Medium-energy $N\pi\pi$ dynamics. I

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New formal apparatus is developed for the dynamics of the $N\pi\pi$ system. The ingredients are the isobar expansion with Bose symmetry, and subenergy unitarity and analyticity. The family of isobar amplitudes considered is substantial in number and realistic for a medium-energy treatment. Dispersion relations are written and the Pasquier inversion procedure is employed, leading to a set of coupled single-variable integral equations. The resulting linear system is suitable for investigating three-body dynamics and for carrying out phenomenological analyses.

I. INTRODUCTION

In a series of recent papers[1-3] we developed a practical treatment of relativistic three-hadron systems. The theory is based simply on the two principles of unitarity and analyticity, as applied to the two-body subenergy channels. In our approach the amplitude to the three-body final state is expanded as a superposition of isobar states in each of the three two-body channels. Each isobar contribution is expressed as the product of the requisite angular functions, the relevant two-body elastic amplitude in the isobar state, and a so-called "isobar factor." The constraints imposed by two-body unitarity in all three subenergy channels are conveniently formulated by means of such an expansion. The full system of constraints on the isobar factors was derived in I for the case of one spin-$\frac{1}{2}$ and two spin-0 hadrons. In the conventional isobar model the isobar factors are taken to be subenergy-independent and depend only on the three-body invariant mass. However, such an ansatz cannot satisfy the unitarity constraints; for this, a dependence of each isobar factor on its subenergy variable is necessary.[4] To incorporate analyticity, the unitarity relations are implemented by means of dispersion relations. The confluence of the isobar expansion with the methods of subenergy unitarity and analyticity leads to coupled linear equations governing the isobar factors which are remarkably tractable because they prove to be integral equations in a single variable.[3] Their solution in any suitably circumscribed context brings to fruition a proper description of the final-state interactions in a three-hadron system.

Although this approach to the relativistic three-body problem, originating[5] historically in the work of Khuri and Treiman[6] on final-state interactions in $K\to 3\pi$, focuses entirely on the *two-body* subsystems and variables, the resulting amplitudes have nevertheless been shown to satisfy *three-body* unitarity[7,8] (which is not an input), at least to good approximation.[9,10] This feature is to be contrasted with the hitherto more actively pursued approaches, such as the relativistic Faddeev[11] or Blankenbecler-Sugar[12,13] methods, in which both two- and three-body unitarity are used as input in constructing the dynamical equations. Earlier S-matrix approaches[14] also sought to incorporate three-body unitarity from the start, and have proved to be impracticable. The input for our three-body theory is certainly "minimal,"[15] yet the integral equations for the isobar factors have a structure at least as rich as that[16] provided in the approaches of Refs. 9–13. Indeed, as regards certain essentially relativistic short-range aspects, the minimal theory appears to be superior to these latter methods, at least to judge from numerical results which have been obtained for the 3π system.[17,18]

In II a system of coupled linear single-variable integral equations was derived for the $N\pi K$ system. All the technical problems associated with unequal-mass kinematics and with spin were solved; however, for the sake of simplicity, one restrictive assumption was made, namely, that of treating only s-wave effects. In the present paper we deal with the remaining complexities of nonzero orbital angular momentum. We derive a set of dynamical equations governing final-state interactions in the $N\pi\pi$ system, in all J^P states and isobar configurations which seem likely to be important at medium energy $W\lesssim 1.5$ GeV. These equations constitute the basis for a number of possible studies of dynamical $N\pi\pi$ problems, a project which we have deferred. Their application to a semiphenomenological investigation of an improved $N\pi\pi$ isobar model has been carried out and will be described in a separate paper.

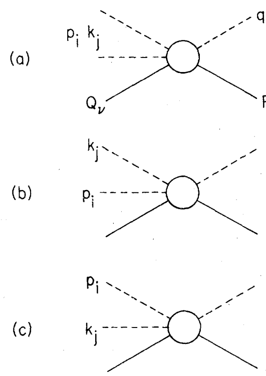

FIG. 1. (a) Bose-symmetric amplitude M_{32}, (b) and (c) unsymmetrized amplitudes $M(Qpk, Pq)$ and $M(Qkp, Pq)$. Momentum indices in (a) refer to pion isospin and nucleon helicity.

II. ISOBAR EXPANSION

The expansion of the final three-body state has been described in terms of isobars by several other authors.[19] We shall adopt the technique and notation of I; equations cited from that reference will be prefixed by I. In fact, large fragments of that paper may be applied directly to the process $N\pi \to N\pi\pi$, once we have incorporated the special feature of Bose symmetrization. It is instructive to discuss this carefully to see how this additional ingredient propagates through the formalism. Of course, the net effect of Bose symmetry is one of simplification; there are fewer amplitudes and correspondingly fewer coupled equations governing them.

In Fig. 1(a) we have the Bose-symmetric amplitude M_{32} defined by the S-matrix element:

$$\langle Q_\nu p_i k_j \,\mathrm{out}\,|P_\lambda q_k \,\mathrm{in}\rangle = i(2\pi)^4 \delta(Q+p+k-P-q) M_{32}/N(QpkPq)\,. \tag{1}$$

It may be written alternatively as

$$M_{32} = \begin{cases} [N(QpPq)\langle Q_\nu p_i \,\mathrm{out}\,| j_j | P_\lambda q_k \,\mathrm{in}\rangle \\ \quad + N(QkPq)\langle Q_\nu k_j \,\mathrm{out}\,| j_i | P_\lambda q_k \,\mathrm{in}\rangle]/\sqrt{2} \\ N(pkPq)\,\bar{u}_{Q_\nu} \langle p_i k_j \,\mathrm{out}\,| f | P_\lambda q_k \,\mathrm{in}\rangle\,. \end{cases} \tag{2}$$

The factors $N(\cdots)$ contain the normalization of states; j and f are the pion and nucleon source operators. We also introduce an unsymmetrized amplitude [Fig. 1(b)]

$$M(Qpk, Pq) = N(QpPq)\langle Q_\nu p_i \,\mathrm{out}\,| j_j | P_\lambda q_k \,\mathrm{in}\rangle \tag{3a}$$

and its companion [Fig. 1(c)]

$$M(Qkp, Pq) = N(QkPq)\langle Q_\nu k_j \,\mathrm{out}\,| j_i | P_\lambda q_k \,\mathrm{in}\rangle\,. \tag{3b}$$

Expression (3a) may be expanded in the manner of Eqs. (I.27)–(I.29). We choose the z axis along the initial momentum P and write

$$\begin{aligned} M(Qpk,Pq) = \sum_{JT} N_J^2 \Big\{ & \sum_{t_1 j_1 m_1 \mu} \mathfrak{A}_{ijk}^{Tt_1} N_{j_1} d^{1/2}_{\nu\mu}(\omega_1) d^{j_1}_{m_1\mu}(\vartheta_1) D^{J*}_{\lambda m_1}(r_1) \langle m_1\mu | M^{Jj_1}(s_1) | \lambda\rangle^{Tt_1} \\ & + \sum_{t_2 j_2 m_2 \mu} \mathfrak{B}_{ijk}^{Tt_2} N_{j_2} d^{1/2}_{\nu\mu}(\omega_2) d^{j_2}_{m_2\mu}(\vartheta_2) D^{J*}_{\lambda m_2}(r_2) e^{i\pi\nu} \langle m_2\mu | M^{Jj_2}(s_2) | \lambda\rangle^{Tt_2} \\ & + \sum_{t_3 l m} \mathfrak{C}_{ijk}^{Tt_3} N_l\, d^l_{-m0}(\vartheta_3) D^{J*}_{\lambda,\nu-m}(r_3) \langle \nu m | M^{Jl}(s_3) | \lambda\rangle^{Tt_3} \Big\}\,. \end{aligned} \tag{4}$$

The phase factor $e^{i\pi\nu}$ in isobar channel 2 was erroneously left out of Eq. (I.28). The isospin projection operators $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ are listed in Appendix A, Eqs. (A1)–(A3). We can show by exchanging the variables $p_i \leftrightarrow k_j$ that $M(Qkp, Pq)$ is the same as (4) except for the additional factor $(-1)^{t_3+l}$ in isobar channel 3. The invariants s_n are as shown in Fig. 2.

Clearly, the symmetrized amplitude is just

$$M_{32} = [M(Qpk, Pq) + M(Qkp, Pq)]/\sqrt{2}\,.$$

If we recall how the three rotations of the final state are related [see Eq. (I.26)],

$$r_1 r_{0\chi_a 0} = r_2 r_{00\pi} r_{0\chi_b 0}{}^{-1} = r_3\,, \tag{5}$$

we can express M_{32} in terms of the Euler angles of the single rotation r_3:

$$\begin{aligned} M_{32} = \sqrt{2} \sum_{TJ\sigma} N_J^2 D^{J*}_{\lambda\sigma}(r_3) \Big\{ & \sum_{t_1 j_1 m_1 \mu} \mathfrak{A}_{ijk}^{Tt_1} N_{j_1} d^{1/2}_{\nu\mu}(\omega_1) d^{j_1}_{m_1\mu}(\vartheta_1) d^J_{m_1\sigma}(\chi_a) \langle m_1\mu | M^{Jj_1}(s_1) | \lambda\rangle^{Tt_1} \\ & + \sum_{t_2 j_2 m_2 \mu} \mathfrak{B}_{ijk}^{Tt_2} N_{j_2} d^{1/2}_{\nu\mu}(\omega_2) d^{j_2}_{m_2\mu}(\vartheta_2) d^J_{\sigma m_2}(\chi_b) e^{i\pi(\nu-m_2)} \langle m_2\mu | M^{Jj_2}(s_2) | \lambda\rangle^{Tt_2} \\ & + \sum_{t_3 l m} \mathfrak{C}_{ijk}^{Tt_3} N_l d^l_{-m0}(\vartheta_3) \delta_{\sigma,\nu-m} \tfrac{1}{2}[1+(-1)^{t_3+l}] \langle \nu m | M^{Jl}(s_3) | \lambda\rangle^{Tt_3} \Big\}\,. \end{aligned} \tag{6}$$

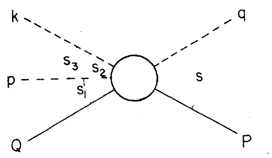

FIG. 2. Subenergy variables for $N\pi \to N\pi\pi$.

We refer the reader to I, especially Fig. 8, for the definition of all the angles. Our next objective is to obtain the unitarity relations satisfied by the isobar amplitudes appearing in these expansions. We emphasize that it is subenergy unitarity which concerns us, because we wish to assemble the isobar structure of the production amplitude according to all of its isobar variables: angular momentum *and* invariant mass in all three isobar channels.

III. DISCONTINUITIES

In Fig. 2 we recall the invariants for the process; we have $s = W^2$ and $s_n = w_n{}^2$, where W is the total energy and w_n the nth subenergy. The discontinuity in each subenergy variable s_n (disc_n, for short) is indicated in Fig. 3. When we evaluate disc_1 and disc_2 we must distinguish the pion variables p_i and k_j; accordingly, we use the expansion of $M(Qpk, Pq)$ to obtain disc_1 and $M(Qkp, Pq)$ to obtain disc_2. For disc_1 the result is the same as Eq. (I.39); for disc_2 the result is the same as Eq. (I.41), with the additional factor $(-1)^{t_3+l}$ in isobar channel 3. To obtain disc_3 we use expansion (6) of M_{32}; the result is the same as Eq. (I.44), with the occurrence on the left-hand side of the extra factor $\frac{1}{2}[1+(-1)^{t_3+l}]$. [It should be noted that the phase correction, mentioned below Eq. (4), propagates into these results as well, in each case accompanying the amplitude in isobar channel 2.] The $N\pi$ phase-space factor appearing in disc_1 is

$$\rho_1 = MQ_1/16\pi^3 w_1\,, \tag{7}$$

and similarly for ρ_2 occurring in disc_2. The $\pi\pi$ phase-space factor in disc_3 is

$$\rho_3 = k_3/64\pi^3 w_3\,. \tag{8}$$

Q_1, Q_2, and k_3 are the momenta in the rest frame of each of the three isobar channels [see I, Fig. 8 and Eqs. (I.70)–(I.72)].

Parity is built into the construction in two steps: Amplitudes of definite $N\pi$ isobar parity j^p are defined as in Eq. (I.50) and amplitudes of definite overall parity J^P are as prescribed in Eqs. (I.B1)

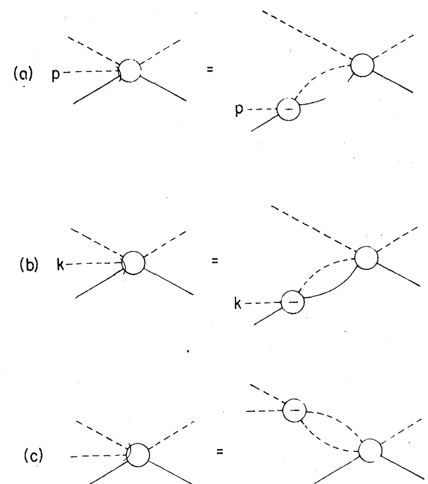

FIG. 3. Subenergy discontinuities: (a) disc_1, (b) disc_2, and (c) disc_3.

and (I.B2). We obtain in the process isobar amplitudes labeled by the following:

total energy, isospin, angular momentum, and parity, and

isobar subenergy, isospin, angular momentum, parity, and multiplicity.

The discontinuity relations simplify as a result of adopting the product form

(isobar amplitude M) = (two-body amplitude M)

× (isobar factor $\mathfrak{M}$).

To express this in detail we have

$$M^{TJ^Pt_1j_1{}^p\kappa_1}(ss_1) = M^{t_1j_1{}^p}(s_1)\mathfrak{M}^{TJ^Pt_1j_1{}^p\kappa_1}(ss_1) \tag{9a}$$

and

$$M^{TJ^Pt_3l\xi}(ss_3) = M^{t_3l}(s_3)\mathfrak{M}^{TJ^Pt_3l\xi}(ss_3)\,. \tag{9b}$$

The multiplicities κ and ξ range over

$\kappa = j, \ldots, \frac{1}{2}$ $(j+\frac{1}{2}$ values),

$\xi = l, \ldots, -l$ $(2l+1$ values).

The discontinuity relations satisfied by the isobar factors are the same as those given in II, Eqs. (II.A1)–(II.A3), with the inclusion of the two additional factors: $(-1)^{t_3+l}$ in the contribution of isobar channel 3 to disc_2, and $\frac{1}{2}[1+(-1)^{t_3+l}]$ on the left-hand side in disc_3. These lengthy relations can be made much more compact.[20] We shall arrange the dependence on the isobar isospin indices into column matrix form and suppress all other labels except for the multiplicities κ and ξ to write

$$\frac{\text{disc}_1\mathfrak{M}_1^{\kappa_1}}{2\pi i} = \rho_1 \int d\cos\vartheta_1 2\pi N_{j_1}\left\{\sum_2 N_{j_2}((e_1^T\alpha e_2),(e_1^T\beta e_2))(\tilde{d}_{\chi_{ab}})DM_2\mathfrak{M}_2^{\kappa_2}(-1)^{1/2-\kappa_2}+\sum_3 N_l\, d^l_{0\,\xi}(\vartheta_3)(e_1^T\hat{d}_{\chi_a})\bar{C}M_3\mathfrak{M}_3^{\xi}\right\}, \tag{10}$$

$$\frac{\mathrm{disc}_2 \mathfrak{M}_2^{\kappa_2}}{2\pi i} = \text{same as (10) with } 1 \leftrightarrow 2 \text{ and } a \leftrightarrow b\,, \tag{11}$$

$$\frac{\mathrm{disc}_3 \mathfrak{M}_3^{\xi}}{2\pi i} \tfrac{1}{2}[1+(-1)^{t_3+l}] = \rho_3 \int d\cos\vartheta_3 2\pi N_l \Big\{ \sum_1 N_{j_1} d^l_{0\,\xi}(\vartheta_3)(e_1^T \hat{d}_{\chi_a}) C M_1 \mathfrak{M}_1^{\kappa_1} + \sum_2 N_{j_2} d^l_{0\,\xi}(\pi-\vartheta_3)(e_2^T \hat{d}_{\chi_b}) C M_2 \mathfrak{M}_2^{\kappa_2}(-1)^{t_3+l} \Big\}\,. \tag{12}$$

C, $\bar{C}$, and D are isospin crossing matrices identified in Appendix A, Eqs. (A10) and (A11). The reader may refer to Ref. 20 for the definition of the two-component column matrices e, $\tilde{d}$, and $\hat{d}$; α and β are two-by-two matrices defined there. In (10) we have defined the angle $\chi_{ab} = \chi_a + \chi_b$. The numerical subscripts in these equations serve to abbreviate complete sets of suppressed isobar quantum numbers.

At this point it is clear that Bose symmetry has run its course. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy identical equations and are therefore given by the same function:

$$\mathfrak{M}_1(s_1) = \mathfrak{M}(s_1) \quad\text{and}\quad \mathfrak{M}_2(s_2) = \mathfrak{M}(s_2)\,. \tag{13}$$

With that observation the isobar factor discontinuities take their final form

$$\frac{1}{2\pi i}\,\mathrm{disc}_1 \mathfrak{M}^{\kappa_1}(s_1) = \frac{M}{16\pi^3 K_1}\left\{ \int_{s_{2<}}^{s_{2>}} ds_2 \sum_2 X_{12} D M(s_2) \mathfrak{M}^{\kappa_2}(s_2)(-1)^{1/2-\kappa_2} + \int_{s_{3<}}^{s_{3>}} ds_3 \sum_3 X_{13} \bar{C} M_3(s_3) \mathfrak{M}_3^{\xi}(s_3) \right\}, \tag{14}$$

$$\frac{1}{2\pi i}\,\mathrm{disc}_3 \mathfrak{M}_3^{\xi}(s_3) = \frac{1}{64\pi^3 K_3}\int_{s_{1<}}^{s_{1>}} ds_1 \sum_1 2 X_{13} C M(s_1) \mathfrak{M}^{\kappa_1}(s_1)\,, \tag{15}$$

in which X_{12} and X_{13} are the angular crossing coefficients[20]:

$$X_{12} = 2\pi N_{j_1} N_{j_2} \big((e_1^T \alpha e_2), (e_1^T \beta e_2)\big)(\tilde{d}_{\chi_{ab}})\,, \tag{16}$$

$$X_{13} = 2\pi N_{j_1} N_l d^l_{0\,\xi}(\vartheta_3)(e_1^T \hat{d}_{\chi_a})\,. \tag{17}$$

To obtain (14) and (15) we have employed Eqs. (7) and (8), along with Eqs. (II.21)–(II.23), and we have used $K_1 = 2WQ_a$ and $K_3 = 2WQ$ [see also Eqs. (I.65), (I.67), and (II.B2)]. The end points of integration in (14) and (15) are indicated in Fig. 4.

It is clear that a coupled system of integral equations, linear in the $\mathfrak{M}$'s, would be the result of inserting these discontinuities into dispersion relations. To make progress it is necessary to confine the scope of the problem somewhat by choosing to couple a family of isobars of manageable size.

IV. MEDIUM-ENERGY ISOBAR SYSTEM

We shall restrict our attention to single pion production at medium energy, say $W \lesssim 1.5$ GeV. We may then invoke orbital angular momentum suppression to select only the following isobar systems:

all s-wave isobars ($N\pi$ S_{11} and S_{31}, $\pi\pi$ S_0 and S_2) in S states,

active p-wave isobars ($N\pi$ P_{11} and P_{33}, $\pi\pi$ P_1) in S states,

and the $N\pi$ P_{33} isobar in P states.

The values of J^P required are $\frac{1}{2}^+$, $\frac{1}{2}^-$, $\frac{3}{2}^+$, and $\frac{3}{2}^-$. (We disregard $J^P = \frac{5}{2}^+$ because, in the initial state, the $N\pi$ f wave has a negligible phase shift.) In Tables I and II we have listed the amplitudes we wish to consider; Fig. 5 provides a guide to the quantum numbers and orbital angular momenta cited in the tables. In the rightmost column of each table we have given the combination of amplitudes $\mathfrak{M}^{j^P\kappa}$ and $\mathfrak{M}_3^{l\,\xi}$ we require for the desired orbital angular momentum dominating at low energy. Jacob and Wick,[21] Appendix B, has been used for this purpose.

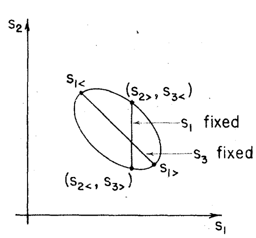

FIG. 4. Paths of integration across the Dalitz plot in Eqs. (14) and (15).

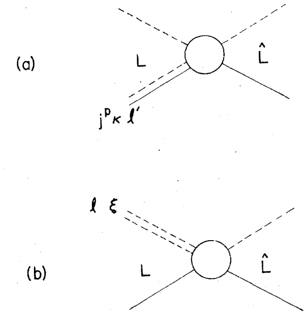

FIG. 5. Orbital angular momenta and isobar quantum numbers: (a) ($N\pi$) isobar + π, (b) ($\pi\pi$) isobar + N.

TABLE I. Contributing states for $(N\pi)$ isobar $+\pi$ [see Fig. 5(a)].

J^P	$\hat{L}$	$(N\pi)$ isobar	j^p	κ	Dominant L	Corresponding amplitude in terms of $\mathfrak{M}(j^p\kappa)$
$\frac{1}{2}^+$	1	S_{11} and S_{31}	$\frac{1}{2}^-$	$\frac{1}{2}$	0	$\mathfrak{M}_S=\mathfrak{M}(\frac{1}{2}^-\frac{1}{2})$
		P_{33}	$\frac{3}{2}^+$	$\frac{1}{2}$ only	1	$\mathfrak{M}_P=\mathfrak{M}(\frac{3}{2}^+\frac{1}{2})$
$\frac{1}{2}^-$	0	P_{11}	$\frac{1}{2}^+$	$\frac{1}{2}$	0	$\mathfrak{M}=\mathfrak{M}(\frac{1}{2}^+\frac{1}{2})$
$\frac{3}{2}^+$	1	P_{33}	$\frac{3}{2}^+$	$\frac{3}{2},\frac{1}{2}$	1	$\mathfrak{M}=-[3\mathfrak{M}(\frac{3}{2}^+\frac{3}{2})+\mathfrak{M}(\frac{3}{2}^+\frac{1}{2})]/\sqrt{20}$
$\frac{3}{2}^-$	2	P_{33}	$\frac{3}{2}^+$	$\frac{3}{2},\frac{1}{2}$	0	$\mathfrak{M}=[\mathfrak{M}(\frac{3}{2}^+\frac{3}{2})+\mathfrak{M}(\frac{3}{2}^+\frac{1}{2})]/2$
$\frac{5}{2}^+$	3	P_{33}	$\frac{3}{2}^+$	$\frac{3}{2},\frac{1}{2}$	1	Disregarded

Now that the scope of the problem has been suitably restricted we see that in Eqs. (14) and (15) there are (apart from isospin multiplicity) three coupled amplitudes for $J^P=\frac{1}{2}^+$, two for $\frac{1}{2}^-$ and for $\frac{3}{2}^-$, and only one for $\frac{3}{2}^+$. To calculate all of the required angular crossing coefficients according to (16) and (17) and transform to the orbital states in Tables I and II is a straightforward, albeit, tedious task. The burden is ameliorated somewhat if we adopt what was referred to in Ref. 20 as the Euclidean approximation, a step which eliminates the occurrence of half-angles. This simplification is, in fact, the setting to zero of the Stapp angle[22] (there are two of them here, called ϵ_1 and ϵ_2 in Refs. 3 and 20). The approximation is commonly made in phenomenological applications of the isobar model.[23]

The discontinuity relations which couple the amplitudes listed in Tables I and II have the same structure as Eqs. (14) and (15):

$$\frac{1}{2\pi i}\,\mathrm{disc}_1\,\mathfrak{M}(s_1)=\frac{M}{16\pi^3K_1}\left[\int ds_2\sum_2 Y_{12}DM(s_2)\mathfrak{M}(s_2)+\int ds_3\,Y_{13}\bar{C}M_3(s_3)\mathfrak{M}_3(s_3)\right], \tag{18}$$

$$\frac{1}{2\pi i}\,\mathrm{disc}_3\,\mathfrak{M}_3(s_3)=\frac{1}{64\pi^3K_3}\int ds_1\sum_1 Y_{31}CM(s_1)\mathfrak{M}(s_1). \tag{19}$$

The angular crossing coefficients Y_{12}, Y_{13}, and Y_{31} are tabulated for each J^P in Table III. We have defined the angles $\chi_{ab}=\chi_a+\chi_b$ and $\omega_{12}=\omega_1+\omega_2$.

The amplitudes of our medium-energy isobar system have definite orbital angular momenta, a feature which dictates their kinematic behavior near threshold. The corresponding kinematic factors must be extracted before we can write dispersion relations.

V. THRESHOLD BEHAVIOR

Each of the amplitudes appearing in the formalism must vanish at threshold with the appropriate powers of the momenta, the powers being given by the orbital angular momenta as sketched in Fig. 5. For the elastic two-body amplitudes in the isobar channels we have

$$M(s_1)\sim Q_1^{2l'} \tag{20}$$

and

$$M_3(s_3)\sim k_3^{2l}. \tag{21}$$

For the isobar amplitudes we require

$$M(ss_1)\sim Q_a^{\,L}Q_1^{\,l'}P^{\hat{L}} \tag{22}$$

and

$$M_3(ss_3)\sim Q^L k_3^{\,l}P^{\hat{L}}, \tag{23}$$

in which Q_a, Q, and P are the momenta in the center-of-mass system of the $(N\pi)$ isobar, the final nucleon, and the initial nucleon, respectively. We have already adopted an amplitude-product form in Eq. (9), where the isobar factors were introduced. Let us go one step further and incorporate into each

TABLE II. Contributing states for $(\pi\pi)$ isobar $+N$ [see Fig. 5(b)].

J^P	$\hat{L}$	$(\pi\pi)$ isobar	l	ξ	$L=0$ amplitude in terms of $\mathfrak{M}_3(l\xi)$
$\frac{1}{2}^+$	1	S_0 and S_2	0	0	$\mathfrak{M}_3=\mathfrak{M}_3(00)$
$\frac{1}{2}^-$	0	P_1	1	1, 0	$\mathfrak{M}_3=[\sqrt{2}\mathfrak{M}_3(11)+\mathfrak{M}_3(10)]/\sqrt{6}$
$\frac{3}{2}^-$	2	P_1	1	1, 0, −1	$\mathfrak{M}_3=[\mathfrak{M}_3(11)+\sqrt{2}\mathfrak{M}_3(10)+\sqrt{3}\mathfrak{M}_3(1-1)]/2\sqrt{3}$

isobar amplitude a factor, call it $\hat{M}(s)$, which describes elastic scattering in the initial $N\pi$ state, with the property that at threshold

$$\hat{M}(s) \sim P^{2\hat{L}} . \tag{24}$$

Because of these considerations we are led to extract kinematic factors and elastic amplitude factors from the isobar amplitudes in the following way:

$$M(ss_1) = \frac{(2WQ_a)^L}{(2w_1Q_1)^{l'}} \frac{\hat{M}(s)}{(2WP)^{\hat{L}}} M(s_1)f(ss_1) \tag{25}$$

[suppressed quantum numbers: $TJ^Pt_1j_1^p$ for $M(ss_1)$ and $f(ss_1)$, TJ^P for $\hat{M}(s)$, and $t_1j_1^p$ for $M(s_1)$] and

$$M_3(ss_3) = \frac{(2WQ)^L}{(2w_3k_3)^l} \frac{\hat{M}(s)}{(2WP)^{\hat{L}}} M_3(s_3)f_3(ss_3) \tag{26}$$

[suppressed quantum numbers: TJ^Pt_3l for $M_3(ss_3)$ and $f_3(ss_3)$, and t_3l for $M_3(s_3)$]. The newly defined functions f and f_3 are the amplitudes for which we wish to obtain the integral equations in their final form.

The isobar factors appearing in Eqs. (18) and (19) are related to f and f_3 as follows:

For $J^P = \frac{1}{2}^+$

$$\mathfrak{M}_S = \frac{\hat{M}^{1/2^+}}{2WP} f_S ,\quad \mathfrak{M}_P = \frac{2WQ_a}{2w_1Q_1}\frac{\hat{M}^{1/2^+}}{2WP} f_P ,\quad \mathfrak{M}_3 = \frac{\hat{M}^{1/2^+}}{2WP} f_3 . \tag{27}$$

For $J^P = \frac{1}{2}^-$

$$\mathfrak{M} = \frac{1}{2w_1Q_1}\hat{M}^{1/2^-} f ,\quad \mathfrak{M}_3 = \frac{1}{2w_3k_3}\hat{M}^{1/2^-} f_3 . \tag{28}$$

For $J^P = \frac{3}{2}^+$

$$\mathfrak{M} = \frac{2WQ_a}{2w_1Q_1}\frac{\hat{M}^{3/2^+}}{2WP} f . \tag{29}$$

For $J^P = \frac{3}{2}^-$

$$\mathfrak{M} = \frac{1}{2w_1Q_1}\frac{\hat{M}^{3/2^-}}{(2WP)^2} f ,\quad \mathfrak{M}_3 = \frac{1}{2w_3k_3}\frac{\hat{M}^{3/2^-}}{(2WP)^2} f_3 . \tag{30}$$

We see in Eqs. (27)–(30) that the kinematic factors themselves have branch cuts in the subenergy variables. Because of this we must acknowledge that along the way to Eqs. (18) and (19) we have been premature in adopting discontinuity notation, a usage which presupposes that we are dealing with functions which have been continued analytically in the cut subenergy planes. Clearly, it is f and f_3 which have the latter property; it is in terms of these functions that dispersion relations may be written. We shall employ Eqs. (18) and (19) and make the replacement

$$\text{disc}\,\mathfrak{M} \to [\text{kinematic factors}]\,\text{disc} f \tag{31}$$

as a procedure equivalent to identifying the f's first and then evaluating their discontinuities.

In accord with (20) and (21) we extract kinematic behavior from the elastic two-body amplitudes by defining

$$M^{j^p}(s_1) = (32\pi^3/M)(2w_1Q_1)^{2l'}\zeta^{j^p}(s_1) \tag{32}$$

and

$$M_3^l(s_3) = 128\pi^3(2w_3k_3)^{2l}\zeta_3^l(s_3) . \tag{33}$$

The new functions ζ and ζ_3 must satisfy unitarity and will be parametrized to fit the available data.

When Eqs. (27)–(33) are assembled into (18) and (19) with the aid of Table III we obtain the final form for the discontinuity relations.

For $J^P = \frac{1}{2}^+$

$$\frac{1}{2\pi i}\,\text{disc}_1 f_S(s_1) = \frac{1}{K_1}\left\{\int ds_2[D\zeta^{1/2^-}(s_2)f_S(s_2) + \sqrt{2}\,R_2D\zeta^{3/2^+}(s_2)f_P(s_2)] + \int ds_3 4M\bar{C}\zeta_3^0(s_3)f_3(s_3)\right\}, \tag{34a}$$

$$\frac{1}{2\pi i}\,\text{disc}_1 f_P(s_1) = \frac{1}{K_1^3}\left\{\int ds_2[\sqrt{2}R_1D\zeta^{1/2^-}(s_2)f_S(s_2) + 2R_{12}D\zeta^{3/2^+}(s_2)f_P(s_2)] + \int ds_3 4\sqrt{2}\,MR_1\bar{C}\zeta_3^0(s_3)f_3(s_3)\right\}, \tag{34b}$$

$$\frac{1}{2\pi i}\,\text{disc}_3 f_3(s_3) = \frac{1}{2MK_3}\int ds_1[C\zeta^{1/2^-}(s_1)f_S(s_1) + \sqrt{2}R_1C\zeta^{3/2^+}(s_1)f_P(s_1)] . \tag{34c}$$

For $J^P = \frac{1}{2}^-$

$$\frac{1}{2\pi i}\,\text{disc}_1 f(s_1) = \frac{1}{K_1}\left[\int ds_2\Omega D\zeta^{1/2^+}(s_2)f(s_2) + \int ds_3 4\sqrt{2}M\Upsilon\bar{C}\zeta_3^1(s_3)f_3(s_3)\right], \tag{35a}$$

$$\frac{1}{2\pi i}\,\text{disc}_3 f_3(s_3) = \frac{1}{2M\sqrt{2}K_3}\int ds_1\Upsilon C\zeta^{1/2^+}(s_1)f(s_1) . \tag{35b}$$

TABLE III. Angular crossing coefficients Y_{12}, Y_{13}, and Y_{31} in Eqs. (18) and (19).

J^P	Discontinuity of	Coupling to	$\mathfrak{M}_S$	$\mathfrak{M}_P$	$\mathfrak{M}_3$
$\frac{1}{2}^+$	$\mathfrak{M}_S$		$\frac{1}{2}$	$\left(\frac{1}{2}\right)^{1/2}\cos\vartheta_2$	$\frac{1}{2}$
	$\mathfrak{M}_P$		$\left(\frac{1}{2}\right)^{1/2}\cos\vartheta_1$	$\cos\vartheta_1\cos\vartheta_2-\frac{1}{4}\sin\vartheta_1\sin\vartheta_2$	$\left(\frac{1}{2}\right)^{1/2}\cos\vartheta_1$
	$\mathfrak{M}_3$		1	$\sqrt{2}\cos\vartheta_1$	
			$\mathfrak{M}$	$\mathfrak{M}_3$	
$\frac{1}{2}^-$	$\mathfrak{M}$		$\frac{1}{2}\cos\omega_{12}$	$\left(\frac{1}{2}\right)^{1/2}\cos(\vartheta_3-\omega_1)$	
	$\mathfrak{M}_3$		$\left(\frac{1}{2}\right)^{1/2}\cos(\vartheta_3-\omega_1)$		
			$\mathfrak{M}$		
$\frac{3}{2}^+$	$\mathfrak{M}$		$\frac{3}{40}\cos(\omega_{12}-\chi_{ab})+\frac{1}{40}\cos\vartheta_1\cos\vartheta_2-\frac{5}{8}\sin\vartheta_1\sin\vartheta_2$		
			$\mathfrak{M}$	$\mathfrak{M}_3$	
$\frac{3}{2}^-$	$\mathfrak{M}$		$\frac{1}{2}\cos\omega_{12}$	$\frac{1}{2}\cos(\vartheta_3-\omega_1)$	
	$\mathfrak{M}_3$		$\cos(\vartheta_3-\omega_1)$		

For $J^P=\frac{3}{2}^+$

$$\frac{1}{2\pi i}\operatorname{disc}_1 f(s_1)=\frac{1}{K_1^{\,3}}\int ds_2 Z D\zeta^{3/2^+}(s_2)f(s_2)\,. \tag{36}$$

For $J^P=\frac{3}{2}^-$

$$\frac{1}{2\pi i}\operatorname{disc}_1 f(s_1)=\frac{1}{K_1}\left[\int ds_2\Omega D\zeta^{3/2^+}(s_2)f(s_2)+\int ds_3 4M\Upsilon\bar{C}\zeta_3^1(s_3)f_3(s_3)\right], \tag{37a}$$

$$\frac{1}{2\pi i}\operatorname{disc}_3 f_3(s_3)=\frac{1}{2MK_3}\int ds_1\Upsilon C\zeta^{3/2^+}(s_1)f(s_1)\,. \tag{37b}$$

In these equations we have defined the quantities

$$\begin{aligned}
R_1&=(2WQ_a)(2w_1Q_1)\cos\vartheta_1\,,\\
R_2&=(2WQ_b)(2w_2Q_2)\cos\vartheta_2\,,\\
R_{12}&=(2WQ_a)(2w_1Q_1)(2WQ_b)(2w_2Q_2)(\cos\vartheta_1\cos\vartheta_2-\tfrac{1}{4}\sin\vartheta_1\sin\vartheta_2)\,,\\
\Omega&=(2w_1Q_1)(2w_2Q_2)\cos\omega_{12}\,,\\
\Upsilon&=(2w_1Q_1)(2w_3k_3)\cos(\vartheta_3-\omega_1)\,,\\
Z&=(2WQ_a)(2w_1Q_1)(2WQ_b)(2w_2Q_2)[\tfrac{3}{20}\cos(\omega_{12}-\chi_{ab})+\tfrac{1}{20}\cos\vartheta_1\cos\vartheta_2-\tfrac{5}{4}\sin\vartheta_1\sin\vartheta_2]\,.
\end{aligned} \tag{38}$$

When these are written out in terms of the invariants they become the lengthy expressions listed in Appendix A, Eqs. (A12)–(A17).

Equations (34)–(37) are discontinuity formulas in a form appropriate for insertion into dispersion relations.

VI. INTEGRAL EQUATIONS

The subenergy dependence of the f's is given by the analytic representation

$$f(s_i)=c+\int_{s_{i0}}^{\infty}\frac{dz_i}{z_i-s_i}\,\frac{1}{2\pi i}\operatorname{disc}_i f(z_i)\,, \tag{39}$$

where the c's have no unitarity cut in s_i. The discontinuities (34)–(37) are linear in the f's and are themselves integrals at fixed z_i in another invariant z_j. As such, expression (39) provides a system of linear coupled integral equations for the f's, but with a double-variable domain of integration. Fortunately, the order of integration over z_i and z_j can be reversed so that the kernel contains an integral over z_i which can be calculated explicitly since it involves only known kinematic factors. This step leads to integral equations in the single variable z_j. The technique for inverting the order of integration is due to Pasquier and Pasquier[8]; it has been described in detail in the context of this kind of problem in II.

When the Pasquier inversion procedure is carried out we obtain the following system of integral equations:

For $J^P = \frac{1}{2}^+$

$$f_S(s_1) = c_S + \int_{-\infty}^{\hat{z}_2} dz_2 H_{SS}(s_1 z_2) D\zeta^{1/2^-}(z_2) f_S(z_2) + \int_{-\infty}^{\hat{z}_2} dz_2 H_{SP}(s_1 z_2) D\zeta^{3/2^+}(z_2) f_P(z_2) + \int_{-\infty}^{\hat{z}_3} dz_3 H_{S3}(s_1 z_3) \overline{C}\zeta_3^0(z_3) f_3(z_3), \qquad (40a)$$

$$f_P(s_1) = c_P + \int_{-\infty}^{\hat{z}_2} dz_2 H_{PS}(s_1 z_2) D\zeta^{1/2^-}(z_2) f_S(z_2) + \int_{-\infty}^{\hat{z}_2} dz_2 H_{PP}(s_1 z_2) D\zeta^{3/2^+}(z_2) f_P(z_2) + \int_{-\infty}^{\hat{z}_3} dz_3 H_{P3}(s_1 z_3) \overline{C}\zeta_3^0(z_3) f_3(z_3), \qquad (40b)$$

$$f_3(s_3) = c_3 + \int_{-\infty}^{\hat{z}_1} dz_1 H_{3S}(s_3 z_1) C\zeta^{1/2^-}(z_1) f_S(z_1) + \int_{-\infty}^{\hat{z}_1} dz_1 H_{3P}(s_3 z_1) C\zeta^{3/2^+}(z_1) f_P(z_1). \qquad (40c)$$

For $J^P = \frac{1}{2}^-$

$$f(s_1) = c + \int_{-\infty}^{\hat{z}_2} dz_2 H_{12}(s_1 z_2) D\zeta^{1/2^+}(z_2) f(z_2) + \int_{-\infty}^{\hat{z}_3} dz_3 H_{13}(s_1 z_3) \overline{C}\zeta_3^1(z_3) f_3(z_3), \qquad (41a)$$

$$f_3(s_3) = c_3 + \int_{-\infty}^{\hat{z}_1} dz_1 H_{31}(s_3 z_1) C\zeta^{1/2^+}(z_1) f(z_1). \qquad (41b)$$

For $J^P = \frac{3}{2}^+$

$$f(s_1) = c + \int_{-\infty}^{\hat{z}_2} dz_2 H(s_1 z_2) D\zeta^{3/2^+}(z_2) f(z_2). \qquad (42)$$

For $J^P = \frac{3}{2}^-$

$$f(s_1) = c + \int_{-\infty}^{\hat{z}_2} dz_2 H'_{12}(s_1 z_2) D\zeta^{3/2^+}(z_2) f(z_2) + \int_{-\infty}^{\hat{z}_3} dz_3 H'_{13}(s_1 z_3) \overline{C}\zeta_3^1(z_3) f_3(z_3), \qquad (43a)$$

$$f_3(s_3) = c_3 + \int_{-\infty}^{\hat{z}_1} dz_1 H'_{31}(s_3 z_1) C\zeta^{3/2^+}(z_1) f(z_1). \qquad (43b)$$

The upper limits of the above integrals are $\hat{z}_1 = \hat{z}_2 = (W-\mu)^2$ and $\hat{z}_3 = (W-M)^2$. Equations (40a)–(43b) are the analogs of Eqs. (II.41)–(II.43); in the present case, the two isobar channels 1 and 2 are identical [cf. Eq. (13)]. The kernel factors $H_{\alpha\beta}(s_i z_j)$ are as follows:

$$H_{SS}(s_1 z_2) = \int_{12} \frac{dz_1}{(z_1 - s_1)K_1}, \qquad (44)$$

$$H_{SP}(s_1 z_2) = \sqrt{2} \int_{12} \frac{dz_1 R_2}{(z_1 - s_1)K_1}, \qquad (45)$$

$$H_{S3}(s_1 z_3) = 4M \int_{13} \frac{dz_1}{(z_1 - s_1)K_1}, \qquad (46)$$

$$H_{PS}(s_1 z_2) = \sqrt{2} \int_{12} \frac{dz_1 R_1}{(z_1 - s_1)K_1^3}, \qquad (47)$$

$$H_{PP}(s_1 z_2) = 2 \int_{12} \frac{dz_1 R_{12}}{(z_1 - s_1)K_1^3}, \qquad (48)$$

$$H_{P3}(s_1 z_3) = 4\sqrt{2}M \int_{13} \frac{dz_1 R_1}{(z_1 - s_1)K_1^3}, \qquad (49)$$

$$H_{3S}(s_3 z_1) = \frac{1}{2M} \int_{31} \frac{dz_3}{(z_3 - s_3)K_3}, \qquad (50)$$

$$H_{3P}(s_3 z_1) = \frac{1}{\sqrt{2}M} \int_{31} \frac{dz_3 R_1}{(z_3 - s_3)K_3}, \qquad (51)$$

$$H_{12}(s_1 z_2) = \int_{12} \frac{dz_1 \Omega}{(z_1 - s_1)K_1} = H'_{12}(s_1 z_2), \qquad (52)$$

$$H_{13}(s_1 z_3) = 4\sqrt{2}M \int_{13} \frac{dz_1 \Upsilon}{(z_1 - s_1)K_1} = \sqrt{2} H'_{13}(s_1 z_3), \qquad (53)$$

$$H_{31}(s_3 z_1) = \frac{1}{2\sqrt{2}M} \int_{31} \frac{dz_3 \Upsilon}{(z_3 - s_3)K_3} = H'_{31}(s_3 z_1)/\sqrt{2}, \qquad (54)$$

$$H(s_1 z_2) = \int_{12} \frac{dz_1 Z}{(z_1 - s_1)K_1^3}. \qquad (55)$$

In Eqs. (44)–(55) the notation $\int_{ij}$ has the following meaning. If we refer to (II.41)–(II.43) and (II.B1), we note that each kernel actually consists of up to three separate pieces, each with its own region of nonvanishing support. Thus, symbolically, we have

$$\int_{12} = \int_{D_{12}} + \theta(-z_2) \int_{L_{12}}, \qquad (56)$$

$$\int_{13} = \int_{D_{13}} + \theta(-z_3) \int_{U_{13}}, \qquad (57)$$

$$\int_{31} = \int_{D_{31}} + \theta((M-\mu)^2 - z_1) \int_{U_{31}} - \theta(-z_1) \int_{L_{31}}, \qquad (58)$$

where the integrals $\int_{T_{ij}}$ are over z_i at fixed z_j, along traversals T_{ij} ($T = D$, U, and L) of kinematic regions in the $(z_i z_j)$ plane, as shown in Fig. 6. In the integrands of (44) and (45) the quantities defined in Eqs. (38) and (B1)–(B6) appear; wherever they do it is understood that the variables are z_i and z_j.

The question of convergence arises for those of the above integrals which are over the unbounded traversals U_{13}, L_{12}, and L_{31}; the treatment of this question will vary according to the problem at hand. For example, in II subtraction constants were introduced [see (II.50)–(II.53)]; these would appear among the fitted parameters in any practical applications of a unitarized isobar model. Further discussion of this point will be made more specific in a subsequent publication in which we apply the present theory to the analytic unitarization of the isobar model for $N\pi \to N\pi\pi$.

Another qualifying comment must be provided in describing possible utilization of these equations. Wherever the root $\sqrt{z_i}$ occurs in the numerator of an integral in $\int_{ij}$, we approximate and make the replacement $\sqrt{z_i} \to \sqrt{s_i}$. This approximation has been discussed in II, in the text between Eqs. (II.58) and (II.59). Without it we cannot calculate the relevant integral without recourse to elliptic integrals. With that proviso, and subject to the above remarks about convergence, all of the expressions (44)–(55) can be explicitly evaluated in terms of the basic kernel integrals $\Delta_{ij}(s_i z_j)$, discussed in II, Appendix B.

VII. DISCUSSION

We have pursued the consequences of applying subenergy unitarity and analyticity to the isobar expansion of the $N\pi\pi$ system, and have obtained a practical theory of medium-energy $N\pi\pi$ dynamics as a result. It may seem disappointing that the search for an isobar model to which these principles are adjoined should lead to something as complex as a full set of three-hadron dynamical equations.[24] We view this situation differently, however. In our approach the search becomes a method for systematically generating a relativistic three-hadron theory. The procedure we have followed is quite general. There would seem to be no essential obstacle in its application to any three-hadron system, for example, to the $NN\pi$ problem including the $D\pi$ channel.

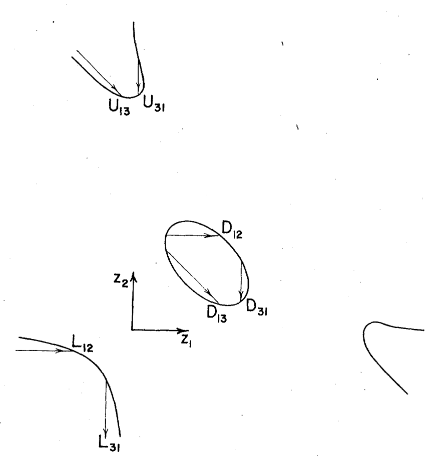

FIG. 6. Paths of integration D_{ij}, U_{ij}, and L_{ij}, defining the integrations in Eqs. (44)–(55) as prescribed in Eqs. (56)–(58).

The distinctive feature of our approach to the relativistic three-body problem is that we work consistently in the two-body subenergy channels throughout. Aaron and Amado[4] have also derived three-body equations starting from the subenergy unitarity constraints; their derivation diverges from ours when the implementation of these constraints is undertaken. Aaron and Amado choose to write dispersion relations in the total invariant energy s, and manipulate the resulting integral equations into the Blankenbecler-Sugar form of their earlier work[13]; we choose to disperse in the subenergy variables. The integral equations which result from these two choices have quite different kernels. Those parts of our kernels which involve integrals over the bounded traversals D_{ij} in Eqs. (56)–(58) are related to the J projection of one-particle-exchange (OPE) processes, as discussed in Appendix 3 of Ref. 1 for the 3π system, and in Appendix C of II for the $N\pi K$ system. Such kernels are common to all approaches to the three-body problem, and correspond physically to easily understood potentials. However, the explicit forms of even these contributions are not identical in all approaches. In a relativistic theory one would expect *a priori* that one-particle-exchange amplitudes would be given by the relevant Feynman-graph expressions. However, in off-shell theories which retain only part of the propagators according to the Blankenbecler-Sugar[12] prescription, it may happen[25] that only some portion of the Feynman amplitude is included and a somewhat *ad hoc* correction has to be invoked. This is not necessary in the present approach.

The Blankenbecler-Sugar prescription does, however, ensure that the exchange terms reconstructed according to it have imaginary parts only in the desired physical region for the three-particle system. Our Eqs. (40a)–(43b) have special properties of this sort which should be emphasized. The full OPE amplitudes (the D_{ij} parts of the kernels) become imaginary in those kinematic regions which allow the exchanged particle to be on the mass shell. If we refer to Fig. 6, with z_2 replaced by z_j and z_1 by s_i, these regions are (a) the bounded region in the center, which is the physical three-particle production or decay region, in which the OPE process of Fig. 7(a) develops an imaginary part corresponding to the real three-particle intermediate state cut by the dotted line in the figure, and (b) the unbounded region in the lower left corner of Fig. 6, for which both s_i and z_j are less than zero, and in which the OPE process also develops an imaginary part, this time associated with an on-shell intermediate *antiparticle*, as shown in Fig. 7(b). This latter imaginary part would cause the solutions of our equations to have the undesirable feature of being complex for subenergy values below threshold.[8,26] One can, of course, exclude by fiat the negative-energy part of the relativistic propagator, as is done in the Blankenbecler-Sugar approach, but this procedure has drawbacks, as we have seen. A unique feature of our integral equations is the presence of *additional* kernel functions other than the OPE kernels, namely those parts indicated in (56)–(58) which involve the remote-region traversals U_{ij} and L_{ij}. These kernels contain the $\Delta_{ij}^{(2)}$ functions of Refs. 1, 3, and 5, and these quantities also become imaginary in the lower left region of Fig. 6, in such a way as to cancel precisely the unwanted imaginary parts of the OPE terms.[27] In fact, $\Delta_{ij}^{(2)}$ is the $J=0$ projection of the process shown in Fig. 7(c), and it develops an imaginary part corresponding to the on-shell intermediate state cut by the dotted line. Such a crossed process would, of course, never appear in any approach of the Faddeev type, which is based on the simple connectedness structure of nonrelativistic potential scattering. Mathematically, the reality of our amplitudes below threshold has been guaranteed by the original subenergy dispersion relation, Eq. (39), and this is a powerful reason for adopting this method of implementing the unitarity constraints. Physically, we are inclined to interpret these additional pieces in

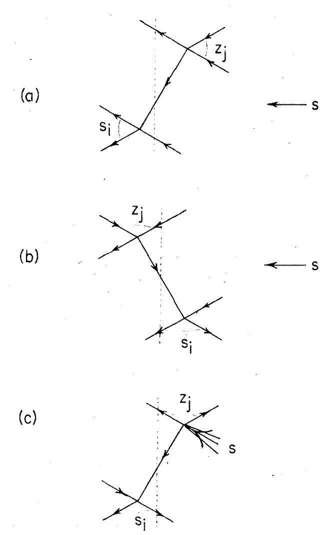

FIG. 7. One-particle-exchange processes associated with the kernels of the integral equations.

our kernel as being associated with short-range forces,[17] insofar as they include crossed processes and enter only in the left-hand integration region of Eqs. (40a)–(43b). Numerical studies of the 3π equations[17,18] indicate that their contributions can be of decisive importance in generating three-hadron resonances.

Mention should also be made of two further alternatives to the Blankenbecler-Sugar and relativistic Faddeev approaches, which have been proposed by Brayshaw, namely, his boundary condition approach,[28] and the more recent relativistic scattering theory.[29] The latter formalism appears to share some of the features of ours; specifically, Brayshaw includes the full OPE contribution. On the other hand, his equations are not free of drawbacks either; in particular, his amplitudes have an unphysical singularity at $s=0$. This is an old problem, and it has been argued that it may be a serious defect.[26,30] Our own amplitudes have no such singularity, but, as discussed above (see Ref. 9), they incorporate a $3\to 3$ amplitude which has some, we hope small, departures from symmetry. It is our belief that there is, as yet, *no* practical relativistic three-body scattering formalism which is free of defects and which includes all the features that one might consider desirable (e.g., full OPE contributions, real analyticity, symmetric $3\to 3$ amplitude, and absence of spurious singularities in s_i and s). Indeed, almost certainly no such theory exists, since the restriction to a fixed number of particles is an intrinsically nonrelativistic one. Be that as it may, the present treatment would seem to provide an interesting approximate solution of the problem, which is systematic, applicable to arbitrary spins (of the individual particles and of the isobars), and, being an on-shell dispersion theory approach, free of any reliance on the mass-shell approximation common to the relativistic Faddeev approach and to Brayshaw's formalism.[29]

The solution of the integral equations for each J^P state contains a Fredholm denominator. This quantity is completely determined by the kernel functions specified in Sec. VI. It depends, of course, only on the three-body variable s and therefore lends itself to the study of three-particle resonance configurations. Such studies of s-channel effects may be interesting to perform in circumstances where local right-hand analytic structure is important, for example, in the vicinity of an inelastic threshold.[25]

To go beyond an evaluation of the Fredholm denominator, one requires aknowledge of the inhomogeneous terms in the integral equations. These will depend on the application considered. For $N\pi \to N\pi\pi$ these terms would be the appropriate Born terms for the production process, as indicated in Ref. 7, Sec. III. The problem of final-state interactions suggests a simpler application, that is, to the evaluation of unitarity corrections to the isobar model. We have performed calculations pertaining to the latter problem, and we shall describe the details and conclusions of that investigation in a separate paper.

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APPENDIX A: ISOSPIN

We use isospin projection operators in the expansion of each of the amplitudes: $N\pi\to N\pi$, $\pi\pi\to\pi\pi$, and $N\pi\to N\pi\pi$. If we index the pion with a Cartesian isovector label then $N\pi_i\to N\pi_j$ has projection operators a^t_{ji} ($t=\frac{1}{2}$ and $\frac{3}{2}$), and $\pi_i\pi_j\to\pi_k\pi_l$ has c^t_{klij} ($t=0$, 1, and 2). Since these are very well known we shall not write them here. We expand $N\pi_k\to N\pi_i\pi_j$ into isobar channels in the manner of I, Fig. 1, as

$$\sum_{Tt_1}\mathcal{A}^{Tt_1}_{ijk}M^{Tt_1}+\sum_{Tt_2}\mathcal{B}^{Tt_2}_{ijk}M^{Tt_2}+\sum_{Tt_3}\mathcal{C}^{Tt_3}_{ijk}M^{Tt_3}\,,$$

in which T is the total isospin and t_n is the isospin in isobar channel n corresponding to the isobar invariants s_n identified in Fig. 2. We list all of these projection operators as follows:

$$\begin{aligned}
\mathcal{A}^{(1/2)(1/2)}_{ijk}&=(i\epsilon_{ijk}+\delta_{ij}\tau_k-\delta_{ik}\tau_j+\delta_{jk}\tau_i)/3\sqrt{3}\,,\\
\mathcal{A}^{(1/2)(3/2)}_{ijk}&=(i\epsilon_{ijk}-2\delta_{ij}\tau_k-\delta_{ik}\tau_j+\delta_{jk}\tau_i)/3\sqrt{6}\,,\\
\mathcal{A}^{(3/2)(1/2)}_{ijk}&=-(i\epsilon_{ijk}+\delta_{ij}\tau_k-\delta_{ik}\tau_j-2\delta_{jk}\tau_i)/3\sqrt{3}\,,\\
\mathcal{A}^{(3/2)(3/2)}_{ijk}&=(5i\epsilon_{ijk}-\delta_{ij}\tau_k+4\delta_{ik}\tau_j-\delta_{jk}\tau_i)/3\sqrt{15}\,,
\end{aligned}\tag{A1}$$

$$\mathcal{B}^{Tt}_{ijk}=\mathcal{A}^{Tt}_{jik}\,,\tag{A2}$$

$$\begin{aligned}
\mathcal{C}^{(1/2)0}_{ijk}&=-\delta_{ij}\tau_k/3\,,\\
\mathcal{C}^{(1/2)1}_{ijk}&=(i\epsilon_{ijk}-\delta_{ik}\tau_j+\delta_{jk}\tau_i)/3\sqrt{2}\,,\\
\mathcal{C}^{(3/2)1}_{ijk}&=(2i\epsilon_{ijk}+\delta_{ik}\tau_j-\delta_{jk}\tau_i)/3\sqrt{2}\,,\\
\mathcal{C}^{(3/2)2}_{ijk}&=(\delta_{ik}\tau_j+\delta_{jk}\tau_i-\tfrac{2}{3}\delta_{ij}\tau_k)/\sqrt{10}\,.
\end{aligned}\tag{A3}$$

In the construction of these we have used the "baryon first" convention; the angular momentum expansion, Eq. (4), is consistent with this.

In order to perform the calculation sketched in Fig. 3, we need to evaluate products of a and c with $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. We indicate the result of these computations as

$$\begin{aligned}
&\sum_l a^{t_1'}_{il}\mathcal{A}^{Tt_1}_{ljk}=\mathcal{A}^{Tt_1}_{ijk}\delta_{t_1t_1'}\,,\\
&\sum_l a^{t_1}_{il}\mathcal{B}^{Tt_2}_{ljk}=\mathcal{A}^{Tt_1}_{ijk}\xi^T_{t_1t_2}\,,\\
&\sum_l a^{t_1}_{il}\mathcal{C}^{Tt_3}_{ljk}=\mathcal{A}^{Tt_1}_{ijk}\xi^T_{t_1t_3}\,,
\end{aligned}\tag{A4}$$

$$\begin{aligned}
&\sum_l a^{t_2}_{jl}\mathcal{A}^{Tt_1}_{ilk}=\mathcal{B}^{Tt_2}_{ijk}\xi^T_{t_2t_1}\,,\\
&\sum_l a^{t_2'}_{jl}\mathcal{B}^{Tt_2}_{ilk}=\mathcal{B}^{Tt_2}_{ijk}\delta_{t_2t_2'}\,,\\
&\sum_l a^{t_2}_{jl}\mathcal{C}^{Tt_3}_{ilk}=\mathcal{B}^{Tt_2}_{ijk}\xi^T_{t_2t_3}\,,
\end{aligned}\tag{A5}$$

$$\begin{aligned}
&\sum_{lm} c^{t_3}_{ijlm}\mathcal{A}^{Tt_1}_{lmk}=\mathcal{C}^{Tt_3}_{ijk}\xi^T_{t_3t_1}\,,\\
&\sum_{lm} c^{t_3}_{ijlm}\mathcal{B}^{Tt_2}_{lmk}=\mathcal{C}^{Tt_3}_{ijk}\xi^T_{t_3t_2}\,,\\
&\sum_{lm} c^{t_3'}_{ijlm}\mathcal{C}^{Tt_3}_{lmk}=\mathcal{C}^{Tt_3}_{ijk}\delta_{t_3t_3'}\,.
\end{aligned}\tag{A6}$$

The ξ's have numerical values which we can tabulate efficiently by introducing two matrices C^T and D^T, such that

$$\xi^T_{t_1t_2}=(D^T)_{t_1t_2}\quad\text{and}\quad\xi^T_{t_1t_3}=(\overline{C}^T)_{t_1t_3}\,,\tag{A7}$$

$$\xi^T_{t_2t_1}=(D^T)_{t_2t_1}\quad\text{and}\quad\xi^T_{t_2t_3}=(-1)^{t_3}(\overline{C}^T)_{t_2t_3}\,,\tag{A8}$$

$$\xi^T_{t_3t_1}=(C^T)_{t_3t_1}\quad\text{and}\quad\xi^T_{t_3t_2}=(-1)^{t_3}(C^T)_{t_3t_2}\,.\tag{A9}$$

In our notation, $\overline{C}^T$ denotes the transpose of C^T. These are the isospin crossing matrices which appear in Eqs. (10) and (12) with their total isospin superscript suppressed. Their numerical entries are

$$C^{1/2}=\frac{1}{\sqrt{3}}\begin{pmatrix}-1&\sqrt{2}\\ \sqrt{2}&1\\ 0&0\end{pmatrix}\quad\text{and}\quad C^{3/2}=\frac{1}{\sqrt{6}}\begin{pmatrix}0&0\\ -1&\sqrt{5}\\ \sqrt{5}&1\end{pmatrix}\,,\tag{A10}$$

with columns labeled $t_1=\frac{1}{2}$, $\frac{3}{2}$ and rows labeled $t_3=0,1,2$:

$$D^{1/2}=-\frac{1}{3}\begin{pmatrix}1&2\sqrt{2}\\ 2\sqrt{2}&-1\end{pmatrix}\quad\text{and}\quad D^{3/2}=\frac{1}{3}\begin{pmatrix}2&\sqrt{5}\\ \sqrt{5}&-2\end{pmatrix}\,,\tag{A11}$$

with rows and columns labeled $t=\frac{1}{2}$, $\frac{3}{2}$.

APPENDIX B: KINEMATICS

The final discontinuity relations, Eqs. (34)–(37), contain a series of quantities arising from the various kinematic and angular factors. They have been identified in Eqs. (38). To put them to use it is necessary that they be expressed in terms of the invariants. We define

$$u_\pm = s + M^2 \pm 2\mu^2 ,$$

$$x_n = s_n + s - \mu^2 ,$$

$$y_n = s_n + M^2 - \mu^2 ,$$

$$z_{12} = s_1 + s_2 - 2\mu^2 ,$$

and

$$y_\pm = (s_1 - \mu^2 \pm WM)^2 ,$$

and list the results as

$$R_1 = 2s_1 z_{12} - x_1 y_1 = 2s_1(s + M^2 - s_3) - x_1 y_1 , \tag{B1}$$

$$R_2 = 2s_2 z_{12} - x_2 y_2 , \tag{B2}$$

$$R_{12} = R_1 R_2 - w_1 w_2 \phi , \tag{B3}$$

$$\Omega = y_1 y_2 - 2M^2 u_- , \tag{B4}$$

$$\Upsilon = -s_3 y_1 - 4Mw_3 s_1 - \frac{W+M}{W}\,\frac{w_3 y_+}{w_3 - W - M} - \frac{W-M}{W}\,\frac{w_3 y_-}{w_3 + W - M} , \tag{B5}$$

$$Z = \tfrac{3}{20}\Omega X + \tfrac{1}{20} R_1 R_2 + (\tfrac{3}{5} MW - 5w_1 w_2)\phi . \tag{B6}$$

In (B6) we have introduced the subsidiary quantity

$$X = (2WQ_a)(2WQ_b)\cos\chi_{ab} = x_1 x_2 - 2su_- . \tag{B7}$$

Finally, we have recalled the Kibble cubic function written in terms of two invariants:

$$\phi = (s_1 + s_2)[(s - \mu^2)(M^2 - \mu^2) - s_1 s_2] + s_1 s_2 u_+ - (sM^2 - \mu^4)u_- . \tag{B8}$$

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