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Validity of the equivalent-photon approximation for virtual photon-photon collisions

C. Carimalo, P. Kessler, and J. Parisi
Laboratoire de Physique Corpusculaire, Collège de France, Paris, France
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For virtual photon-photon collisions in electron storage rings we derive the equivalent-photon approximation from a helicity treatment and present it in two forms, involving, respectively (i) polarized transverse photons ("transverse-photon approximation") and (ii) unpolarized ones ("Williams-Weizsäcker approximation"). We first postulate the conditions of validity of the approximation on the basis of analytic considerations, and then check them numerically in the case of the process $ee \to ee\mu^+\mu^-$. For this check we consider the completely differentiated cross section as far as approximation (i) is concerned, and in the case of approximation (ii), the cross section differentiated with respect to all variables except the azimuthal angles. Our results are given in the form of tables showing the lower and higher limit of the error involved in the approximation for a large variety of kinematic configurations (i.e., energy losses and scattering angles of both electrons). These tables are discussed in detail, and conclusions are drawn as to the applicability of the equivalent-photon approximation to future experiments to be performed with high-energy electron storage rings.

I. INTRODUCTION

Since the time it was suggested to use electron storage rings for producing virtual photon-photon collisions,[1] this idea has always been closely associated with the application of the equivalent-photon or Williams-Weizsäcker approximation method.[2] The advantages of that approximation procedure are obvious and well known to everyone: It allows one to obtain rough numerical predictions by means of very simple formulas, thus sparing effort and time, and even achieving better physical transparency. Moreover, it may considerably simplify the analysis of experiments, since it provides the possibility of extracting directly an approximate cross section for $\gamma\gamma \to X$ from measurements of $ee \to eeX$.

The question then unavoidably arises: How good is the approximation? Many authors tried to answer that question,[3] but obviously a general answer cannot be given, as the conditions of measurement will vary widely from one experiment to another. The problem has become all the more acute since we know that in general, with present and future electron storage rings of very high energy (such as PETRA, PEP, or LEP), electron tagging at 0° will hardly be possible, so that the photons involved in $\gamma\gamma$ collisions will not be very close to the mass shell.

Our purpose in this paper is as follows:

(i) To provide a better understanding of the equivalent-photon approximation by deriving it from a helicity treatment, and to postulate its *a priori* conditions of validity.

(ii) To perform a numerical check of its validity, on the basis of the completely differentiated cross section of $ee \to eeX$ (to begin with), in order to reach conclusions which are independent of experimental acceptance and efficiency conditions.

For the numerical check, we use the reaction $ee \to ee\mu^+\mu^-$. That QED process is both important by itself (for a calibration experiment) and typical for hadron production (i.e., $ee \to eeq\bar{q}$).

In Sec. II we briefly show how the helicity formula is set up for a process $ee \to eeX$. In Sec. III we define the *a priori* conditions to be postulated for the "transverse-photon approximation" (or polarized-equivalent-photon approximation), i.e., for neglecting all longitudinal terms in the helicity formula. That approximation is numerically checked in Sec. IV. Thereafter, eliminating some more terms (the transverse-polarization terms) by integrating over the azimuthal angles, we stay with the one-term approximation which is the Williams-Weizsäcker (or unpolarized-equivalent-photon) approximation; its numerical check is shown in Sec. V. A brief conclusion is drawn in Sec. VI. Full details of calculation are contained in an Appendix.

II. THE HELICITY FORMULA $ee \to eeX$

The matrix element for the Feynman diagram of Fig. 1 is written

$$\mathfrak{M} \sim l_\mu c^{\mu\nu} r_\nu \equiv l_\mu g^{\mu\rho} c_{\rho\sigma} g^{\sigma\nu} r_\nu \qquad (2.1)$$

where l_μ, $c^{\mu\nu}$, and r_ν stand, respectively, for the electromagnetic current at the left-hand, central, and right-hand vertex. We now want to define for either photon a system of polarization vectors, i.e., ϵ_m and ϵ'_n, respectively (with m, n taking the values ± 1 or 0, and with the usual properties of normali-

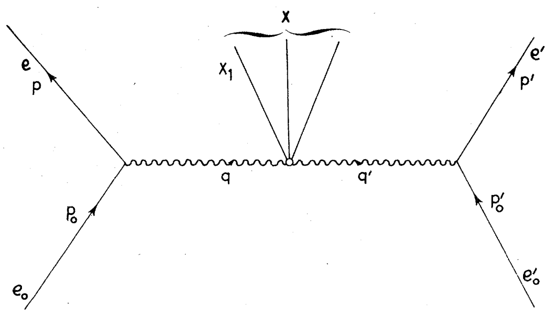

FIG. 1. Feynman diagram for the process $ee \to eeX$ occurring via the $\gamma\gamma$ collision mechanism.

zation and orthogonality), associated with the $\gamma\gamma$ center-of-mass frame. For complete specification of such a system, calling the z axis the $\gamma\gamma$ collision axis, we must still define the y axis. At the central vertex, we choose that axis orthogonal to the three-momentum of some selected outgoing particle X_1; at the left-hand vertex, orthogonal to the three-momentum of particle e (or e_0); at the right-hand vertex, orthogonal to the three-momentum of e' (or e'_0).

Calling ϵ_m and ϵ'_n the polarization vectors introduced specifically at the central vertex, their components are given as follows (in order: x, y, z, and 0 component in the $\gamma\gamma$ c.m. frame):

$$\begin{aligned} &\epsilon_\pm^\mu = -\epsilon_\pm'^\mu = \mp \frac{1}{\sqrt{2}}(1, \pm i, 0, 0), \\ &\epsilon_0^\mu = (0, 0, q_0, \bar{q}) \\ &\epsilon_0'^\mu = (0, 0, q'_0, \bar{q}') \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} &q_0 = \frac{1}{2M_X}(M_X{}^2 - Q^2 + Q'^2), \\ &q'_0 = \frac{1}{2M_X}(M_X{}^2 - Q'^2 + Q^2), \\ &\bar{q} = -\bar{q}' = (q_0{}^2 + Q^2)^{1/2} = (q_0'^2 + Q'^2)^{1/2}, \end{aligned} \tag{2.3}$$

where we define M_X as the invariant mass of X, and $Q^2 = -q^2$ ($\equiv \bar{q}^2 - q_0{}^2$), $Q'^2 = -q'^2$. The polarization vectors used at the left-hand and right-hand vertex are then respectively $\epsilon_m \exp(im\varphi_1)$ and $\epsilon'_n \exp[in(\varphi - \varphi_1)]$, defining (always in the $\gamma\gamma$ c.m. frame) φ_1 as the azimuthal angle between e (or e_0) and X_1, and φ as the azimuthal angle between e (or e_0) and e' (or e'_0).

We now define the "helicity amplitudes"

$$\begin{aligned} &l_m = (-)^m l_\mu \epsilon_m^{\mu *} e^{-im\varphi_1}, \quad c_{mn} = \epsilon_m^\mu c_{\mu\nu} \epsilon_n'^\nu, \\ &r_n = (-)^n r_\nu \epsilon_n'^{\nu *} e^{-in(\varphi-\varphi_1)}. \end{aligned} \tag{2.4}$$

Introducing the closure relations

$$\begin{aligned} &g^{\mu\rho} = \sum_m (-)^m \epsilon_m^{*\mu} \epsilon_m^\rho + \frac{q^\mu q^\rho}{q^2}, \\ &g^{\sigma\nu} = \sum_n (-)^n \epsilon_n'^\sigma \epsilon_n'^{*\nu} + \frac{q'^\nu q'^\sigma}{q'^2}, \end{aligned} \tag{2.5}$$

substituting those relations into (2.1), using gauge invariance at each vertex, and taking account of the definitions (2.4), one is led to

$$\mathfrak{M} \sim l_m c^{mn} r_n e^{i[m\varphi_1 + n(\varphi - \varphi_1)]} \tag{2.6}$$

(with implicit summation over m,n). Defining now

$$\begin{aligned} &L_{m\bar{m}} = \sum l_m l_{\bar{m}}^*, \\ &C^{m\bar{m},n\bar{n}} = \sum c^{mn} c^{\bar{m}\bar{n}*}, \; R_{n\bar{n}} = \sum r_n r_{\bar{n}}^*, \end{aligned} \tag{2.7}$$

where $\sum$ means summation over the spin states of external particles at the vertex concerned, one gets

$$\sum |\mathfrak{M}|^2 \sim L_{m\bar{m}} C^{m\bar{m},n\bar{n}} R_{n\bar{n}} \, e^{i[(m-\bar{m})\varphi_1 + (n-\bar{n})(\varphi - \varphi_1)]} \tag{2.8}$$

(with implicit summation over m, $\bar{m}$, n, $\bar{n}$), and finally for the cross section:

$$\begin{aligned} \frac{d\sigma}{dP_{\rm LI}} = &\frac{e^8}{32E_0{}^2Q^4Q'^4} L_{m\bar{m}} C^{m\bar{m},n\bar{n}} R_{n\bar{n}} \\ &\times e^{i[(m-\bar{m})\varphi_1 + (n-\bar{n})(\varphi - \varphi_1)]}, \end{aligned} \tag{2.9}$$

where E_0 is the initial energy of either electron in the overall c.m. frame (supposed to be the laboratory frame), and the Lorentz-invariant phase space is defined as usual

$$dP_{\rm LI} = (2\pi)^4 \delta^4\Big(p_0 + p'_0 - p - p' - \sum_i p_{Xi}\Big) \frac{1}{2(2\pi)^3} \frac{d^3p}{E} \frac{1}{2(2\pi)^3} \frac{d^3p'}{E'} \prod_i \frac{1}{2(2\pi)^3} \frac{d^3p_{Xi}}{E_{Xi}} \tag{2.10}$$

using the generic name X_i for the particles composing the system X, and the symbol E for energy components of the various momentum four-vectors.

Using the trivial symmetry properties of $L_{m\bar{m}}$ and $R_{n\bar{n}}$, and the Hermiticity of the central tensor, i.e., $C_{m\bar{m},n\bar{n}} = C^*_{\bar{m}m,\bar{n}n}$, one gets the helicity formula in its most general form

$$\frac{32E_0^2Q^4Q'^4}{e^8}\frac{d\sigma}{dP_{\rm LI}} = L_{++}(C_{++,++}+C_{++,--}+C_{--,++}+C_{--,--})R_{++}$$
$$+2L_{++}[(\mathrm{Re}C_{++,+-}+\mathrm{Re}C_{--,+-})\cos 2(\varphi-\varphi_1)-(\mathrm{Im}C_{++,+-}+\mathrm{Im}C_{--,+-})\sin 2(\varphi-\varphi_1)]R_{+-}$$
$$+2L_{+-}[(\mathrm{Re}C_{+-,++}+\mathrm{Re}C_{+-,--})\cos 2\varphi_1-(\mathrm{Im}C_{+-,++}+\mathrm{Im}C_{+-,--})\sin 2\varphi_1]R_{++}$$
$$+2L_{+-}[\mathrm{Re}C_{+-,+-}\cos 2\varphi+\mathrm{Re}C_{+-,-+}\cos 2(2\varphi_1-\varphi)-\mathrm{Im}C_{+-,+-}\sin 2\varphi-\mathrm{Im}C_{+-,-+}\sin 2(2\varphi_1-\varphi)]R_{+-}$$
$$+\text{longitudinal terms}\,, \qquad (2.11)$$

where we call "longitudinal terms" all terms with at least one "0" helicity subscript.

If—following Carlson and Tung[4]—one sticks to the case of only two particles composing the final system X, or of an inclusive measurement of "X_1 plus anything" within that system, one can use the additional relation provided by parity conservation and rotational invariance $(-)^{m+\bar{m}+n+\bar{n}}C_{m\bar{m},n\bar{n}}=C_{-m-\bar{m},-n-\bar{n}}$. Formula (2.11) is then reduced to the form

$$\frac{32E_0^2Q^4Q'^4}{e^8}\frac{d\sigma}{dP_{\rm LI}} = 2L_{++}(C_{++,++}+C_{++,--})R_{++}+4L_{++}(\mathrm{Re}C_{++,+-})R_{+-}\cos 2(\varphi-\varphi_1)+4L_{+-}(\mathrm{Re}C_{+-,++})R_{++}\cos 2\varphi_1$$
$$+2L_{+-}C_{+-,+-}R_{+-}\cos 2\varphi+2L_{+-}C_{+-,-+}R_{+-}\cos 2(2\varphi_1-\varphi)+\text{longitudinal terms.}$$

The explicit form of the longitudinal terms can be found in Ref. 4.

III. THE TRANSVERSE-PHOTON APPROXIMATION

As is well known, the basic procedure of the equivalent-photon approximation consists in neglecting the longitudinal contributions [explicitly, there are still 15 longitudinal terms left in formula (2.12)]. Indeed, for the sake of gauge invariance, they tend to vanish when Q, Q' go to zero. But since these parameters cannot become strictly equal to zero in the process considered, the question arises: How small should they be? One obviously must fix some scale; that can be done in the following way.

Always assuming that we stay in the $\gamma\gamma$ c.m. frame, and calling z the $\gamma\gamma$ collision axis, the longitudinal polarization vector of the left-hand photon can be expressed as

$$\epsilon_\mu^{(0)}=\frac{Q}{M_X q_3}\left(p_{X\mu}+\frac{p_X q}{Q^2}q_\mu\right), \qquad (3.1)$$

where p_X is the total four-momentum of the system X. (We here write the helicity index in parentheses, in order to avoid confusion with ordinary covariant or contravariant subscripts). The gauge-invariance condition for the electromagnetic current at the central vertex, with respect to the left-hand photon, is explicitly written as

$$q_\mu c_\nu^\mu \equiv q_0 c_{0\nu}-q_3 c_{3\nu}=0\,. \qquad (3.2)$$

Defining, for the sake of our demonstration, helicity amplitudes such that helicity is fixed only for the left-hand (not for the right-hand) photon, i.e., $c_\nu^{(m)}=\epsilon_\mu^{(m)}c_\nu^\mu$, we get from (3.1) and (3.2)

$$c_\nu^{(0)}=\frac{Q}{M_X q_3}p_{X\mu}c_\nu^\mu=\frac{Q}{q_3}c_{0\nu}=\frac{Q}{q_0}c_{3\nu}\,. \qquad (3.3)$$

The magnitude of the ratio between longitudinal and transverse helicity amplitudes is then given by

$$\left|\frac{c_\nu^{(0)}}{c_\nu^{(\pm)}}\right|=\frac{Q}{|q_0|}\left|\frac{c_{3\nu}}{c_{\pm,\nu}}\right|, \qquad (3.4)$$

where $c_{\pm,\nu}=-(1/\sqrt{2})(c_{1\nu}\pm ic_{2\nu})$.

Now we introduce our fundamental assumption, namely that in general the electromagnetic current for the process $\gamma\gamma\to X$ should not be too anisotropic in ordinary three-space, i.e., $|c_{3\nu}|\approx|c_{1\nu}|$, $|c_{2\nu}|$. If that assumption is true, the condition for neglecting the longitudinal helicity amplitudes becomes $Q\ll|q_0|$. Similarly, defining helicity amplitudes with respect to the right-hand photon, we shall find the condition $Q'\ll|q_0'|$.

In conclusion, using (2.3) and assuming our fundamental assumption to be true, all longitudinal terms in formula (2.11) or (2.12) may be neglected, in first approximation, when the inequalities

$$2QM_X\ll|M_X^2-Q^2+Q'^2|\,,$$
$$2Q'M_X\ll|M_X^2-Q'^2+Q^2| \qquad (3.5)$$

are both satisfied.

Actually, there are several different configurations of relative values of Q, Q', M_X satisfying both inequalities (3.5). Here we are only interested in one of them, namely $Q,Q'\ll M_X/2$. Defining $\epsilon=2Q/M_X$ and $\epsilon'=2Q'/M_X$, we thus assume the transverse-photon approximation to be valid for $\epsilon,\epsilon'\ll 1$.

Our assumption of "approximate isotropy" of the electromagnetic current for $\gamma\gamma\to X$ in ordinary three-space is purely empirical and remains to be checked.

IV. NUMERICAL CHECK OF THE TRANSVERSE-PHOTON APPROXIMATION

As said above, our checking will bear on the reaction $ee \to ee\mu^+\mu^-$ (particle X_1 of Fig. 1 being identified, for instance, with μ^+). We here consider the completely differential cross section, i.e.,

$$\bar{\sigma} \equiv \frac{d\sigma}{d\,E d\Omega dE' d\Omega' d\Omega_1} = K\,\frac{d\sigma}{dP_{\rm LI}}\,,$$

where all variables used (E, E' are energies of e, e', respectively; $\Omega, \Omega', \Omega_1$ are solid angles of e, e', and μ^+, respectively) are defined in the laboratory frame. K is a kinematic factor, given in the Appendix [formulas (A15), (A16)].

Always in the laboratory frame, we define the following angles:

θ, ϕ: orbital and azimuthal angle of e,
$\pi - \theta', \phi'$: orbital and azimuthal angle of e',
ψ, ϕ_1: orbital and azimuthal angle of μ^+.

We also introduce $x = (E_0 - E)/E_0$, $x' = (E_0 - E')/E_0$. In addition, we find it convenient to replace our parameter ϵ by

$$\bar{\epsilon} = \frac{Q}{E_0(xx')^{1/2}} = \epsilon + O(\epsilon^2)\,,$$

which is more closely related to the experimental parameters. Similarly we introduce $\bar{\epsilon}' = Q'/(E_0\sqrt{xx'})$. The condition $\bar{\epsilon}, \bar{\epsilon}' \ll 1$ are equivalent to

$$\theta \ll \left(\frac{xx'}{1-x}\right)^{1/2}, \quad \theta' \ll \left(\frac{xx'}{1-x'}\right)^{1/2}$$

(θ, θ' in radians).

Our check is performed for $\bar{\epsilon}, \bar{\epsilon}' = \frac{1}{100}, \frac{1}{30}, \frac{1}{10}, \frac{1}{3}$, and for $x, x' = 0.1, 0.4, 0.7$. As for ψ and the various azimuthal angles, we let them go through a wide variety of values (ψ being, however, limited to the range $30° \leq \psi \leq 150°$). For each configuration of values of $x, x', \bar{\epsilon}, \bar{\epsilon}'$, we notice the lower and higher limit thus found for the error Δ defined as

$$\Delta = (\bar{\sigma}_{\rm approx} - \bar{\sigma}_{\rm exact})/\bar{\sigma}_{\rm exact}\,.$$

We find it interesting, in addition, to try two different types of approximation:

(i) approximation I, where we simply neglect all longitudinal terms in formula (2.12).

(ii) approximation II where, in addition to neglecting all longitudinal terms, we let ϵ, ϵ' go to zero in the remaining terms.

Approximation II involves considerably simplified expressions of the tensor elements ($L_{m\bar{m}}, R_{n\bar{n}}, C_{m\bar{m}, n\bar{n}}$) and the kinematic correlations used (see Appendix, part 3); it may also appear logically more coherent. However, although in principle all terms neglected in approximation II are still of order ϵ or ϵ', a close inspection shows that the order of magnitude of some of them is given by ϵ or ϵ' times some factor which may become large in given kinematic situations. Therefore, we anticipate that errors will in general be larger in approximation II than in approximation I.

As for the exact calculation, instead of using (2.12), we preferred, for simplicity, to apply the standard method of Feynman-diagram calculation in QED. That method requires no particular comment.

The beam energy was chosen as $E_0 = 15$ GeV. Actually, above $E_0 \approx 10$ GeV (since in all configurations considered one has $4m_e^2 \ll Q^2, Q'^2, 4m_\mu^2 \ll M_X^2$), the lepton masses practically vanish from the calculation, whether approximate or exact (see Appendix); then, since there is no longer a mass scale, our results become independent of E_0, i.e., they depend only on the dimensionless quantities used: $x, x', \bar{\epsilon}, \bar{\epsilon}'$ (or θ, θ') and the other angles.

These results are shown in Table I. They suggest the following comments:

(a) In both approximations I and II, as expected, the error limits tend to increase systematically, in magnitude, with $\bar{\epsilon}$ and $\bar{\epsilon}'$. As functions of either of those parameters, keeping the other one constant, they tend to increase rapidly when the other one is small, and much more slowly when the other one is relatively large.

(b) Comparing approximation I with II, it is noticed that—again, as expected—I is better than II in the average (but not systematically).

(c) Comparing the various parts of Table I with each other, one notices that approximation I becomes gradually better when x, x' are increased. That fact is explained by the "extinction effect" occurring in the transverse terms, due to polarization [see formula (A26) in the Appendix; as shown there, that effect is quite strong at small values of x, x', and is gradually reduced when they are increased]. The extinction effect here appears specifically associated with the dynamics of the process $\gamma\gamma \to \mu^+\mu^-$; however, similar effects may occur in other reactions. As for approximation II, one also notices some improvement when x, x' are both increased. On the other hand, however, one can see that the approximation becomes considerably worse when one goes to unsymmetric configurations ($x' > x$), and there especially when $\bar{\epsilon}'$ becomes large. The latter fact is basically due to large errors in the expression of the $\gamma\gamma$ c.m. emission angle χ [formula (A22) or (A23)] and to the angular distribution in $\gamma\gamma \to \mu^+\mu^-$, which happens to be particularly sensitive to such errors at small values of $\sin\chi$.

(d) Generally speaking, whether in approximation

TABLE I. Range of $\Delta = (\bar{\sigma}_{\text{approx}} - \bar{\sigma}_{\text{exact}})/\bar{\sigma}_{\text{exact}}$ in the process $ee \to ee\mu^+\mu^-$, according to approximation I or II, for various kinematic configurations with fixed values of the electron variables $x, x'; \bar{\epsilon}, \bar{\epsilon}'$ (or θ, θ'), letting the muon orbital angle ψ and all azimuthal angles go through a wide varity of values ($30° < \psi < 150°$). Beam energy $E_0 = 15$ GeV.

		$\bar{\epsilon} = \frac{1}{100}$	$\bar{\epsilon} = \frac{1}{30}$	$\bar{\epsilon} = \frac{1}{10}$	$\bar{\epsilon} = \frac{1}{3}$
		$x = 0.1$, $x' = 0.1$			
$\bar{\epsilon}' = \frac{1}{100}$	Approx I	$-7\% < \Delta < +7\%$	$-13\% < \Delta < +16\%$	$-31\% < \Delta < +43\%$	$-75\% < \Delta < +227\%$
	Approx II	$-20\% < \Delta < +5\%$	$-23\% < \Delta < +14\%$	$-36\% < \Delta < +40\%$	$-83\% < \Delta < +224\%$
$\bar{\epsilon}' = \frac{1}{30}$	Approx I	$-13\% < \Delta < +16\%$	$-19\% < \Delta < +24\%$	$-36\% < \Delta < +54\%$	$-77\% < \Delta < +254\%$
	Approx II	$-23\% < \Delta < +14\%$	$-23\% < \Delta < +17\%$	$-38\% < \Delta < +42\%$	$-83\% < \Delta < +228\%$
$\bar{\epsilon}' = \frac{1}{10}$	Approx I	$-31\% < \Delta < +43\%$	$-36\% < \Delta < +54\%$	$-48\% < \Delta < +87\%$	$-81\% < \Delta < +342\%$
	Approx II	$-36\% < \Delta < +40\%$	$-38\% < \Delta < +42\%$	$-45\% < \Delta < +53\%$	$-85\% < \Delta < +241\%$
$\bar{\epsilon}' = \frac{1}{3}$	Approx I	$-75\% < \Delta < +227\%$	$-77\% < \Delta < +254\%$	$-81\% < \Delta < +342\%$	$-91\% < \Delta < +770\%$
	Approx II	$-83\% < \Delta < +224\%$	$-83\% < \Delta < +228\%$	$-85\% < \Delta < +241\%$	$-87\% < \Delta < +288\%$
		$x = 0.1$, $x' = 0.4$			
$\bar{\epsilon}' = \frac{1}{100}$	Approx I	$-7\% < \Delta < +6\%$	$-13\% < \Delta < +12\%$	$-30\% < \Delta < +29\%$	$-68\% < \Delta < +101\%$
	Approx II	$-8\% < \Delta < +8\%$	$-11\% < \Delta < +9\%$	$-26\% < \Delta < +18\%$	$-63\% < \Delta < +66\%$
$\bar{\epsilon}' = \frac{1}{30}$	Approx I	$-13\% < \Delta < +11\%$	$-19\% < \Delta < +18\%$	$-35\% < \Delta < +37\%$	$-69\% < \Delta < +116\%$
	Approx II	$-23\% < \Delta < +25\%$	$-24\% < \Delta < +24\%$	$-27\% < \Delta < +29\%$	$-64\% < \Delta < +74\%$
$\bar{\epsilon}' = \frac{1}{10}$	Approx I	$-29\% < \Delta < +32\%$	$-34\% < \Delta < +40\%$	$-46\% < \Delta < +62\%$	$-72\% < \Delta < +164\%$
	Approx II	$-58\% < \Delta < +83\%$	$-59\% < \Delta < +81\%$	$-60\% < \Delta < +84\%$	$-57\% < \Delta < +98\%$
$\bar{\epsilon}' = \frac{1}{3}$	Approx I	$-62\% < \Delta < +137\%$	$-64\% < \Delta < +152\%$	$-68\% < \Delta < +199\%$	$-80\% < \Delta < +440\%$
	Approx II	$-95\% < \Delta < +373\%$	$-96\% < \Delta < +369\%$	$-95\% < \Delta < +368\%$	$-87\% < \Delta < +391\%$
		$x = 0.1$, $x' = 0.7$			
$\bar{\epsilon}' = \frac{1}{100}$	Approx I	$-3\% < \Delta < +4\%$	$-6\% < \Delta < +7\%$	$-15\% < \Delta < +16\%$	$-40\% < \Delta < +51\%$
	Approx II	$-11\% < \Delta < +10\%$	$-12\% < \Delta < +10\%$	$-18\% < \Delta < +14\%$	$-35\% < \Delta < +44\%$
$\bar{\epsilon}' = \frac{1}{30}$	Approx I	$-7\% < \Delta < +7\%$	$-10\% < \Delta < +11\%$	$-18\% < \Delta < +20\%$	$-42\% < \Delta < +55\%$
	Approx II	$-29\% < \Delta < +34\%$	$-30\% < \Delta < +33\%$	$-30\% < \Delta < +32\%$	$-33\% < \Delta < +53\%$
$\bar{\epsilon}' = \frac{1}{10}$	Approx I	$-17\% < \Delta < +18\%$	$-19\% < \Delta < +22\%$	$-26\% < \Delta < +32\%$	$-48\% < \Delta < +66\%$
	Approx II	$-71\% < \Delta < +115\%$	$-72\% < \Delta < +114\%$	$-72\% < \Delta < +112\%$	$-67\% < \Delta < +137\%$
$\bar{\epsilon}' = \frac{1}{3}$	Approx I	$-41\% < \Delta < +53\%$	$-43\% < \Delta < +56\%$	$-47\% < \Delta < +63\%$	$-61\% < \Delta < +78\%$
	Approx II	$-92\% < \Delta < +559\%$	$-94\% < \Delta < +557\%$	$-92\% < \Delta < +552\%$	$-88\% < \Delta < +609\%$
		$x = 0.4$, $x' = 0.4$			
$\bar{\epsilon}' = \frac{1}{100}$	Approx I	$-5\% < \Delta < +6\%$	$-10\% < \Delta < +11\%$	$-23\% < \Delta < +29\%$	$-57\% < \Delta < +112\%$
	Approx II	$-5\% < \Delta < +5\%$	$-11\% < \Delta < +11\%$	$-27\% < \Delta < +33\%$	$-72\% < \Delta < +127\%$
$\bar{\epsilon}' = \frac{1}{30}$	Approx I	$-10\% < \Delta < +11\%$	$-14\% < \Delta < +17\%$	$-27\% < \Delta < +36\%$	$-59\% < \Delta < +123\%$
	Approx II	$-11\% < \Delta < +11\%$	$-12\% < \Delta < +14\%$	$-29\% < \Delta < +34\%$	$-72\% < \Delta < +129\%$
$\bar{\epsilon}' = \frac{1}{10}$	Approx I	$-23\% < \Delta < +29\%$	$-27\% < \Delta < +36\%$	$-36\% < \Delta < +57\%$	$-64\% < \Delta < +158\%$
	Approx II	$-27\% < \Delta < +33\%$	$-29\% < \Delta < +34\%$	$-32\% < \Delta < +42\%$	$-74\% < \Delta < +134\%$
$\bar{\epsilon}' = \frac{1}{3}$	Approx I	$-57\% < \Delta < +112\%$	$-59\% < \Delta < +123\%$	$-64\% < \Delta < +158\%$	$-77\% < \Delta < +285\%$
	Approx II	$-72\% < \Delta < +127\%$	$-72\% < \Delta < +129\%$	$-74\% < \Delta < +134\%$	$-76\% < \Delta < +195\%$
		$x = 0.4$, $x' = 0.7$			
$\bar{\epsilon}' = \frac{1}{100}$	Approx I	$-3\% < \Delta < +3\%$	$-5\% < \Delta < +6\%$	$-12\% < \Delta < +12\%$	$-34\% < \Delta < +33\%$
	Approx II	$-11\% < \Delta < +6\%$	$-12\% < \Delta < +8\%$	$-17\% < \Delta < +19\%$	$-31\% < \Delta < +68\%$
$\bar{\epsilon}' = \frac{1}{30}$	Approx I	$-5\% < \Delta < +6\%$	$-8\% < \Delta < +8\%$	$-14\% < \Delta < +15\%$	$-36\% < \Delta < +35\%$
	Approx II	$-14\% < \Delta < +13\%$	$-15\% < \Delta < +14\%$	$-19\% < \Delta < +21\%$	$-31\% < \Delta < +69\%$
$\bar{\epsilon}' = \frac{1}{10}$	Approx I	$-13\% < \Delta < +13\%$	$-15\% < \Delta < +16\%$	$-21\% < \Delta < +22\%$	$-41\% < \Delta < +44\%$
	Approx II	$-35\% < \Delta < +41\%$	$-36\% < \Delta < +40\%$	$-37\% < \Delta < +45\%$	$-35\% < \Delta < +77\%$
$\bar{\epsilon}' = \frac{1}{3}$	Approx I	$-37\% < \Delta < +39\%$	$-39\% < \Delta < +42\%$	$-42\% < \Delta < +52\%$	$-54\% < \Delta < +95\%$
	Approx II	$-85\% < \Delta < +169\%$	$-86\% < \Delta < +166\%$	$-86\% < \Delta < +168\%$	$-82\% < \Delta < +214\%$

TABLE I. *(continued)*

		$\bar{\epsilon}=\frac{1}{100}$	$\bar{\epsilon}=\frac{1}{30}$	$\bar{\epsilon}=\frac{1}{10}$	$\bar{\epsilon}=\frac{1}{3}$
		$x=0.7$, $x'=0.7$			
$\bar{\epsilon}'=\frac{1}{100}$	Approx I	$-2\%<\Delta<+3\%$	$-4\%<\Delta<+5\%$	$-10\%<\Delta<+11\%$	$-26\%<\Delta<+29\%$
	Approx II	$-8\%<\Delta<+5\%$	$-9\%<\Delta<+9\%$	$-20\%<\Delta<+24\%$	$-51\%<\Delta<+90\%$
$\bar{\epsilon}'=\frac{1}{30}$	Approx I	$-4\%<\Delta<+5\%$	$-6\%<\Delta<+7\%$	$-12\%<\Delta<+13\%$	$-28\%<\Delta<+32\%$
	Approx II	$-9\%<\Delta<+9\%$	$-10\%<\Delta<+10\%$	$-21\%<\Delta<+24\%$	$-52\%<\Delta<+90\%$
$\bar{\epsilon}'=\frac{1}{10}$	Approx I	$-10\%<\Delta<+11\%$	$-12\%<\Delta<+13\%$	$-17\%<\Delta<+19\%$	$-32\%<\Delta<+38\%$
	Approx II	$-20\%<\Delta<+24\%$	$-21\%<\Delta<+24\%$	$-23\%<\Delta<+29\%$	$-54\%<\Delta<+95\%$
$\bar{\epsilon}'=\frac{1}{3}$	Approx I	$-26\%<\Delta<+29\%$	$-28\%<\Delta<+32\%$	$-32\%<\Delta<+38\%$	$-43\%<\Delta<+54\%$
	Approx II	$-51\%<\Delta<+90\%$	$-52\%<\Delta<+90\%$	$-54\%<\Delta<+95\%$	$-53\%<\Delta<+135\%$

I or II, if one wishes to keep the errors smaller than, let us say, a factor of 2, one must restrict $\bar{\epsilon},\bar{\epsilon}'$ to values lower than some limit $\bar{\epsilon}_{\max}\lesssim\frac{1}{10}$. That restriction means that one should stick to rather small electron scattering angles; for instance, when $x=x'=0.1$, one thus gets the limit $\theta_{\max}=\theta'_{\max}\simeq 10$ mrad.

We conclude that, for predicting or analyzing experimental results, the transverse-photon approximation (involving the possibility of separating the various polarization terms in the analysis provided one has relatively high statistics) can only be applied validly, in practice, when an electron-tagging device at 0° is used.

V. NUMERICAL CHECK OF THE WILLIAMS-WEIZSÄCKER APPROXIMATION

If we wish to go one step further in our approximation, i.e., to keep only the first, "unpolarized," term in formula (2.11) or (2.12), there is an obvious condition to be satisfied: Namely, we should be able to integrate fully, between 0 and 2π, over both relative azimuthal angles φ and φ_1. Since in first approximation those angles are equal to the corresponding relative azimuthal angles in the laboratory frame [formulas (A20), (A24)], the following experimental condition must be fulfilled: At least both electron-tagging devices—and not necessarily the "central detector"—should have cylindrical symmetry, i.e., uniform efficiency and no limitation of acceptance.

With that assumption, we shall consider the differential cross section

$$\tilde{\sigma}=\int \bar{\sigma}\,d\phi\,d\phi'$$

$$=\frac{d\sigma}{dE\,d(\cos\theta)dE'\,d(\cos\theta')d\Omega_1}$$

to which the one-term (Williams-Weizsäcker) approximation can be applied. That approximation allows one to extract directly the unpolarized differential $\gamma\gamma$ cross section from an experimental measurement. It should however be remarked that such a cross section is for off-shell photons, which means that it may be different from the cross section of two colliding free photons. Indeed, if (in addition to M_X) another mass scale is occurring in the $\gamma\gamma$ process —for instance, according to the vector-dominance model—one may have to introduce a "form factor"

$$\sigma_{\gamma\gamma}(Q^2,Q'^2)=\sigma_{\gamma\gamma}(0,0)F(Q^2,Q'^2)\,.$$

Possibly, in the analysis of an experiment, the value of $\sigma_{\gamma\gamma}$ on shell might then be determined by extrapolation.

In the Table II, where the Williams-Weizsäcker approximation will be numerically checked, we use the same values of $x,x',\bar{\epsilon},\bar{\epsilon}'$ as in Sec. IV. For each configuration of those parameters, we let ψ vary between 30° and 150°, and we notice the lower and higher limit found for the error, i.e., for

$$\Delta=(\tilde{\sigma}_{\text{approx}}-\tilde{\sigma}_{\text{exact}})/\tilde{\sigma}_{\text{exact}}\,.$$

Again, we consider two different approximations: approximation I where we simply neglect all terms other than the first one in formula (2.12), and approximation II where, in addition, we let ϵ and ϵ' go to zero in the remaining term. Actually, it must be noticed that approximation I cannot be used for analyzing an experiment, because the integration over azimuthal angles of the electrons destroys the exact kinematic correlations occurring in the Lorentz transformation between the laboratory frame and the $\gamma\gamma$ c.m. frame. Therefore approximation I is not very useful by itself. Nevertheless, it may be interesting to compare both approximations with each other, in order to realize to what extent the errors involved in the Williams-Weizsäcker approximation are due, respectively, to the neglect of longitudinal terms

TABLE II. Range of $\Delta = (\bar{\sigma}_{\text{approx}} - \bar{\sigma}_{\text{exact}})/\bar{\sigma}_{\text{exact}}$ in the process $ee \to ee\mu^+\mu^-$, according to approximation I or II, for various kinematic configurations with fixed values of $x, x'; \bar{\epsilon}, \bar{\epsilon}'$ (or θ, θ'), letting ψ go through a wide variety of values ranging between 30° and 150°. Beam energy $E_0 = 15$ GeV.

		$\bar{\epsilon}=\frac{1}{100}$	$\bar{\epsilon}=\frac{1}{30}$	$\bar{\epsilon}=\frac{1}{10}$	$\bar{\epsilon}=\frac{1}{3}$
			$x=0.1,\ x'=0.1$		
$\bar{\epsilon}'=\frac{1}{100}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-7\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-4\%<\Delta<+1\%$	$-33\%<\Delta<+5\%$
$\bar{\epsilon}'=\frac{1}{30}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-7\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<0$	$-3\%<\Delta<+1\%$	$-33\%<\Delta<+4\%$
$\bar{\epsilon}'=\frac{1}{10}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-2\%<\Delta<0$	$-7\%<\Delta<0$
	Approx II	$-4\%<\Delta<+1\%$	$-3\%<\Delta<+1\%$	$-3\%<\Delta<0$	$-33\%<\Delta<+2\%$
$\bar{\epsilon}'=\frac{1}{3}$	Approx I	$-7\%<\Delta<0$	$-7\%<\Delta<0$	$-7\%<\Delta<0$	$-7\%<\Delta<+1\%$
	Approx II	$-33\%<\Delta<+5\%$	$-33\%<\Delta<+4\%$	$-33\%<\Delta<+2\%$	$-26\%<\Delta<-6\%$
			$x=0.1,\ x'=0.4$		
$\bar{\epsilon}'=\frac{1}{100}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-7\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-4\%<\Delta<+7\%$
$\bar{\epsilon}'=\frac{1}{30}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-7\%<\Delta<0$
	Approx II	$-2\%<\Delta<+1\%$	$-2\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-4\%<\Delta<+6\%$
$\bar{\epsilon}'=\frac{1}{10}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-7\%<\Delta<+2\%$
	Approx II	$-12\%<\Delta<+1\%$	$-12\%<\Delta<+1\%$	$-12\%<\Delta<0$	$-5\%<\Delta<+3\%$
$\bar{\epsilon}'=\frac{1}{3}$	Approx I	$-6\%<\Delta<+1\%$	$-6\%<\Delta<+1\%$	$-6\%<\Delta<+1\%$	$-23\%<\Delta<+1\%$
	Approx II	$-54\%<\Delta<+1\%$	$-54\%<\Delta<+1\%$	$-51\%<\Delta<+1\%$	$-28\%<\Delta<-3\%$
			$x=0.1,\ x'=0.7$		
$\bar{\epsilon}'=\frac{1}{100}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-5\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$0<\Delta<+7\%$
$\bar{\epsilon}'=\frac{1}{30}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-5\%<\Delta<+1\%$
	Approx II	$-3\%<\Delta<+1\%$	$-3\%<\Delta<+1\%$	$-2\%<\Delta<+1\%$	$0<\Delta<+6\%$
$\bar{\epsilon}'=\frac{1}{10}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-5\%<\Delta<+3\%$
	Approx II	$-21\%<\Delta<+1\%$	$-21\%<\Delta<+1\%$	$-20\%<\Delta<+1\%$	$-12\%<\Delta<+4\%$
$\bar{\epsilon}'=\frac{1}{3}$	Approx I	$-8\%<\Delta<+1\%$	$-8\%<\Delta<+1\%$	$-8\%<\Delta<+2\%$	$-18\%<\Delta<+6\%$
	Approx II	$-49\%<\Delta<+45\%$	$-48\%<\Delta<+46\%$	$-47\%<\Delta<+47\%$	$-32\%<\Delta<+66\%$
			$x=0.4,\ x'=0.4$		
$\bar{\epsilon}'=\frac{1}{100}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-6\%<\Delta<+1\%$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-2\%<\Delta<+2\%$	$-19\%<\Delta<+7\%$
$\bar{\epsilon}'=\frac{1}{30}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-6\%<\Delta<+1\%$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-2\%<\Delta<+1\%$	$-19\%<\Delta<+7\%$
$\bar{\epsilon}'=\frac{1}{10}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-2\%<\Delta<0$	$-7\%<\Delta<+1\%$
	Approx II	$-2\%<\Delta<+2\%$	$-2\%<\Delta<+1\%$	$-2\%<\Delta<+1\%$	$-18\%<\Delta<+6\%$
$\bar{\epsilon}'=\frac{1}{3}$	Approx I	$-6\%<\Delta<+1\%$	$-6\%<\Delta<+1\%$	$-7\%<\Delta<+1\%$	$-11\%<\Delta<0$
	Approx II	$-19\%<\Delta<+7\%$	$-19\%<\Delta<+7\%$	$-18\%<\Delta<+6\%$	$-11\%<\Delta<+1\%$
			$x=0.4,\ x'=0.7$		
$\bar{\epsilon}'=\frac{1}{100}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<0$	$-5\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+2\%$	$-1\%<\Delta<+8\%$
$\bar{\epsilon}'=\frac{1}{30}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-5\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+8\%$
$\bar{\epsilon}'=\frac{1}{10}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-5\%<\Delta<0$
	Approx II	$-4\%<\Delta<+2\%$	$-4\%<\Delta<+2\%$	$-3\%<\Delta<+2\%$	$0<\Delta<+6\%$
$\bar{\epsilon}'=\frac{1}{3}$	Approx I	$-7\%<\Delta<+1\%$	$-7\%<\Delta<+1\%$	$-7\%<\Delta<+1\%$	$-9\%<\Delta<+5\%$
	Approx II	$-35\%<\Delta<+7\%$	$-35\%<\Delta<+7\%$	$-33\%<\Delta<+7\%$	$-18\%<\Delta<+8\%$
			$x=0.7,\ x'=0.7$		
$\bar{\epsilon}'=\frac{1}{100}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-4\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+2\%$	$-3\%<\Delta<+10\%$
$\bar{\epsilon}'=\frac{1}{30}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-4\%<\Delta<0$
	Approx II	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+2\%$	$-3\%<\Delta<+10\%$
$\bar{\epsilon}'=\frac{1}{10}$	Approx I	$-1\%<\Delta<+1\%$	$-1\%<\Delta<+1\%$	$-1\%<\Delta<0$	$-5\%<\Delta<0$
	Approx II	$-1\%<\Delta<+2\%$	$-1\%<\Delta<+2\%$	$0<\Delta<+2\%$	$-2\%<\Delta<+10\%$
$\bar{\epsilon}'=\frac{1}{3}$	Approx I	$-4\%<\Delta<0$	$-4\%<\Delta<0$	$-5\%<\Delta<0$	$-8\%<\Delta<0$
	Approx II	$-3\%<\Delta<+10\%$	$-3\%<\Delta<+10\%$	$-2\%<\Delta<+10\%$	$+8\%<\Delta<+13\%$

and transverse-polarization terms, or to the additional simplifications made inside the remaining term.

It is easily seen from our analytic formulas that now all terms neglected are of second order in $\bar{\epsilon}, \bar{\epsilon}'$, so that we may expect considerable improvements with respect to Table I. On the other hand, it may be anticipated here again that approximation II will in general be worse than I, because the additional simplifications contained in II imply, in some kinematic situations, that one neglects terms of the order of $\bar{\epsilon}^2$, or $\bar{\epsilon}'^2$, times some large factor.

The results shown in Table II suggest the following comments:

(a) There is indeed a striking improvement in both approximations with respect to Table I.

(b) Approximation I is excellent everywhere. Approximation II is generally somewhat worse, and becomes much worse—here again—in cases where $x'>x$ and $\bar{\epsilon}'$ is large. Again, this is due to relatively large errors involved in the expression of χ, which reflect themselves quite strongly in particular when $\sin\chi$ is small.

(c) Sticking to the more useful approximation II, we conclude that, if we wish to keep the errors within a factor of 2, we may now go up to $\bar{\epsilon}_{max}=\bar{\epsilon}'_{max}\simeq\frac{1}{3}$; for instance, when $x=x'=0.1$, that means: $\theta_{max}=\theta'_{max}\simeq 2°$.

VI. CONCLUSION

We hope that, in this work, we have somewhat clarified the understanding of the nature and mechanism of the equivalent-photon or Williams-Weizsäcker approximation. It is a powerful but delicate tool, and must be handled with great caution. Any misuse might lead to considerable errors of prediction or analysis.

It appears that, from the point of view of the quality of the approximation, there is a decisive advantage in favor of electron-tagging systems at 0°, i.e., at angles less than a few milliradians. Tagging devices at larger angles will probably not allow one to perform a detailed analysis (of angular distributions, in particular) in the extremely simple way provided by the Williams-Weizsäcker approximation.

Certainly, the limits of error will be reduced (through cancellation between positive and negative values of Δ) by integrating over the full range of the central detector. How far they will be reduced depends on the specific acceptance and efficiency parameters in a given experiment.

It is clear that our study is by no means exhaustive. We here considered only the case of double-tagging experiments. As far as single-tagging or no-tagging measurements may be contemplated (in spite of background problems and other difficulties), it would be interesting, as well, to check the approximation for such measurements. It might be useful, on the other hand, to examine processes other than muon pair production, with a dynamic structure as different as possible (e.g., resonance production).

As a last remark, we would like to stress the increasing importance of radiative corrections (mainly at the electron vertices) with growing machine energies; those corrections were ignored here. As far as virtual-photon and soft-photon corrections are concerned, they will only change the vertex functions (the tensor elements $L_{m\bar{m}}$ and $R_{n\bar{n}}$ in this paper). On the other hand, when the emission of hard photons must be included, some of the kinematic correlations used here will be disrupted, and a different study will be required.

To combine the equivalent approximation with radiative corrections is a difficult but urgent task for the future. A step in that direction has been done recently.[5]

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APPENDIX: DETAILS OF CALCULATION

1. Dynamics

We here use the four-dimensional helicity scheme set up by one of us (P.K.).[6] In that scheme, the full process $ee\to eeX$ (with one of the outgoing particles, X_1, especially selected) is represented by the diagram of Fig. 2 where the four "vertex planes" (L), (R), (C), (C') correspond respectively to the "elementary processes" $e_0\to e\gamma$, $e'_0\to e'\gamma'$, $\gamma\gamma'\to X$, $X\to X_1+\cdots$. The rotation angles α, α', and χ are connecting respectively, as shown in the figure, (L) with (C), (R) with (C), and (C) with (C'). One notices that α, α' are space-time rotation angles (they are the arguments of hyperbolic instead of ordinary cosines and sines), whereas χ is to be identified with the emission angle of X_1 in the $\gamma\gamma$ c.m. frame. It may also be remarked that the azimuthal angles φ and φ_1 have an extremely

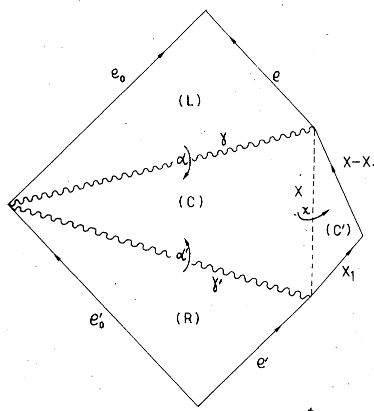

FIG. 2. Four-space representation of the process $ee \to eeX$ occurring via the $\gamma\gamma$ collision mechanism.

simple interpretation in our scheme. They are the rotation angles connecting respectively hyperplane (L, C) with hyperplane (C, R), and hyperplane (L, C) with hyperplane (C, C').

The tensor elements $L_{m\bar{m}}$ and $R_{n\bar{n}}$ needed here (see Sec. II) are easily calculated:

$$L_{++} = Q^2(\cosh^2\alpha + 1) + 4m_e^2 \sinh^2\alpha\,, \quad L_{+-} = -(Q^2 + 4m_e^2)\sinh^2\alpha\,, \tag{A1}$$

$$R_{++} = Q'^2(\cosh^2\alpha' + 1) + 4m_e^2 \sinh^2\alpha'\,, \quad R_{+-} = -(Q'^2 + 4m_e^2)\sinh^2\alpha'\,. \tag{A2}$$

The expressions (in terms of lab variables) of Q^2, Q'^2, and of the hyperbolic functions of α, α' will be given below [formulas (A7), (A8)].

Considering the case $X \equiv \mu^+\mu^-$ ($X_1 \equiv \mu^+$), we also give the expressions of the tensor elements $C_{m\bar{m},n\bar{n}}$ occurring in formula (2.12). Introducing the definitions

$$z = M_X^2 + Q^2 + Q'^2\,, \quad \Lambda = z^2 - 4Q^2Q'^2\,,$$
$$\beta = (1 - 4m_\mu^2/M_X^2)^{1/2}\,, \quad u = \tfrac{1}{4}M_X^2\beta^2\sin^2\chi\,,$$
$$w = \tfrac{1}{4}(z^2 - \beta^2\Lambda\cos^2\chi)\,,$$

we get

$$C_{++,++} + C_{++,--} = \frac{4}{w^2}\left[w(z^2 - w) - (w - 2uz)^2 - 2uz^2(Q^2 + Q'^2) + (2w - z^2)Q^2Q'^2\right],$$

$$C_{++,+-} = -\frac{4}{w^2}uz[2(w - uz) - Q^2z]\,,$$

$$C_{+-,++} = -\frac{4}{w^2}uz[2(w - uz) - Q'^2z]\,, \tag{A3}$$

$$C_{+-,+-} = -\frac{8}{w^2}[(w - uz)^2 - wQ^2Q'^2]\,,$$

$$C_{+-,-+} = -\frac{8}{w^2}u^2z^2\,.$$

2. Kinematics

(a) Expression of the dynamic variables used as functions of the lab variables

The laboratory variables being defined as in Sec. IV, one gets

$$Q^2 = Q_0^2 + 4E_0E\sin^2\theta/2\,, \quad Q_0^2 = m_e^2\frac{(E_0 - E)^2}{E_0E}\,, \tag{A4}$$

$$Q'^2 = Q_0'^2 + 4E_0E'\sin^2\theta'/2\,, \quad Q_0'^2 = m_e^2\frac{(E_0 - E')^2}{E_0E'}\,, \tag{A5}$$

$$M_X^2 = 4(E_0 - E)(E_0 - E') - 2EE'[1 - \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)]\,, \tag{A6}$$

$$\sinh^2\alpha = \frac{4(Q^2 - Q_0^2)}{Q^2 + 4m_e^2}\,\frac{4E_0(E_0 - E')[4E_0(E_0 - E') - M_X^2 - Q^2 + Q'^2] - M_X^2Q'^2}{\Lambda}\,, \quad \cosh^2\alpha = \sinh^2\alpha + 1\,, \tag{A7}$$

$$\sinh^2\alpha' = \frac{4(Q'^2 - Q_0'^2)}{Q'^2 + 4m_e^2}\,\frac{4E_0(E_0 - E)[4E_0(E_0 - E) - M_X^2 - Q'^2 + Q^2] - M_X^2Q^2}{\Lambda}\,, \quad \cosh^2\alpha' = \sinh^2\alpha' + 1\,, \tag{A8}$$

$$\sin\varphi = \frac{16E_0^2EE'\sin\theta\sin\theta'\sin(\phi' - \phi)}{\Lambda^{1/2}(Q^2 + 4m_e^2)^{1/2}(Q'^2 + 4m_e^2)^{1/2}\sinh\alpha\sinh\alpha'}\,,$$

$$\cos\varphi = \left[\frac{z\cosh\alpha\cosh\alpha'}{QQ'} - \frac{8E_0(E + E') + M_X^2 - Q^2 - Q'^2}{(Q^2 + 4m_e^2)^{1/2}(Q'^2 + 4m_e^2)^{1/2}}\right]\Big/ 2\sinh\alpha\sinh\alpha'\,. \tag{A9}$$

Finally, we must give the expressions, in laboratory variables, of the cos and sin functions of χ and φ_1. They will involve the parameter E_1 (the energy of μ^+ in the laboratory frame) which is not an independent variable. From energy-momentum conservation, one gets

$$E_1 = \frac{E_XM_X^2 \pm \tilde{p}_X[M_X^4 - 4m_\mu^2(E_X^2 - \tilde{p}_X^2)]^{1/2}}{2(E_X^2 - \tilde{p}_X^2)} \tag{A10}$$

using the definitions

$$E_X = 2E_0 - E - E',$$
$$\tilde{p}_X = -E[\cos\theta\cos\psi + \sin\theta\sin\psi\cos(\phi_1 - \phi)] + E'[\cos\theta'\cos\psi - \sin\theta\sin\psi\cos(\phi' - \phi_1)]. \quad \text{(A11)}$$

When $M_X^4 > 4m_\mu^2 E_X^2$, only solution + is acceptable for E_1; when $M_X^4 < 4m_\mu^2 E_X^2$ and $\tilde{p}_X > 0$, both solutions $\pm$ must be taken into account. [We notice that there is no solution in the case $M_X^2 < 4m_\mu^2 E_X^2, \tilde{p}_X < 0$, and also, obviously, in the case $M_X^4 < 4m_\mu^2(E_X^2 - \tilde{p}_X^2)$.]

Defining also $\beta_1 = (1 - m_\mu^2/E_1^2)^{1/2}$, one gets

$$\cos\chi = \frac{M_X^2 - Q^2 + Q'^2 - 4E_1\{E_0(1 - \beta_1\cos\psi) - E[1 - \beta_1\cos\theta\cos\psi - \beta_1\sin\theta\sin\psi\cos(\phi_1 - \phi)]\}}{\beta\Lambda^{1/2}},$$
$$\sin\varphi_1 = \frac{4E_0EE_1X}{\Lambda^{1/2}(Q^2 + 4m_e^2)^{1/2}u^{1/2}\sinh\alpha}, \quad \text{(A12)}$$

with

$$X = -(1 - \cos\theta)\beta_1\sin\psi E'\sin\theta'\sin(\phi' - \phi_1) + \sin\theta(1 - \beta_1\cos\psi)E'\sin\theta'\sin(\phi' - \phi)$$
$$+ \sin\theta\,\beta_1\sin\psi[2E_0 - E'(1 + \cos\theta')]\sin(\phi_1 - \phi),$$

$$\cos\varphi_1 = \frac{Y}{2\Lambda^{1/2}(Q^2 + 4m_e^2)^{1/2}\sinh\alpha}, \quad \text{(A13)}$$

with

$$Y = 2\Lambda^{1/2}E\{2E_0 - E'[1 + \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\phi' - \phi)] - 2E_1[1 - \beta_1\cos\theta\cos\psi - \beta_1\sin\theta\sin\psi\cos(\phi_1 - \phi)]\}$$
$$- \beta\cos\chi[4E_0(E_0 - E')(M_X^2 - Q^2 + Q'^2) - M_X^2 z].$$

(b) Phase space

The Lorentz-invariant phase space for the four outgoing particles is transformed (taking account of energy-momentum conservation) into

$$dP_{\mathrm{LI}} = K\,dE\,d\Omega\,dE'\,d\Omega'\,d\Omega_1, \quad \text{(A14)}$$

using

$$K = \frac{EE'}{16(2\pi)^8}\,\frac{\beta_1^2 E_1}{2\beta_1 E_0 - bE - b'E'}, \quad \text{(A15)}$$

where we define

$$b = \beta_1 - \cos\theta\cos\psi - \sin\theta\sin\psi\cos(\phi_1 - \phi),$$
$$b' = \beta_1 + \cos\theta'\cos\psi - \sin\theta'\sin\psi\cos(\phi' - \phi_1). \quad \text{(A16)}$$

It is to be noticed that, in our kinematic formulas, as well as in the dynamic expressions (A1)–(A3), trivial simplifications occur when $4m_e^2 \ll Q^2, Q'^2$ on the one hand, and $4m_\mu^2 \ll M_X^2$ on the other hand.

3. Simplifications for $\epsilon, \epsilon' \to 0$

Using in particular the simplified definitions

$$z = \Lambda^{1/2} = M_X^2, \quad w = \tfrac{1}{4}M_X^4(1 - \beta^2\cos^2\chi),$$

formula (A13) now becomes

$$C_{++,++} + C_{++,--} = \frac{8[1 + 2\beta^2\sin^2\chi - \beta^4(1 + \sin^4\chi)]}{(1 - \beta^2\cos^2\chi)^2},$$
$$C_{++,+-} = C_{+-,++} = -\frac{8\beta^2(1 - \beta^2)\sin^2\chi}{(1 - \beta^2\cos^2\chi)^2}, \quad \text{(A17)}$$

$$C_{+-,+-} = -\frac{8(1-\beta^2)^2}{(1-\beta^2\cos^2\chi)^2}\,,$$

$$C_{+-,-+} = -\frac{8\beta^4\sin^4\chi}{(1-\beta^2\cos^2\chi)^2}\,.$$

Formula (A6) simply becomes

$$M_X{}^2 = 4(E_0 - E)(E_0 - E')\,. \tag{A18}$$

Formulas (A7), (A8) are now transformed into

$$\sinh^2\alpha = \frac{4(Q^2 - Q_0{}^2)E_0E}{(Q^2 + 4m_e{}^2)(E_0 - E)^2}\,, \quad \sinh^2\alpha' = \frac{4(Q'^2 - Q_0'^2)E_0E'}{(Q'^2 + 4m_e{}^2)(E_0 - E')^2}\,. \tag{A19}$$

As for formula (A9), it is simply replaced by

$$\varphi = \phi' - \phi\,. \tag{A20}$$

Instead of (A11), we now have

$$E_X = 2E_0 - E - E'\,, \quad \tilde{p}_X = (E' - E)\cos\psi\,. \tag{A21}$$

Formula (A12) is replaced by

$$\beta\cos\chi = 1 - \frac{E_1}{E_0 - E'}(1 - \beta_1\cos\psi)\,, \tag{A22}$$

or by the equivalent and slightly simpler relation

$$\beta\sin\chi = \frac{2E_1}{M_X}\beta_1\sin\psi\,. \tag{A23}$$

Instead of (A13), one simply gets

$$\varphi_1 = \phi_1 - \phi\,. \tag{A24}$$

Finally, (A16) is replaced by

$$b = \beta_1 - \cos\theta\,, \quad b' = \beta_1 + \cos\theta'\,. \tag{A25}$$

To conclude, it should be remarked that, here again, considerable additional simplifications occur when $4m_e{}^2 \ll Q^2, Q'^2$, $4m_\mu{}^2 \ll M_X{}^2$. Assuming both these conditions realized, one gets the transverse-photon approximation in the form

$$\begin{aligned}\bar{\sigma} &= \frac{e^8}{32E_0{}^2Q^4Q'^4} K[2L_{++}(C_{++,++} + C_{++,--})R_{++} + 2L_{+-}C_{+-,-+}R_{+-}\cos 2(2\phi_1 - \phi - \phi')] \\ &= \frac{\alpha^4}{\pi^4}\frac{1}{E_0{}^2[2E_0 - E(1-\cos\psi) + E'(1+\cos\psi)]^2\theta^2\theta'^2xx'} \\ &\quad\times\left[\left(1 - x + \frac{x^2}{2}\right)\left(1 - x' + \frac{x'^2}{2}\right)\frac{2-\sin^2\chi}{\sin^2\chi} - (1-x)(1-x')\cos 2(2\phi_1 - \phi - \phi')\right]\,,\end{aligned} \tag{A26}$$

with

$$\sin^2\chi = \frac{4(E_0 - E)(E_0 - E')}{[2E_0 - E(1-\cos\psi) - E'(1+\cos\psi)]^2}\sin^2\psi\,. \tag{A27}$$

In formula (A26), one clearly observes the "extinction effect" mentioned in Sec. IV; it occurs at small values of x, x', when $\chi \simeq 90°$ and $\cos 2(2\phi_1 - \phi - \phi') \simeq 1$.

The Williams-Weizsäcker formula is then obtained in the form

$$\tilde{\sigma} = \frac{4\alpha^4}{\pi^2}\frac{(1 - x + x^2/2)(1 - x' + x'^2/2)}{E_0{}^2[2E_0 - E(1-\cos\psi) - E'(1+\cos\psi)]^2\theta^2\theta'^2xx'}\,\frac{2-\sin^2\chi}{\sin^2\chi}\,. \tag{A28}$$

It can be checked that one gets this formula as well by setting

$$E_0{}^2\theta\theta'\tilde{\sigma} = \frac{d\sigma}{dx\,dx'\,d\theta\,d\theta'\,d\Omega_1} = N(x,\theta)N'(x',\theta')\frac{d\sigma_{\gamma\gamma}}{d\omega_1}\frac{d\omega_1}{d\Omega_1} \tag{A29}$$

(where ω_1 is the solid angle of μ^+ in the $\gamma\gamma$ c.m. frame), using the standard expressions of the Williams-Weizsäcker spectra

$$N(x,\theta)=\frac{2\alpha}{\pi}\,\frac{1-x+x^2/2}{\theta x}\,,\quad N'(x',\theta')=\frac{2\alpha}{\pi}\,\frac{1-x'+x'^2/2}{\theta' x'}\,, \tag{A30}$$

and in addition

$$\frac{d\sigma_{\gamma\gamma}}{d\omega_1}=\frac{\alpha^2}{M_X{}^2}\,\frac{2-\sin^2\chi}{\sin^2\chi}\,, \tag{A31}$$

$$\frac{d\omega_1}{d\Omega_1}=\frac{M_X{}^2}{[2E_0-E(1-\cos\psi)-E'(1+\cos\psi)]^2}\,. \tag{A32}$$

As a last remark, we notice that when χ goes to zero $\sin^2\chi$ must obviously be replaced by $\sin^2\chi+O(m_\mu{}^2)$ in the denominator of formula (A26) or (A28).

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