Instantaneous Coulomb interaction in quantum chromodynamics

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The instantaneous Coulomb interaction is studied in the SU(2) Yang-Mills theory. The Coulomb Green's function and quark-antiquark static potential are evaluated in a background gauge field that is spherically symmetric and of short range. Expansion of the Coulomb Green's function in vector spherical harmonic functions reduces the problem to a radial problem. The radial problem can be solved if the background field is taken to be sharply cut off at a range ρ. Translation invariance is restored by averaging the potential over superpositions of the background field in the dilute-gas approximation. The resulting potential $V(R)$ is asymptotically proportional to R^{-1}. The strength of the potential is examined as a function of the magnitude of the background field. It is argued that instantons produce vacuum fluctuations similar to the background field that is used here.

I. INTRODUCTION

A quantity of theoretical interest in quantum chromodynamics (QCD) is the quark-antiquark static potential $V(R)$, which is the potential energy of an infinitely heavy quark-antiquark pair in a color-singlet combination separated by a distance R. It is expected that $V(R)$ diverges as $R\to\infty$ and that this implies quark confinement in QCD. The potential $V(R)$ has been calculated in the first few orders of perturbation theory,[1] and some nonperturbative contributions to $V(R)$ due to large gauge fields have also been considered.[2-5]

The purpose of this paper is to examine a contribution to the $q\bar{q}$ potential $V(R)$, namely that due to the instantaneous Coulomb interaction of the quarks, described below. This contribution was also considered in Refs. 3-5. Here this interaction and its contribution to $V(R)$ will be analyzed in detail for a class of gauge fields that are spherically symmetric and of short range. The gauge fields that will be considered are similar to the vacuum fluctuations produced by instanton effects. The analysis will be restricted to the pure gauge theory with gauge group SU(2).

In the Coulomb gauge, the Hamiltonian for an SU(2) gauge theory with a pair of static $I=\frac{1}{2}$ quark color charges in a color-singlet $I=0$ combination is[6]

$$H=\frac{1}{2}\int d^3x\,[(E^i_{aT})^2+(B^i_a)^2]+\frac{1}{2}\int d^3x(E^i_{aL})^2\,, \quad (1.1)$$

where

$$B^i_a=\frac{1}{2}\epsilon_{ijk}(\partial^j A^k_a-\partial^k A^j_a+g\,\epsilon_{abc}A^j_bA^k_c)\,. \quad (1.2)$$

Here A^i_a is the gauge field, which is transverse by the Coulomb gauge condition, $\partial_i A^i_a=0$; E^i_{aT} is the transverse part of the color-electric field, and is the canonical momentum of A^i_a with canonical commutation relation

$$[E^i_{aT}(\vec{x}),A^j_b(\vec{y})]=-i\delta_{ab}\,\delta^{ij}_T(\vec{x}-\vec{y})\,, \quad (1.3)$$

where $\delta^{ij}_T(\vec{x})$ is the transverse δ function. The longitudinal part of the color-electric field E^i_{aL} is constrained by Gauss's law, and can be defined by

$$E^i_{aL}(\vec{x})=\partial_i\int d^3x'\,G_{ab}(\vec{x},\vec{x}')\rho_b(\vec{x}')\,, \quad (1.4)$$

where $G_{ab}(\vec{x},\vec{x}')$ is the Coulomb Green's function[6] and $\rho_a(\vec{x})$ is the color charge density. The Coulomb Green's function G_{ab} is a functional of A^i_a defined by

$$-D_{abi}(\vec{x})\partial_iG_{bc}(\vec{x},\vec{x}')=\delta_{ac}\,\delta^3(\vec{x}-\vec{x}')\,, \quad (1.5)$$

where

$$D_{abi}(\vec{x})=\delta_{ab}\partial_i-g\epsilon_{abc}A^i_c(\vec{x})\,. \quad (1.6)$$

The color charge density ρ_a includes contributions from both the static $q\bar{q}$ pair and the gluon fields

$$\rho_a(\vec{x})=g[\tfrac{1}{2}\sigma_a\delta^3(\vec{x}-\vec{r})+\tfrac{1}{2}\sigma'_a\delta^3(\vec{x}-\vec{r}')] +g\epsilon_{abc}A^i_b(\vec{x})E^i_{cT}(\vec{x})\,, \quad (1.7)$$

where $\vec{r}$ and $\vec{r}'$ are the positions of the quark and antiquark, and $\frac{1}{2}\sigma_a$ and $\frac{1}{2}\sigma'_a$ are the $I=\frac{1}{2}$ generators of SU(2) in their color spaces. These definitions ensure that Gauss's law holds:

$$D_{abi}E^i_b(\vec{x})=g[\tfrac{1}{2}\sigma_a\delta^3(\vec{x}-\vec{r})+\tfrac{1}{2}\sigma'_a\delta^3(\vec{x}-\vec{r}')]\,. \quad (1.8)$$

The term in H due to the field E^i_{aL} will be referred to as the instantaneous Coulomb interaction.

The $q\bar{q}$ static potential is the energy of the lowest-energy state of H. In order to restrict attention to the contribution to the $q\bar{q}$ potential due to the instantaneous Coulomb interaction of the $q\bar{q}$ pair, two simplifying assumptions will be made. First, the charge density due to gluons in Eq. (1.7) will be dropped. Second, the lowest-energy state of the theory with a static $q\bar{q}$ pair will be replaced simply by the vacuum state $|\Omega\rangle$ of the pure gauge theory, and the $q\bar{q}$ energy identified with $\langle\Omega|H|\Omega\rangle$. These two approximations ignore the effects of the gluon self-couplings, which change the charge density and the lowest-energy state in the presence of external charges. In an Abelian theory, in which the gauge fields have no self-couplings, these assumptions would be true. These assumptions are similar to semiclassical approximations in that the lowest-order influence of the quantum fluctuations of the gauge fields is included by the A_a^i dependence of G_{ab}, but effects due to changes of the fluctuations arising from the presence of the external charges are ignored.

With these assumptions, the potential energy $H_{q\bar{q}}$ of the $q\bar{q}$ pair becomes simply

$$H_{q\bar{q}}=\tfrac{1}{2}g^2\int d^3x\langle\Omega|[\partial_i G_{ab}(\vec{x},\vec{r})\tfrac{1}{2}\sigma_b + \partial_i G_{ab}(\vec{x},\vec{r}\,')\tfrac{1}{2}\sigma_b']^2|\Omega\rangle, \quad (1.9)$$

where vacuum-energy terms have been discarded. Since the $q\bar{q}$ pair is taken to be in an $I=0$ combination, the products of the SU(2) generators $\frac{1}{2}\sigma_a$ and $\frac{1}{2}\sigma_a'$ are

$$\begin{aligned}&\tfrac{1}{4}\sigma_a\sigma_b=\tfrac{1}{4}\sigma_a'\sigma_b'=\tfrac{1}{4}\delta_{ab},\\ &\tfrac{1}{4}\sigma_a\sigma_b'=-\tfrac{1}{4}\delta_{ab}.\end{aligned} \quad (1.10)$$

Thus $H_{q\bar{q}}$ reduces to

$$H_{q\bar{q}}=\tfrac{1}{2}g^2\int d^3x\langle\Omega|\tfrac{1}{4}[\partial_i G_{ab}(\vec{x},\vec{r})]^2+\tfrac{1}{4}[\partial_i G_{ab}(\vec{x},\vec{r}\,')]^2 -\tfrac{1}{2}\partial_i G_{ab}(\vec{x},\vec{r})\partial_i G_{ab}(\vec{x},\vec{r}\,')|\Omega\rangle. \quad (1.11)$$

The first two terms in Eq. (1.11) are quark self-energies, and they may be dropped. The $q\bar{q}$ static potential $V(\vec{r}-\vec{r}\,')$ is thus

$$V(\vec{r}-\vec{r}\,')=\langle\Omega|u(\vec{r},\vec{r}\,')|\Omega\rangle, \quad (1.12)$$

where

$$u(\vec{r},\vec{r}\,')=-\tfrac{1}{4}g^2\int d^3x\,\partial_i G_{ab}(\vec{x},\vec{r})\partial_i G_{ab}(\vec{x},\vec{r}\,'). \quad (1.13)$$

The potential $u(\vec{r},\vec{r}\,')$, which is a functional of the background gauge field A_a^i that defines G_{ab} by Eq. (1.5), will be called the instantaneous Coulomb potential of the quarks. The background field A_a^i may be thought of as a vacuum fluctuation of the gauge-field operator. The $q\bar{q}$ potential $V(\vec{r}-\vec{r}\,')$, which is independent of A_a^i, is the expectation value of $u(\vec{r},\vec{r}\,')$ in the vacuum.

In order to clarify the nature of the approximations leading to Eq. (1.12) for $V(\vec{r}-\vec{r}\,')$, and to compare with the temporal-gauge formulation of quantum chromodynamics (QCD),[5] this formula is rederived in Appendix A in the temporal gauge.

In this paper, the Coulomb Green's function G_{ab} and the instantaneous Coulomb potential $u(\vec{r},\vec{r}\,')$ will be computed for a spherically symmetric background gauge field $A_a^i(\vec{x})$ of the form

$$A_a^i(\vec{x})=\frac{2}{g}\epsilon_{aij}\frac{x^j}{x}\frac{b(x)}{x}, \quad (1.14)$$

where $x=|\vec{x}|$ and the dimensionless function $b(x)$ will be taken to be of short range, $b(x)\to 0$ as $x\to\infty$. The field A_a^i is transverse, as it must be in the Coulomb gauge. This spherically symmetric form of A_a^i makes it possible to solve Eq. (1.5) by expanding $G_{ab}(\vec{x},\vec{x}\,')$ in vector spherical harmonic (VSH) functions.[7] This expansion separates radial and angular variables. The equation for the resulting radial Green's function can be solved explicitly if $b(x)$ is taken to be sharply cut off at an arbitrary range ρ: $b(x)=0$ if $x\geq\rho$. The Green's function G_{ab} and potential $u(\vec{r},\vec{r}\,')$ that result from the sharply cut-off field configuration should be qualitatively similar to those that would result from any short-range configuration for which $b(x)\to 0$ as $x\to\infty$.

The potential $u(\vec{r},\vec{r}\,')$ is not translationally invariant because it depends on the field configuration $A_a^i(\vec{x})$ which is centered at the origin. The $q\bar{q}$ potential $V(\vec{r}-\vec{r}\,')$ is translationally invariant because the vacuum expectation value (1.12) includes the contributions from arbitrary field configurations. In this paper, translation invariance will be restored in a heuristic way by including the effects of all configurations that are superpositions of the field $A_a^i(\vec{x})$ of Eq. (1.14), in the dilute-gas approximation.[2,8]

When translation invariance is restored in this way, the result is a $q\bar{q}$ potential $V(R)$ that is proportional to R^{-1} as $R=|\vec{r}-\vec{r}\,'|\to\infty$. Since the potential is asymptotically Coulombic, the large-distance effect of the field A_a^i is equivalent to a charge renormalization. The effective renormalized charge $g_{\rm eff}^2$ is greater than the bare charge g^2 and so the background field A_a^i increases the force between quarks with large separations. Furthermore, $g_{\rm eff}^2$ diverges for special choices of the radial function $b(x)$ that defines A_a^i, namely those for which $-D_{abi}\partial_i$ has a normalizable zero mode.[3]

The background field $A_a^i(\vec{x})$ of the form (1.14) is similar to the vacuum fluctuations that result from instanton effects[2,9] in QCD. It has been shown that the effect of instantons on the $q\bar{q}$ potential is equivalent to a charge renormalization at large distances.[2] The calculations of this paper suggest that this is so because instantons produce only short-range vacuum fluctuations. The contribution to the $q\bar{q}$ potential due to long-range fluctuations, which would be produced by merons[2] for instance, may give a confining potential.[4,5,10,11]

The outline of the paper is as follows. In Sec. II A the Coulomb Green's function G_{ab} for a field configuration of the form (1.14) will be expanded in VSH functions, and the problem of computing G_{ab} reduced to a radial problem. The instantaneous Coulomb potential $u(\vec{r},\vec{r}\,')$ will be expressed as a sum of terms due to different VSH modes. In Sec. II B it will be shown how to restore translation invariance in a heuristic way by including the effects of superpositions of configurations of the form (1.14) in the dilute-gas approximation. In Sec. III the radial problem will be solved for the case of a sharply cut-off field for which $b(x)=0$ for $x\geq\rho$. The resulting $q\bar{q}$ potential $V(R)$ will be calculated for $R\geq 2\rho$ and shown to be proportional to R^{-1} at large R. In Sec. IV it will be argued that instantons produce short-range vacuum fluctuations similar to the background field considered in Secs. II and III. In Sec. V the calculations are summarized. Finally, in Appendix A the equation (1.12) for $V(R)$ will be derived in the temporal-gauge formulation of QCD, and in Appendix B the definitions and formulas involving VSH functions will be given.

II. INSTANTANEOUS COULOMB INTERACTION

A. Separation of radial and angular variables

The purpose of this section is to reduce to a radial equation the equation (1.5) for the Coulomb Green's function G_{ab} in the spherically symmetric field configuration A_a^i of Eq. (1.14). The separation of angular and radial variables is accomplished by expanding G_{ab} in vector spherical harmonic (VSH) functions.

To see that an expansion in VSH functions is useful, consider the eigenvalue equation that corresponds to the Green's function Eq. (1.5):

$$\lambda\psi_a=-D_{abi}\partial_i\psi_b\,. \tag{2.1}$$

The index a of the eigenfunction $\psi_a(\vec{x})$ takes the values $a=1,2,3$, so $\psi_a(\vec{x})$ can be treated as a three-component vector function $\vec{\psi}(\vec{x})$. The eigenvalue equation with the configuration A_a^i of Eq. (1.14) can be written in vector form as

$$\lambda\vec{\psi}=-\nabla^2\vec{\psi}+2i\,\frac{b(x)}{x^2}\,\vec{L}\times\vec{\psi}\,, \tag{2.2}$$

where $x=|\vec{x}|$ and $\vec{L}$ is the angular momentum operator

$$\vec{L}=-i\vec{x}\times\vec{\nabla}\,. \tag{2.3}$$

Separation of variables in Eq. (2.2) is possible because the vector operators $-\nabla^2$ and $\vec{L}\times$ commute and so may be simultaneously diagonalized by writing $\vec{\psi}$ as the product of a VSH function and a radial wave function.

The VSH functions[7] will be denoted $\vec{D}^{\sigma}_{nm}(\theta\varphi)$. The indices σ,n,m take the following values:

$$\begin{aligned}&\sigma=+1,0,-1\,,\\&n=0,1,2,\ldots\text{ for }\sigma=+1\,,\\&n=1,2,3,\ldots\text{ for }\sigma=0,\ -1\,,\\&m=0,\pm1,\pm2,\ldots,\pm(n+\sigma)\,.\end{aligned} \tag{2.4}$$

The definitions of the VSH functions $\vec{D}^{\sigma}_{nm}$ are given in Appendix B along with the set of formulas involving them needed for the calculations.

If the eigenfunction $\vec{\psi}$ of Eq. (2.2) is written as

$$\vec{\psi}(\vec{x})=R_n^{\sigma}(x)\vec{D}^{\sigma}_{nm}(\theta,\varphi)\,, \tag{2.5}$$

where (x,θ,φ) are the spherical coordinates of $\vec{x}$, then by Eqs. (B2) and (B3) of Appendix B, the radial wave function $R_n^{\sigma}(x)$ obeys

$$\lambda R_n^{\sigma}(x)=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{n(n+1)}{x^2}-\frac{2b(x)}{x^2}f_n^{\sigma}\right]R_n^{\sigma}(x)\,, \tag{2.6}$$

where

$$f_n^{\sigma}=\begin{cases}-n & \text{if }\sigma=1\\ 1 & \text{if }\sigma=0\\ n+1 & \text{if }\sigma=-1\,.\end{cases} \tag{2.7}$$

Thus R_n^{σ} obeys the radial Schrödinger equation for a particle with angular momentum n in the potential $-2b(x)f_n^{\sigma}/x^2$.

It should be noted that the eigenfunctions with $n=0$, which must also have $\sigma=+1$ by Eq. (2.4), are not affected by the presence of the background field $A_a^i(\vec{x})$ because $f_0^1=0$. These eigenfunctions obey the free (i.e., $A_a^i=0$) radial equation.

By the completeness of the VSH functions, Eq. (B10), the Coulomb Green's function G_{ab} can be expanded as

$$G_{ab}(\vec{x},\vec{x}\,')=\sum_{\sigma,n,m}K^{\sigma}_{nm}D^{\sigma}_{nma}(\theta,\varphi)\,g_n^{\sigma}(x,x')[D^{\sigma}_{nmb}(\theta',\varphi')]^*\,, \tag{2.8}$$

where the normalization factor K^{σ}_{nm} is defined in Eq. (B5). The radial Green's function $g_n^{\sigma}(x,x')$ is defined by the equation

$$\frac{1}{x^2}\,\delta(x-x')=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{n(n+1)}{x^2}-\frac{2b(x)}{x^2}\,f_n^{\sigma}\right]\times g_n^{\sigma}(x,x')\,, \tag{2.9}$$

in accordance with Eq. (2.6). In Sec. III this radial equation will be solved for the special case in which the potential term $-2b(x)f_n^{\sigma}/x^2$ is taken to be a square-well potential.

The instantaneous Coulomb potential $u(\vec{r},\vec{r}\,')$ in the background field A_a^i is defined in Eq. (1.12). If the expansion (2.8) of G_{ab} is substituted into that equation, then the orthogonality relations (B7) of the VSH functions lead to an expansion of $u(\vec{r},\vec{r}\,')$ as a sum of contributions from modes with each angular dependence

$$u(\vec{r},\vec{r}\,')=-\tfrac{1}{4}g^2\sum_{\sigma,n,m}K^{\sigma}_{nm}\vec{D}^{\sigma}_{nm}(\Omega)\cdot[\vec{D}^{\sigma}_{nm}(\Omega')]^*X^{\sigma}_n(r,r')\,, \tag{2.10}$$

where Ω and Ω' stand for the angular variables of $\vec{r}$ and $\vec{r}\,'$, and the radial function X_n^σ is

$$X^{\sigma}_n(r,r')=\int_0^\infty x^2dx\,g^{\sigma}_n(x,r)\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{n(n+1)}{x^2}\right]\times g^{\sigma}_n(x,r')\,. \tag{2.11}$$

The sum over the quantum number m in Eq. (2.10) is given by the addition theorem (B8), so the formula for $u(\vec{r},\vec{r}\,')$ simplifies to

$$u(\vec{r},\vec{r}\,')=-\tfrac{1}{4}g^2\sum_{\sigma,n}\frac{1}{4\pi}[2(n+\sigma)+1]P_n(\cos\omega)X^{\sigma}_n(r,r')\,, \tag{2.12}$$

where

$$\cos\omega=\vec{r}\cdot\vec{r}\,'/rr' \tag{2.13}$$

and P_n is the Legendre polynomial of degree n.[12] Finally, the definition (2.9) of the radial Green's function g_n^σ implies that $X_n^\sigma(r,r')$ can be rewritten

$$X^{\sigma}_n(r,r')=g^{\sigma}_n(r,r')+\int_0^\infty dx\,g^{\sigma}_n(x,r)2b(x)f^{\sigma}_n g^{\sigma}_n(x,r')\,. \tag{2.14}$$

Equations (2.12)–(2.14) reduce the problem of calculating $u(\vec{r},\vec{r}\,')$ to that of calculating the radial Green's function g_n^σ.

If the background field $A_a^i(\vec{x})$ is zero, i.e., $b(x)=0$, the equations reduce to those for the ordinary Abelian Coulomb interaction. Then the radial Green's function g_n^σ is simply $g_n^{(0)}$:

$$g^{\sigma}_n(x,x')=g^{(0)}_n(x,x')\equiv\frac{1}{2n+1}\left[\theta(x-x')\frac{(x')^n}{x^{n+1}}+\theta(x'-x)\frac{x^n}{(x')^{n+1}}\right]. \tag{2.15}$$

Also X_n^σ is just equal to $g_n^{(0)}$, and thus the potential $u(\vec{r},\vec{r}\,')$ is the ordinary Coulomb potential $u^{(0)}(\vec{r}-\vec{r}\,')$:

$$u(\vec{r},\vec{r}\,')=u^{(0)}(\vec{r}-\vec{r}\,')=-\frac{3}{4}\frac{g^2}{4\pi|\vec{r}-\vec{r}\,'|}\,. \tag{2.16}$$

If the background field $A_a^i(\vec{x})$ is of short range, i.e., decreases more rapidly than $|\vec{x}|^{-1}$ as $|\vec{x}|\to\infty$, then $b(x)\to 0$ as $x\to\infty$. In that case, the potential term $-2b(x)f_n^\sigma/x^2$ in Eq. (2.9) can be neglected with respect to the centrifugal term $n(n+1)/x^2$ if x is sufficiently large. It follows that as $x\to\infty$ the x dependence of $g_n^\sigma(x,x')$ approaches that of $g_n^{(0)}(x,x')$, although the x' dependences can differ; that is, $g_n^\sigma(x,x')\sim x^{-n-1}q_n^\sigma(x')$ as $x\to\infty$ where $q_n^\sigma(x')$ is independent of x. Also, the radial function $X_n^\sigma(r,r')$ can be written

$$X^{\sigma}_n(r,r')=g^{(0)}_n(r,r')+\Delta X^{\sigma}_n(r,r')\,, \tag{2.17}$$

where ΔX_n^σ contains the dependence on the background field, and $\Delta X_n^\sigma(r,r')$ is proportional to r^{-n-1} as $r\to\infty$ with r' fixed. Therefore the instantaneous Coulomb potential $u(\vec{r},\vec{r}\,')$ for a short-range background field $A_a^i(\vec{x})$ can be written

$$u(\vec{r},\vec{r}\,')=u^{(0)}(\vec{r}-\vec{r}\,')+\Delta u(\vec{r},\vec{r}\,')\,, \tag{2.18}$$

where

$$\Delta u(\vec{r},\vec{r}\,')=-\tfrac{1}{4}g^2\sum_{\sigma}\sum_{n\geq 1}\frac{1}{4\pi}[2(n+\sigma)+1]\times P_n(\cos\omega)\Delta X^{\sigma}_n(r,r')\,; \tag{2.19}$$

and the sum over n does not include $n=0$ because the $n=0$, $\sigma=1$ mode is not affected by the presence of the background field, as explained earlier.

The contribution of the mode with $n=0$, $\sigma=1$ to the Green's function G_{ab} is the contribution which decreases most slowly at large distance. The mode with $n=0$, $\sigma=1$ produces the asymptotic behavior $u^{(0)}(\vec{r}-\vec{r}\,')\sim|\vec{r}|^{-1}$ as $|\vec{r}|\to\infty$ of the ordinary Coulomb potential. Since the sum in Eq. (2.19) for $\Delta u(\vec{r},\vec{r}\,')$ does not have a contribution with $n=0$, $\Delta u(\vec{r},\vec{r}\,')$ decreases more rapidly than $u^{(0)}(\vec{r}-\vec{r}\,')$ at large distances; specifically, $\Delta u(\vec{r},\vec{r}\,')\sim|\vec{r}|^{-2}$ as $|\vec{r}|\to\infty$. This property of $\Delta u(\vec{r},\vec{r}\,')$, which will be verified explicitly in the calculation of Sec. III in which the field $A_a^i(\vec{x})$ is sharply cut off at a range ρ, is necessary in order to restore translation invariance in the heuristic way that will be described in the next part of this section.

B. Restoration of translation invariance

The instantaneous Coulomb potential $u(\vec{r},\vec{r}\,')$ is a functional of the background field $A_a^i(\vec{x})$ and is thus not translationally invariant. In particular, the configuration of Eq. (1.14) is centered at the origin so $u(\vec{r},\vec{r}\,')$ depends explicitly on both of the distances r and r' from the origin. Translation invariance of the $q\bar{q}$ potential $V(\vec{r}-\vec{r}\,')$ results from the inclusion of the effects of arbitrary superpositions of the configuration (1.14) centered at an arbitrary point $\vec{c}$, $A_a^i(\vec{x}-\vec{c})$.

In the approximation (1.12) for $V(\vec{r}-\vec{r}\,')$, one may restore translation invariance in a simple, heuristic way by calculating the contributions of arbitrary superpositions of configurations of the form (1.14) in the dilute-gas approximation[2,8] and averaging $u(\vec{r},\vec{r}')$ over all such configurations. The question of the validity of this approximation will not be addressed in this paper. The $q\bar{q}$ potential V that is calculated in this way might be thought of as a model of the contribution to the exact potential due to short-range fields.

It is convenient to insert a complete set of eigenstates $|A_a^i\rangle$ of the gauge-field operator into the formula (1.12) for $V(\vec{r}-\vec{r}')$, and write

$$V(\vec{r}-\vec{r}')=\frac{1}{N}\int dA_a^i\,|\langle A_a^i|\Omega\rangle|^2 u(\vec{r},\vec{r}')\,, \qquad (2.20)$$

where $\langle A_a^i|\Omega\rangle$ is the vacuum functional, i.e., the probability amplitude for finding the gauge field in the configuration A_a^i, and $\int dA_a^i$ indicates a functional integral over all configurations. The normalization factor N is

$$N=\int dA_a^i|\langle A_a^i|\Omega\rangle|^2\,. \qquad (2.21)$$

This expression for $V(\vec{r}-\vec{r}')$ shows that the background field configuration A_a^i that defines $u(\vec{r},\vec{r}')$ can be thought of as a vacuum fluctuation of the gauge field, and that $V(\vec{r}-\vec{r}')$ is the average of $u(\vec{r},\vec{r}')$ over all configurations weighted by the probability $|\langle A_a^i|\Omega\rangle|^2$.

The vacuum functional $\langle A_a^i(\vec{x}-\vec{c})|\Omega\rangle$ for a field of the form (1.14) centered at $\vec{c}$ must be independent of $\vec{c}$ by translation invariance of the vacuum state $|\Omega\rangle$. Let the probability $|\langle A_a^i|\Omega\rangle|^2$ for such a configuration be denoted by ξ. In the dilute-gas approximation, the probability for a superposition of n such configurations $\sum_{i=1}^n A_a^i(\vec{x}-\vec{c}_i)$ is just ξ^n because the centers $\vec{c}_i$ are assumed to be sufficiently far apart that the elements of the superposition are independent. This approximation depends on the fact that $A_a^i(\vec{x})$ is a short-range field.

If the functional integral (2.20) is approximated by including only field configurations $A_a^i(\vec{x})$ that are dilute superpositions of configurations of the form (1.14), then the potential $V(\vec{r}-\vec{r}')$ becomes

$$V(\vec{r}-\vec{r}')=\frac{1}{N}\sum_{n=0}\frac{1}{n!}\,\xi^n\int d^3c_1\cdots d^3c_n \times u(\vec{r},\vec{r}';\vec{c}_1\cdots\vec{c}_n)\,, \qquad (2.22)$$

where n is the number of elements in the superposition, $u(\vec{r},\vec{r}';\vec{c}_1\cdots\vec{c}_n)$ is the instantaneous Coulomb potential (1.13) in the background field $\sum_{i=1}^n A_a^i(\vec{x}-\vec{c}_i)$, and the centers $\vec{c}_i$ are treated as collective coordinates.[13,14] The factor $1/n!$ is the Boltzmann counting factor, which is required since configurations that differ only by a permutation of the $\vec{c}_i$'s are identical. In the same approximation the normalization factor N is

$$N=\sum_{n=0}^{\infty}\frac{1}{n!}\,\xi^n\int d^3c_1\cdots d^3c_n\,. \qquad (2.23)$$

For a dilute superposition of n configurations of the form (1.14), the potential $u(\vec{r},\vec{r}';\vec{c}_1\cdots\vec{c}_n)$ can be approximated by computing the influence of each configuration separately. Since the field (1.14) is of short range, if either $\vec{r}$ or $\vec{r}'$ is far from all of the centers $\vec{c}_i$, $u(\vec{r},\vec{r}')$ is approximately $u^{(0)}(\vec{r}-\vec{r}')$ [see Eq. (2.18)]; and when $\vec{r}$ and $\vec{r}'$ are both near $\vec{c}_i$, the other elements of the superposition can be neglected. Thus, in the dilute-gas approximation

$$u(\vec{r},\vec{r}';\vec{c}_1\cdots\vec{c}_n)=u^{(0)}(\vec{r}-\vec{r}')+\sum_{i=1}^{n}\Delta u(\vec{r}-\vec{c}_i,\vec{r}'-\vec{c}_i)\,, \qquad (2.24)$$

where $\Delta u(\vec{r}-\vec{c},\vec{r}'-\vec{c})$, which was defined in Eq. (2.18), is the change in the potential $u(\vec{r},\vec{r}')$ due to the presence of the background field $A_a^i(\vec{x}-\vec{c})$ centered at $\vec{c}$. The validity of Eq. (2.24) in the dilute-gas approximation depends on the fact that $\Delta u(\vec{r},\vec{r}')$ decreases to zero more rapidly than $u^{(0)}(\vec{r}-\vec{r}')$ as either $r\to\infty$ or $r'\to\infty$, which holds provided $A_a^i(\vec{x})$ is of short range.

With these approximations, the normalization factor N is

$$N=e^{\xi V}\,, \qquad (2.25)$$

where $V=\int d^3c$ is the volume of space, kept finite for the moment. The potential $V(\vec{r}-\vec{r}')$ is

$$V(\vec{r}-\vec{r}')=\frac{1}{N}\left[e^{\xi V}u^{(0)}(\vec{r}-\vec{r}')+e^{\xi V}\xi\int d^3c\,\Delta u(\vec{r}-\vec{c},\vec{r}'-\vec{c})\right]. \qquad (2.26)$$

Thus

$$V(\vec{r}-\vec{r}')=u^{(0)}(\vec{r}-\vec{r}')+\mathcal{V}_c(\vec{r}-\vec{r}')\,, \qquad (2.27)$$

where the correction $\mathcal{V}_c(\vec{r}-\vec{r}')$ to the ordinary Coulomb potential is

$$\mathcal{V}_c(\vec{r}-\vec{r}')=\xi\int d^3c\,\Delta u(\vec{r}-\vec{c},\vec{r}'-\vec{c})\,. \qquad (2.28)$$

Equation (2.27) provides a heuristic way to restore the translation invariance of the $q\bar{q}$ potential.

The formula (2.28) for the correction term $\mathcal{V}_c(\vec{r}-\vec{r}')$ of the potential, i.e., the term that depends on the field $A_a^i(\vec{x})$, is incomplete. First, the parameter ξ, which can be interpreted as the number density of configurations of the form (1.14) in the vacuum,[15] was not determined. Second, in the evaluation of the functional integral (2.20), field configurations not of the form of a superposition of the basic configuration (1.14) were ignored. Therefore $\mathcal{V}_c(\vec{r}-\vec{r}\,')$ must be regarded as the contribution to the $q\bar{q}$ potential due to the special class of configurations of the form (1.14). The complete potential is an average over general forms, which would include for example scale transformations or nonspherical deformations of (1.14). The potential $V(\vec{r}-\vec{r}\,')$ of Eq. (2.27) would be a qualitative model for the complete $q\bar{q}$ potential if either the configuration (1.14) is the dominant form of vacuum fluctuation or the contribution from other forms is qualitatively similar to that of the form (1.14).

In Sec. IV it will be shown that instanton effects lead to vacuum fluctuations that are qualitatively similar to the configuration (1.14). The considerations of that section will suggest how to define the density parameter ξ and to extend the treatment of the functional integral (2.20) to more general configurations. But the main point of this paper is a calculation of $\mathcal{V}_c(\vec{r}-\vec{r}\,')$ for the special configuration of the form (1.14).

Finally, the formula (2.28) for $\mathcal{V}_c(\vec{r}-\vec{r}\,')$ can be simplified. The short-range part $\Delta u(\vec{r},\vec{r}\,')$ of the instantaneous Coulomb potential is given in Eq. (2.19) as a sum over contributions from modes with each angular dependence. If this form of Δu is used in Eq. (2.28), then $\mathcal{V}_c(\vec{r}-\vec{r}\,')$ becomes

$$\mathcal{V}_c(\vec{r}-\vec{r}\,')=-\tfrac{1}{4}g^2\sum_{\sigma}\sum_{n\geq 1}\frac{1}{4\pi}[2(n+\sigma)+1]\mathcal{V}_n^\sigma(\vec{r}-\vec{r}\,'),\tag{2.29}$$

where

$$\mathcal{V}_n^\sigma(\vec{r}-\vec{r}\,')=\xi\int d^3c\,P_n(\cos\omega)\Delta X_n^\sigma(|\vec{r}-\vec{c}|,|\vec{r}\,'-\vec{c}|)\tag{2.30}$$

and now

$$\cos\omega=(\vec{r}-\vec{c})\cdot(\vec{r}\,'-\vec{c})/|\vec{r}-\vec{c}||\vec{r}\,'-\vec{c}|.\tag{2.31}$$

The formula can be simplified by changing variables of integration in Eq. (2.30) from $\vec{c}$ to $|\vec{r}-\vec{c}|$, $|\vec{r}\,'-\vec{c}|$, and the azimuthal angle relative to the axis $\vec{r}-\vec{r}\,'$. The integrand is independent of the angle, so the integral over that variable is 2π. Let $r_1=|\vec{r}-\vec{c}|$ and $r_2=|\vec{r}\,'-\vec{c}|$. In terms of r_1 and r_2,

$$\cos\omega=\frac{r_1^2+r_2^2-R^2}{2r_1r_2},\tag{2.32}$$

where

$$R=|\vec{r}-\vec{r}\,'|.\tag{2.33}$$

The potentials $\mathcal{V}_c$ and $\mathcal{V}_n^\sigma$ are functions of the magnitude R only, and the contribution $\mathcal{V}_n^\sigma(R)$ to $\mathcal{V}_c(R)$ can be written

$$\mathcal{V}_n^\sigma(R)=\xi\frac{2\pi}{R}\int_D dr_1dr_2r_1r_2P_n\left(\frac{r_1^2+r_2^2-R^2}{2r_1r_2}\right)\times\Delta X_n^\sigma(r_1,r_2),\tag{2.34}$$

where the domain of integration D is defined by

$$\left.\begin{aligned}&r_1,r_2\geq 0,\\&r_1+r_2\geq R,\\&r_1+R\geq r_2,\\&r_2+R\geq r_1.\end{aligned}\right\}\quad\text{for }(r_1,r_2)\in D.\tag{2.35}$$

Or, since $\Delta X_n^\sigma(r_1,r_2)$ is symmetric under the interchange of r_1 and r_2,

$$\mathcal{V}_n^\sigma(R)=\xi\frac{4\pi}{R}\int_{D/2}dr_1dr_2P_n\left(\frac{r_1^2+r_2^2-R^2}{2r_1r_2}\right)\times r_1r_2\Delta X_n^\sigma(r_1,r_2),\tag{2.36}$$

where $D/2$ is that half of the domain D in which $r_2\geq r_1$.

Equations (2.27), (2.29), and (2.36) are the final results for the translationally invariant contribution to the $q\bar{q}$ static potential $V(R)$ due to the spherically symmetric field A_a^i of Eq. (1.14). In Sec. III the radial Green's function $g_n^\sigma(x,x')$ and the potential $\mathcal{V}_c(R)$ will be evaluated for a configuration A_a^i that is sharply cut off at an arbitrary radius ρ.

III. SOLUTION FOR A SHARPLY CUT-OFF CONFIGURATION

In this section the radial problem posed in Sec. II will be solved for a configuration A_a^i of the form (1.14) that is sharply cut off at an arbitrary range ρ, i.e., $A_a^i(\vec{x})=0$ if $|\vec{x}|\geq\rho$. Specifically, the function $b(x)$ that defines $A_a^i(\vec{x})$ will simply be taken to be

$$b(x)=b_0\frac{x^2}{\rho^2}\theta(\rho-x),\tag{3.1}$$

where b_0 is a dimensionless parameter that sets the magnitude of A_a^i. Since $b(x)$ is proportional to x^2, $A_a^i(\vec{x})$ is well behaved at $\vec{x}=0$.

The discontinuity of $A_a^i(\vec{x})$ at $|\vec{x}|=\rho$ is unphysical, but does not produce any divergences in the quantities that will be calculated. In fact, the qualitative features of the effects of A_a^i would be the same for any other short-range field, even one that decreased as slowly as $|\vec{x}|^{-2}$ as $|\vec{x}|\to\infty$. A sharply cut-off field configuration is used to illustrate effects that arise from the short-range nature of the configuration and are independent of its precise asymptotic behavior.

With $b(x)$ given by Eq. (3.1) the equation (2.9) that defines the radial Green's function $g_n^\sigma(x,x')$ is

$$\frac{1}{x^2}\delta(x-x')=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{n(n+1)}{x^2}-\frac{1}{\rho^2}B_n^\sigma\theta(\rho-x)\right]\times g_n^\sigma(x,x')\,, \tag{3.2}$$

where

$$B_n^\sigma=2b_0f_n^\sigma\,. \tag{3.3}$$

Thus g_n^σ is the radial Green's function for the Schrödinger equation of a particle with angular momentum n in a square-well potential. If $B_n^\sigma>0$ the square well is attractive, if $B_n^\sigma<0$ it is repulsive.

Equation (3.2) is solved by introducing two linearly independent zero modes $\varphi_\pm(x)$ that obey

$$0=\left[-\frac{d^2}{dx^2}-\frac{2}{x}\frac{d}{dx}+\frac{n(n+1)}{x^2}-\frac{B}{\rho^2}\theta(\rho-x)\right]\varphi_\pm(x)\,. \tag{3.4}$$

The dependence of B and $\varphi_\pm(x)$ on n and σ will not be explicitly indicated. $\varphi_+(x)$ is defined as the zero mode that is regular at $x=\infty$, $\varphi_-(x)$ that regular at $x=0$. For special choices of B, a *normalizable* zero mode $\varphi(x)$ exists; then $\varphi_+=\varphi_-=\varphi$, the differential operator that defines g_n^σ is singular, and g_n^σ is not defined. The existence of normalizable zero modes of the eigenvalue equation (2.1) was pointed out by Gribov.[3]

The Green's function $g_n(x,x')$ of the differential operator in Eq. (3.4) is written in terms of the zero modes $\varphi_\pm$ as

$$g_n(x,x')=\kappa\,[\theta(x-x')\varphi_+(x)\varphi_-(x')+\theta(x'-x)\varphi_-(x)\varphi_+(x')]\,, \tag{3.5}$$

where κ is a constant given by

$$\frac{1}{\kappa}=x^2W(\varphi_+,\varphi_-) \tag{3.6}$$

and $W(\varphi_+,\varphi_-)$ is the Wronskian

$$W(\varphi_+,\varphi_-)=\varphi_+(x)\varphi_-'(x)-\varphi_+'(x)\varphi_-(x)\,. \tag{3.7}$$

The square-well problem (3.4) for the zero modes $\varphi_\pm(x)$ can be solved explicitly. The exterior solutions $(x\geqslant\rho)$ are

$$\begin{aligned}\varphi_+(x)&=\left(\frac{\rho}{x}\right)^{n+1}\\ \varphi_-(x)&=\alpha\left(\frac{x}{\rho}\right)^n+(1-\alpha)\left(\frac{\rho}{x}\right)^{n+1}.\end{aligned}\qquad(x\geqslant\rho) \tag{3.8}$$

The interior solutions $(x\leqslant\rho)$ are

$$\begin{aligned}\varphi_+(x)&=c_1j_n(qx)+c_2y_n(qx)\\ \varphi_-(x)&=cj_n(qx)\,,\end{aligned}\qquad(x\leqslant\rho) \tag{3.9}$$

where $q^2=B/\rho^2$ and j_n and y_n denote spherical Bessel functions.[12] If B is negative, the interior solutions can be obtained from Eq. (3.9) by analytic continuation in the argument qx. The solutions are normalized according to $\varphi_\pm(\rho)=1$ for convenience. The constants c_1, c_2, and α, c are determined by the requirement that $\varphi_\pm(x)$ and $\varphi_\pm'(x)$ be continuous at $x=\rho$; this leads to

$$c=\frac{1}{j_n(z)}\,,\quad\alpha=\frac{zj_{n-1}(z)}{(2n+1)j_n(z)}\,, \tag{3.10}$$

where $z=q\rho=B^{1/2}$. The constants c_1,c_2 will not be needed. As required by their definitions, $\varphi_+(x)$ is regular at $x=\infty$, $\varphi_-(x)$ at $x=0$. Finally, the constant κ defined in Eq. (3.6) is

$$\kappa=\frac{j_n(z)}{\rho zj_{n-1}(z)}\,. \tag{3.11}$$

The radial function $X_n^\sigma(r,r')$ needed to calculate $\mathcal{V}_c(R)$ is given in terms of $g_n^\sigma(r,r')$ in Eq. (2.14). For the square well,

$$X_n(r,r')=g_n(r,r')+\frac{B}{\rho^2}\int_0^\rho x^2dx\,g_n(x,r)g_n(x,r')\,, \tag{3.12}$$

where again the dependence of X, g, and B on σ has been suppressed. In the domain $r,r'>\rho$, the function $X_n(r,r')$ takes a particularly simple form. To be definite, let $r>r'>\rho$. Then the formula (3.5) for $g_n(x,x')$ implies

$$X_n(r,r')=\kappa\varphi_+(r)\varphi_-(r')+\varphi_+(r)\varphi_+(r')\frac{B\kappa^2}{\rho^2}\int_0^\rho dx\,\varphi_-^2(x)\,. \tag{3.13}$$

Substitution of the explicit form (3.8) for the exterior solutions $\varphi_\pm$ gives

$$X_n(r,r')=\frac{1}{2n+1}\frac{(r')^n}{r^{n+1}}+\Delta X_n(r,r')\,, \tag{3.14}$$

where

$$\Delta X_n(r,r')=\frac{1}{\rho}M_n(B)\left(\frac{\rho^2}{rr'}\right)^{n+1} \tag{3.15}$$

and

$$M_n(B)=\kappa\rho(1-\alpha)+\frac{B\kappa^2}{\rho}\int_0^\rho dx\,x^2\varphi_-^2(x)\,. \tag{3.16}$$

The first term on the right-hand side of Eq. (3.14) is the free Green's function $g_n^{(0)}(r,r')$. Thus $\Delta X_n(r,r')$ contains the effects of the background field A_a^i and is the proper function to use in Eqs. (2.29) and (2.30) for $\mathcal{V}_c(R)$. To be precise, $M_n=0$ if $B=0$.

The formula for the constant M_n, which is dimensionless and depends only on B, can be simplified if the explicit forms for α, κ, and $\varphi_-(x)$ are substituted into Eq. (3.16). It can be shown that

$$M_n(B)=\frac{1}{2}\frac{j_n^{\,2}(z)}{j_{n-1}^{\,2}(z)}-\frac{(2n-1)j_{n+1}(z)}{2(2n+1)j_{n-1}(z)}, \qquad (3.17)$$

where again the argument z is $z=B^{1/2}$. If B is negative, then analytic continuation leads to

$$M_n(B)=-\frac{1}{2}\frac{i_n^{\,2}(\zeta)}{i_{n-1}^{\,2}(\zeta)}+\frac{(2n-1)i_{n+1}(\zeta)}{2(2n+1)i_{n-1}(\zeta)}, \qquad (3.18)$$

where $\zeta=(-B)^{1/2}$ and i_n denotes the modified spherical Bessel function.[12]

Equation (3.15) for $\Delta X_n(r,r')$ holds if $n\geq 1$ and $r,r'>\rho$. The equation shows that $\Delta X_n(r,r')$ is proportional to r^{-n-1} as $r\to\infty$ with r' fixed. For $n=0$, the parameter B is $B=2b_0f_0^1=0$ so $\Delta X_0=0$. Thus $\Delta u(\vec{r},\vec{r}\,')$, which is given in terms of $\Delta X_n^\sigma(r,r')$ in Eq. (2.19), decreases as $|\vec{r}|^{-2}$ as $|\vec{r}|\to\infty$, and is of shorter range than the ordinary Coulomb potential $u^{(0)}(\vec{r}-\vec{r}\,')$. These results are in accordance with the arguments of Sec. II A following Eq. (2.17), and would hold for any short-range field, not just a sharply cut-off field.

Equation (3.15) gives $\Delta X_n(r,r')$ for $r,r'\geq\rho$. For r or $r'<\rho$, $\Delta X_n(r,r')$ depends on the interior solutions of $\varphi_\pm(x)$ and is of a more complicated form. For subsequent calculations it suffices to point out that if $r'\geq\rho$ then for any r, $\Delta X_n(r,r')$ is of the form

$$\Delta X_n(r,r')=(r')^{-n-1}\chi_n(r), \qquad (3.19)$$

where $\chi_n(r)$ is independent of r'. This result is easily derived from the equations for ΔX_n and g_n. It shows that for the square well, $\Delta X_n(r,r')$ is proportional to $(r')^{-n-1}$ for $r'\geq\rho$, not just for $r'\to\infty$ as would be true for any short-range potential.

Equations (2.18), (2.19), and (3.15) determine the instantaneous Coulomb potential $u(\vec{r},\vec{r}\,')$ for $r,r'\geq\rho$ in the background field $A_a^i(\vec{x})$. The evaluation of $u(\vec{r},\vec{r}\,')$ was possible because of the spherical symmetry of $A_a^i(\vec{x})$, and because the function $b(x)$ was given the sharply cut-off form (3.1).

Now translation invariance will be restored in the manner described in Sec. II B, by computing the potential $\mathcal{V}_c(R)$ defined in Eqs. (2.27) and (2.28). The contribution $\mathcal{V}_n^\sigma(R)$ to $\mathcal{V}_c(R)$ due to modes with angular quantum numbers (n,σ) is given in Eq. (2.36). The large-distance behavior, that is for $R\geq 2\rho$, is particularly simple.

For $R\geq 2\rho$, the domain of integration $D/2$ of Eq. (2.36) can be divided into two parts: region I where $r_1\leq\frac{1}{2}R$ and region II where $r_1\geq\frac{1}{2}R$. The integral over region I is equal to zero. Region I of $D/2$ is defined by the inequalities $R-r_1\leq r_2\leq R+r_1$ and $0\leq r_1\leq\frac{1}{2}R$. Thus $r_2\geq\frac{1}{2}R\geq\rho$ so $\Delta X_n(r_1,r_2)$ is of the form (3.19). Thus the integral that occurs in Eq. (2.36) is

$$\begin{aligned}I&=\int_I dr_1dr_2P_n\left(\frac{r_1^{\,2}+r_2^{\,2}-R^2}{2r_1r_2}\right)r_1r_2\chi_n(r_1)r_2^{-n-1}\\&=\int_0^{R/2}dr_1r_1\chi_n(r_1)F_n(r_1),\end{aligned} \qquad (3.20)$$

where

$$F_n(r_1)=\int_{R-r_1}^{R+r_1}dr_2P_n\left(\frac{r_1^{\,2}+r_2^{\,2}-R^2}{2r_1r_2}\right)r_2^{-n}. \qquad (3.21)$$

But $F_n(r_1)$ is equal to zero as can be shown as follows: The integral defining $F_n(r_1)$ can be rewritten by a change of variable of integration as

$$\begin{aligned}F_n(r_1)&=r_1\int_{-1}^{1}du\,P_n(u)\{ur_1+[R^2-r_1^{\,2}(1-u^2)]^{1/2}\}^{-n}\left\{1+\frac{ur_1}{[R^2-r_1^{\,2}(1-u^2)]^{1/2}}\right\}\\&=\frac{(-1)^nr_1}{(R^2-r_1^{\,2})^n}\int_{-1}^{1}du\,P_n(u)\{ur_1-[R^2-r_1^{\,2}(1-u^2)]^{1/2}\}^n\left\{1+\frac{ur_1}{[R^2-r_1^{\,2}(1-u^2)]^{1/2}}\right\}.\end{aligned} \qquad (3.22)$$

If n is even (odd) then $P_n(u)$ is an even (odd) function of u so only the even (odd) part of the product of the curly brackets in Eq. (3.22) need be retained. But it can be seen that this quantity is a polynomial in u of degree $n-2$. Therefore $F_n(r_1)$ vanishes because $P_n(u)$ is orthogonal to any polynomial of degree less than n when integrated over the domain $-1\leq u\leq 1$.

Thus for $R\geq 2\rho$ only region II where $r_1\geq R/2$ contributes to the integral in Eq. (2.36) for $\mathcal{V}_n^\sigma(R)$. In region II both $r_1\geq\rho$ and $r_2\geq\rho$ so $\Delta X_n(r_1,r_2)$ has the simple form (3.15). Therefore $\mathcal{V}_n^\sigma(R)$ can be written

$$\mathcal{V}_n^\sigma(R)=\xi\frac{4\pi}{R}\int_{R/2}^{\infty}dr_1\int_{r_1}^{r_1+R}dr_2P_n\left(\frac{r_1^{\,2}+r_2^{\,2}-R^2}{2r_1r_2}\right)\times r_1r_2\frac{M_n^\sigma}{\rho}\left(\frac{\rho^2}{r_1r_2}\right)^{n+1}, \qquad (3.23)$$

where $M_n^\sigma=M_n(B_n^\sigma)$; $M_n(B)$ is defined in Eqs. (3.17), (3.18) and B_n^σ in Eq. (3.3). Thus

$$\mathcal{V}_n^\sigma(R)=\xi\rho^3\frac{4\pi}{R}C_nM_n^\sigma\left(\frac{2\rho}{R}\right)^{2n-2}, \qquad (3.24)$$

where C_n is the numerical constant

$$C_n = \int_1^\infty \frac{dx_1}{x_1^n} \int_{x_1}^{x_1+2} \frac{dx_2}{x_2^n} P_n\left(\frac{x_1^2 + x_2^2 - 4}{2x_1x_2}\right). \quad (3.25)$$

The complete correction $\mathcal{V}_c(R)$ to the ordinary Coulomb potential is given in Eq. (2.29) as a sum over contributions from modes with angular quantum numbers (n,σ). For $R \geq 2\rho$, $\mathcal{V}_c(R)$ is thus

$$\mathcal{V}_c(R) = -\tfrac{1}{4} g^2 \xi \rho^3 \sum_{n\geq 1} \frac{Z_n(b_0)}{R}\left(\frac{2\rho}{R}\right)^{2n-2}, \quad (3.26)$$

where the constant $Z_n(b_0)$, which is dimensionless and depends only on the parameter b_0, is

$$Z_n(b_0) = \sum_{\sigma=0,\pm 1} [2(n+\sigma)+1] C_n M_n(B_n^\sigma), \quad (3.27)$$

where $B_n^\sigma = 2b_0 f_n^\sigma$. The sum in Eq. (3.26) does not include $n=0$ because $\mathcal{V}_n^\sigma = 0$ for $n=0$.

For $R < 2\rho$, $\mathcal{V}_n^\sigma(R)$ and $\mathcal{V}_c(R)$ are relatively slowly varying functions of R. In particular, they are nonsingular in the region $R < 2\rho$, in contrast to the ordinary Coulomb potential $u^{(0)}(R)$ which diverges at $R=0$. Evaluation of $\mathcal{V}_c(R)$ for $R < 2\rho$ will not be undertaken here; the large-R behavior of the potential is of most interest.

Finally, the large-separation behavior of the correction $\mathcal{V}_c(R)$ to the Coulomb potential can be examined. It can be shown that the sum over n in Eq. (3.26) for $\mathcal{V}_c(R)$ with $R \geq 2\rho$ is convergent. For $R \gg 2\rho$, terms in the sum with $n \geq 2$ can be neglected, so asymptotically

$$\mathcal{V}_c(R) \sim -\tfrac{1}{4} g^2 \xi \rho^3 \frac{Z_1}{R}. \quad (3.28)$$

For $n=1$, C_n and $M_n(B)$ are given by

$$C_1 = 1 \quad (3.29)$$

and

$$M_1(B) = \tfrac{1}{6} - \frac{1}{2z}\cot z + \tfrac{1}{2}\cot^2 z, \quad (3.30\text{a})$$

$$M_1(B) = \tfrac{1}{6} + \frac{1}{2\zeta}\coth\zeta - \tfrac{1}{2}\coth^2\zeta, \quad (3.30\text{b})$$

where $z^2 = B$ and $\zeta^2 = -B$, and (3.30a) or (3.30b) is used if $B>0$ or $B<0$, respectively. The renormalization constant $Z_1(b_0)$ is

$$Z_1(b_0) = \sum_{\sigma=0,\pm 1} (3+2\sigma) M_1(2b_0 f_1^\sigma)$$
$$= 5M_1(-2b_0) + 3M_1(2b_0) + M_1(4b_0). \quad (3.31)$$

The $q\bar{q}$ potential $V(R)$ due to the sharply cut-off background field A_a^i is the sum of the ordinary Coulomb potential $u^{(0)}(R)$ and the correction term $\mathcal{V}_c(R)$ which contains the A_a^i dependence. Since $\mathcal{V}_c(R)$ is asymptotically proportional to R^{-1}, the effect of A_a^i on the large-R behavior of $V(R)$ is effectively a charge renormalization. That is, as $R \to \infty$

$$V(R) \sim -\tfrac{3}{4}\frac{g^2}{4\pi R}[1 + \tfrac{4}{3}\pi\rho^3\xi Z_1(b_0)] = -\tfrac{3}{4}\frac{g_{\text{eff}}^2}{4\pi R}. \quad (3.32)$$

Since the parameter ξ can be interpreted as the number density of configurations of the form (1.14) in the vacuum,[15] the quantity $\frac{4}{3}\pi\rho^3\xi$ is the fraction of space occupied by vacuum fluctuations of the form (1.14).

The renormalization constant Z_1, which depends only on the parameter b_0 that determines the magnitude of $A_a^i(\vec{x})$, is positive over most of the range of b_0, $-\infty < b_0 < \infty$. For $b_0 \ll 1$, the following expansions hold:

$$M_1(2b_0 f) \sim \tfrac{2}{45}(2b_0 f) + \tfrac{6}{945}(2b_0 f)^2, \quad (3.33)$$

$$Z_1(b_0) \sim \tfrac{32}{105} b_0^2. \quad (3.34)$$

Thus $Z_1(b_0)$ is positive near the origin. The following lower bounds can be proved by consideration of Eqs. (3.30a), (3.30b), and (3.31):

$$\begin{aligned} &Z_1(b_0) > 0, \quad \text{if } -\tfrac{1}{2}\pi^2 \leq b_0 \leq \tfrac{1}{2}\pi^2 \\ &Z_1(b_0) \geq -1, \quad \text{if } b_0 > \tfrac{1}{2}\pi^2 \\ &Z_1(b_0) \geq -\tfrac{1}{2}, \quad \text{if } b_0 < -\tfrac{1}{2}\pi^2. \end{aligned} \quad (3.35)$$

Furthermore, $Z_1(b_0) \to +\infty$ whenever b_0 approaches a value for which there is an $n=1$ normalizable zero mode of the eigenfunction equation (2.1). The quantity $M_1(B)$ diverges for $B = N^2\pi^2$ where N is a positive integer; these are the values of B for which there exists a normalizable zero mode of the radial equation (3.4) with $n=1$.[16] Thus $Z_1(b_0)$ is infinite for

$$\begin{aligned} &b_0 = \pm\tfrac{1}{2}N^2\pi^2, \\ &b_0 = +\tfrac{1}{4}N^2\pi^2. \end{aligned} \quad (3.36)$$

Since $Z_1(b_0)$ tends to $+\infty$ at these points, $Z_1(b_0)$ must be large and positive over much of the range of b_0. It should be noted that if b_0 is of order 1, then $A_a^i(x)$ is a large gauge field[9] in that A_a^i is proportional to $1/g$ by Eq. (1.14).

The fact that $Z_1(b_0)$ is positive implies that the presence of the vacuum fluctuations A_a^i produces an attractive $q\bar{q}$ force that is stronger at large separations than the ordinary Coulomb force. This increased attraction is in accordance with the expected infrared slavery of QCD. The potential $V(R)$ is still proportional to R^{-1} as $R \to \infty$, so the short-range background field A_a^i does not produce confinement. However, it is interesting that a short-range field, one that is even sharply cut off at a fixed range ρ, affects the long-distance component of $V(R)$, i.e., the component proportional to R^{-1}. Specifically, Eq. (3.26) for $\mathcal{V}_c(R)$ is exact for $R \geq 2\rho$; but for $R \geq 2\rho$, the quark and antiquark cannot both be within the same vacuum fluctuation since ρ is the radius of the fluctuations. Thus the effect on the long-distance behavior of $V(R)$ is a nonlocal one, and does not require both quarks to be in the same local fluctuation.

The conclusion of this section is that although the short-range vacuum fluctuations do not produce a confining potential $V(R)$, they do affect the large-distance behavior of $V(R)$. The influence of dilute superpositions of short-range configurations $A_a^i(\vec{x})$, even sharply cut-off ones, produces an effective charge renormalization in that $V(R)$ is proportional to R^{-1} as $R \to \infty$. The renormalized charge can be large compared to the bare charge if $A_a^i(\vec{x})$ is a large gauge field, i.e., is proportional to $1/g$.

In the next section it will be shown that large gauge-field fluctuations similar to the background fields considered in this section are produced by instanton effects in QCD.

IV. SHORT-RANGE VACUUM FLUCTUATIONS DUE TO INSTANTONS

In Secs. II and III the influence of short-range background fields on the $q\bar{q}$ potential was studied by an evaluation of the instantaneous Coulomb potential due to a particular form of the field. The field $A_a^i(\vec{x})$ was spherically symmetric and sharply cut off at radius ρ. Translation invariance was restored heuristically by averaging over superpositions of the configuration $A_a^i(\vec{x}-\vec{c})$ with arbitrary center $\vec{c}$, in the dilute-gas approximation. The effect of the presence of A_a^i was shown to be effectively a positive charge renormalization at large distances.

The purpose of this final section is to show that the background field used in Secs. II and III is similar to the vacuum fluctuations produced by the vacuum tunneling events that are described by instanton solutions in the semiclassical approximation.[2,8,9]

The starting point for this discussion will be a formal expression for the vacuum functional $|\langle A_a^i|\Omega\rangle|^2$ in terms of a Feynman path integral. Consider the Euclidean path integral

$$F(T;A_a^i)=\int_{P(T;A_a^i)}(DA'\cdots)\exp[-S_{\text{eff}}(A')]\,, \quad (4.1)$$

where $A_a^i(\vec{x})$ is a time-independent field configuration and the set of paths $P(T;A_a^i)$ is the set of paths $A_a'^i(\vec{x},\tau)$ that begin and end at $A_a'^i=0$ at $\tau=\pm T$, and pass through $A_a^i(\vec{x})$ at $\tau=0$:

$$A_a'^i(\vec{x},\tau)=\begin{cases}0 & \text{at } \tau=\pm T\,,\\ A_a^i(\vec{x}) & \text{at } \tau=0\,.\end{cases} \quad (4.2)$$

The limit $T\to\infty$ will eventually be taken. In Eq. (4.1) $S_{\text{eff}}(A')$ is the Euclidean action plus the gauge-fixing and Faddeev-Popov ghost terms appropriate to the Coulomb gauge, and the integration $(DA'\cdots)$ is over gauge fields and Faddeev-Popov ghost fields. Since any path in $P(T;A_a^i)$ can be split into a path from 0 to A_a^i followed by another from A_a^i to 0, the Feynman path-integral formula[17] implies that $F(T;A_a^i)$ is the product of two Green's functions,

$$F(T;A_a^i)=\langle A=0|e^{-HT}|A_a^i\rangle\langle A_a^i|e^{-HT}|A=0\rangle\,, \quad (4.3)$$

where $|A_a^i\rangle$ is the eigenstate of the gauge field with eigenvalue $A_a^i(\vec{x})$. The Green's function that occurs in Eq. (4.3) can be expanded in the complete set of eigenstates $|n\rangle$ of H as

$$\langle A_a^i|e^{-HT}|A=0\rangle=\sum_n e^{-E_nT}\langle A_a^i|n\rangle\langle n|A=0\rangle\,. \quad (4.4)$$

When $T\to\infty$ only the lowest-energy eigenstate in the sum over $|n\rangle$ survives, which is the vacuum state $|\Omega\rangle$ with energy $E_\Omega=0$. Thus

$$\lim_{T\to\infty}F(T;A_a^i)=|\langle A=0|\Omega\rangle|^2|\langle A_a^i|\Omega\rangle|^2\,. \quad (4.5)$$

Since the first factor on the right-hand side of Eq. (4.5) is independent of A_a^i, the vacuum functional $|\langle A_a^i|\Omega\rangle|^2$ can be written as the Euclidean path integral

$$|\langle A_a^i|\Omega\rangle|^2=N\int_{P(A_a^i)}(DA'\cdots)\exp[-S_{\text{eff}}(A')]\,, \quad (4.6)$$

where N is a normalization factor and $P(A_a^i)$ is the set of paths $A_a'^i(\vec{x},\tau)$ that begin and end at $A_a'^i=0$ at $\tau=\pm\infty$ and pass through $A_a^i(\vec{x})$ at $\tau=0$.

In principle the expression (4.6) provides a means of determining the probability that the gauge field is in the configuration $A_a^i(\vec{x})$ in the vacuum state. In the semiclassical approximation, the dominant contribution to the path integral is from paths $A_a'^i$ that are near the local minima of the action $S(A')$, i.e., the instanton solutions.[8] Then the vacuum functional $|\langle A_a^i|\Omega\rangle|^2$ is small unless an instanton solution passes near $A_a^i(\vec{x})$ at $\tau=0$.

A closed-form expression for the instanton solution in the Coulomb gauge is not known. In fact it is known that the instanton solution is discontinuous in the Coulomb gauge.[18] Thus it is useful to consider instead the vacuum functional in the temporal gauge, i.e., the gauge in which the time components A_a^0 of the gauge fields are set equal to zero. In the temporal-gauge formulation, the vacuum functional is required to be invariant under the residual gauge freedom associated with invariance under time-independent gauge transformations.[5] By arguments similar to those that led to Eq. (4.6), a path-integral formula can again be derived,

$$|\langle A_a^i|\Omega_0\rangle|^2 = N\int_{P_0(A_a^i)} DA' \exp[-S(A')] , \qquad (4.7)$$

where the subscript 0 indicates the temporal-gauge theory. Here $P_0(A_a^i)$ is the set of paths that begin and end at pure-gauge configurations at $\tau = \pm\infty$ and pass through $A_a^i(\vec{x})$ at $\tau = 0$. The end points of the paths in $P_0(A_a^i)$ are left arbitrary to ensure that $|\langle A_a^i|\Omega_0\rangle|^2$ is invariant under gauge transformations of A_a^i. The vacuum functionals of the two gauge choices are related by

$$\langle A_a^i|\Omega_0\rangle = \langle A_{aT}^i|\Omega\rangle , \qquad (4.8)$$

where A_{aT}^i is transverse and A_a^i is any time-independent gauge transformation of A_{aT}^i.

In the temporal gauge the instanton solution[19] with scale size ρ and position $(\vec{0}, \tau_0)$ can be written

$$A_a^i(\vec{x}, \tau - \tau_0) = \frac{2}{g}\left[\frac{x^a x^i}{x^2}\frac{a}{x} + \epsilon_{aij}\frac{x^j}{x}\frac{b}{x} + \left(\delta_{ai} - \frac{x^a x^i}{x^2}\right)\frac{c}{x}\right], \qquad (4.9)$$

where

$$a = \frac{\rho^2}{x^2+\rho^2}\left[\frac{(\tau-\tau_0)x}{(\tau-\tau_0)^2+x^2+\rho^2} + \alpha\right],$$

$$b = \frac{1}{2}\frac{x^2(1+\cos 2\alpha) + [(\tau-\tau_0)^2+\rho^2](1-\cos 2\alpha) + 2(\tau-\tau_0)x\sin 2\alpha}{(\tau-\tau_0)^2+x^2+\rho^2}, \qquad (4.10)$$

$$c = \frac{1}{2}\frac{[(\tau-\tau_0)^2+\rho^2-x^2]\sin 2\alpha + 2(\tau-\tau_0)x\cos 2\alpha}{(\tau-\tau_0)^2+x^2+\rho^2},$$

and

$$\alpha = \frac{x}{(x^2+\rho^2)^{1/2}}\left[\arctan\frac{\tau-\tau_0}{(x^2+\rho^2)^{1/2}} + \frac{\pi}{2}\right] \qquad (4.11)$$

and $x = |\vec{x}|$. For $\tau = 0$ the field $A_a^i(\vec{x}, -\tau_0)$ is of short range, i.e., decreases as $|\vec{x}|^{-2}$ as $|\vec{x}| \to \infty$.

The path-integral formulas (4.6) and (4.7) show that the instanton produces vacuum fluctuations near $A_a^i(\vec{x}, -\tau_0)$. The field $A_a^i(\vec{x}, -\tau_0)$, and also its Coulomb gauge version $A_{aT}^i(x, -\tau_0)$, bears some resemblance to the field $A_a^i(\vec{x})$ considered in Secs. II and III. It is proportional to $1/g$. It is spherically symmetric, although not precisely of the form (1.14) because of the terms a and c. It is of short range, although it is not sharply cut off but decreases as $|\vec{x}|^{-2}$ as $|\vec{x}| \to \infty$. In spite of the differences, the instantaneous Coulomb potential $u(\vec{r}, \vec{r}\,')$ due to $A_{aT}^i(\vec{x}, -\tau_0)$ should be qualitatively similar to that of $A_a^i(\vec{x})$. In particular it can be written as the sum of the ordinary Coulomb potential $u^{(0)}$ and a correction Δu that depends on A_{aT}^i. Furthermore, it is known that there exists a value of the time parameter τ_0 such that $-D_{abi}\partial_i$ has a zero mode,[18] and this zero mode occurs in the mode with $n=1$, $\sigma = -1$.

Contributions to the path integral (4.6) from the instanton solution with position $(\vec{c}, \tau_0)$ will produce equivalent vacuum fluctuations centered at $\vec{c}$. Contributions from solutions corresponding to a dilute gas of instantons and anti-instantons will produce superpositions of the vacuum fluctuations produced by one instanton.

A full semiclassical treatment of the path integral (4.6) would determine the density parameter ξ appropriate for the fields $A_a^i(\vec{x})$ that lie on instanton solutions.[20] The restoration of translation invariance brought about by the inclusion of multi-instanton solutions would be similar to the heuristic treatment of Sec. II B. In addition the scale parameters ρ and time parameters τ_0 of the instantons would be treated as collective coordinates so that the final result would be an average over different forms of the fluctuations, rather than just the contribution due to a specific form as in Secs. II and III.

The effect of instantons on the $q\bar{q}$ potential has been calculated in the dilute-gas approximation using the Wilson-loop formula.[2] The resulting potential $V(R)$ decreases as R^{-1} as $R \to \infty$; the effect of the instantons is equivalent to a charge renormalization at large distances. The calculations of Secs. II and III suggest that this result holds because instantons produce only short-range vacuum fluctuations.

There have been suggestions that the mechanism responsible for quark confinement is the presence in the vacuum of long-range fields $A_a^i(\vec{x})$ that decrease as $|\vec{x}|^{-1}$ as $|\vec{x}| \to \infty$.[4,5,10] In the covariant derivative $D_{abi} = \delta_{ab}\partial_i - g\epsilon_{abc}A_c^i$ for such fields, the gradient term and the field-dependent term both decrease as $|\vec{x}|^{-1}$ as $|\vec{x}| \to \infty$ so the field has a large effect on the large-distance behavior of the potential: For instance, the instantaneous Coulomb potential $u(\vec{r}, \vec{r}\,')$ no longer necessarily approaches the ordinary Coulomb potential $u^{(0)}(\vec{r} - \vec{r}\,')$ at large separations.

The path-integral formula (4.6) may rule out the existence of long-range fields A_a^i in the vacuum, because the action $S(A')$ is infinite (when written in terms of the *bare* coupling constant) for any path in $P(A_a^i)$ if A_a^i is of long range.[21] An example of such a path is the meron solution.[2] In the temporal gauge or the Coulomb gauge the meron solution with position $(\vec{0}, \tau_0)$ is

$$A_a^i(\vec{x}, \tau - \tau_0) = \frac{2}{g}\epsilon_{aij}\frac{x^j}{x^2}\left\{\frac{1}{2} + \frac{1}{2}\frac{\tau - \tau_0}{[(\tau-\tau_0)^2 + x^2]^{1/2}}\right\}. \quad (4.12)$$

This configuration is a pure gauge at $\tau = \pm\infty$, while at $\tau = 0$ it decreases as x^{-1} as $x = |\vec{x}| \to \infty$. In particular for $\tau = \tau_0 = 0$ it has the magnetic monopole form.[22] If paths such as (4.12) do contribute to the path integral (4.6) in spite of their infinite action, then long-range fluctuations would be present in the vacuum and would affect the $q\bar{q}$ potential in an important way. However, the analysis of Secs. II and III cannot simply be repeated for a long-range field because the restoration of symmetry used there relied on approximations such as Eq. (2.24) that hold only if the background field is of short range.

V. SUMMARY

The aim of this paper was to study the Coulomb Green's function G_{ab} and instantaneous Coulomb potential $u(\vec{r}, \vec{r}\,')$ in a background field A_a^i for which the equations defining these quantities could be solved. The spherically symmetric form (1.14) for A_a^i was chosen because it allows separation of angular and radial variables simply by an expansion of G_{ab} in VSH functions. The sharply cut-off field of Eq. (3.1) was chosen in order that the radial problem could be solved explicitly. The potential $u(\vec{r}, \vec{r}\,')$ resulting from these choices was shown to be the sum of the ordinary Coulomb potential $u^{(0)}(\vec{r} - \vec{r}\,')$ and a correction term $\Delta u(\vec{r}, \vec{r}\,')$ which decreases more rapidly as $r \to \infty$ or $r' \to \infty$ than $u^{(0)}(\vec{r} - \vec{r}\,')$. This would be true for any field $A_a^i(\vec{x})$ that decreases more rapidly than $|\vec{x}|^{-1}$ as $|\vec{x}| \to \infty$.

Translation invariance was restored in a heuristic way by approximating the vacuum as a dilute superposition of vacuum fluctuations of the form (1.14) centered at arbitrary points. Although this approximation is certainly incomplete, the resulting potential $V(R)$ might be viewed as a model for the complete $q\bar{q}$ potential. It was found that even though $A_a^i(\vec{x})$ is sharply cut off at range ρ, the long-range R^{-1} tail of $V(R)$ is affected by the presence of A_a^i. This effect is equivalent to a charge renormalization. The renormalization constant $Z_1(b_0)$ was shown to be large and positive over much of the range of the parameter b_0 that determines the magnitude of A_a^i; $Z_1(b_0)$ can be large if b_0 is of order 1 in which case the gauge field A_a^i is large, i.e., is proportional to $1/g$. In particular, $Z_1(b_0) \to +\infty$ whenever b_0 is such that the operator $-D_{abi}\partial_i$ has a normalizable zero model with angular quantum number $n = 1$.

Finally to motivate the attention given to short-range fields of the form (1.14) it was shown that instantons produce vacuum fluctuations of a similar form.

The potential $V(R)$ decreases as R^{-1} as $R \to \infty$ and does not confine quarks. It may be that a confining potential in QCD requires the presence of long-range vacuum fluctuations such as those produced by merons or those present in the magnetic-monopole vacuum described by Mandelstam.[10]

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APPENDIX A

The purpose of this appendix is to clarify the approximations used to derive the formula (1.12), (1.13) for $V(R)$ by rederiving this formula in the temporal-gauge formulation of QCD, the gauge in which the time components A_a^0 of the gauge fields are set equal to zero.[5]

Reference 5 concerns the calculation of the $q\bar{q}$ potential in the temporal gauge. There, an arbitrary gauge-invariant state containing an antiquark at r' and quark at r in a color-singlet combination is written

$$|q\bar{q}\rangle = q^\dagger(\vec{r})K(A)\bar{q}^\dagger(\vec{r}\,')|\Omega_0\rangle \ , \quad (A1)$$

where $q^\dagger$ and $\bar{q}^\dagger$ are nonrelativistic quark and antiquark fields and $|\Omega_0\rangle$ is the vacuum state of the temporal-gauge pure gauge theory. The operator $K(A)$ is a functional of the gauge field A_a^i and is constructed so that $q^\dagger(\vec{r})\,K(A)\,\bar{q}^\dagger(\vec{r}\,')$ is gauge invariant. For a gauge transformation $g(x)$, the gauge field $A^i(\vec{x}) \equiv -ig\frac{1}{2}\sigma_a A_a^i(\vec{x})$ transforms as

$$A'^i(\vec{x}) = g(\vec{x})[A^i(\vec{x}) + \partial^i]g^\dagger(\vec{x}) \ ; \quad (A2)$$

and the functional $K(A)$ must transform as

$$K(A') = g(\vec{r})K(A)g^\dagger(\vec{r}\,') \ . \quad (A3)$$

It is shown in Ref. 5 that the energy of the state $|q\bar{q}\rangle$ is

$$H_{q\bar{q}} = \left\langle \Omega_0 \left| \frac{1}{4}\int d^3x \,\mathrm{Tr}\,\frac{\delta K^\dagger}{\delta A_a^i(\vec{x})}\,\frac{\delta K}{\delta A_a^i(\vec{x})}\right| \Omega_0\right\rangle . \quad (A4)$$

The $q\bar{q}$ potential energy is the energy of the state with $K(A)$ chosen to minimize $H_{q\bar{q}}$.

The condition (A3) obeyed by $K(A)$ can be solved formally by a gauge-fixing procedure that takes the theory to the Coulomb gauge. Let $U(\vec{x};A)$ be the gauge transformation that takes A_a^i to the Coulomb gauge:

$$A_T^i = U^\dagger(A)[A^i + \partial^i]U(A) , \tag{A5}$$

where A_T^i is transverse, $\partial_i A_T^i = 0$. The subtle point arising from the multiplicity of choices of U related to the Gribov ambiguity[3] will be ignored here. A formal solution of the condition (A3) is

$$K(A) = U(\vec{r};A)k(A_T)U^\dagger(\vec{r}';A) , \tag{A6}$$

where $k(A_T)$ is an arbitrary functional of A_{aT}^i. The functional $K(A)$ satisfies (A3) because A_{aT}^i is the same for all gauge-equivalent fields A_a^i and because

$$U(\vec{x};A') = g(\vec{x})U(\vec{x};A) . \tag{A7}$$

If the functional $k(A_T)$ is left arbitrary and the energy $H_{q\bar{q}}$ is reexpressed in terms of A_{aT}^i and k, then $H_{q\bar{q}}$ becomes the energy of an arbitrary state containing a static $q\bar{q}$ pair in the Coulomb-gauge formulation of QCD.

The most natural approximate choice for $K(A)$ is that which results from setting $k(A_T) = 1$. In that approximation, it can be shown that

$$\frac{\delta K(A)}{i\,\delta A_a^i(\vec{x})} = V_{ab}(\vec{x})[g\partial_i G_{bc}(\vec{x},\vec{r}) - g\partial_i G_{bc}(\vec{x},\vec{r}')] \times U(\vec{r})\tfrac{1}{2}\sigma_c U^\dagger(\vec{r}') , \tag{A8}$$

where

$$V_{ab}(\vec{x}) = 2\,\mathrm{Tr}\tfrac{1}{2}\sigma_a U(\vec{x})\tfrac{1}{2}\sigma_b U^\dagger(\vec{x}) \tag{A9}$$

and G_{ab} is the Coulomb Green's function for the Coulomb-gauge field gauge equivalent to A_a^i. Therefore, the energy $H_{q\bar{q}}$ is

$$H_{q\bar{q}} = \langle \Omega_0 | \tfrac{1}{8}g^2 \int d^3x [\partial_i G_{ab}(\vec{x},\vec{r}) - \partial_i G_{ab}(\vec{x},\vec{r}')]^2 | \Omega_0 \rangle . \tag{A10}$$

Finally, when $|\Omega_0\rangle$ is replaced by the Coulomb-gauge vacuum state $|\Omega\rangle$ [see Eq. (4.8)], $H_{q\bar{q}}$ becomes precisely the energy in Eq. (1.11).

Since the approximation that leads to (A10) is to set $k(A_T) = 1$, it can be said that this approximation does not take into account the changes in the transverse (i.e., Coulomb gauge) gluon fields due to the presence of the static quarks.

APPENDIX B

The following definitions and formulas involving vector spherical harmonic (VSH) functions were used in deriving the results of this paper.

The three VSH functions $\vec{D}_{nm}^\sigma(\theta,\varphi)$, $\sigma = 1, 0, -1$, are defined by[7]

$$\begin{aligned} \vec{D}_{nm}^1 &= x\vec{\nabla}Y_{n+1}^m + (n+1)\hat{r}\,Y_{n+1}^m , \\ \vec{D}_{nm}^0 &= \vec{\nabla}\times(\vec{x}\,Y_n^m) , \\ \vec{D}_{nm}^{-1} &= x\vec{\nabla}\,Y_{n-1}^m - n\,\hat{r}\,Y_{n-1}^m , \end{aligned} \tag{B1}$$

where $x = |\vec{x}|$ and $Y_n^m(\theta,\varphi)$ are the scalar spherical harmonic functions. The indices σ, n, m take the values given in Eq. (2.4). The index n may be thought of as the orbital angular momentum quantum number, and the quantity $n+\sigma$ as the total angular momentum (orbital plus spin) of the spin-1 vector field.

The VSH functions are eigenfunctions of the vector operators $-\nabla^2$ and $\vec{L}\times$ where $\vec{L} = -i\vec{x}\times\vec{\nabla}$:

$$-\nabla^2\vec{D}_{nm}^\sigma = n(n+1)\frac{1}{x^2}\vec{D}_{nm}^\sigma \tag{B2}$$

and

$$\vec{L}\times\vec{D}_{nm}^\sigma = if_n^\sigma\,\vec{D}_{nm}^\sigma , \tag{B3}$$

where

$$f_n^1 = -n, \quad f_n^0 = 1, \quad f_n^{-1} = n+1 . \tag{B4}$$

Two VSH functions are orthogonal unless their indices are equal. A normalization factor K_{nm}^σ is defined by

$$K_{nm}^\sigma = \frac{1}{4\pi}\,\frac{(n+\sigma-m)!}{(n+\sigma+m)!}\,\frac{1}{n(n+1)}\,k_n^\sigma , \tag{B5}$$

where

$$k_n^1 = n, \quad k_n^0 = 2n+1, \quad k_n^{-1} = n+1 . \tag{B6}$$

The normalized orthogonality relation is

$$K_{nm}^\sigma \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta\, d\theta\, \vec{D}_{nm}^\sigma(\theta,\varphi)\cdot[\vec{D}_{n'm'}^{\sigma'}(\theta,\varphi)]^* = \delta_{\sigma\sigma'}\delta_{nn'}\delta_{mm'} . \tag{B.7}$$

The following addition theorems were used to derive Eq. (2.12):

$$\sum_{m=-n-\sigma}^{n+\sigma} K_{nm}^\sigma \vec{D}_{nm}^\sigma(\theta,\varphi)\cdot[\vec{D}_{nm}^\sigma(\theta',\varphi')]^* = \frac{1}{4\pi}[2(n+\sigma)+1]P_n(\cos\omega) , \tag{B.8}$$

where P_n is the Lengendre polynomial of degree n, and the angles θ, θ', and ω form a spherical triangle

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi') . \qquad (B.9)$$

The VSH functions are complete. The completeness relation is

$$\sum_{\sigma,n,m} K^{\sigma}_{nm} D^{\sigma}_{nma}(\theta,\varphi)[D^{\sigma}_{nmb}(\theta',\varphi')]^* = \delta_{ab}\delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi') . \qquad (B.10)$$

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[14]It is not necessary to introduce a collective coordinate representing the relative orientation of the configuration in the group space because V is already gauge invariant since the $q\bar{q}$ pair is in a color-singlet combination.

[15]The parameter ξ, which has units (length)$^{-3}$, is a measure of the probability of finding the field fluctuation A^i_a in the vacuum, so can be interpreted as the density of such fluctuations in the vacuum. Also, the dominant contribution to the sum over n in Eq. (2.22) is from the term with $n = \xi V$.

[16]For arbitrary n, the quantity $M_n(B)$ diverges because of the presence of a normalizable zero mode when $B = j_{n-1,N}^2$ where $j_{n-1,N}$ is the Nth zero of the Bessel function $j_{n-1}(z)$.

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