^2,g) = \left[O\exp\left(-\int_{Q_0^2}^{Q^2}\frac{dx}{x}\gamma[g(x,Q_0^2)]\right)\right]_{ij} \times E_\sigma^j(Q_0^2, g(Q^2,Q_0^2))\,, \tag{A9}$$

where $g(Q_0^2, Q_0^2) = g$ and the O denotes an ordered Q^2 integral.[32] We shall also use the notation

$$E_\sigma^i(Q^2,g) = R_{ij}^\sigma(Q_0^2,Q^2)E_\sigma^j(Q_0^2,g(Q^2,Q_0^2))\,.$$

R has the property that $R^\sigma(Q_0^2,Q^2) = R^\sigma(Q_0^2,Q_1^2)R^\sigma(Q_1^2,Q^2)$. If $g(Q_1^2,Q^2)$ is small, one can use the lowest-order calculation of γ for evaluating $R(Q_1^2,Q^2)$. Defining

$$\tilde v_\sigma^i = v_\sigma^j R_{ji}^\sigma(Q_0^2,Q_1^2)\,, \tag{A10}$$

one has

$$\int (2\pi)^3 2E\frac{d\bar\sigma}{d^3p}\,\omega^{\sigma-2}d\omega = \frac{8\pi^2\alpha^2}{3(Q^2)^2}\sum_{j,i}\tilde v_\sigma^i R_{ji}^\sigma(Q_1^2,Q^2)E_\sigma^i(Q_0^2,g(Q^2,Q_0^2))\,. \tag{A11}$$

In the nonsinglet sector

$$R_{ij}^\sigma(Q_1^2,Q^2) = \delta_{ij}[\ln(Q^2/Q_1^2)]^{-A_\sigma}\,,$$

where

$$A_\sigma = \frac{3C_2(R)}{22C_2(G)-8T(R)}\times\left(1-\frac{2}{(\sigma-1)(\sigma-2)} + 4\sum_{l=1}^{\infty}\frac{\sigma-2}{(l+1)(l+\sigma-1)} - \frac{4}{\sigma-1}\right)\,. \tag{A12}$$

For differences of cross sections it is possible to eliminate the singlet pieces. For example,

$$\int (2\pi)^3 2E\left(\frac{d\bar\sigma^{\pi^+}}{d^3p} - \frac{d\sigma^{\pi^0}}{d^3p}\right)\omega^{\sigma-2}d\omega = \frac{32\pi^3\alpha^2}{9(Q^2)^2}(\tilde v_\sigma^{O_3} + \tilde v_\sigma^{\bar O_3})[\ln(Q^2/Q_1^2)]^{-A_\sigma}\,, \tag{A13}$$

where $\tilde v_\sigma^{O_3}$ refers to π^+ matrix elements and we have used

$$E_\sigma^{O_3}(Q_0^2, g(Q^2,Q_0^2)) = \tfrac{4}{3}\pi$$

from lowest-order perturbation theory with the v_σ^a normalized so that they are equal to the number of colors for elementary external particles.

An alternative form to (A10) is to define

$$v^i(\omega) = \frac{1}{2\pi i}\int_{L-i\infty}^{L+i\infty}\omega^{-\sigma}v_\sigma^i d\sigma \tag{A14}$$

and

$$v^i(\omega,Q^2) = \frac{1}{2\pi i}\int_{L-i\infty}^{L+i\infty}\omega^{-\sigma}\sum_j v_\sigma^j R_{ji}^\sigma(Q_0^2,Q^2)\,.$$

$$= \frac{1}{2\pi i}\int_{L-i\infty}^{L+i\infty}\omega^{-\sigma}v_\sigma^i(Q^2)\,. \tag{A15}$$

Then

$$(2\pi)^3 2E\frac{d\bar{\sigma}}{d^3p} = \frac{8\pi^2}{3(Q^2)^2}\sum_i \omega v^i(\omega, Q^2)E^i\,, \qquad \text{(A16)}$$

where $E^G = 0$, $E^S = E^{\bar{S}} = \frac{8}{3}\pi(\frac{2}{3})^{1/2}$, $E^{O_3} = E^{\bar{O}_3} = \frac{4}{3}\pi$, and $E^{O_8} = E^{\bar{O}_8} = 4\pi/(3\sqrt{3})$. The form (A16) is closest to the parton model differing only in that v^i has a Q^2 dependence. This form depends crucially on the property of asymptotic freedom while the form (A10) does not.

APPENDIX B

First we prove a simple theorem.

Theorem: Any rank-two tensor $T_{\alpha\beta}$ which is made up of $x^i = (p^1, p^2, \ldots, p^n q)$ and satisfies

(i) $q_\alpha T_{\alpha\beta} = 0$ and $q_\beta T_{\alpha\beta} = 0$,

(ii) $T_{\alpha\beta} = T_{\beta\alpha}$,

can be written as

$$T_{\alpha\beta} = [A_{\alpha\beta}(q)]_{\alpha'\beta'} T'_{\alpha'\beta'}\,,$$

where

$$[A_{\alpha\beta}(q)]_{\alpha'\beta'} = [g_{\alpha\beta}q_{\alpha'}q_{\beta'} + g_{\alpha\alpha'}g_{\beta\beta'}q^2 - g_{\alpha\alpha'}q_\beta q_{\beta'} - g_{\beta\beta'}q_\alpha q_{\alpha'}] \qquad \text{(B1)}$$

and $T'_{\alpha'\beta'}$ satisfies

$$[T'_{\alpha'\beta'} - T'_{\beta'\alpha'}][g_{\mu\alpha'}g_{\nu\beta'}g_{\sigma\delta} - g_{\alpha'\sigma}g_{\beta'\nu}g_{\mu\delta} - g_{\alpha'\mu'}g_{\beta'\sigma}g_{\nu\delta}] = 0\,.$$

In the above p^i can be any four-vector; it can include γ's. The fermion indices on γ will be suppressed.

Proof: Condition (ii) implies that there are no $\epsilon_{\alpha\beta\gamma\delta}$ terms in the tensor. Therefore, no term becomes zero by itself under contraction with q_α or q_β, but they cancel with some other term. Therefore,

$$T_{\alpha\beta} = g_{\alpha\beta}q^2 f_1 + \sum_{ij} x^i_\alpha x^j_\beta f_{ij}\,. \qquad \text{(B2)}$$

Write (B2) as

$$T_{\alpha\beta} = (g_{\alpha\beta}q^2 - q_\alpha q_\beta)f_1 + \sum_{ij} x^i_\alpha x^j_\beta f'_{ij}\,, \qquad \text{(B3)}$$

where

$$f'_{ij} = f_{ij} \quad \text{for } i \neq n+1\,,\ j \neq n+1$$

and

$$f'_{n+1,n+1} = f_{n+1,n+1} + f_1\,.$$

$(g_{\alpha\beta}q^2 - q_\alpha q_\beta)f_1$ satisfies conditions (i) and (ii); so must $\sum_{ij} x^i_\alpha x^j_\beta f'_{ij}$. Also $(g_{\alpha\beta}q^2 - q_\alpha q_\beta)f_1$ is a special case of the result (B1), where $T'_{\alpha'\beta'} = \frac{1}{2}g_{\alpha'\beta'}f_1$. So, now we only have to prove the theorem for a tensor $T'_{\alpha\beta}$ which is free of $g_{\alpha\beta}$ terms. From now on we will drop the prime, remembering that $g_{\alpha\beta}$ terms do not now exist in $T_{\alpha\beta}$.

Writing

$$T_{\alpha\beta} = \sum_i C^i_{\alpha\beta} \qquad \text{(B4)}$$

such that upon contraction with q_α, a given $C^i_{\alpha\beta}$ exactly cancels say $C^j_{\alpha\beta}$ and upon contraction with q_β cancels $C^k_{\alpha\beta}$, where $k \neq j \cdot k \neq j$ because this is possible only for the special case which we have already taken care of. This can be achieved by writing a given term as a sum of many terms, in case the term cancels with more than one term.

Now we define a procedure called clipping of the tensor in minimally current-conserving tensors (MCC's). Rearrange the terms such that $C^1_{\alpha\beta}$ cancels $C^2_{\alpha\beta}$ when contracted with q_α and cancels with $C^3_{\alpha\beta}$ when contracted with q_β.

Now *assume* that there exists a term $K_{\alpha\beta}$ such that

$$q_\beta C^2_{\alpha\beta} = q_\beta K_{\alpha\beta}\,, \qquad \text{(B5)}$$

$$q_\alpha C^3_{\alpha\beta} = q_\alpha K_{\alpha\beta}\,. \qquad \text{(B6)}$$

Then adding $K_{\alpha\beta}$ and $-K_{\alpha\beta}$, break $T_{\alpha\beta}$ into parts

$$T_{\alpha\beta} = C^1_{\alpha\beta} + C^2_{\alpha\beta} + C^3_{\alpha\beta} - K_{\alpha\beta} + \sum_{i=4}^{n} C^i_{\alpha\beta} + K_{\alpha\beta} = T'_{\alpha\beta} + T^R_{\alpha\beta}\,,$$

where

$$T'_{\alpha\beta} = C^1_{\alpha\beta} + C^2_{\alpha\beta} + C^3_{\alpha\beta} - K_{\alpha\beta}\,,$$

$$T^R_{\alpha\beta} = \sum_{i=4}^{n} C^i_{\alpha\beta} + K_{\alpha\beta}\,.$$

It is easy to verify that $T'_{\alpha\beta}$ and $T^R_{\alpha\beta}$ both now separately satisfy current conservation. This process can be continued until all the pieces clipped have just four terms. Rewriting

$$T_{\alpha\beta} = \sum_i T^i_{\alpha\beta}\,, \qquad \text{(B7)}$$

$$T^i_{\alpha\beta} = C^{i1}_{\alpha\beta} + C^{i2}_{\alpha\beta} + C^{i3}_{\alpha\beta} - K^i_{\alpha\beta}\,. \qquad \text{(B8)}$$

Here the C's have been relabeled.

Now let us prove the existence of K^i by explicit construction.

$C^{i1}_{\alpha\beta} = x^{k_1}_\alpha x^{k_2}_\beta f$, where k_1 and k_2 are functions of the i index. For convenience, we write

$$f = F/x^{k_1} \cdot q x^{k_2} \cdot q\,,$$

so that

$$q_\alpha C^{i1}_{\alpha\beta} = x^{k_2}_\beta F/x^{k_2} \cdot q\,. \qquad \text{(B9)}$$

$C^2_{\alpha\beta}$ cancels $C^1_{\alpha\beta}$ when contracted by q_β. Therefore,

$$C^2_{\alpha\beta} = -\frac{x_\alpha^{k_3} x_\beta^{k_2} F}{x^{k_3}\cdot q x^{k_2}\cdot q}\,. \tag{B10}$$

Similarly for $C^3_{\alpha\beta}$ we get

$$C^3_{\alpha\beta} = -\frac{x_\alpha^{k_1} x_\beta^{k_4} F}{x^{k_1}\cdot q x^{k_2}\cdot q}\,. \tag{B11}$$

Then

$$C^{i1}_{\alpha\beta} + C^{i2}_{\alpha\beta} + C^{i3}_{\alpha\beta} = [g_{\alpha\gamma} g_{\beta\delta} q_\epsilon q_\xi - g_{\alpha\epsilon} g_{\beta\delta} q_\gamma q_\xi - g_{\alpha\gamma} g_{\beta\xi} q_\epsilon q_\delta] f_{\gamma\delta\epsilon\xi}\,,$$

with

$$f_{\gamma\delta\epsilon\xi} = \frac{x_\gamma^{k_1} x_\delta^{k_2} x_\epsilon^{k_3} x_\xi^{k_4} F}{(x^{k_1}\cdot q)(x^{k_2}\cdot q)(x^{k_3}\cdot q)(x^{k_4}\cdot q)}\,. \tag{B12}$$

Clearly $K^i_{\alpha\beta} = g_{\alpha\epsilon} g_{\beta\xi} q_\gamma q_\delta f_{\gamma\delta\epsilon\xi}$ satisfies requirements (B5) and (B6). Hence, we have proved that a decomposition like (B7) is valid. But

$$T^i_{\alpha\beta} = \sum_{i=1}^{3} C^i_{\alpha\beta} + K_{\alpha\beta} = [B(q)_{\alpha\beta}]_{\gamma\delta\epsilon\xi} f_{\gamma\delta\epsilon\xi}\,,$$

where

$$[B(q)_{\alpha\beta}]_{\gamma\delta\epsilon\xi} = [g_{\alpha\gamma} g_{\beta\delta} q_\epsilon q_\xi - g_{\alpha\epsilon} g_{\beta\delta} q_\gamma q_\xi - q_{\alpha\gamma} q_{\beta\xi} q_\epsilon q_\delta + g_{\alpha\epsilon} g_{\beta\xi} q_\gamma q_\delta]$$

can be written as

$$T^i_{\alpha\beta} = \frac{1}{q^2}[A_{\alpha\beta}(q)]_{\alpha'\beta'} \times\left(\frac{x_{\alpha'}^{k_1} x_{\beta'}^{k_2}}{(x^{k_1}\cdot q)(x^{k_2}\cdot q)} + \frac{x_{\alpha'}^{k_3} x_{\beta'}^{k_4}}{(x^{k_3}\cdot q)(x^{k_4}\cdot q)} - \frac{x_{\alpha'}^{k_3} x_{\beta'}^{k_2}}{(x^{k_3}\cdot q)(x^{k_2}\cdot q)} - \frac{x_{\alpha'}^{k_1} x_{\beta'}^{k_4}}{(x^{k_1}\cdot q)(x^{k_4}\cdot q)}\right)F\,,$$

$$T^i_{\alpha\beta} = [A_{\alpha\beta}(q)]_{\alpha'\beta'} T'^i_{\alpha'\beta'}\,.$$

Hence (B7) can be written as

$$T_{\alpha\beta} = [A_{\alpha\beta}(q)]_{\alpha'\beta'} \sum_i T'^i_{\alpha'\beta'} = [A_{\alpha\beta}(q)]_{\alpha'\beta'} T'_{\alpha'\beta'}\,. \tag{B13}$$

This completes the first part of the proof.

The condition on $T'_{\alpha'\beta'}$ is obtained by demanding condition (ii). As $[A_{\alpha\beta}(q)]_{\alpha'\beta'} \neq [A_{\beta\alpha}(q)]_{\alpha'\beta'}$, we demand

$$[A_{\alpha\beta}(q)]_{\alpha'\beta'} T'_{\alpha'\beta'} = [A_{\beta\alpha}(q)]_{\alpha'\beta'} T'_{\alpha'\beta'}\,. \tag{B14}$$

Substituting for $[A_{\alpha\beta}(q)]_{\alpha'\beta'}$ from (B1) and doing a little algebra one obtains the desired result. Hence the theorem. Eq. (86) is a direct application of this theorem.

Corollary 1. Let $T_{\alpha\beta\gamma\delta\epsilon\cdots}$ be any tensor satisfying

$$q_\alpha T_{\alpha\beta\gamma\cdots} = q_\beta T_{\alpha\beta\gamma\cdots} = 0 \text{ and } T_{\alpha\beta\gamma\cdots} = T_{\beta\alpha\gamma\cdots}\,.$$

Then,

$$T_{\alpha\beta\gamma\delta\epsilon\cdots} = [A'(q)_{\alpha\beta}]_{\alpha'\beta'} T'_{\alpha'\beta'\gamma\delta\epsilon\cdots}\,.$$

Proof. Introduce another set of 4 vectors R_i, $i=1,\ldots,n$. Define $R_{3\gamma} R_{4\delta} R_{5\epsilon} R_{6\xi} \cdots T_{\alpha\beta\gamma\delta\epsilon\xi\cdots} = T'_{\alpha\beta}[R]$. Now $T'_{\alpha\beta}[R]$ can be written as

$$T'_{\alpha\beta}[R] = [A'(q)_{\alpha\beta}]_{\alpha'\beta'} B_{\alpha'\beta'}[R] = [A'(q)_{\alpha\beta}]_{\alpha'\beta'} T'_{\alpha\beta\gamma\delta\epsilon\xi} \times R_{3\gamma} R_{4\delta} R_{5\epsilon} R_{6\xi}\cdots\,. \tag{B15}$$

Therefore,

$$T_{\alpha\beta\gamma\delta\epsilon\cdots} = [A'(q)_{\alpha\beta}]_{\alpha'\beta'} T'_{\alpha'\beta'\gamma\delta\epsilon\cdots}\,.$$

Corollary 2. Let $T_{i_1j_1;i_2j_2;i_3j_3\cdots i_nj_n}(k_1,k_2,k_3,\ldots,k_n)$ be a tensor which satisfies current conservation in indices i_1j_1 with respect to a vector k_1, in indices i_2,j_2 with respect to a vector k_2 and so on. Further suppose T is symmetric under interchange of i_l and j_l. Then

$$T_{i_1j_1;i_2j_2;i_3,j_3;\cdots}(k_1,k_2\cdots) = [A(k_1)_{i_1j_1}]_{i_1'j_1'}[A(k_2)_{i_2j_2}]_{i_2'j_2'}\cdots \times T'_{i_1'j_1';i_2'j_2';i_3'j_3'\cdots}\,. \tag{B16}$$

(B16) is an immediate consequence of (B15) applied repeatedly. Equation (95) is an application of (B16) where $n=2$, $i_1\to\alpha$, $j_1\to\beta$, $i_2\to\gamma$, $j_2\to\delta$.

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