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Hadronic corrections to the annihilation rate of heavy vector mesons to lepton pairs

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Hadronic corrections to the annihilation rate of heavy vector mesons to lepton pairs are considered to leading nontrivial order in gluon exchange and in the quark-confining potential. It is shown that, to an excellent approximation, the rate is $\Gamma(V\to l^+l^-)\simeq(16\pi\alpha^2 e_Q{}^2/M^2)|\phi_0(r=1/m)|^2$, where $M\simeq 2m$ is the vector-meson mass and $\phi_0(r)$ is the solution of the zeroth-order nonrelativistic quark-antiquark bound-state problem. This has the effect of reducing the Van Royen–Weisskopf prediction for the rate.

I. THE PROBLEM

An important feature of new-particle spectroscopy is the annihilation of heavy-quark–antiquark ($Q\overline{Q}$) bound states into lepton pairs and hadrons. Appelquist and Politzer[1] have given a plausible argument that this class of processes may be described by decay amplitudes written in a factorized form when the quark mass m_Q is sufficiently large, i.e., in the limit $m_Q\to\infty$. In a schematic notation the decay rates for these bound-state decays take the limiting form

$$\Gamma_0=(\text{kinematical factors})\,|\phi_0(0)|^2\times(\text{annihilation})^2\,, \tag{1.1}$$

where $\phi_0(0)=$ the wave function at the origin for the zero-order nonrelativistic bound-state problem, and $(\text{annihilation})^2$ is extracted from the amplitude for $Q\overline{Q}$ annihilation of mass-shell quarks to photons and/or gluons, computed to lowest order. Equation (1.1) has formed the basis of a number of phenomenological analyses, which have been thoroughly reviewed.[2]

In order to apply (1.1) to realistic $Q\overline{Q}$ systems, it is important to understand the nature and magnitude of the corrections to (1.1) for finite m_Q. It has been emphasized many times that the natural expansion parameter of quantum chromodynamics (QCD), α_s, is not proportional to the average quark velocity if the average size of the $Q\overline{Q}$ system exceeds those distance scales for which QCD perturbation theory is valid, which typically might be

$$r_0\lesssim(2\text{ GeV})^{-1}\simeq 0.1\text{ f}\,. \tag{1.2}$$

Since the average size of ground-state charmonium is approximately 0.5 f, and that of ground-state Υ approximately 0.25 f, the average velocity of the quarks in these states is strongly affected by the quark-confinement mechanism, which complicates the discussion of corrections to (1.1) in applications to charmonium or Υ states. Therefore, corrections to (1.1) even for *nonrelativistic* $Q\overline{Q}$ systems divide into two distinct (phenomenological) cases depending on the characteristic distance scale of the bound-state relative to (1.2):

(1) The low-lying $Q\overline{Q}$ bound-state spectroscopy is dominated by distances $r<r_0$, in which case lowest-order radiative corrections are similar to those of positronium.

(2) The bound-state spectroscopy of the $Q\overline{Q}$ states is dominated by the quark-confining potential, in which case nonperturbative effects must be considered.

Case (2) will require model-dependent considerations, since quark confinement is not a feature that can be derived from an underlying Lagrangian. As a result, some ambiguity occurs in the separation of *residual* short-distance effects of QCD from those of the quark-confining mechanism.

It is the purpose of this paper to present the results of an investigation of the simplest of the $Q\overline{Q}$ annihilation processes, the annihilation of a vector meson to lepton pairs for both these cases. We begin in the next section with a discussion of the corrections to (1.1) for $V\to l^+l^-$ in case (1) when QCD perturbation theory *is* applicable. In this case one is tempted to read off this correction from the analogous one-photon contribution to the $^3S_1$ positronium ground-state energy[3-5] (with the vacuum-polarization correction to the annihilation-photon propagator omitted). The difficulty is that in quantum electrodynamics (QED) the renormalized electron charge is defined at zero momentum transfer and the electron mass is an empirically measurable quantity. Because of the infrared singularities present in Yang-Mills theory, one *cannot* renormalize the gauge coupling constant at zero momentum transfer. Similarly the quark mass must be defined so that it is also free of infrared singularities. This essential difference in renormalization conventions between QCD and QED is significant in that $\alpha_s\ln\alpha_s$ corrections to (1.1) may appear in QCD even though no analogous corrections appear in QED. [The absence of $O(\alpha \ln\alpha)$ effects in QED has been explained by noting that nonmoving charges cannot radiate in electrodynamics.[3] This argument is not applicable to QCD, where the charges carry color.] If one chooses a renormalization mass of $O(m_Q)$ in defining α_s, then in case (1) a correction of $O(\alpha_s \ln\alpha_s)$ appears because of the required invariance of the physical decay rate under changes of the renormalization mass scale. A more detailed discussion of this issue is presented in Sec. II.

As we argue in Sec. II, the $O(\alpha_s \ln\alpha_s)$ correction is due to the gluon and light-quark vacuum-polarization correction to the Coulomb potential (in Coulomb gauge). If the renormalization mass is chosen at the typical momentum transfer to the quarks in the $Q\overline{Q}$ bound state, then the $O(\alpha_s \ln\alpha_s)$ may be absorbed in $|\phi_0(0)|^2$ by a redefinition of the zeroth-order problem to include the vacuum-polarization corrections in the Coulomb potential. This course of action is not useful for charmonium and Υ spectroscopy, since this will involve distances for which QCD perturbation theory is not valid, and for which the (vacuum-polarization-corrected) Coulomb potential is negligible compared to the confining forces. That is, the radius of the light-quark vacuum-polarization cloud is $r_{\text{cloud}} \simeq \frac{1}{3}$ F, which is well outside the domain of perturbation theory. This means that the results of Sec. II are not applicable to ψ or Υ annihilation. These decays belong to case (2), a situation for which the vacuum-polarization corrections to the Coulomb part of the potential probably should *not* be included. Rather, screening should be treated nonperturbatively and associated with the $Q\overline{Q}$ confining potential.[6]

Experience with the phenomenology of charmonium and other meson spectroscopy suggests that the correct zeroth-order problem is obtained from the potential[2]

$$V(r) = -\frac{4}{3}\frac{\alpha_M}{r} + V_s(r)\,, \tag{1.3}$$

where $V_s(r) =$ the $Q\overline{Q}$ confining potential (including the long-distance screening) and $\alpha_M =$ the QCD fine-structure constant renormalized at mass scale $M \simeq 2m$. [That is, the strong-coupling constant is normalized so that $\alpha_s(q^2=-M^2) \equiv \alpha_M$, with $m \simeq M/2$, the quark *constituent* mass.] Notice that this residual Coulomb interaction is renormalized at distances of the order M^{-1}, which is less than the QCD Bohr radius, for reasons discussed above. In a sense, this is the major difference between case (1) and case (2). With this difference in mind, we consider the corrections to the process $V \to l^+l^-$ for case (2) in Secs. III and IV, using the basic tool of the bound-state problem, the Bethe-Salpeter equation.[7] Since the portion of the Bethe-Salpeter kernel describing the confinement of $Q\overline{Q}$ pairs is infrared singular, we also verify that all infrared singularities are absorbed into $|\phi_0(0)|^2$. In Sec. IV we obtain the infrared-finite result,

$$\Gamma(V \to l^+l^-) \simeq \frac{16\pi\alpha^2 e_Q^2}{M^2}|\phi_0(0)|^2\left[1 - \frac{16\alpha_M}{3\pi} - 2\left(\frac{\lambda}{m^2}\right) + \cdots\right] \tag{1.4}$$

to lowest order in α_M and λ, where

$$6\lambda \simeq -m V_s(0)\,.$$

Further, we show that

$$\phi_0(r) \simeq \phi_0(0)\{1 - \tfrac{4}{3}(\tfrac{1}{2}\alpha_M m)r - [\lambda - \tfrac{1}{12}m^2(\tfrac{4}{3}\alpha_M)^2]r^2 + O(r^3)\} \tag{1.5a}$$

$$\simeq \phi_0(0)e^{-\lambda r^2}\exp[-\tfrac{4}{3}(\tfrac{1}{2}\alpha_M m)r] + O(r^3) + O(\alpha_M{}^2)\,. \tag{1.5b}$$

Equation (1.4) may therefore be rewritten

$$\begin{aligned}\Gamma(V \to l^+l^-) &\simeq \frac{16\pi\alpha^2 e_Q^2}{M^2}\left|\phi_0\left(r=\frac{1}{m}\right)\right|^2 \times[1 - (4/\pi - 1)(\tfrac{4}{3}\alpha_M) + \cdots]\\ &= \frac{16\pi\alpha^2 e_Q^2}{M^2}\left|\phi_0\left(r=\frac{1}{m}\right)\right|^2 \times[1 - (0.2732)(\tfrac{4}{3}\alpha_M) + \cdots]\end{aligned} \tag{1.6}$$

to the accuracy of our calculation. In going from (1.4) to (1.6) one has made a selective (nonperturbative) summation of the effects of quark confinement, and included them in $\phi_0(r=1/m)$, so that, in principle, (1.6) contains more information than perturbation theory. Physically (1.6) suggests that $Q\overline{Q}$ annihilation takes place at an average distance of $r \simeq 1/m$, rather than at $r=0$ (cf. Appendix B).

If (1.6) correctly sums up the dominant effects of quark confinement, then the Van Royen–Weisskopf formula[8] may possibly be used even when perturbation theory is not valid if one makes the replacement $|\phi_0(0)|^2 \to |\phi_0(r=1/m)|^2$. Perhaps (1.6) has some degree of validity for lighter quarks as well. A similar correction for the annihilation processes of $Q\overline{Q} \to$ gluons is also expected. It seems clear that (1.6) is preferable to (1.4) in applications to practical problems, since the correction in (1.4) is large even for $\alpha_s \simeq 0.2$ and $(v/c)^2 \simeq \frac{1}{6}$. By contrast, the radiative corrections have shifted the effective annihilation radius to $r \simeq 1/m$, with a small residual correction.

A more detailed discussion of these problems, and a derivation of results is presented in Secs.

II to IV. Certain technical details are given in Appendices.

II. QCD PERTURBATION THEORY

Suppose that one is dealing with a hypothetical $Q\overline{Q}$ bound state with the mass m of the quark Q sufficiently large so that the low-lying bound-state spectrum is determined from one-gluon exchange. The zeroth-order bound-state problem for these low-lying states is given, to a good approximation, by the solution to the Schrödinger equation, with quark confinement neglected. Then

$$\left(-\frac{\nabla^2}{m}-\frac{4}{3}\frac{\alpha_\mu}{r}\right)\phi_c(r)=-\epsilon\phi_c(r)\ , \qquad (2.1)$$

where the QCD fine-structure constant is defined at an arbitrary mass scale μ. [Highly excited states will be sensitive to the quark-confining potential, and one must use (1.3) for *those* bound states.] The quark mass in (2.1) is the *constituent* quark mass defined at a mass scale such that[9]

$$2m=M+\epsilon\ , \qquad (2.2)$$

where M is the ground-state vector-meson mass and

$$\epsilon=\tfrac{1}{4}(\tfrac{4}{3}\alpha_\mu)^2 m\ . \qquad (2.3)$$

In other words, in every respect Eqs. (2.1)–(2.3) are identical to the analogous positronium problem, *except* that the gauge coupling is defined at the arbitrary mass scale μ.

The lowest-order formula for the decay width $V\to l^+l^-$ for the $^3S_1$ ground state is

$$\Gamma(V\to l^+l^-)=\frac{16\pi\alpha^2 e_Q{}^2}{M^2}\,|\phi_c(0;\alpha_\mu)|^2\ , \qquad (2.4)$$

where $\phi_c(r;\alpha_\mu)$ is the solution to (2.1) with the gauge coupling renormalized at $\vec{q}^{\,2}=-\mu^2$, as discussed above. As is well known,

$$|\phi_c(0;\alpha_\mu)|^2=\frac{1}{\pi}\left[\frac{m}{2}(\tfrac{4}{3}\alpha_\mu)\right]^3\ . \qquad (2.5)$$

Since the binding energy is $O(\alpha_\mu{}^2 m)$, to leading order in α_μ,

$$\Gamma(V\to l^+l^-)=\tfrac{1}{2}(\alpha^2 e_Q{}^2 m)(\tfrac{4}{3}\alpha_\mu)^3\ . \qquad (2.6)$$

The trouble with (2.6) is that the decay rate is not invariant to changes in the renormalization mass μ and higher-order corrections must be considered to ensure renormalization-group invariance of the experimental rate. Recall, however, that the solution to (2.1) gives

$$\phi_c(0;\alpha_\mu)\sim(\alpha_\mu)^{3/2}\ , \qquad (2.7)$$

which is *nonperturbative* in character and arises from the infinite number of Coulomb exchanges between the $Q\overline{Q}$ pair. We therefore anticipate that if we are to obtain the renormalization-group invariance of the decay rate, we must suitably modify the zeroth-order problem, so as to guarantee that $|\phi_c(0)|^2$ is renormalization-group invariant to the appropriate order in perturbation theory. [We are tacitly assuming that the quark mass is already renormalized at *its* characteristic mass scale by (2.2). If we had chosen a different mass scale for the quark mass, then we would also have to consider the renormalization-group properties of the quark mass.[9] Thus, by our choice of quark mass as the *constituent* mass we presumably have avoided this particular issue.]

We now discuss how one may understand the renormalization-group invariance of $\Gamma(V\to l^+l^-)$ to one-loop order. The static Hamiltonian in Coulomb gauge for the Q-$\overline{Q}$ interactions has been obtained in Yang-Mills theory, to one-loop accuracy, from calculation of all fourth-order diagrams. For color-singlet $Q\overline{Q}$ states the static interaction Hamiltonian in momentum space is[10]

$$H_{\rm int}=+v(|\vec{q}|)[1+O(\vec{q}^{\,2}/m^2)+O((\vec{q}^{\,2}/m^2)\ln\vec{q}^{\,2}/m^2)]\ , \qquad (2.8)$$

where, for SU(3) of color

$$v(|\vec{q}|)=-\tfrac{4}{3}(4\pi)\frac{\alpha_\mu}{\vec{q}^{\,2}}\left[1-\frac{1}{\pi}\alpha_\mu\left(\frac{2n}{3}-11\right)\ln\left(\frac{|\vec{q}|^2}{\mu^2}\right)+\cdots\right]. \qquad (2.9)$$

[One may renormalization-group "improve" (2.9), but this is not needed for our discussion.] In the above, n = number of *light* quarks in the theory. Only those quarks whose vacuum-polarization cloud is *larger* than the average-size of the $Q\overline{Q}$ system need be included in (2.9). Heavy-quark vacuum polarization and terms of $O(\vec{q}^{\,2}/m^2)$ in (2.8) will give corrections of $O(\alpha_\mu{}^2)$ or $O(\alpha_\mu{}^2\ln\alpha_\mu)$ to (2.6). Here we focus on a possible $O(\alpha_\mu\ln\alpha_\mu)$ correction to the Van Royen–Weisskopf formula.

Let us modify the zero-order problem to include the vacuum-polarization-corrected Coulomb potential given by (2.9). Then

$$\left[-\frac{\nabla^2}{m}+v(r)\right]\phi_0(r)=-\epsilon\phi_0(r)\ , \qquad (2.10)$$

where $v(r)$ is the Fourier transform of (2.9). We now argue that the rate for $V\to l^+l^-$ will be renormalization-group invariant, to one-loop level, with this starting point. Suppose that the average momentum flowing through the Coulomb propagator is γ, where

$$\gamma\simeq\tfrac{1}{2}m(\tfrac{4}{3}\alpha_\mu)+\cdots \qquad (2.11)$$

to leading order. Then

$$v(|\vec{q}|)=\frac{-4\pi}{|\vec{q}^2|}(\tfrac{4}{3}\alpha_\gamma)\left[1-\frac{1}{\pi}\left(\frac{2n}{3}-11\right)\alpha_\gamma\ln\left(\frac{|\vec{q}|^2}{\gamma^2}\right)+\cdots\right], \tag{2.12}$$

where

$$\alpha_\gamma=\alpha_\mu\left[1-\frac{1}{\pi}\left(\frac{2n}{3}-11\right)\alpha_\mu\ln\left(\frac{\gamma^2}{\mu^2}\right)+\cdots\right] \tag{2.13}$$

is the gauge-coupling renormalized at $\vec{q}^2=-\gamma$. The important point is that the term $(1/|\vec{q}^2|)\ln(|\vec{q}^2|/\gamma^2)$ does not contribute to the decay rate in $O(\alpha_\gamma\ln\alpha_\gamma)$ because of the special choice of renormalization mass at $\mu=\gamma$. Therefore, one can now solve the Coulomb problem

$$\left(-\frac{\nabla^2}{m}-\frac{4}{3}\frac{\alpha_\gamma}{r}\right)\phi_c(r;\alpha_\gamma)=-\epsilon\phi_c(r;\alpha_\gamma) \tag{2.14}$$

to obtain the prediction

$$\Gamma(V\to l^+l^-)=\tfrac{1}{2}\alpha^2e_Q{}^2m(\tfrac{4}{3}\alpha_\gamma)^3[1+O(\alpha_\gamma)+\cdots]. \tag{2.15}$$

Equivalently,

$$\Gamma(V\to l^+l^-)=\tfrac{1}{2}\alpha^2e_Q{}^2m(\tfrac{4}{3}\alpha_\mu)^3\times\left[1-3\left(\frac{1}{\pi}\right)\left(\frac{2n}{3}-11\right)\alpha_\mu\ln\left(\frac{\gamma}{\mu}\right)+O(\alpha_\mu)\right]. \tag{2.16}$$

Equations (2.15) and (2.16) are equivalent renormalization-group-invariant predictions to one-loop order.[11]

There are two different choices of renormalization mass μ which are of special interest. If one chooses $\mu=\gamma$, then the prediction for the decay rate is given by (2.15) when effects of confinement are negligible. It is also popular to consider renormalization at mass scale $\mu=M$, in which case

$$\Gamma(V\to l^+l^-)=\tfrac{1}{2}\alpha^2e_Q{}^2m(\tfrac{4}{3}\alpha_M)^3\times\left[1+\frac{3}{\pi}\left(\frac{2n}{3}-11\right)\alpha_M\ln(\alpha_M{}^{-1})+O(\alpha_M)\right]. \tag{2.17}$$

(Note the sign of the correction.) It is also possible to renormalization-group "improve" (2.16) or (2.17) by considering the renormalization-group improvement of (2.12), which is equivalent to considering the one-particle irreducible (1PI) vacuum-polarization-corrected Coulomb propagation.

To recapitulate, one may state the result of this section more generally. If the renormalization mass for the Coulomb exchange is chosen to be $\mu\sim O(\alpha m)$, then the prediction for the leptonic decay rate will be of the form (2.15). If the renormalization mass is chosen so that $\mu\sim O(m)$, then an $O(\alpha_m\ln\alpha_m)$ correction will be present, as in (2.17). It should be reemphasized that (2.16) or (2.17) are not applicable to presently available $Q\bar{Q}$ systems, since their validity depends on the premise that the average size of the $Q\bar{Q}$ system is $r\sim(\alpha m)^{-1}$, and sufficiently short-ranged to be in the domain of validity of perturbation theory. This excludes the ψ and Υ problems.

A similar discussion applies to the annihilation of a bound $Q\bar{Q}$ state $\to n$ gluons. The only subtlety is that although the typical momentum transfer to the Coulomb gluon is $O(\alpha m)$, the typical momentum transfer to an *annihilation* gluon is $O(m)$. Therefore, for example, a one-loop renormalization-group-invariant prediction for the annihilation $(Q\bar{Q})$ bound state $\to 2$ gluons is given by

$$\Gamma(Q\bar{Q}\to 2\text{ gluons})=(\text{const})m\alpha_\gamma{}^3\alpha_m{}^2 \tag{2.18}$$

to leading-order, when Eqs. (2.10)–(2.12) give an accurate approximation to the bound-state problem, with the factorization of (2.15) and (2.18) achieved by the separation of the two distinct mass scales. All terms of $O(\alpha\ln\epsilon)$ have thus been absorbed into the bound-state wave function by an appropriate choice of zeroth-order problem. (Recall $\epsilon m\sim\gamma^2$). The conjecture of Appelquist and Politzer,[1] that the factorized form for $\Gamma(V\to l^+l^-)$ and $\Gamma(Q\bar{Q}\to\text{gluons})$ holds even when the confining potential is significant, as described schematically by (1.1), requires further analysis.

The remainder of this paper is devoted to a discussion of the corrections to the Van Royen–Weisskopf formula[8] when confinement is important. As discussed in Sec. I, we must begin with the appropriate zeroth-order problem. Consider the potential

$$V(r)=-\frac{4}{3}\frac{\alpha_M}{r}+V_s(r), \tag{2.19}$$

with the *residual* one-gluon exchange given with the gauge-coupling normalized at $\mu=M$, *and* with *all* vacuum polarization effects *already* absorbed in the vacuum polarization corrected confining potential $V_s(r)$.[6] We do not include vacuum-polarization corrections to the Coulomb interaction to avoid possible double counting. Adopting this point of view, corrections of $O(\alpha_M\ln\alpha_M)$ do not appear in the phenomenological situation. Some authors[12] do permit vacuum-polarization corrections to the Coulomb interaction by considering $\alpha_M(\vec{q}^2)/\vec{q}^2$, with $\alpha(\vec{q}^2)$ a *running* coupling constant. In this case, the $O(\alpha_M\ln\alpha_M)$ corrections of (2.17) are required, but our discussion in Sec. I and II suggests that one should *not* use a running coupling constant in (2.19)[13] if the confining potential includes the screening due to light quarks.[6]

III. FORMULATION OF PHENOMENOLOGICAL PROBLEM

The vector-meson decay into lepton pairs is obtained from the one-photon annihilation contribution to the vector-meson energy shift by considering the decay to a virtual photon,

$$\Gamma(V\to\gamma^*)=-2\,\mathrm{Im}(\Delta E)_{1\gamma}\,, \tag{3.1}$$

with the photon vacuum-polarization contribution to $(\Delta E)_{1\gamma}$ omitted. The energy shift $(\Delta E)_{1\gamma}$ can be calculated by a bound-state formalism,[7] such as the Bethe-Salpeter equation, evaluated in a perturbation expansion around the nonrelativistic limit. In this section we follow the strategy of Karplus-Klein[5] in their evaluation of $(\Delta E)_{1\gamma}$ for positronium, but modify the Bethe-Salpeter kernel to account for the hypothetical quark confinement. Since these bound-state methods are now fairly standard, we will only emphasize those features which distinguish nonrelativistic $Q\overline{Q}$ annihilation from the analogous positronium case. Our presentation will be semiphenomenological, since assumptions concerning that part of the effective kernel giving rise to quark confinement will be required.

The two-particle color-singlet $Q_1\overline{Q}_2$ Green's function $G(x_1,x_2;x_1',x_2')$ satisfies the Bethe-Salpeter equation

$$(i\gamma\cdot\partial-m)_1(i\gamma\cdot\partial-m)_2^{(t)}G(x_1,x_2;x_1',x_2')-\int d^4x_3d^4x_4K(x_1,x_2;x_3,x_4)G(x_3,x_4;x_1',x_2')=\delta^4(x_1-x_1')\delta^4(x_2-x_2')\,, \tag{3.2}$$

where the superscript (t) denotes the transposed operator. In our work $K(x_1,x_2;x_3,x_4)$ is the *effective* color-singlet $Q\overline{Q}$ kernel. A discussion of the definition of the quark masses m_1 and m_2 in (3.2) is required since they cannot be directly measured. In this work m_1 and m_2 are renormalized constituent quark masses defined at the mass scale μ, such that[9]

$$m_1(\mu)+m_2(\mu)=M+\epsilon\,, \tag{3.3}$$

where M is the ground-state vector-meson mass and ϵ is the Coulomb binding energy,

$$\epsilon=\frac{1}{2}\,\frac{m_1(\mu)m_2(\mu)}{m_1(\mu)+m_2(\mu)}\left(\frac{4}{3}\,\alpha_M\right)^2. \tag{3.4}$$

Therefore, for the $Q\overline{Q}$ system (3.3) implies

$$2m(\mu)=M+\tfrac{1}{4}\,m(\mu)(\tfrac{4}{3}\alpha_M)^2\,, \tag{3.5}$$

so that

$$m(\mu)=\tfrac{1}{2}M[1-\tfrac{1}{8}(\tfrac{4}{3}\alpha_M)^2]^{-1} \tag{3.6a}$$

$$\simeq\tfrac{1}{2}M[1+\tfrac{1}{8}(\tfrac{4}{3}\alpha_M)^2]\,, \tag{3.6b}$$

where (3.6b) is valid for sufficiently small α_M. Notice that the QCD coupling constant α_M is defined at mass scale M, while the constituent mass $m(\mu)$ is defined at mass scale μ.

However, for heavy quarks, $\epsilon/M\ll 1$ and

$$\mu\simeq M \tag{3.7}$$

as emphasized by Georgi and Politzer.[9] Our definition of constituent mass includes the lowest-order "Coulomb" binding energy in (3.3), which will allow us to make direct use of results from electrodynamics, avoiding some lengthy calculations.

Since we are focusing on the *nonrelativistic* $Q\overline{Q}$ problem, we assume that the effective color-singlet kernel is approximately local, so that

$$K(x_1,x_2;x_3,x_4)\simeq\delta(x_1-x_3)\delta(x_2-x_4)K_1(x_1-x_2)\,. \tag{3.8}$$

Our goal is to calculate (3.1) in perturbation theory, keeping the corrections to lowest nontrivial order in α_M *and* the average velocity of the quarks, considered as distinct expansion parameters. To begin this perturbation expansion one must specify the zeroth-order problem which forms the basis of the perturbative expansion. The zeroth-order wave functions in the $Q\overline{Q}$ center of mass is defined to be the ground-state solution of the Schrödinger equation

$$\left(-\frac{\nabla^2}{m}+V(r)\right)\phi_0(r)=-\epsilon\phi_0(r)\,, \tag{3.9}$$

where

$$V(r)=-\frac{4}{3}\,\frac{\alpha_M}{r}+V_s(r)\,, \tag{3.10}$$

with $V_s(r)$ the $Q\overline{Q}$ confining potential. In (3.9) the energy scale is chosen so that the energy eigenvalue for the *ground state* is

$$\epsilon=\tfrac{1}{4}m(\tfrac{4}{3}\alpha_M)^2$$

exactly as in Eq. (3.4). There is no contradiction between (3.9), (3.10), and (3.4) since one can always redefine the confining potential so that

$$V_s(r)\to V_s(r)+\text{constant}\,.$$

In other words, $V_s(0)$ is chosen so that the binding energy computed from (3.9) coincides with (3.4), which means that the *absolute* energy scale of (3.9) and (3.10) is adjusted to coincide with that of (2.3) and (3.4) by adding a suitable constant to $V(r)$. [We also shall require $V_s(r)$ to be finite as $r\to 0$.] The detailed r dependence of $V_s(r)$ will play no role in our considerations, and need not be specified further.

One can develop the zeroth-order problem in terms of the Bethe-Salpeter equation as well. For the nonrelativistic system we further approximate $K_1(x)$ by the *instantaneous* kernel K_s where we divide the one-particle kernel into an instantaneous part and a correction $\delta K_1(x)$. That is,

$$K_1(x)=\delta(x^0)K_s(\vec{r})+\delta K_1(x)\,. \tag{3.11}$$

Since considerable circumstantial evidence exists to suggest that the confining potential transforms as a Lorentz scalar,[14] we take

$$K_s(r)=i[(\gamma_0)_1(\gamma_0)_2(-\tfrac{4}{3}\alpha_M/r)+V_s(r)]\,, \tag{3.12}$$

which relates the instantaneous kernel to $V(r)$ defined in (3.10). The perturbative expansion for (3.1) will be computed in terms of $\delta K_1(x)$, defined by Eqs. (3.8), (3.10)–(3.12). Therefore, δK_1 includes both the retardation and transverse gluon contributions to the lowest-order result. Since the treatment of the effective confinement portion of the kernel as a relativistic *propagating* local interaction leads to problems with unitarity, we do not have a clear idea of the retardation effect of confinement. Such technical complications prevent us from including that correction in this discussion, so that we only feel safe in discussing the *static* effects of confinement, and we omit the retardation correction to confinement in our work.

As is evident from (3.1)

$$\Gamma(V\to l^+l^-)=(\text{kinematical factors})(\Delta E)_{1\gamma} \tag{3.13}$$

computed to lowest order in $\alpha=e^2/4\pi$. Following Karplus and Klein[5] we may write

$$(\Delta E)_{1\gamma}=\frac{-\pi\alpha}{m^2}\{\mathrm{Tr}[\overline{\Psi}(0)\gamma_\mu C]\}\{\mathrm{Tr}[C^{-1}\gamma^\mu\Psi(0)]\}\,, \tag{3.14}$$

where C is the charge-conjugation matrix and Ψ is the matrix

$$\Psi_{\alpha\beta}(0)=\psi(0)\chi_{\alpha\beta}\,, \tag{3.15}$$

with $\chi_{\alpha\beta}$ a (4×4) constant spin matrix. Equations (3.14) and (3.15) *define* $\psi(0)$, with

$$\psi(0)=\phi_0(0)(1+\cdots)\,, \tag{3.16}$$

where $\phi_0(0)$ is as in (3.9). If the effects of confinement and vacuum polarization are neglected, then

$$\psi(0)=\phi_c(0)\left(1-\frac{8\alpha_M}{3\pi}+\cdots\right)\,, \tag{3.17}$$

where $\phi_c(0)$ is the solution of the *Coulomb* problem and *not* (3.9). We shall compute $\psi(0)$ in the next section, keeping the leading effects of quark confinement in addition to the correction exhibited in (3.17). From this result we will present the leptonic decay width of the vector mesons as the corrected Van Royen–Weisskopf formula

$$\Gamma(V\to e^+e^-)=\frac{16\pi\alpha^2 e_Q{}^2|\psi(0)|^2}{M^2}\,, \tag{3.18}$$

where e_Q is the quark charge.

IV. LOWEST-ORDER CORRECTION

Following Karplus and Klein,[5] the one-photon annihilation contribution to the $^3S_1$ $Q\overline{Q}$ mass is

$$\begin{aligned}(\Delta E)_{1\gamma}=&-i(Z_1)^{-1}\int d^4x\,d^4y\,\phi_L(x)K_A(x,y)\phi_L(y)\\&-i(Z_1)^{-1}\int d^4x\,d^4y\,d^4z\,d^4w\,\phi_0(x)[\delta K_1(x,z)S_c(z,w)K_A(w,y)+K_A(x,z)S_c(z,w)\delta K_1(w,z)]\phi_0(y)\,,\end{aligned} \tag{4.1}$$

where K_A = one-photon annihilation kernel, δK_1 is as in (3.11),

$$[S_c(z,w)]^{-1}=(i\gamma\cdot\partial_z-m)_1^{(t)}(i\gamma\cdot\partial_w-m)_2\,,\qquad \phi_0(y)=\phi_0(\vec{y})\chi_{\alpha\beta}\,, \tag{4.2}$$

with $\phi_0(\vec{y})$ obtained from (3.9), and $\chi_{\alpha\beta}$ as in (3.15), and $(Z_1)^{1/2}$ the vertex renormalization, which must cancel the ultraviolet divergence of the vertex. (Of course the one-gluon part of δK_1 must be regulated in a gauge-invariant fashion.) As Karplus and Klein show,[5] to the accuracy of our calculation,

$$\phi_L(x)\simeq i\int d^4y\,S_c(x,y)K_1(\vec{y})\delta(y^0)\phi_0(\vec{y})\,. \tag{4.3}$$

In fact, we require only

$$\phi_L(x=0)\cong i\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{[\gamma\cdot(\frac{1}{2}P-p_1)-m_1]^{(t)}}\,\frac{1}{\gamma\cdot(\frac{1}{2}P+p_2)-m_2}\int\frac{d^3q}{(2\pi)^3}K_F(-(\vec{q}-\vec{p})^2)\phi_0(\vec{q})\,, \tag{4.4}$$

where we have reexpressed (4.3) in momentum space, with $P^2=M^2$ = (vector-meson mass)^2, and

$$K_1(\vec{r})\delta(t)=\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}K_F(-\vec{k}^2)\,. \tag{4.5}$$

Specializing to $m_1 = m_2 = m$, one extracts $\psi(0)$ from (3.1) with the aid of (3.14), (4.2)–(4.5). The result is

$$\mathrm{Tr}[C^{-1}\gamma^\mu \Psi(0)] = -i(Z_1)^{-1/2}\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\frac{1}{[\gamma\cdot(\frac{1}{2}P-p)-m]^{(t)}}C^{-1}\gamma^\mu \frac{1}{\gamma\cdot(\frac{1}{2}P+p)-m}\int\frac{d^3q}{(2\pi)^3}K_F(-(\vec{q}-\vec{p})^2)\phi_0(\vec{q})\right\}$$

$$-i(Z_1)^{-1/2}\int \frac{d^4p}{(2\pi)^3}\mathrm{Tr}\left\{\frac{1}{[\gamma\cdot(\frac{1}{2}P-p)-m]^{(t)}}C^{-1}\gamma^\mu \frac{1}{\gamma\cdot(\frac{1}{2}P+p)-m}\int\frac{d^3q}{(2\pi)^3}\delta K_1(p_0,(\vec{p}-\vec{q}))\phi_0(\vec{q})\right\}. \quad (4.6)$$

As indicated in Sec. III, we are omitting retardation corrections to the *confining* kernel, owing to difficulties with unitarity. Therefore, δK_1 includes only vector-gluon contributions. Since $K_F(-\vec{k}^2)$ is the sum of two terms, due to single-gluon exchange and the confining potential, one may divide Eq. (4.6) according to these exchanges. The vertex correction due to the confining potential is ultraviolet finite, and we set $\delta Z_1 = 0$ for this portion of the vertex correction. (Possible *finite* corrections to δZ_1 due to the confining potential are omitted for reasons identical to the omission of the retardation of the effective confining kernel.) Since we are using a confining potential which transforms as a Lorentz scalar,[14] to the order we are working

$$\mathrm{Tr}[C^{-1}\gamma^\mu\Psi(0)]_{\mathrm{tot}} = \mathrm{Tr}[C^{-1}\gamma^\mu\Psi(0)]_{\mathrm{scalar}} + \mathrm{Tr}[C^{-1}\gamma^\mu\Psi(0)]_{\mathrm{vector}}, \quad (4.7)$$

where

$$\mathrm{Tr}[C^{-1}\gamma^\mu\Psi(0)]_{\mathrm{scalar}} = -i\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\frac{1}{[\gamma\cdot(\frac{1}{2}P-p)-m]^{(t)}}C^{-1}\gamma^\mu\frac{1}{\gamma\cdot(\frac{1}{2}P+p)-m}\int d^4x e^{-ip\cdot x}D_F(x)\phi_0(\vec{r})\right\} \quad (4.8)$$

and

$$\mathrm{Tr}[C^{-1}\gamma^\mu\psi(0)]_{\mathrm{vector}} = -i(Z_1)^{-1/2}\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\left[(\gamma^\lambda)^{(t)}\frac{1}{[\gamma\cdot(\frac{1}{2}P-p)-m]^{(t)}}(C^{-1}\gamma^\mu)\frac{1}{\gamma\cdot(\frac{1}{2}P+p)-m}\gamma_\lambda\right]\right.$$
$$\left.\times\int d^4x e^{ip\cdot x}\mathfrak{D}_F^\Lambda(x)\phi_0(\vec{r})\right\}. \quad (4.9)$$

We have defined

$$D_F(x) = V_s(r)\delta(t_0) \quad (4.10)$$

in order to write the confining potential as the static portion of an effective propagator. Similarly, $\mathfrak{D}_F^\Lambda(x)$ is the regulated version of the Fourier transform of the vector-gluon propagator

$$\mathfrak{D}_F(p) = \frac{1}{p^2 - i\epsilon}. \quad (4.11)$$

Equation (4.9) is identical to the vertex correction of positronium, except for the replacement $\phi_c(r) \to \phi_0(r)$. However, since we show (Appendix A) that

$$\phi_0(r) \underset{r\to 0}{\sim} \phi_0(0)\left[1 - \frac{4}{3}\left(\frac{\alpha_M m}{2}\right)r + O(r^2)\right], \quad (4.12)$$

the vertex renormalization constant Z_1 is identical to that of electrodynamics. We can further subdivide

$$\Psi(0)_{\mathrm{vector}} = \Psi(0)_{\mathrm{QCD}} + \Psi(0)_{\mathrm{correction}}, \quad (4.13)$$

where $\Psi(0)_{\mathrm{QCD}}$ is obtained from (4.9) by means of the replacement $\phi_0(\vec{r}) \to \phi_c(\vec{r})$, and $\Psi(0)_{\mathrm{correction}}$ is obtained from (4.9) by means of the replacement $\phi_0(\vec{r}) \to [\phi_0(\vec{r}) - \phi_c(\vec{r})]$. Since $\Psi(0)_{\mathrm{correction}}$ is ultraviolet finite as a result of (4.12), we set $Z_1 = 1$ in this term since we calculate only to *leading order* in α_M. In Sec. III we very carefully chose the mass scales of the problem so that the binding energy has its lowest-order QCD value

$$\epsilon = (\tfrac{4}{3}\alpha_M)^2 m,$$

with the Coulomb propagator renormalized at mass scale $\mu = M$. Hence, writing $\Psi(0)_{\mathrm{QCD}} = \psi(0)_{\mathrm{QCD}}\chi_{\alpha\beta}$, we have

$$\psi(0)_{\mathrm{QCD}} = \phi_c(0)\left[1 - \frac{2}{\pi}\left(\frac{4}{3}\alpha_M\right) + \cdots\right] \quad (4.14)$$

directly from the positronium result,[4,5] without a new calculation being required. We now turn to the evaluation of $\psi(0)_{\mathrm{scalar}}$ and $\psi(0)_{\mathrm{correction}}$, which do require some calculation.

In the rest system of the vector meson, where $P^\mu = (M, 0)$,

$$\psi(0)_{\mathrm{scalar}} = -i\int\frac{d^4p}{(2\pi)^4}\frac{\frac{1}{3}\vec{p}^2 + [(\frac{1}{2}M+m)^2 - p_0^2]}{[(\frac{1}{2}M-p_0)^2 - E^2][\frac{1}{2}M+p_0)^2 - E^2]}\int d^3r e^{+i\vec{p}\cdot\vec{r}}V_s(r)\phi_0(r) \quad (4.15)$$

with $E^2 = p^2 + m^2$. Similarly

$$\psi(0)_{\text{correction}} = i\int \frac{d^4p}{(2\pi)^4} \frac{\frac{1}{3}\vec{p}^2 + (\frac{1}{2}M + m)^2 - p_0{}^2}{[(\frac{1}{2}M - p_0)^2 - E^2][(\frac{1}{2}M + p_0)^2 - E^2]} (\tfrac{4}{3}\alpha_M) \int d^3r e^{i\vec{p}\cdot\vec{r}} \left(\frac{-1}{r}\right)[\phi_0(r) - \phi_c(r)]. \tag{4.16}$$

Note that only the static, longitudinal part of the gluon exchange appears in (4.16) to leading order in α_M, and that (4.16) vanishes as $\lambda \to 0$, with λ defined in Appendix A. Let us proceed with the evaluation of $\psi(0)_{\text{scalar}}$. The momentum-space version of Eq. (3.9) is

$$\int d^3r e^{i\vec{p}\cdot\vec{r}} V_s(r)\phi_0(r) = -\left(\epsilon + \frac{\vec{p}^2}{m}\right)\phi_0(\vec{p}) + \tfrac{4}{3}\alpha_M \int d^3r \frac{e^{i\vec{p}\cdot\vec{r}}}{r} \phi_0(\vec{r}). \tag{4.17}$$

This can be inserted in (4.15), and the p_0 integration can be performed in both (4.15) and (4.16). It is then useful to reassemble the various pieces of (4.7), obtaining

$$\begin{aligned}\psi(0)_{\text{tot}} = {}& \psi(0)_{\text{QCD}} + \frac{1}{8m} I_1 - \tfrac{1}{3}\int \frac{d^3p}{(2\pi)^3}\left(\frac{p^2}{E}\right)\frac{1}{p^2 + m\epsilon}\int d^3r e^{i\vec{p}\cdot\vec{r}} V_s(\vec{r})\phi_0(\vec{r}) \\ & - \tfrac{1}{8}(\tfrac{4}{3}\alpha_M)[2m(M+2m)]\int \frac{d^3p}{(2\pi)^3}\frac{1}{E(p^2+m\epsilon)}\int d^3r \frac{e^{i\vec{p}\cdot\vec{r}}}{r}\phi_c(r) \\ & - \tfrac{1}{3}(\tfrac{4}{3}\alpha_M)\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{E(p^2+m\epsilon)}\int d^3r \frac{e^{i\vec{p}\cdot\vec{r}}}{r}[\phi_c(r) - \phi_0(r)],\end{aligned} \tag{4.18}$$

where $\psi(0)_{\text{QCD}}$ is given by (4.14) and

$$I_1 = 2m(M+2m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{(m^2+p^2)^{1/2}}\phi_0(\vec{p}). \tag{4.19}$$

Since $\phi_0(\vec{p}) \underset{p^2\to\infty}{\simeq} -m(\vec{p}^2)^{-2}\phi_0(0)$, Eq. (4.19) is both infrared and ultraviolet finite. It does not appear possible to evaluate (4.19) exactly; however, for the $Q\bar{Q}$ bound state in question, the dominant three-momentum is $\vec{p}^2 \simeq 6\lambda$ (and *not* $-m\epsilon$) [see Eq. (B8)]. Therefore,

$$I_1 \simeq \frac{8m - 2\epsilon}{(1 + 6\lambda/m^2)^{1/2}}\int \frac{d^3p}{(2\pi)^3}\phi_0(\vec{p}) \simeq 8m\left(1 - \frac{3\lambda}{m^2} + \cdots\right)\phi_0(0) \tag{4.20}$$

to the order we are interested. We also obtain Eq. (4.20) by a different method in Appendix B when the gluon correction is neglected, which verifies the evaluation of (4.19).

The remaining integrals in (4.18) are easily evaluated

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3}\left(\frac{p^2}{E}\right)\left(\frac{1}{p^2+m\epsilon}\right)\int d^3r e^{i\vec{p}\cdot\vec{r}} V_s(r)\phi_0(r) &\simeq \int \frac{d^3p}{(2\pi)^3}\frac{1}{E}\int d^3r e^{i\vec{p}\cdot\vec{r}} V_s(r)\phi_0(r) + O(\epsilon) \\ &\simeq \frac{1}{m}\int d^3r \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} V_s(r)\phi_0(r) = \frac{1}{m}V_s(0)\phi_0(0)\end{aligned} \tag{4.21}$$

and

$$\int d^3r \frac{e^{i\vec{p}\cdot\vec{r}}}{r}\phi_c(r) = \frac{4\pi\phi_c(0)}{\vec{p}^2 + \frac{1}{4}m(\frac{4}{3}\alpha_M)^2}, \tag{4.22}$$

from the explicit behavior of the Coulomb wave function. To leading order in α_M and zeroth order in λ, we have

$$\phi_c(r) - \phi_0(r) \simeq [\phi_c(0) - \phi_0(0)]\exp[-\tfrac{4}{3}(\tfrac{1}{2}\alpha_M m)r] + O(r^2) \tag{4.23}$$

which is all that is required to evaluate the last integral of (4.18) to the accuracy of our calculation.

At this stage we insert (4.19)–(4.23) into (4.18) to obtain

$$\begin{aligned}\psi(0)_{\text{tot}} = {}& \psi(0)_{\text{QCD}} + \left(1 - \frac{3\lambda}{m^2}\right)\phi_0(0) - \frac{1}{3m}V_s(0)\phi_0(0) - 4\pi\left(\frac{4}{3}\alpha_M\right)\phi_c(0)\int \frac{d^3p}{(2\pi)^3}\frac{m^2}{E}\frac{1}{(p^2+m\epsilon)^2} \\ & - \frac{4\pi}{3}\left(\frac{4}{3}\alpha_M\right)[\phi_c(0) - \phi_0(0)]\int \frac{d^3p}{(2\pi)^3}\left(\frac{p^2}{E}\right)\frac{1}{(p^2+m\epsilon)^2} + \cdots.\end{aligned} \tag{4.24}$$

It is trivial to see, by standard perturbation arguments, that, as $\lambda \to 0$,

$$[\phi_c(0) - \phi_0(0)] \sim 0 . \tag{4.25}$$

Since the last term in (4.24) is dominated by the ultraviolet region, it cannot have an infrared divergence, and therefore it may be neglected as a higher-order effect. Further

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{E} \frac{1}{(p^2+m\epsilon)^2} \simeq \frac{1}{m} \int \frac{d^3p}{(2\pi)^3} \frac{1}{[p^2+\frac{1}{4}m^2(\frac{4}{3}\alpha_M)]^2} , \tag{4.26}$$

which is trivially integrated. Equation (4.24) now reads

$$\psi(0)_{\text{tot}} = \phi_0(0)\left[1 - \frac{2}{\pi}(\tfrac{4}{3}\alpha_M) - \left(\frac{3\lambda}{m^2} + \frac{1}{3m}V_s(0)\right)\right] + [\phi_0(0) - \phi_c(0)]\left[\frac{2}{\pi}\left(\frac{4}{3}\alpha_M\right)\right] + \cdots . \tag{4.27}$$

However, from Appendix A, Eq. (A5),

$$mV_s(0) = -[6\lambda + m\epsilon] . \tag{4.28}$$

Since the last term in (4.27) is higher order, the final result is

$$\psi(0)_{\text{tot}} = \phi_0(0)\left[1 - \frac{2}{\pi}\left(\frac{4}{3}\alpha_M\right) - \left(\frac{\lambda}{m^2}\right) + \cdots\right] = \phi_0(0)\left[1 - \frac{2}{\pi}\left(\frac{4}{3}\alpha_M\right) + \frac{V_s(0)}{6m} + \cdots\right] . \tag{4.29}$$

As a check, we have verified (4.29) in the limit $\alpha_M = 0$ in Appendix B, with the integrations carried out in a somewhat different way, which confirms (4.29). We conjecture that the retardation correction to the confining potential for a linear potential is of $O(V_s'(0)/m^2) \sim O(v^4)$, which is higher order than the terms retained in (4.29). If so, (4.29) is the complete correction to $\psi(0)$ to $O(\alpha_M) + O(\lambda)$.

V. DISCUSSION

Combining Eq. (3.18) with (4.29) one obtains the modified Van Royen–Weisskopf formula

$$\Gamma(V \to e^+e^-) = \frac{16\pi\alpha^2 e_Q^{\,2}}{M^2} |\phi_0(0)|^2 \times \left[1 - \frac{4}{\pi}\left(\frac{4}{3}\alpha_M\right) - 2\left(\frac{\lambda}{m^2}\right) + \cdots\right] . \tag{5.1}$$

Since

$$\phi_0(r) \simeq \phi_0(0)\left\{1 - \frac{4}{3}\left(\frac{\alpha_M m}{2}\right) r - \left[\lambda - \frac{m^2}{12}\left(\frac{4}{3}\alpha_M\right)^2\right] r^2 + O(r^3)\right\} \tag{5.2a}$$

$$\simeq \phi_0(0) \exp[-\tfrac{4}{3}\alpha_M(m/2)r] \exp(-\lambda r^2) + O(r^3) + O(\alpha_M^{\,2}) \quad \text{for small } r . \tag{5.2b}$$

Therefore,[15] accurate to $O(\alpha_M) + O(\lambda)$

$$\Gamma(V \to e^+e^-) = \frac{16\pi\alpha^2 e_Q^{\,2}}{M^2} \left|\phi_0\left(r = \frac{1}{m}\right)\right|^2 \times \left[1 - \tfrac{4}{3}\alpha_M\left(\frac{4}{\pi} - 1\right) + \cdots\right] . \tag{5.3}$$

At a more fundamental level one may inquire whether the factorized form (1.1) is valid to all orders in QCD, with the factor (annihilation)^2, computable and infrared finite in terms of the short-distance of the theory. Superficially Eq. (1.1) is of the same form as the factorization proved for the parton model. However, here we are concerned with *threshold* processes, where the understanding of the factorization is on a weaker footing. The only careful general discussion, aside from specific calculations, is that of Appelquist and Politzer,[1] who considered the limit $m_Q \to \infty$. It would be useful to our understanding of bound-state annihilation processes if the analysis of threshold processes in QCD could be given a firmer theoretical basis than presently available. This study is a beginning in that direction.

Finally we remark that Eq. (5.3) predicts that the ratio of the leptonic rates

$$\frac{\Gamma(\psi' \to e^+e^-)}{\Gamma(\psi \to e^+e^-)} < 1 \tag{5.4}$$

even for a pure linear potential. (By contrast this ratio is 1 if only the naive QCD correction[4] is employed.) We obviously also predict

$$\frac{\Gamma(\Upsilon' \to e^+e^-)}{\Gamma(\Upsilon \to e^+e^-)} < 1 \tag{5.5}$$

and its analog for other heavy $Q\bar{Q}$ systems. The actual ratio in both (5.4) and (5.5) is approximately 0.9 for a pure linear potential.[16] Further, since

$$|\phi_0(r = 1/m)|^2 < |\phi_0(0)|^2 , \tag{5.6}$$

the naive absolute rate is reduced accordingly. (In charmonium this is roughly a 20% effect.[16])

Note added. The procedure of Karplus and Klein is valid only for corrections dominated by the relativistic region $\vec{p}^2 \sim m^2$, and does not treat correctly effects for which $\vec{p}^2 \ll m^2$. As a result, if we consider a confining potential of the form

$$V_s(r) = V_s(0) + ar , \tag{5.7}$$

then our calculation is accurate only to $O(\alpha_M)$ and $O(\lambda/m^2)$, and to *zero-order* in β, where

$$\beta = (a/m^2)^{1/3} . \tag{5.8}$$

It has been estimated[17] that

$$\langle p^4/m^4\rangle \sim O(\beta^4) + O(\alpha_M\beta^3) \tag{5.9}$$

and hence the correction

$$\sum_n \langle \nabla^2 V_s(r)/m^2\rangle/\Delta\epsilon_n \sim \beta^2 . \tag{5.10}$$

Therefore, the omitted contributions to $\delta\phi_0(0)/\phi_0(0)$, coming from these corrections to the Bethe-Salpeter kernel, are of $O(\beta^2)$, and hence negligible to the order we are working. Moreover, Eq. (5.10) is related to the difficult question of retarding a confining potential. By contrast, the correction of $O(\lambda/m^2) \sim O(V_s(0)/m)$ is dominated by the ultraviolet region $\vec{p}^2 \sim m^2$ (cf. Appendix B), while corrections related to (5.9) and (5.10) require a more careful treatment of the region $\vec{p}^2 \ll m^2$ than is possible in the Karplus-Klein method. Notice that the average size of a heavy $Q\overline{Q}$ system such as charmonium may be estimated to be[17]

$$\langle r\rangle \sim (\beta m)^{-1} \gg m^{-1} \tag{5.11}$$

since $\beta^2 \sim 0.15$ for charmonium. Hence, our restriction to zeroth order in β, but first order in λ/m^2 and α_M, is consistent with our formulation of the problem, where

$$\lambda/m^2 = -V_s(0)/6m + O(\alpha_M{}^2) . \tag{5.12}$$

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APPENDIX A: SHORT-DISTANCE BEHAVIOR OF WAVE FUNCTIONS

Given,

$$\left(-\frac{\nabla^2}{m} + V(r)\right)\phi_0(r) = -\epsilon\,\phi_0(r) \tag{A1}$$

with

$$V(r) = -\frac{4}{3}\,\frac{\alpha_M}{r} + V_s(r) , \tag{A2}$$

we compute $\phi_0(r)$ for $r \to 0$. Let us expand

$$\phi_0(r) = \phi_0(0)(1 - Ar - Br^2 + \cdots) . \tag{A3}$$

Substituting in (A1) we find

$$\phi_0(r) \underset{r\to 0}{\sim} \phi_0(0)\left\{1 - \frac{4}{3}\,\alpha_M\left(\frac{m}{2}\right) r + \frac{mr^2}{6}\left[-\epsilon + V_s(0) + \frac{m}{2}\left(\frac{4}{3}\,\alpha_M\right)^2\right] + O(r^3)\right\} \tag{A4a}$$

$$\simeq \phi_0(0)\exp\left[-\frac{m}{2}\left(\frac{4}{3}\alpha_M\right)r\right]e^{-\lambda r^2} + O(r^3) + O(\alpha_M{}^2) , \tag{A4b}$$

where we have defined

$$\lambda = -\frac{m}{6}\left[\epsilon + V_s(0)\right] . \tag{A5}$$

The momentum-space version of (A1) is

$$\left(\epsilon + \frac{\vec{p}^2}{m}\right)\phi_0(\vec{p}) = -\int \frac{d^3q}{(2\pi)^3}\left[V_s(\vec{p}-\vec{q}) - \frac{\frac{4}{3}(4\pi\alpha_M)}{(\vec{p}-\vec{q})^2}\right]\phi_0(\vec{q}). \tag{A6}$$

Therefore,

$$\phi_0(\vec{p}) = \frac{-m}{\vec{p}^2 + m\epsilon}\int \frac{d^3q}{(2\pi)^3}\left[V_s(\vec{p}-\vec{q}) - \frac{4\pi(\frac{4}{3}\alpha_M)}{(\vec{p}-\vec{q})^2}\right]\phi_0(\vec{q}) . \tag{A7}$$

We assume that $V_s(r=0) \neq \infty$, so that

$$\vec{p}^2 V_s(\vec{p}^2) \xrightarrow[\vec{p}^2\to\infty]{} 0 ,$$

with the result that

$$\phi_0(\vec{p}) \xrightarrow[\vec{p}^2\to\infty]{} \frac{m}{(\vec{p}^2)^2}\int \frac{d^3q}{(2\pi)^3}\,\phi_0(\vec{q}) = \frac{m}{(\vec{p}^2)^2}\,\phi_0(r=0) . \tag{A8}$$

Both (A4) and (A8) assert that the leading behavior of $\phi_0(\vec{r})$ as $r \to 0$ is identical to that of the Coulomb wave function.

Finally, standard Schrödinger-perturbation-theoretic arguments imply that

$$[\phi_0(0) - \phi_c(0)] \xrightarrow[\substack{\lambda\to 0\\ \alpha_m\,\text{fixed}}]{} 0 . \tag{A9}$$

APPENDIX B: ALTERNATIVE CALCULATION OF $\psi(0)_{\text{scalar}}$

Let us evaluate $\psi(0)_{\text{scalar}}$, defined in Eq. (4.15) in the limit that $\alpha_M = 0$. Combining (4.15) with (A6) when $\alpha_M = 0$ gives

$$\psi(0)_{\text{scalar}} = i\int \frac{d^4p}{(2\pi)^4}\,\frac{\frac{1}{3}\vec{p} + (\frac{1}{2}M + m)^2 - p_0{}^2}{[(\frac{1}{2}M - p_0)^2 - E^2 + i\epsilon][\frac{1}{2}M + p_0)^2 - E^2 + i\epsilon]}\left(\epsilon + \frac{\vec{p}^2}{m}\right)\phi_0(\vec{p}) . \tag{B1}$$

The p_0 integration is easily performed, with the result

$$\psi(0)_{\text{scalar}} = \frac{1}{m} \int \frac{d^3p}{(2\pi)^3} \frac{m^2 + \frac{1}{3}\vec{p}^2}{E} \phi_0(\vec{p}), \qquad (B2)$$

where terms of $O(\epsilon)$ are neglected. Since

$$\phi_0(\vec{p}) \underset{\vec{p}^2 \to \infty}{\sim} (\vec{p}^2)^{-2}$$

from (A8), the integrand of (B2) can be expanded near $\vec{p}^2 \sim 0$ and

$$\psi(0)_{\text{scalar}} \simeq \int \frac{d^3p}{(2\pi)^3} \left(1 - \frac{1}{6}\frac{\vec{p}^2}{m^2} + \cdots\right)\phi_0(\vec{p})$$

$$\simeq \phi_0(0) + \frac{\nabla^2\phi_0(0)}{6m^2} + \cdots . \qquad (B3)$$

It is important to note that one *cannot* expand Eq. (4.19) in $\vec{p}^2$ when $\alpha_M \neq 0$, since in this case $\lim_{r\to 0} \nabla^2\phi_0(r)$ diverges linearly, as can be verified from Eqs. (A1) and (A2). Therefore, Eq. (B3) is valid only when $\alpha_M = 0$. From Eq. (A4) and (A5) we find

$$\psi(0)_{\text{scalar}} = \phi_0(0)\left(1 - \frac{\lambda}{m^2} + \cdots\right) \qquad (B4)$$

for $\alpha_M = 0$ with $6\lambda = -m[V_s(0) + \epsilon]$ from (A5). The result of this Appendix gives an independent check of our result (4.29). The computation in the body of the text must be performed somewhat carefully, since $\alpha_M \neq 0$.

We can give an alternative interpretation of Eqs. (B2)–(B4). From (B3)

$$\psi(0)_{\text{scalar}} \simeq \int d^3p\left(1 - \frac{\vec{p}^2}{6m^2} + \cdots\right)\phi_0(\vec{p})$$

$$\cong \int \frac{d\Omega_{\vec{r}}}{4\pi} \int d^3\vec{p}\, e^{i\vec{p}\cdot\vec{r}}\, \phi_0(\vec{p})\Big|_{r^2 = 1/m^2} . \qquad (B5)$$

Therefore,

$$\psi(0)_{\text{scalar}} = \left(\int \frac{d\Omega}{4\pi}\, \phi_0(\vec{r})\right)_{r^2 = 1/m^2} . \qquad (B6)$$

However, the ground-state wave function has no angular dependence, so that

$$\psi(0)_{\text{scalar}} = \phi_0(r = 1/m) \qquad (B7)$$

to the accuracy of our calculation. This conclusion does not depend on the details of $V_s(r)$. Further, we note from (B3) and (B4) that

$$\lambda = \tfrac{1}{6} \int \frac{d^3p}{(2\pi)^3} (\vec{p}^2)\phi_0(\vec{p}), \qquad (B8)$$

which relates λ to the typical (velocity)^2 of the state. This will also be approximately valid when $\alpha_M \neq 0$ as long as the confining potential determines the average velocity of the state, which is the case for both ψ and T states.

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