Low-energy photo- and electroproduction for physical pions. I. Ward-identity and chiral-symmetry-breaking structure

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The Ward identities of current algebra are combined with gauge-invariance constraints, on-shell partial conservation of the axial-vector current, and the Bjorken limit to obtain the low-energy expressions of the pion photo- and electroproduction invariant amplitudes for physical pions.

I. INTRODUCTION

Owing to the recent success of the experimental,[1] phenomenological,[2] and theoretical[3] low-energy πN scattering program, it is of interest to extend the low-energy pion photon- and electroproduction program in a similar manner. To this end we apply the Ward identities of current algebra to the Compton-type process $\gamma_\nu + N \to A_\mu + N$, where γ_ν is a vector photon while A_μ is an axial-vector isovector. Then employing PCAC (partial conservation of the axial-vector current) while simultaneously exploiting the gauge condition on the off-shell electroproduction amplitude, we are able to derive expressions for all the low-energy but on-mass-shell amplitudes. In a later work[4] (referred to as II), we shall examine the phenomenological implications of this analysis, including an estimate of the chiral-symmetry-breaking nonstrange-quark mass $\hat{m}$.

In order that the main results of this analysis do not get buried in the morass of algebra that follows, we list the final form for the 18 Fubini-Nambu-Wataghin[5] electroproduction amplitudes:

$$A_1^{(+,0)}(\nu,t,k^2) = -\frac{k\cdot q g F_1^{V,S}(k^2)}{s_m u_m} - \frac{g_A(\mu^2)F_2^{V,S}(k^2)}{4mf_\pi} - \frac{\Sigma_1^{(+,0)}(t)}{f_\pi m_V{}^2} + B_1^{(+,0)}(\nu,t,k^2), \quad (1a)$$

$$A_2^{(+,0)}(\nu,t,k^2) = -\frac{gF_1^{V,S}(k^2)}{s_m u_m} - \frac{\Sigma_2^{(+,0)}(t)}{f_\pi m_V{}^2} + B_2^{(+,0)}(\nu,t,k^2), \quad (1b)$$

$$A_3^{(+,0)}(\nu,t,k^2) = -\frac{\nu g F_2^{V,S}(k^2)}{m s_m u_m} - \frac{\Sigma_3^{(+,0)}(t)}{f_\pi m_V{}^2} + B_3^{(+,0)}(\nu,t,k^2), \quad (1c)$$

$$A_4^{(+,0)}(\nu,t,k^2) = \frac{k\cdot q g F_2^{V,S}(k^2)}{2m s_m u_m} - \frac{\Sigma_4^{(+,0)}(t)}{f_\pi m_V{}^2} + B_4^{(+,0)}(\nu,t,k^2), \quad (1d)$$

$$A_5^{(+,0)}(\nu,t,k^2) = -\frac{\Sigma_5^{(+,0)}(t)}{f_\pi m_V{}^2} + B_5^{(+,0)}(\nu,t,k^2), \quad (1e)$$

$$A_6^{(+,0)}(\nu,t,k^2) = -\frac{\Sigma_6^{(+,0)}(t)}{f_\pi m_V{}^2} + B_6^{(+,0)}(\nu,t,k^2), \quad (1f)$$

$$A_1^{(-)}(\nu,t,k^2) = \frac{2\nu g F_1^V(k^2)}{s_m u_m} + B_1^{(-)}(\nu,t,k^2), \quad (1g)$$

$$A_2^{(-)}(\nu,t,k^2) = \frac{2\nu g F_1^V(k^2)}{k\cdot q s_m u_m} + B_2^{(-)}(\nu,t,k^2), \quad (1h)$$

$$A_3^{(-)}(\nu,t,k^2) = \frac{k\cdot q g F_2^V(k^2)}{2m s_m u_m} + \frac{F_1^V(k^2)G(t)}{f_\pi} + B_3^{(-)}(\nu,t,k^2), \quad (1i)$$

$$A_4^{(-)}(\nu,t,k^2) = -\frac{\nu g F_2^V(k^2)}{m s_m u_m} + B_4^{(-)}(\nu,t,k^2) \quad (1j)$$

$$A_5^{(-)}(\nu,t,k^2) = \frac{gF_1^V(k^2)}{k\cdot q(t-\mu^2)} - \frac{2g\mathfrak{F}(k^2)}{t-\mu^2} - \frac{2mG(t)+\mu^2\overline{H}(t)}{f_\pi}\mathfrak{F}_V(k^2) + B_5^{(-)}(\nu,t,k^2), \quad (1k)$$

$$A_6^{(-)}(\nu,t,k^2) = \frac{F_1^V(k^2)G(t) - g_A(t)\mathfrak{F}_V(k^2)}{2f_\pi} + B_6^{(-)}(\nu,t,k^2), \quad (1l)$$

with g the $\pi^0 pp$ coupling constant, m the nucleon and m_V the vector-meson mass, $f_\pi \approx 93$ MeV, and invariants s_m, u_m, ν, $k\cdot q$, t, and k^2 defined in Sec. II. The unfamiliar form factors $G(t)$, $\overline{H}(t)$, and $\mathfrak{F}(k^2)$, $\mathfrak{F}_V(k^2)$, $\mathfrak{F}_\pi(k^2)$ are defined in Sec. IV. The chiral-symmetry breaking "Σ terms" are given approximately in (1) and are explained in detail in Sec. V. Lastly, the background terms $B_i^{(+,0,-)}(\nu,t,k^2)$ are resonance-dominated axial-vector amplitudes. All of the terms in (1) will be derived in this work except the B_i. The latter are somewhat model-dependent (as similar terms are in πN scattering) and will be considered in II.

With the advent of new low-energy photo- and electroproduction data, it is hoped that (1) will prove useful in the near future. An analytic subthreshold expansion analogous to the one recently developed for πN scattering[2] would be extremely illuminating in this regard. At the present time the state of the art combines experimental multipole amplitudes with the Fubini-Furlan on-shell expansion in the Breit frame.[6-8] To be sure, the latter approach based on charge commutators and extrapolations of the dynamical Low equation is extremely revealing, but one would still like to explore the analytic Ward-identity photoproduction analog of the πN analysis. Furthermore, it is expected that (1) will generate new low-energy predictions.

We divide the present analysis into four parts. The kinematics of photo- and electroproduction and the nucleon pole terms, M^N, are worked out in Sec. II. In Sec. III the soft-pion limit is rederived and the non-gauge-invariant nature of this off-shell result is stressed. Then in Sec. IV the current-algebra Ward identities are used to obtain the on-shell amplitudes (1), consistent with the off-shell gauge condition and the soft limit of Sec. III. To accomplish this, we compare the amplitudes $q^\mu M_{\mu\nu}$ with $q^\mu M^N_{\mu\nu}$ and $M_{\mu\nu}k^\nu$ with $M^N_{\mu\nu}k^\nu$, taking into account the inherent pole structure of the axial-vector amplitude $M_{\mu\nu}$. Finally in Sec. V the gauge-invariant chiral-$SU_2\times SU_2$-breaking Σ terms are found by invoking the Bjorken limit for the electroproduction (and not the photoproduction) amplitude. The infinite-momentum frame and SU_6 symmetry are then used to relate these Σ terms to the nonstrange current quark mass $\hat{m}$ as given by (78) and (81).

II. NOTATION AND KINEMATICS

The general electroproduction of isotopic pions is symbolically represented by $\gamma_\nu(k)+N(p)\to\pi^i(q)+N(p')$ with an S-matrix element[9]

$$(S-1)_{fi}=i(2\pi)^4\delta(p'+q-p-k)e\pi^i(q)T^i_\nu\epsilon^\nu(\vec{k}), \qquad (2)$$

$$T^i_\nu=-i\int d^4x e^{iq\cdot x}(\Box+\mu^2)\times\langle N(\vec{p}\,')|T\{\phi^i_\pi(x),V^\gamma_\nu(0)\}|N(\vec{p})\rangle \qquad (3a)$$

$$=\bar{N}(\vec{p}\,')M^i_\nu N(\vec{p}). \qquad (3b)$$

Strictly speaking, T_ν does not represent the correct electroproduction amplitude off the pion mass shell (see, e.g., Ref. 9), but it will suffice for our analysis since we are interested only in the on-shell result (1). Given (3), we note that M_ν must satisfy the following off-shell gauge condition,[10] derived in Appendix I:

$$M^i_\nu k^\nu=-(q^2-\mu^2)i\epsilon^{i3k}\phi^k_\pi(0). \qquad (4)$$

As expected, M_ν is gauge invariant for $q^2=\mu^2$.

The general M function in (3b) has the isotopic decomposition

$$M^i_\nu=M^{(+)}_\nu I^i_+ + M^{(-)}_\nu I^i_- + M^{(0)}_\nu I^i_0, \qquad (5a)$$

where

$$I^i_+=\delta^{i3},\quad I^i_-=i\epsilon^{i3k}\tau^k,\quad I^i_0=\tau^i. \qquad (5b)$$

Then each of the isotopic M functions has the Fubini-Nambu-Wataghin (FNW) covariant decomposition:

$$M^{(+,-,0)}_\nu=\sum_{j=1}^{6}A^{(+,-,0)}_j(\nu,t,k^2)K^j_\nu \qquad (6)$$

for on-mass-shell pions with

$$K^1_\nu=\tfrac{1}{2}[\not{k},\gamma_\nu]\gamma_5, \qquad (7a)$$

$$K^2_\nu=2(k\cdot qP_\nu-k\cdot Pq_\nu)\gamma_5, \qquad (7b)$$

$$K^3_\nu=(k\cdot q\gamma_\nu-\not{k}q_\nu)\gamma_5, \qquad (7c)$$

$$K^4_\nu=2(k\cdot P\gamma_\nu-\not{k}P_\nu)\gamma_5-2mK^1_\nu, \qquad (7d)$$

$$K^5_\nu=(k\cdot qk_\nu-k^2q_\nu)\gamma_5, \qquad (7e)$$

$$K^6_\nu=(\not{k}k_\nu-k^2\gamma_\nu)\gamma_5, \qquad (7f)$$

where $P=\frac{1}{2}(p'+p)$, $\Delta=p'-p=k-q$, $t=\Delta^2$, $\nu=k\cdot p=q\cdot p=(s-u)/4$, and m is the nucleon mass, μ the pion mass. The first four covariants in (7) are the Chew-Goldberger-Low-Nambu[11] (CGLN) set for $k^2=0$ photoproduction in our metric.[9] Under crossing, ν becomes $-\nu$ with $A^{(+,0)}_{1,2,4}$ and $A^{(-)}_{3,5,6}$ even and $A^{(-)}_{1,2,4}$ and $A^{(+,0)}_{3,5,6}$ odd.

For future reference we list the following identities:

$$2\epsilon_{\nu\alpha\beta\gamma}P^\alpha k^\beta\gamma^\gamma=K^3_\nu+K^6_\nu, \qquad (8a)$$

$$\epsilon_{\nu\alpha\beta\gamma}q^\alpha k^\beta P^\gamma=\tfrac{1}{2}tK^1_\nu-\tfrac{1}{2}\left(1-\frac{k^2}{k\cdot q}\right)K^2_\nu+mK^4_\nu-\frac{k\cdot P}{k\cdot q}K^5_\nu, \qquad (8b)$$

$$\epsilon_{\nu\alpha\beta\gamma}q^\alpha k^\beta\gamma^\gamma=K^4_\nu, \qquad (8c)$$

$$\tfrac{1}{2}i\epsilon_{\nu\alpha\beta\gamma}k^\alpha\sigma^{\beta\gamma}=K^1_\nu, \qquad (8d)$$

where $\sigma^{\alpha\beta}=\frac{1}{2}i[\gamma^\alpha,\gamma^\beta]$. Note that the apparent kinematic singularity in (8b) is removed by employing a seventh covariant

$$\frac{1}{2}\frac{k^2}{k\cdot q}K^2_\nu-\frac{k\cdot P}{k\cdot q}K^5_\nu=(k^2P_\nu-k\cdot Pk_\nu)\gamma_5\equiv K^7_\nu. \qquad (9)$$

In terms of the six covariants (7), the invariant electroproduction amplitudes are constrained from (9) as[12]

$$2\nu A_2+k^2A_5\to\text{finite as } k\cdot q\to 0. \qquad (10)$$

At $k^2=0$, $\epsilon^\nu k_\nu=0$, the constraint (9) disappears and the four photoproduction amplitudes are then free of kinematic singularities and constraints (zeros).

Lastly it will be useful to identify the pole contributions to the various amplitudes. For *pseudoscalar* coupling of pions to nucleons, the effective Hamiltonian is

$$H = g\bar{N}\gamma_5\tau^i N\pi^i, \tag{11}$$

and the electromagnetic vertices $\bar{N}\Gamma_\nu NA^\nu$ and $\pi^*\Gamma_\nu\pi A^\nu$ are

$$\Gamma_\nu^N = \tfrac{1}{2}[F_1^S(k^2) + \tau^3 F_1^V(k^2)]\gamma_\nu + \tfrac{1}{2}[F_2^S(k^2) + \tau^3 F_2^V(k^2)]\frac{i\sigma_{\nu\lambda}k^\lambda}{2m}, \tag{12a}$$

$$\Gamma_\nu^\pi = i\epsilon^{i3k}(q_\nu - \Delta_\nu)F_\pi(k^2) = i\epsilon^{i3k}(2q_\nu - k_\nu)F_\pi(k^2). \tag{12b}$$

The sum of the s- and u-channel nucleon poles and t-channel pion pole, $M_\nu^P = M_\nu^N + M_\nu^\pi$, is for off-mass-shell pions,

$$M_\nu^{(+,0)P} = \frac{-g(q^2)}{(s-m^2)(u-m^2)}\left[F_1^{V,S}(k^2)(k\cdot qK_\nu^1 + K_\nu^2) + \frac{F_2^{V,S}(k^2)}{2m}(2\nu K_\nu^3 - k\cdot qK_\nu^4)\right], \tag{13a}$$

$$M_\nu^{(-)P} = \frac{g(q^2)}{(s-m^2)(u-m^2)}\left[2\nu F_1^V(k^2)\left(K_\nu^1 + \frac{1}{k\cdot q}K_\nu^2\right) + \frac{F_2^V(k^2)}{2m}(k\cdot qK_\nu^3 - 2\nu K_\nu^4)\right]$$
$$+\frac{g(q^2)F_1^V(k^2)}{k\cdot q(t-q^2)}K_\nu^5 + \frac{g(q^2)[F_1^V(k^2) - F_\pi(k^2)]}{t-q^2}(2q_\nu - k_\nu)\gamma_5, \tag{13b}$$

where $F_1^S(0) = F_1^V(0) = F_\pi(0) = 1$ and $F_2^S(0) = -0.12$, $F_2^V(0) = 3.7$. As noted by FNW, $M_\nu^{(+,0)P}$ is gauge invariant, but $M_\nu^{(-)P}$ is not. For $k^2 = 0$, however, the entire pole photoproduction amplitude is gauge invariant because $F_1^V(0) - F_\pi(0) = 0$ in (13b). The notation $s_m = s - m^2$, $u_m = u - m^2$, $\nu = k\cdot P = q\cdot P$ with $s_m u_m = (k\cdot q)^2 - 4\nu^2$ is used when converting (13) to the language of (1).

III. CURRENT ALGEBRA, PCAC, AND THE SOFT-PION THEOREM

We review the standard soft-pion theorem for electroproduction, but now in the context of the off-shell gauge condition (4). Along with the pion amplitude (3), consider the two-current axial-vector amplitude

$$T_{\mu\nu}^i = i\int d^4x\; e^{iq\cdot x}\langle N'|T\{A_\mu^i(x), V_\nu^\gamma(0)\}|N\rangle \tag{14}$$

and also the divergence amplitude

$$\tilde{T}_\nu^i = i\int d^4x e^{iq\cdot x}\langle N'|T\{\partial\cdot A^i(x), V_\nu^\gamma(0)\}|N\rangle. \tag{15}$$

Then the PCAC relation

$$\partial\cdot A^i = f_\pi\mu^2\phi_\pi^i \tag{16}$$

can be expressed in terms of the single-current M functions M_ν of (3) and $\tilde{M}_\nu$ in $\tilde{T}_\nu = \bar{N}'\tilde{M}_\nu N$ as

$$\tilde{M}_\nu^i = \frac{f_\pi\mu^2}{q^2-\mu^2}M_\nu^i. \tag{17}$$

The relevant current-algebra relations, on and off the pion mass shell, all follow from (17) and the $SU_2\times SU_2$ current-algebra commutation relations at equal times,

$$\delta(x_0)[A_\mu^i(0), V_0^j(x)] = i\epsilon^{ijk}A_\mu^k(0)\delta^4(x), \tag{18a}$$

$$\delta(x_0)[\partial\cdot A^i(0), V_0^j(x)] = i\epsilon^{ijk}\partial\cdot A^k(0)\delta^4(x), \tag{18b}$$

$$\delta(x_0)[A_0^i(x), V_\nu^j(0)] = i\epsilon^{ijk}A_\nu^k(0)\delta^4(x). \tag{18c}$$

In particular, contracting $\tilde{M}_\nu$ with k^ν, shifting the momentum dependence in (15), integrating by parts and using (18b) and $\partial\cdot V^\gamma = 0$ gives

$$\tilde{M}_\nu^i k^\nu = \int d^4x ik^\nu e^{-ik\cdot x}T\{\partial\cdot A^i(0), V_\nu^\gamma(x)\}$$
$$= -\int d^4x e^{-ik\cdot x}\delta(x_0)[\partial\cdot A(0), V_0^\gamma(x)]$$
$$= -i\epsilon^{i3k}\partial\cdot A^k(0). \tag{19}$$

Then applying (16) and (17) to (19) immediately leads to the gauge condition (4).

In order to determine $\tilde{M}_\nu$ and therefore M_ν by (17), we turn to the axial-vector current Ward identity obtained by contracting the M function $M_{\mu\nu}$ in $T_{\mu\nu} = \bar{N}'M_{\mu\nu}N$ with q^μ and applying the current-algebra relation (18c). This yields

$$q^\mu M_{\mu\nu}^i = -i\epsilon^{i3k}A_\nu^k(0) + i\tilde{M}_\nu^i. \tag{20}$$

Then to find $\tilde{M}_\nu$ in the soft-pion limit $q\to 0$, one isolates from $M_{\mu\nu}$ the nucleon pole terms $M_{\mu\nu}^N$ between on-shell nucleon spinors,

$$-M_{\mu\nu}^{iN}(q\to 0) = \tfrac{1}{2}ig_A(0)[\tau^i\gamma_\mu\gamma_5(\not{p}' + \not{q} - m)^{-1}\Gamma_\nu$$
$$+\Gamma_\nu(\not{p} - \not{q} - m)^{-1}\tau^i\gamma_\mu\gamma_5]. \tag{21}$$

This is the only part of $q^{\mu}M_{\mu\nu}$ which survives as $q\to 0$,

$$f_\pi^{-1}q^{\mu}M^{\,i}_{\mu\nu}(q\to 0)=\frac{ig(0)}{2m}(\tau^i\gamma_5\Gamma_\nu+\Gamma_\nu\gamma_5\tau^i) -iM^{iN}_{\nu}(q\to 0)\,, \qquad (22)$$

where the M^{iN}_{ν} are the pseudoscalar nucleon-pole contributions (all but the F_π term) in (13) for $q\to 0$, and we have used the Goldberger-Treiman identity $mg_A(0)=f_\pi g(0)$ in this limit in (22). Substituting (22) into (20) and again applying (17), now as $q\to 0$, we obtain the standard soft-pion theorem for electroproduction,

$$M^{\,i}_{\nu}(q\to 0)=M^{iN}_{\nu}(q\to 0)+\frac{g(0)}{2m}F^V_1(k^2)I^{\,i}_{-}\gamma_\nu\gamma_5 - \frac{g(0)}{4m^2}[F^S_2(k^2)I^{\,i}_{0}+F^V_2(k^2)I^{\,i}_{+}]K^1_\nu - \frac{1}{2f_\pi}I^{\,i}_{-}[g_A(k^2)\gamma_\nu\gamma_5+h_A(k^2)k_\nu\gamma_5]\,, \qquad (23)$$

where the axial-vector nucleon form factors g_A and h_A are defined by

$$\langle N'|A^i_\mu(x)|N\rangle =\overline{N}(\vec{p}\,')\tfrac{1}{2}\tau^i i[g_A(t)\gamma_\mu\gamma_5+h_A(t)\Delta_\mu\gamma_5]N(\vec{p})e^{i\Delta\cdot x}\,, \qquad (24)$$

with $\Delta=p'-p$ and $t=\Delta^2$.

The usual check on the soft-pion theorem (23) is to note that as $q\to 0$ *and* $k\to 0$, only the pseudoscalar nucleon pole terms survive in (23),

$$M^{\,i}_{\nu}\epsilon^{\nu}(\vec{k}) \underset{q,k\to 0}{\to} -\frac{g(0)}{2m}i\vec{\sigma}\cdot\vec{\epsilon}I^{\,i}_{-}\,, \qquad (25)$$

which is the well known Kroll-Ruderman-Klein limit.[13] Then examining (23) for on-shell photons $k^2=0$, $\epsilon\cdot k=0$ (but not $k\to 0$) with the second and fourth terms cancelling, we may add back in the vanishing pion pole ($\propto 2q_\nu-k_\nu$) and obtain the Fubini-Furlan-Rossetti (FFR)[14] soft-pion theorem for the background photoproduction ($k^2=0$) amplitude

$$M_\nu=M^P_\nu+\overline{M}_\nu\,, \qquad (26)$$

$$\overline{M}^{\,i}_{\nu}(q\to 0)=-\frac{g(0)}{4m^2}(\kappa^S I^{\,i}_{0}+\kappa^V I^{\,i}_{+})K^1_\nu\,, \qquad (27)$$

where $\kappa^{S,V}=\kappa^p\pm\kappa^n$ are the isoscalar and isovector anomalous moments of the nucleon. Furthermore, since the pion pole contains only $I^{\,i}_{-}$, (27) can be extended to electroproduction ($k^2\neq 0$) for $\overline{M}^{(+,0)}$. Lastly, the soft-pion theorem (23) has been used to probe the $g_A(k^2)$, Nambu-Schrauner term.[15] Recently, however, it has been suggested[16] that such a structure as (23) is inconsistent with dispersion theory because it contains the $g_A(k^2)$ axial-vector form factor instead of the pion form factor $F_\pi(k^2)$ from the pion pole term. In order to justify (23), it then appears necessary to alter the dispersion formulation by including a fixed J-plane pole which would contribute to the high as well as the low-energy dispersion amplitudes, for photoproduction as well as for electroproduction.[16]

It is our view that both (23) and the usual dispersion approach are consistent and correct and that no fixed J-plane poles are needed in high-energy photoproduction on the basis of current algebra. Rather, the problem is that (23) is *misleading* because the pion is off its mass shell. However, it is worth noting that (23) is still consistent with the gauge condition (4) as $q\to 0$. First recall that

$$(t-\mu^2)\phi^i_\pi(0)=j^{\,i}_\pi(0)=g(t)\tau^i\gamma_5 \qquad (28)$$

between nucleon spinors $\overline{N}'$ and N, and this converts (4) to ($t\to k^2$ as $q\to 0$)

$$M^{\,i}_{\nu}k^{\nu} \underset{q\to 0}{\to} \frac{\mu^2}{k^2-\mu^2}g(k^2)I^{\,i}_{-}\gamma_5\,. \qquad (29)$$

This result is consistent with (23) contracted with k_ν,

$$M^{\,i}_{\nu}(q\to 0)k^{\nu}=-\frac{1}{2f_\pi}I^{\,i}_{-}[2mg_A(k^2)+k^2h_A(k^2)]\gamma_5\,, \qquad (30)$$

owing to the cancellation of the first two terms in (23) and the implicit pion pole in $h_A(k^2)$ (to be discussed in detail in the next section). In either case, $M_\nu(q\to 0)$ [as given by (3)] is not gauge invariant even for photoproduction. Such a situation can only cast doubt on the applicability of the soft-pion theorem to the real-world gauge-invariant photoproduction and electroproduction amplitudes. It turns out, however, that the correct photoproduction ($k^2=0$) amplitude [not (3)] is gauge invariant even if the pion is off-shell [as noted after (3)]. Consequently, the photoproduction FFR limit (27) should yield the correct low-energy $q^2=\mu^2=0$ invariant amplitudes and this is borne out by our final result, (1a). The electroproduction Nambu-Schrauner $g_A(k^2)$ term should be examined more closely, however, and the Breit-frame analysis of Furlan and coworkers[6-8] which avoids the gauge condition altogether may be appreciated in this context.

All of the previous comments go beyond the analogous problem for low-energy πN scattering—viz., to find the physical pion mass, chiral-symmetry breaking, and background resonance corrections to the soft-pion theorems. Our goal is to link the former two types of corrections directly to the off-shell gauge condition (4), thus avoiding any gauge-invariance ambiguities in the on-shall photoproduction or electroproduction amplitudes.

IV. ON-SHELL WARD IDENTITY AND THE GAUGE CONDITION

We begin, as in the on-shell approach to πN scattering, by applying the entire axial-vector–nucleon vertex (24) for the external pion momentum while isolating the pion-pole contribution to $h_A(t)$ as follows [Δ in (24) becomes $-q$ here]:

$$\Gamma_\mu^{iA} = \tfrac{1}{2}\tau^i i[g_A(q^2)\gamma_\mu\gamma_5 - h_A(q^2)q_\mu\gamma_5] \qquad (31a)$$

$$= \tfrac{1}{2}\tau^i i\left[f_\pi g(q^2)\left(\frac{\gamma_\mu}{m} + \frac{2q_\mu}{q^2-\mu^2}\right) - \frac{\overline{h}_A(q^2)}{2m}(\gamma_\mu q^2 - \not{q}q_\mu)\right]\gamma_5 . \qquad (31b)$$

The equivalence between (31a) and (31b) requires that

$$mg_A(q^2) + \tfrac{1}{2}q^2\overline{h}_A(q^2) = f_\pi g(q^2)\,, \qquad (32a)$$

$$h_A(q^2) = -2f_\pi g(q^2)(q^2-\mu^2)^{-1} + \overline{h}_A(q^2)\,, \qquad (32b)$$

(32a) being the generalization of the Goldberger-Treiman relation $mg_A(0) = f_\pi g(0)$.

Next separate off the full axial-vector–nucleon contribution $M_{\mu\nu}^{iN}$ from the complete two-current amplitude $M_{\mu\nu}^{i}$ corresponding to (14):

$$M_{\mu\nu}^{i} = M_{\mu\nu}^{iN} + M_{\mu\nu}^{\prime i}\,, \qquad (33)$$

$$-M_{\mu\nu}^{iN} = \Gamma_\mu^{iA}(\not{p}+\not{q}-m)^{-1}\Gamma_\nu + \Gamma_\nu(\not{p}-\not{q}-m)^{-1}\Gamma_\mu^{iA} . \qquad (34)$$

To obtain a handle on $M'_{\mu\nu}$, we first consider the vector Ward identity which follows from (18a) and $\partial\cdot V^\gamma = 0$,

$$T_{\mu\nu}^{i}k^\nu = -i\epsilon^{i3k}\langle N'|A_\mu^k(0)|N\rangle \qquad (35a)$$

or equivalently in terms of M functions,

$$M_{\mu\nu}^{iN}k^\nu + M_{\mu\nu}^{\prime i}k^\nu = -\tfrac{1}{2}iI_-^{i}[g_A(t)\gamma_\mu\gamma_5 + h_A(t)\Delta_\mu\gamma_5]\,, \qquad (35b)$$

where (34) implies

$$M_{\mu\nu}^{iN}k^\nu = -\tfrac{1}{2}iI_-^{i}F_1^V(k^2)[g_A(q^2)\gamma_\mu\gamma_5 - h_A(q^2)q_\mu\gamma_5]\,. \qquad (36)$$

Then defining

$$G(t,q^2) = \frac{g_A(t)-g_A(q^2)}{t-q^2}\,, \qquad (37a)$$

$$H(t,q^2) = \frac{h_A(t)-h_A(q^2)}{t-q^2}\,, \qquad (37b)$$

$$\mathfrak{F}_V(k^2) = \frac{F_1^V(k^2)-1}{k^2}\,, \qquad (37c)$$

(35b) and (36) require that

$$M_{\mu\nu}^{\prime i}k^\nu = -\tfrac{1}{2}iI_-^{i}[F_1^V(k^2)G(t,q^2)(k^2-2k\cdot q)\gamma_\mu - g_A(t)\mathfrak{F}_V(k^2)k^2\gamma_\mu - F_1^V(k^2)H(t,q^2)(k^2-2k\cdot q)q_\mu + h_A(t)\mathfrak{F}_V(k^2)k^2q_\mu + h_A(t)k_\mu]\gamma_5 . \qquad (38)$$

From (38) we see that the advantage of employing the full axial-vector–nucleon vertex (31) in the Born terms (34) is that the nonsingular functions (37a) and (37c) occur in a natural way. Removing k^ν from both sides of (38) we write

$$M_{\mu\nu}^{\prime i} = \tfrac{1}{2}iI_-^{i}[F_1^V(k^2)G(t,q^2)\gamma_\mu(2q_\nu-k_\nu) + g_A(t)\mathfrak{F}_V(k^2)\gamma_\mu k_\nu - F_1^V(k^2)H(t,q^2)q_\mu(2q_\nu-k_\nu) - h_A(t)\mathfrak{F}_V(k^2)q_\mu k_\nu - h_A(t)g_{\mu\nu}]\gamma_5 + R_{\mu\nu}^{i}\,, \qquad (39)$$

where $R_{\mu\nu}^{i}$ is a yet-to-be determined gauge-invariant term, $R_{\mu\nu}^{i}k^\nu = 0$. It must also include all the resonance dynamics, as we shall note shortly.

Next consider the axial-vector Ward identity for $M_{\mu\nu}^{i}$, (20), for $q \not\to 0$. For the complete axial-vector–nucleon poles (34), the relation analogous to (22), but for $q \not\to 0$, is

$$q^\mu M_{\mu\nu}^{iN} = \tfrac{1}{2}ig_A(q^2)[\tau^i\gamma_5\Gamma_\nu + \Gamma_\nu\gamma_5\tau^i] - \frac{i}{2g(q^2)}[2mg_A(q^2) + q^2h_A(q^2)]M_\nu^{iN}\,, \qquad (40)$$

where again M_ν^{iN} is the pseudoscalar nucleon pole contribution in (13). Contracting (39) with q^μ and using (33), (40), and (32), we find that (20) implies

$$q^\mu R_{\mu\nu}^{i} = \tfrac{1}{2}iI_-^{i}\{F_1^V(k^2)G(t,q^2)(K_\nu^6 + 2K_\nu^3) - g_A(t)\mathfrak{F}_V(k^2)K_\nu^6 + [F_1^V(k^2)(2mG(t,q^2) + q^2H(t,q^2)) + h_A(t)](2q_\nu-k_\nu)\gamma_5 + [2mg_A(t) + q^2h_A(t)]\mathfrak{F}_V(k^2)k_\nu\gamma_5\} + i\tilde{M}_\nu^{i} - \frac{i}{4m}g_A(q^2)[F_2^S(k^2)I_0^{i} + F_2^V(k^2)I_+^{i}]K_\nu^1 - \frac{if_\pi\mu^2}{q^2-\mu^2}M_\nu^{iN}\,. \qquad (41)$$

By construction $R_{\mu\nu}$ does not contain any remnant of the s- or u-channel nucleon poles—but it can have a t-channel pole piece. Removing the pion pole in $(q^2-\mu^2)^{-1}$ as on the left-hand side of the Ward identity (20), we write the on-shell PCAC relation as

$$R^i_{\mu\nu}=\frac{if_\pi q_\mu}{q^2-\mu^2}(\overline{M}{}^i_\nu+C^i_\nu)+\overline{M}{}^i_{\mu\nu}\,, \tag{42}$$

where the bar in (42) refers to non-Born (nucleon or t-channel pion) parts of the electroproduction amplitude

$$M_\nu=M^N_\nu+M'_\nu=M^P_\nu+\overline{M}_\nu\,. \tag{43}$$

Note that to obtain $\overline{M}$ from M' one must remove the pion pole in $h_A(t)$ from (39). We stress that the quantity C_ν in (42) includes the terms necessary to satisfy the gauge condition (4) and also includes chiral-symmetry-breaking terms. That is, $R_{\mu\nu}k^\nu=\overline{M}_{\mu\nu}k^\nu=0$ requires from (4) and (13) that

$$\begin{aligned}C^i_\nu k^\nu&=-\overline{M}{}^i_\nu k^\nu=M^{iP}_\nu k^\nu-M^i_\nu k^\nu\\&=-I^i_-\left(g(q^2)[F^V_1(k^2)-F_\pi(k^2)]+\frac{q^2-\mu^2}{2f_\pi\mu^2}[2mg_A(t)+th_A(t)]\right)\gamma_5\,.\end{aligned} \tag{44}$$

We will see shortly that C_ν has a complicated pole structure in $(t-q^2)^{-1}$ and $(t-\mu^2)^{-1}$. Finally, contracting (42) with q^μ and equating the result to (41) and using (17) and (43), we obtain the electroproduction background amplitude

$$\begin{aligned}if_\pi\overline{M}{}^i_\nu=\tfrac{1}{2}iI^i_-\{&F^V_1(k^2)G(t,q^2)[K^6_\nu+2K^3_\nu]-g_A(t)\mathfrak{F}_V(k^2)K^6_\nu\\&+[F^V_1(k^2)(2mG(t,q^2)+q^2H(t,q^2))+h_A(t)](2q_\nu-k_\nu)\gamma_5+[2mg_A(t)+q^2h_A(t)]\mathfrak{F}_V(k^2)k_\nu\gamma_5\}\\&-\frac{i}{4m}g_A(q^2)[F^S_2(k^2)I^i_0+F^V_2(k^2)I^i_+]K^1_\nu+if\pi\frac{\mu^2}{q^2-\mu^2}M^{i\pi}_\nu-\frac{if_\pi q^2}{q^2-\mu^2}C^i_\nu-q^\mu\overline{M}{}^i_{\mu\nu}\,,\end{aligned} \tag{45}$$

where $M^{i\pi}_\nu$ is the pion t-channel pole part of M^{iP}_ν in (13).

To proceed further, we invoke the (obvious) dynamical requirement that the nonpole electroproduction amplitude $\overline{M}{}^i_\nu$ contains no extraneous poles at $q^2=\mu^2$, $t=\mu^2$, or $t=q^2$. To this end we display the poles in $h_A(t)$ and $H(t,q^2)$ given by (32b) and

$$H(t,q^2)=-\frac{2f_\pi}{t-q^2}\left(\frac{g(t)}{t-\mu^2}-\frac{g(q^2)}{q^2-\mu^2}\right)+\overline{H}(t,q^2)\,, \tag{46}$$

and also solve (44) for C^i_ν as

$$C^i_\nu=I^i_-\frac{2q_\nu-k_\nu}{t-q^2}g(q^2)[F^V_1(k^2)-F_\pi(k^2)]\gamma_5-\frac{q^2-\mu^2}{f_\pi\mu^2}Y^i_u\,, \tag{47}$$

where Y_ν is defined by its divergence,

$$Y^i_\nu k^\nu=\tfrac{1}{2}I^i_-[2mg_A(t)+th_A(t)]\gamma_5\,. \tag{48}$$

Then (45) becomes

$$\begin{aligned}if_\pi\overline{M}{}^i_\nu=\tfrac{1}{2}iI^i_-\Big\{&F^V_1(k^2)G(t,q^2)[K^6_\nu+2K^3_\nu]-g_A(t)\mathfrak{F}_V(k^2)K^6_\nu\\&+[F^V_1(k^2)(2mG(t,q^2)+q^2\overline{H}(t,q^2))+\overline{h}_A(t)](2q_\nu-k_\nu)\gamma_5\\&+[2mg_A(t)+q^2\overline{h}_A(t)]\mathfrak{F}_V(k^2)k_\nu\gamma_5\\&-\frac{2f_\pi g(t)F^V_1(k^2)q^2}{(t-q^2)(t-\mu^2)}(2q_\nu-k_\nu)\gamma_5-\frac{2f_\pi g(t)}{t-\mu^2}(2q_\nu-k_\nu)\gamma_5\\&-\frac{2f_\pi g(t)\mathfrak{F}_V(k^2)q^2}{t-\mu^2}k_\nu\gamma_5+\frac{2f_\pi g(q^2)F_\pi(k^2)}{t-q^2}(2q_\nu-k_\nu)\gamma_5\Big\}\\&-\frac{i}{4m}g_A(q^2)[F^S_2(k^2)I^i_0+F^V_2(k^2)I^i_+]K^1_\nu+i\frac{q^2}{\mu^2}Y^i_\nu-q^\mu\overline{M}{}^i_{\mu\nu}\,,\end{aligned} \tag{49}$$

where we have explicitly displayed the pole dependence of all the terms in (49) except Y^i_ν. The choice for C_ν, (47), is the only solution of (44) which simultaneously eliminates poles in $(q^2-\mu^2)^{-1}$ for both F^V_1 and F_π in (49), while having at most a t-channel pole dependence as required by the discussion after (41).

We are now in a position to determine part of the unknown Y_ν term in the background electroproduction amplitude (49). Since (49) must be free of *all* poles in $(t-\mu^2)^{-1}$ and $(t-q^2)^{-1}$, as well as in $(q^2-\mu^2)^{-1}$—at least on the pion mass shell—we write

$$Y_\nu^i = Y_\nu^{iP} + (\overline{Y}_\nu^i)_{\rm NGI} + (\overline{Y}_\nu^i)_{\rm GI}\,, \tag{50}$$

where Y_ν^P is the pole piece of Y_ν and $\overline{Y}_\nu$ is its nonpole piece, part of which is not gauge invariant, $(\overline{Y}_\nu)_{\rm NGI}$, and part gauge invariant, $(\overline{Y}_\nu)_{\rm GI}$. To guarantee that Y^P cancels the poles in (49), we must have

$$Y_\nu^{iP} = I_-^i \frac{f_\pi \mu^2}{q^2}\left\{\left[\frac{g(t)F_1^V(k^2)q^2}{(t-\mu^2)(t-q^2)} + \frac{g(t)}{t-\mu^2} - \frac{g(q^2)F_\pi(k^2)}{t-q^2}\right](2q_\nu - k_\nu)\gamma_5 + \frac{g(t)\mathfrak{F}_V(k^2)q^2}{t-\mu^2}k_\nu\gamma_5\right\}. \tag{51}$$

Next, to obtain $(\overline{Y}_\nu^i)_{\rm NGI}$ we multiply (50) by k^ν and use (48) and (51),

$$\begin{aligned}(\overline{Y}_\nu^i)_{\rm NGI}k^\nu &= (Y_\nu^i - Y_\nu^{iP})k^\nu\\ &= \tfrac{1}{2}I_-^i \frac{\mu^2}{q^2}\{[2mG(t,q^2) + q^2\overline{H}(t,q^2) + \overline{h}_A(t)](k^2 - 2q\cdot k)\gamma_5 - 2f_\pi g(q^2)\mathfrak{F}_\pi(k^2)k^2\gamma_5\}\\ &\quad + \tfrac{1}{2}I_-^i \frac{q^2-\mu^2}{q^2}[2mg_A(t) + th_A(t)]\gamma_5\,,\end{aligned} \tag{52}$$

where $\mathfrak{F}_\pi(k^2)k^2 = F_\pi(k^2) - 1$. Removing a factor of k^ν from (52) we have

$$\begin{aligned}(\overline{Y}_\nu^i)_{\rm NGI} &= -\tfrac{1}{2}I_-^i \frac{\mu^2}{q^2}\{[2mG(t,q^2) + q^2\overline{H}(t,q^2) + \overline{h}_A(t)](2q_\nu - k_\nu)\gamma_5 + 2f_\pi g(q^2)\mathfrak{F}_\pi(k^2)k_\nu\gamma_5\}\\ &\quad - \tfrac{1}{2}I_-^i \frac{(q^2-\mu^2)}{q^2(t-q^2)}[2mg_A(t) + th_A(t)](2q_\nu - k_\nu)\gamma_5\,.\end{aligned} \tag{53}$$

Finally, combining (50), (51), and (53) with (49), we achieve the desired expression for the background electroproduction amplitude

$$\begin{aligned}f_\pi \overline{M}_\nu^i &= \tfrac{1}{2}I_-^i\{2f_\pi g(q^2)\mathfrak{F}(k^2)k_\nu\gamma_5 + F_1^V(k^2)G(t,q^2)[2K_\nu^3 + K_\nu^6]\\ &\qquad - g_A(t)\mathfrak{F}_V(k^2)K_\nu^6 - 2\mathfrak{F}_V(k^2)[2mG(t,q^2) + q^2\overline{H}(t,q^2)]K_\nu^5\}\\ &\quad - \frac{1}{4m}g_A(q^2)[F_2^S(k^2)I_0^i + F_2^V(k^2)I_+^i]K_\nu^1 + \frac{q^2}{\mu^2}(\overline{Y}_\nu^i)_{\rm GI} + iq^\mu \overline{M}_{\mu\nu}^i\\ &\quad - \tfrac{1}{2}I_-^i \frac{(q^2-\mu^2)}{\mu^2(t-q^2)}[2mg_A(t) + th_A(t)](2q_\nu - k_\nu)\gamma_5\,,\end{aligned} \tag{54}$$

where

$$\mathfrak{F}(k^2) = \frac{1}{k^2}[F_1^V(k^2) - F_\pi(k^2)] = \mathfrak{F}_V(k^2) - \mathfrak{F}_\pi(k^2) \tag{55}$$

contains no poles in k^2.

This result can be expressed in a form which manifests the gauge condition by adding in the pole terms (13) and combining the non-gauge-invariant first term of (54) with the last term of (13). Thus, for general q^2, we have the complete electroproduction amplitude,

$$\begin{aligned}M_\nu^i &= M_\nu^{iP} + \overline{M}_\nu^i\\ &= \frac{g(q^2)}{(s-m^2)(u-m^2)}\left\{\left[-(F_1^S(k^2)I_0^i + F_1^V(k^2)I_+^i) + \frac{2\nu}{k\cdot q}F_1^V(k^2)I_-^i\right](k\cdot qK_\nu^1 + K_\nu^2)\right.\\ &\qquad\left. - \frac{1}{2m}[F_2^S(k^2)I_0^i + F_2^V(k^2)I_+^i](2\nu K_\nu^3 - k\cdot qK_\nu^4) + \frac{1}{2m}F_2^V(k^2)I_-^i(k\cdot qK_\nu^3 - 2\nu K_\nu^4)\right\}\\ &\quad + \frac{g(q^2)I_-^i}{t-q^2}\left[\frac{1}{k\cdot q}F_1^V(k^2) - 2\mathfrak{F}(k^2)\right]K_\nu^5 - \frac{g_A(q^2)}{4mf_\pi}[F_2^S(k^2)I_0^i + F_2^V(k^2)I_+^i]K_\nu^1\\ &\quad + f_\pi^{-1}I_-^i\{F_1^V(k^2)G(t,q^2)K_\nu^3 - \mathfrak{F}_V(k^2)[2mG(t,q^2) + q^2\overline{H}(t,q^2)]K_\nu^5 + \tfrac{1}{2}[F_1^V(k^2)G(t,q^2) - g_A(t)\mathfrak{F}_V(k^2)]K_\nu^6\}\\ &\quad + \frac{q^2}{f_\pi \mu^2}(\overline{Y}_\nu^i)_{\rm GI} + f_\pi^{-1}iq^\mu \overline{M}_{\mu\nu}^i - \tfrac{1}{2}I_-^i \frac{q^2-\mu^2}{f_\pi \mu^2}\,\frac{2mg_A(t) + th_A(t)}{t-q^2}(2q_\nu - k_\nu)\gamma_5\,.\end{aligned} \tag{56}$$

All that remains to be calculated are the gauge-invariant but model-dependent terms $(\overline{Y}_\nu)_{\rm GI}$ and $\overline{M}_{\mu\nu}$. The former is due to chiral-symmetry breaking, to be discussed in the next section; the latter $\overline{M}_{\mu\nu}$ represents the contribution of dynamical resonances to the two-current amplitude (14) and will be considered in II.

A number of points concerning our solution (56) are worth noting:

(i) On the pion mass shell, $q^2=\mu^2$, M_ν^i becomes gauge invariant, as required. Our final result (1) is just this limit with $(\overline{Y}_\nu)_{\rm GI}$ converted to the $\Sigma(t)$, as described in the next section, and $\overline{M}_{\mu\nu}$ converted to B [defined in (83)] with $G(t,\mu^2)\equiv G(t)$, $\overline{H}(t,\mu^2)\equiv\overline{H}(t)$.

(ii) Except for the dynamical nucleon and pion poles along with the constraint condition (10), which is obeyed by the pole parts of the invariant amplitudes, the remaining terms in (56) are analytic in ν, t and k^2 on the pion mass shell $q^2=\mu^2$. For $q^2\neq\mu^2$, the last term in (56) has an extraneous, unphysical pole signifying again that M_ν as defined by (3) represents the correct electroproduction amplitude only on the pion mass shell. The last term also generates the gauge condition (4) using (16) and (24).

(iii) As a check on (56), it is straightforward to verify that it becomes the standard soft-pion result (23) as $q\to 0$. The FFR theorem for $A_1^{(+,0)}$ has only $O(\mu^2)$ corrections for both $k^2=0$ and $k^2\neq 0$ with $q^2=\mu^2$. The Nambu-Schrauner $g_A(k^2)$ term appears to be present in $A_6^{(-)}$ in the limit $q^2=\mu^2$, $\nu=0$, $t=k^2$, the on-shell values of the invariants which correspond to $q\to 0$. A new g_A' term[17] appears in $A_3^{(-)}$ for $t\sim q^2\sim\mu^2$.

(iv) Our method of first extracting the nucleon axial-vector Born terms in $M_{\mu\nu}$ and then requiring the absence of nondynamical pole structures $(t-\mu^2)^{-1}$, $(q^2-\mu^2)^{-1}$ and $(t-q^2)^{-1}$—the latter nonexistent as least for $q^2=\mu^2$—acts as a powerful constraint (along with the gauge condition) for determining the auxiliary amplitudes $R_{\mu\nu}$, C_ν, Y_ν^P, and $(\overline{Y}_\nu)_{\rm NGI}$. Weisberger[10] proceeds (in the $q\to 0$ limit only) in a different fashion by first extracting the pion-pole term from $M_{\mu\nu}$.

(v) Our method shows no initial preference for pseudoscalar over axial-vector nucleon pole couplings in M_ν, yet the former terms appear in the final form (56). Also the pion-pole form factor $F_\pi(k^2)$ is contained in the $\mathfrak{F}(k^2)$ term. Thus we conclude that (56) is consistent with the usual dispersion-theoretic approach to photo- and electroproduction. On the basis of this analysis we see no necessity for the introduction of fixed J-plane poles in the dispersion-theoretic amplitudes.

(vi) The solution (56) in effect resolves the "factor of 2" problem.[10,17,18] That is, the pion-pole structure of $(t-\mu^2)^{-1}(2q_\nu-k_\nu)$ is not quite matched by the h_A term in the current-commutator part of the Ward identity (20), the latter being proportional to $(t-\mu^2)^{-1}(q_\nu-k_\nu)$. The factor of 2 difference in q_ν does not, of course, affect the soft-pion $q\to 0$ limit (23).

V. CHIRAL-SYMMETRY BREAKING AND THE BJORKEN LIMIT

Now we concentrate on obtaining the chiral-symmetry-breaking part $(\overline{Y}_\nu)_{\rm GI}$ in (56). First we observe that for Y_ν defined by (47) and (48),

(i) The non-gauge-invariant parts of Y_ν are pure I_-: $Y_\nu^{(-)P}$, $\overline{Y}_\nu^{(-)}$ as given by (51) and (53). These terms are independent of chiral-symmetry-breaking models.

(ii) The gauge-invariant parts of Y_ν are in I_+ and I_0: $\overline{Y}_\nu^{(+)}$, $\overline{Y}_\nu^{(0)}$. These terms depend upon a model of chiral-symmetry breaking; e.g., in the $(3,\overline{3})$ model they will be proportional to the nonstrange $SU_2\times SU_2$-breaking quark mass $\hat{m}$.

Next we recall that since $\tilde{T}_\nu=\overline{N}'\tilde{M}_\nu N$ as defined by (15) is proportional to $\partial\cdot A$, the chiral limit $\partial\cdot A\to 0$ must correspond to $\tilde{M}_\nu\to 0$. This leads us naturally to a consideration of the Bjorken-Johnson-Low limit,[19] $q_0\to i\infty$ with $\vec{q}$ fixed,

$$\tilde{M}_\nu^i \xrightarrow[q_0\to i\infty]{} -\frac{1}{q_0}\int d^4x e^{iq\cdot x}\delta(x_0)[\partial\cdot A^i(x), V_0^\gamma(0)]. \tag{57}$$

Using the PCAC relation (17), we may convert (57) to a constraint upon the amplitude of interest, (56),

$$M_\nu^i \xrightarrow[q_0\to i\infty]{} -\frac{q_0}{f_\pi\mu^2}\int d^4x e^{iq\cdot x}\delta(x_0)[\partial\cdot A^i(x), V_\nu^\gamma(0)], \tag{58}$$

where $q_0\to i\infty$ and $\vec{q}$ fixed means $q^2\to q_0^{\,2}\to -\infty$.

To check the model independence of (58), we set the index $\nu=0$ in order to use the model-independent current-algebra commutator (18b) in (58) for $M_0^{(-)}$.

This yields

$$I_-^i M_0^{(-)} \xrightarrow[q_0\to i\infty]{} -\frac{q_0}{f_\pi\mu^2} i\epsilon^{ijk}\partial\cdot A^k(0) \tag{59a}$$

$$=\tfrac{1}{2}I_-^i\frac{q_0}{f_\pi\mu^2}[2mg_A(t)+th_A(t)]\gamma_5, \tag{59b}$$

where we have applied (24) between on-shell nucleon spinors to convert (59a) to (59b). On the other hand, the last (non-gauge-invariant $Y^{(-)}$) term in (56) dominates the $\nu=0$ amplitude $M_0^{(-)}$. To keep the nucleons on mass shell, momentum conservation, $p+k=p'+q$, dictates that k_0 must become large with $\vec{k}$ fixed when q_0 becomes large with $\vec{q}$ fixed. That is, $k_0, q_0 \to i\infty$ with $t=(p'-p)^2$ fixed in the Bjorken limit and this necessitates off-shell electroproduction (and not photoproduction) kinematics. Thus, setting $\nu=0$ in the last term of (56) with $2q_0 - k_0 \to q_0$, $q^2 \to -\infty$ with t fixed, we are again led to (59b). The other terms in (56) are damped to zero because k^2, $q^2 \to -\infty$ and $F_{1,2}^{V,S}(k^2)$, $F_\pi(k^2) \to 0$.

Having verified the consistency of (58) with (56) for $\nu=0$, we investigate the model-dependent current commutator in (58) for $\nu=1-3$. This will determine the quantities of interest, $Y^{(+,0)}$, since these terms are multiplied by q^2 in (56). Anticipating this result we write the general equal-time commutator in (58) as

$$\delta(x_0)[\partial\cdot A^i(x), V_\nu^\gamma(0)] = g_{\nu 0} i\epsilon^{i3k}\partial\cdot A^k(0)\delta^4(x) + \delta^{i3}L_\nu^{(+)}\delta^4(x) + \tau^i L_\nu^{(0)}\delta^4(x) \quad (60)$$

with $L_\nu^{(+,0)}$ model dependent. Then equating the isospin-even terms in (58) to those in (56) as $q_0 \to i\infty$, we see that between nucleon spinors,

$$\overline{Y}_\nu^{(+,0)} \underset{q_0\to i\infty}{\longrightarrow} -\frac{1}{q_0}L_\nu^{(+,0)}. \quad (61)$$

Now $\overline{Y}_\nu^{(+,0)}$ must be gauge invariant and so we write

$$\overline{Y}_\nu^{(+,0)} = \sum_j \mathfrak{A}_j^{(+,0)}(q^2)K_\nu^j \quad (62)$$

summed over the six gauge-invariant electroproduction covariants with the dependence upon ν, t, and k^2 suppressed. But inspecting the K_ν in (7), we see that they all become large, at least $O(q_0)$ as $k_0 \to q_0 \to i\infty$. Thus, since L_ν as defined by (60) is independent of q_0, we must have L_ν and the K_ν at $q_0 = i\infty$ related by

$$L_\nu^{(+,0)} = -\frac{1}{q_0}\sum_j \Sigma_j^{(+,0)}(t)K_\nu^j(q_0 \to i\infty). \quad (63)$$

That is, while L_ν is not manifestly gauge invariant, neither are the K_ν^j as $q_0 \to i\infty$, yet they must be related as indicated in (63). The $\Sigma_j(t)$ are c-number "Σ terms" which contain all the chiral-symmetry-breaking dynamics.

To obtain the final form of the low-energy theorems (1) from (56) and the above Bjorken limit, we observe that (61)–(63) imply that

$$q^2\mathfrak{A}_j^{(+,0)}(q^2) \underset{q^2\to -\infty}{\longrightarrow} \Sigma_j^{(+,0)}(t). \quad (64)$$

Thus, $\mathfrak{A}_j^{(+,0)} \to 0$ as $q^2 \to -\infty$ allows us to assume the unsubtracted dispersion relations [recall from (51) the $\overline{Y}_\nu^{(+,0)}$ have no poles],

$$\mathfrak{A}_j^{(+,0)}(q^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}\mathfrak{A}_j^{(+,0)}(q'^2)}{q'^2 - q^2}\,dq'^2. \quad (65)$$

Then (64) corresponds to the sum rule

$$\frac{1}{\pi}\int \mathrm{Im}\mathfrak{A}_j^{(+,0)}(q'^2)dq'^2 = -\Sigma_j^{(+,0)}(t). \quad (66)$$

At this point we can follow the discussion of Ref. 8, but reformulated in covariant language. One expects that $\mathrm{Im}\mathfrak{A}_j(q'^2)$ is largest in the region of the vector-meson mass $k'^2 = m_V^2$, where k' and q' enter into the four-momentum relation, $k'+p = q'+p'$, as do k and q, $k+p=q+p'$—i.e., the momentum transfer remains fixed, $\Delta = k-q = k'-q' = p'-p$ and $t=\Delta^2$. Thus, we may approximate $\mathrm{Im}\mathfrak{A}_j$ by the resonance (narrow) width form

$$\mathrm{Im}\mathfrak{A}_j^{(+,0)}(q'^2) = \lambda_j^{(+,0)}\delta(q'^2 - m_V^2 - \mu^2), \quad (67a)$$

where we have assumed for simplicity[8] that m_V for $\lambda_{3,8}$ corresponding to the isovector and isoscalar vector-meson states is given by

$$m_V^2 = \begin{cases} m_\rho^2 \approx 30\mu^2 \text{ for } I_+ \\ m_8^2 = \frac{1}{3}(4m_{K^*}^2 - m_\rho^2) \approx 44\mu^2 \text{ for } I_0. \end{cases} \quad (67b)$$

The form (67a) is not obvious and so we devote Appendix B to its justification.

Finally, evaluating (62) on mass shell, $\mathfrak{A}_j^{(+,0)}(\mu^2)$ can be found by substituting (67) into (65) and (66), eliminating the λ_j between the two equations to find

$$\overline{Y}_\nu^{(+,0)} \underset{q^2\to\mu^2}{\longrightarrow} -\frac{1}{m_V^2}\sum_j \Sigma_j^{(+,0)}(t)K_\nu^j. \quad (68)$$

Given (68), (1) follows from (56) evaluated on-shell in a straightforward manner.

The next step is to find an explicit form for the Σ terms in (1) via a model-dependent theory of chiral-symmetry breaking. Needless to say, the quark model is most compelling at present. Then with quark currents defined in terms of quark fields as

$$\partial\cdot A^i(x) = -\hat{m}\,\overline{q}(x)\lambda^i\gamma_5 q(x), \quad (69a)$$

$$V_\nu^\gamma(x) = \tfrac{1}{2}\overline{q}(x)(\lambda^3 + 3^{-1/2}\lambda^8)\gamma_\nu q(x), \quad (69b)$$

$$J_{\alpha\beta}^i(x) = \tfrac{1}{2}\overline{q}(x)\lambda^i\sigma_{\alpha\beta}q(x), \quad (69c)$$

the standard $(3,\overline{3})$ commutation relations give[8]

$$\delta(x_0)[\partial\cdot A^i(x), V_\nu^\gamma(0)] = ig_{\nu 0}\epsilon^{i3k}\partial\cdot A^k(0)\delta^4(x) + i\hat{m}\epsilon_{0\nu}{}^{\alpha\beta}\left\{(1/\sqrt{3})\delta^{i3}[\sqrt{2}J_{\alpha\beta}^0(0) + J_{\alpha\beta}^8(0)] + \tfrac{1}{3}J_{\alpha\beta}^i(0)\right\}\delta^4(x). \quad (70)$$

We see again that the I^i_- term in (70) is model independent, while the $L_\nu^{(+,0)}$ terms of (60) are model dependent, proportional in (70) to the nonstrange quark mass $\hat{m}$. To extract the $\Sigma_j^{(+,0)}(t)$ from (70) via (63), we note that $\epsilon_{0\nu\alpha\beta}$ is the Bjorken limit of the gauge-invariant covariant $q_0^{-1}\epsilon_{\mu\nu\alpha\beta}k^\mu$. The L_ν terms in (70) are then proportional to

$$L_\nu \propto \epsilon_{0\nu}{}^{\alpha\beta}J_{\alpha\beta} = -\frac{1}{q_0}\epsilon_{\nu\mu}{}^{\alpha\beta}k^\mu J_{\alpha\beta}, \tag{71}$$

and once we know the momentum covariants in (71), we will be able to express (71) in the form (63) using the identities (8).

Following the notation of Ref. 8, we now expand the tensor current (69c) in terms of form factors $G_j^{V,S}(t)$, analogous to the form factors $F_{1,2}^{V,S}(t)$ for the vector current. Between nucleon spinors $\bar{N}(\vec{p}\,')$ and $N(\vec{p})$ with $\Delta = p' - p$, $2P = p' + p$, we write ($G_1 \to -G_1$ of Ref. 8).

$$\begin{aligned} J^i_{\alpha\beta}(0) = {} & \sigma_{\alpha\beta}G_1^i(t) + i(\gamma_\alpha\Delta_\beta - \Delta_\alpha\gamma_\beta)G_2^i(t)/2m \\ & + i(\Delta_\alpha P_\beta - P_\alpha\Delta_\beta)G_3^i(t)/m^2 \\ & + i(\gamma_\alpha P_\beta - P_\alpha\gamma_\beta)G_4^i(t)/m. \end{aligned} \tag{72}$$

The SU_3 decomposition of these form factors is

$$\langle B'|G^i|B\rangle = 2\langle p|G^3|p\rangle(dd_{B'iB} + fif_{B'iB}), \tag{73}$$

where $d+f=1$. The U_3-singlet component G^0 and scale $\langle p|G^3|p\rangle$ in (73) are yet to be determined. Once found, the SU_2 decomposition of the form factors $G^{i,\gamma}$ in (70) is

$$G^{i,\gamma} = \delta^{i3}G^V + \tau^i G^S \tag{74a}$$

where

$$G^S = \tfrac{1}{3}G^3, \quad G^V = (\tfrac{2}{3})^{1/2}G^0 + (\tfrac{1}{3})^{1/2}G^8. \tag{74b}$$

Finally, we use the $\epsilon_{\mu\nu\alpha\beta}$ identities (8) to express (71) in terms of the six FNW covariants (7), thus picking off the Σ terms of (63) in the Bjorken limit for the tensor current (70):

$$\Sigma_1^{(+,0)}(t) = \hat{m}[2G_1^{V,S}(t) - G_3^{V,S}(t)t/m^2], \tag{75a}$$

$$\Sigma_2^{(+,0)}(t) = 0, \tag{75b}$$

$$\Sigma_3^{(+,0)}(t) = -\hat{m}G_4^{V,S}(t)/m, \tag{75c}$$

$$\Sigma_4^{(+,0)}(t) = \hat{m}[G_2^{V,S}(t) - 2G_3^{V,S}(t)]/m, \tag{75d}$$

$$\Sigma_5^{(+,0)}(t) = 0, \tag{75e}$$

$$\Sigma_6^{(+,0)}(t) = -\hat{m}G_4^{V,S}(t)/m, \tag{75f}$$

where m is the nucleon mass and $\hat{m}$ the non-strange-quark mass $\frac{1}{2}(m_u + m_d)$ occurring in the quark-model, chiral-symmetry-breaking Hamiltonian $H' = m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s$.

If the tensor-current form factors $G_j(t)$ could be measured, the chiral-symmetry-breaking part of the problem would be solved. Unfortunately, this is not the case, so we must appeal to SU_6-symmetry arguments to determine $G^S(t)$ and $G^V(t)$. Following Ref. 8, one explores the U_3 structure of G_1 by working in the baryon rest frame, where $SU_{6,W}$ requires $J^i_{lm} \to -\epsilon_{lmn}A^i_n$, both being proportional to the spin matrix σ_n in the nonrelativistic static ($t \to 0$) SU_6 limit. In this case, $d/f = \frac{3}{2}$ in (73) with scale $2\langle p|G_1^3|p\rangle = g_A(0) = \frac{5}{3}$. Then (73) gives (for proton matrix elements)

$$\begin{aligned} & G_1^3(0) = \tfrac{1}{2}g_A(0) = \tfrac{5}{6}, \\ & G_1^8(0) = \tfrac{1}{10}\sqrt{3}\,g_A(0) = \tfrac{1}{6}\sqrt{3}, \end{aligned} \tag{76a}$$

$$\begin{aligned} G_1^0(0) & = d\langle N|\tfrac{1}{2}\lambda_0|N\rangle g_A(0) \\ & = \tfrac{1}{2}(\tfrac{2}{3})^{1/2}\tfrac{3}{5}g_A(0) = \tfrac{1}{2}(\tfrac{2}{3})^{1/2}. \end{aligned} \tag{76b}$$

Applying (76) to (74), one finds[8]

$$\begin{aligned} & G_1^S(0) = \tfrac{1}{6}g_A(0) = \tfrac{5}{18}, \\ & G_1^V(0) = \tfrac{3}{10}g_A(0) = \tfrac{1}{2}. \end{aligned} \tag{77}$$

Substituting (77) back into (75a) we see that

$$\Sigma_1^{(+)}(0) = \hat{m}, \quad \Sigma_1^{(0)}(0) = \tfrac{5}{9}\hat{m}. \tag{78}$$

Since we have formulated the chiral-symmetry-breaking problem for electroproduction in the $q_0 \to i\infty$ limit, the U(3)-SU(6) result (78) can be most easily understood and generalized by working directly in the quark model. That is, the chiral-symmetry breaking commutators can be expressed as[20]

$$\begin{aligned} [i\partial\cdot A_3(\vec{x}), \vec{V}_V(0)] & = -\tfrac{1}{2}i\hat{m}[v_3(\vec{x}), \bar{q}\lambda_3\vec{\gamma}q] \\ & = -\hat{m}(\bar{u}\vec{\sigma}u + \bar{d}\vec{\sigma}d)\delta^3(\vec{x}), \end{aligned} \tag{79a}$$

$$\begin{aligned} [i\partial\cdot A_3(\vec{x}), \vec{V}_S(0)] & = -i\hat{m}[v_3(\vec{x}), \bar{q}\lambda_8\vec{\gamma}q/2\sqrt{3}\,] \\ & = -\tfrac{1}{3}\hat{m}(\bar{u}\vec{\sigma}u - \bar{d}\vec{\sigma}d)\delta^3(\vec{x}). \end{aligned} \tag{79b}$$

Next we use the fact that the totally antisymmetric SU(6) (color) $\frac{1}{2}^+$ proton wave function composed of spin-$\frac{1}{2}$ quarks has valence fractions of quark spins parallel and antiparallel to the proton spin summing up to $\frac{5}{3} + \frac{1}{3} = 2$ for up and $\frac{1}{3} + \frac{2}{3} = 1$ for down quarks. Then the *net* helicity distributions parallel to the proton lead to the quark-proton overlaps

$$\langle p|\bar{u}\vec{\sigma}u|p\rangle = (\tfrac{5}{3} - \tfrac{1}{3})\bar{u}_p\vec{\sigma}u_p, \tag{80a}$$

$$\langle p|\bar{d}\vec{\sigma}d|p\rangle = (\tfrac{1}{3} - \tfrac{2}{3})\bar{u}_p\vec{\sigma}u_p. \tag{80b}$$

The sum and difference between (80a) and (80b) therefore convert (79) to

$$\langle p|[i\partial\cdot A_3(\vec{x}), \vec{V}_V(0)]|p\rangle = -\hat{m}\bar{u}_p\vec{\sigma}u_p, \tag{81a}$$

$$\langle p|[i\partial\cdot A_3(\vec{x}), \vec{V}_S(0)]|p\rangle = -\tfrac{5}{9}\hat{m}\bar{u}_p\vec{\sigma}u_p. \tag{81b}$$

Finally, we can identify the right-hand side of

(81) in terms of the K_1 covariant for $k_0 = q_0 \to i\infty$ and $t = |\vec{k}| = |\vec{q}| = 0$, for then $K_1 \to \not{k}\vec{\gamma}\gamma_5 \to -iq_0\vec{\sigma}$. Application of (60) and (63) to (81) then reproduces (78). Furthermore, this method also tells us that

$$\Sigma_2^{(+,0)} = \Sigma_3^{(+,0)} = \Sigma_4^{(+,0)} = \Sigma_5^{(+,0)} = \Sigma_6^{(+,0)} = 0 . \qquad (82)$$

As to the t dependence of $\Sigma_1^{(+,0)}(t)$, models such as[21] U(6, 6) have it suppressed as $\Sigma_1^{(+,0)} 1 + O(t/m^2)$ and therefore negligible.

VI. CONCLUSION

We have justified the on-shell expansion (1) in terms of the expressions (56) and (68). Equation (56) is a consequence of the Ward identities of current algebra (20) and (35), the PCAC relation (17), and the PCAC expansion of the background (42), plus a subtle interplay of gauge-invariance constraints with the pole structures $(t-q^2)^{-1}$, $(t-\mu^2)^{-1}$, and $(q^2-\mu^2)^{-1}$ in the axial-vector amplitude $M_{\mu\nu}$. The form factors $G(t)$, $\overline{H}(t)$, $\mathfrak{F}(k^2)$, $\mathfrak{F}_V(k^2)$, and $\mathfrak{F}_\pi(k^2)$ defined in Sec. IV are nonsingular functions of their argument, t or k^2. As a check on the form (1), we stress again that (56) becomes the soft-pion result (23) as $q \to 0$, recovering both the FFR[14] and Nambu-Schrauner[15] terms. There is no ambiguity between the dispersive pole and current-algebra terms.

Chiral-symmetry breaking is embodied in the C_ν term of (42) leading to the Y_ν term of (47). The existence of a C_ν term is in turn a consequence of the gauge condition (4). In Sec. V, $\overline{Y}_\nu$ is related to the chiral-symmetry-breaking equal time commutator $[\partial \cdot A^i, V_\nu^\gamma]$ in the Bjorken limit. In this limit the isospin I_- part is consistent with the on-shell scheme of Sec. IV. The isospin $I_{+,0}$ parts can be expanded in terms of the six gauge-invariant FNW covariants and chiral-symmetry-breaking Σ terms, vector-meson dominated as in (68). We invoke the quark model to evaluate these Σ terms in the $q_0 \to i\infty$ limit, then leading to (78) and (82).

What remains to be done is to find the background amplitudes B_i in (1) or $q^\mu \overline{M}_{\mu\nu}$ in (56), related by

$$i f_\pi^{-1} q^\mu \overline{M}_{\mu\nu}^{(+,-,0)} = \sum_j B_j^{(+,-,0)}(\nu, t, k^2) K_\nu^j . \qquad (83)$$

The dominant resonances in (83) are $P_{33}\Delta(1230)$, $D_{13}N^*(1520)$, $P_{11}N^*(1470)$, and $S_{11}N^*(1535)$. In II[4] we shall evaluate (83) both in the soft-pion limit as in Ref. 22 and also for on-shell pions for threshold photoproduction. As the reader shall see, the data analysis is in very good agreement with the soft theorems and also the on-shell low-energy expansions (1). Furthermore, we will be able to extract the nonstrange-quark mass $\hat{m}$ in a variety of different ways, all giving *very roughly* the same estimate of $\hat{m} \sim 1\mu$.

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APPENDIX A

The gauge condition (4) was derived in the text as a consequence of PCAC (16) and the vector Ward identity (19). Since (4) plays an essential role in our analysis, we wish to stress here that its existence is independent of current algebra or PCAC. To show this, we first perform the indicated differentiation in (3a) and then use translation invariance to shift the space-time dependence from the pion operators to the electromagnetic current to obtain

$$M_\nu^i = i \int d^4x e^{-ik\cdot x}\{T(j_\pi^i(0)V_\nu^\gamma(x)) + iq_0\delta(x_0)[\phi_\pi^i(0), V_\nu^\gamma(0)] - \delta(x_0)[\dot{\phi}_\pi^i(0), V_\nu^\gamma(x)]\} . \qquad (A1)$$

We contract (A1) with k^ν by applying the operator $i\partial^\nu$ to the exponential and integrating by parts. Then using $\partial \cdot V^\gamma = 0$ along with translation invariance (to shift the space-time dependence back to the pion quantities), we integrate by parts a second time in order to remove the derivatives from the δ functions, obtaining

$$k^\nu M_\nu^i = -\int d^4x e^{iq\cdot x}\delta(x_0)\{[j_\pi^i(x), V_0^\gamma(0)] + q_0^2[\phi_\pi^i(x), V_0^\gamma(0)] + [\ddot{\phi}_\pi^i(x), V_0^\gamma(0)]\} , \qquad (A2)$$

or equivalently

$$k^\nu M_\nu^i = -(q^2-\mu^2) \int d^4x e^{iq\cdot x}\delta(x_0)[\phi_\pi^i(x), V_0^\gamma(0)] . \qquad (A3)$$

Combining the result (A3) (space-time shifted) with the model-independent charge-type equal-time commutator

$$\delta(x_0)[\varphi_\pi^i(0), V_0^\gamma(x)] = i\epsilon^{i3k}\phi_\pi^k(0)\delta^4(x), \qquad (A4)$$

we immediately obtain the gauge condition (4).

APPENDIX B

We wish to estimate the dispersion integral in (65), making use of the fact that $\text{Im}\mathcal{A}_j(q'^2)$ is large when $k'^2 = m_V^2$. Here k' and q' obey the same momentum-conservation relation, $p + k' = p' + q'$, as do the external photon and pion momenta k and q, $p + k = p' + q$. Put another way, we wish to fix the momentum-transfer invariant $t = \Delta^2$ with $\Delta = p' - p = k - q = k' - q'$ also fixed in the Bjorken limit.

In order to determine $q'^2 = \zeta^2$ when $k'^2 = m_V^2$ with $q^2 = \mu^2$, $p'^2 = p^2 = m^2$, Δ and k^2 fixed, we work in the c.m. frame with $\vec{p} = -\vec{k} = -\vec{k}'$ and $\vec{p}' = -\vec{q} = -\vec{q}'$. In the Bjorken limit, we take the pion and photon off their mass shells by varying their energies with their three-momenta constrained as above. Then $\Delta_0 = p_0' - p_0 = k_0 - q_0 = k_0' - q_0'$ fixed becomes in the c.m. frame

$$p_0' - p_0 = (q_0^2 - \mu^2 + m^2)^{1/2} - (k_0^2 - k^2 + m^2)^{1/2}. \qquad (B1)$$

Since $p_0' - p_0 = k_0 - q_0$ we can solve (B1) for q_0 and subtracting it from k_0 we find

$$\Delta_0 = \frac{1}{2}\left(\frac{\mu^2 - k^2}{m^2 - k^2}\right)[k_0 - (k_0^2 - k^2 + m^2)^{1/2}]. \qquad (B2)$$

In a similar fashion, Δ_0 can be expressed in terms of the k' and q' variables as

$$\Delta_0 = \frac{1}{2}\left(\frac{\zeta^2 - m_V^2}{m^2 - m_V^2}\right)[k_0' - (k_0'^2 - m_V^2 + m^2)^{1/2}]. \qquad (B3)$$

Equating (B2) to (B3) we obtain

$$\zeta^2 = m_V^2 + (\mu^2 - k^2)\left(\frac{1 - m_V^2/m^2}{1 - k^2/m^2}\right) \times \frac{k_0 - (k_0^2 + m^2 - k^2)^{1/2}}{(k_0^2 + m_V^2 - k^2)^{1/2} - (k_0^2 + m^2 - k^2)^{1/2}}. \qquad (B4)$$

For k^2 small, $k^2 \ll m_V^2$, we see from (B4) that $\zeta^2 - m_V^2 \sim O(\mu^2)$ for any value of k_0. Thus we are led to the Lorentz-invariant result $\zeta^2 = m_V^2 + \mu^2$, used in (67a) with further $O(\mu^2)$ terms neglected because they contribute to chiral-symmetry-breaking corrections to $O(\hat{m}\mu^2)$, which is small.

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