$$\tfrac{1}{2}[G((m_A-\mu)^2)+G((m_A+\mu)^2)]$$
$$=\frac{f_A}{f_\pi}+\frac{1}{2m_\rho}\frac{f_\rho}{f_\pi}\left[\frac{\beta((m_A-m_\rho)^2)}{m_A-m_\rho}-\frac{\beta((m_A+m_\rho)^2)}{m_A+m_\rho}\right]. \quad \text{(B3)}$$

The contributions are graphically depicted in Fig. 3. If the pion contribution is negelcted in the spirit of the arguments of Sec. IV, relation (4.10) is obtained.

Let us use

$$[Q_5{}^\alpha, D_i{}^\beta]=D_i{}^{5\gamma}, \quad \text{(B4)}$$

where $D_i{}^\beta$, $D_i{}^{5\gamma}$ are dipoles of vector and axial-vector currents defined as in (2.39).

Sandwiching again between an A_1 state at rest and the vacuum, one obtains

$$\frac{1}{2}\left[\frac{G((m_A-\mu)^2)}{m_A-\mu}+\frac{G((m_A+\mu)^2)}{m_A+\mu}\right]=\frac{f_A}{f_\pi m_A}$$
$$+\frac{f_\rho}{2m_\rho{}^2 f_\pi}\left[\frac{\beta((m_A-m_\rho)^2)}{m_A-m_\rho}+\frac{\beta((m_A+m_\rho)^2)}{m_A+m_\rho}\right]. \quad \text{(B5)}$$

PHYSICAL REVIEW D VOLUME 2, NUMBER 6 15 SEPTEMBER 1970

Set of Crossing-Symmetric Dynamical Equations and Infinitely Rising Regge Trajectories*

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An equation based on unitarity and analyticity is written down. It is a relativistic generalization of a Schrödinger equation with a local energy-dependent potential of the Yukawa type, in contrast to the usual Mandelstam equations, which correspond to a nonlocal energy-independent potential. A prescription is given for our potential, which makes the amplitude exactly crossing symmetric. It is argued that a first guess at the potential might simply be to take the low-energy crossed-channel resonances of the Veneziano model. The cutoff on the number of resonances is then an undetermined parameter, which, however, merely serves to fix the energy scale in the limit of a vanishingly small pion mass. A determinantal approximation to our scheme (which differs from the determinantal approximation to the usual N/D equations) is then found to lead to Regge trajectories which rise indefinitely. The $I=1$ output resonances agree approximately with the input ones, average duality is found to be satisfied, and the P-wave scattering length comes out close to the Weinberg value. These results do not change much, at least at low engeries, even if a strong dose of inelasticity is introduced.

I. INTRODUCTION

An amplitude has been proposed by Veneziano which has linearly rising Regge trajectories, crossing, and duality.[1] Unfortunately, it depends on a number of arbitrary parameters. Indeed, this number becomes infinite if satellites are admitted. Since the Veneziano model does not satisfy unitarity, however, it might be hoped that this condition could be used to determine some, if not all, of these parameters. Now, in practice it is generally difficult to do this without sacrificing crossing. But it is always possible to vary the parameters of a unitary model until at least approximate partial crossing is achieved. One can then see how closely the resulting amplitude resembles the Veneziano model.

Numerous relativistic models based on unitarity and analyticity have been proposed over the years. Most of them are based on the Mandelstam representation with a finite number of subtractions[2] and differ only in their treatment of short-range effects. The most ambitious has been the strip approximation of Chew and Frautschi,[3] which satisfies crossing exactly, at least in principle. The difficulty with practically all of these models is that they do not lead to trajectories which rise very high. It is generally hoped that the addition of higher channels might change this (although doubts have recently been thrown even on this[4]). But this is difficult to implement in practice. In some sense, it is done in the multi-Regge integral equation approach,[5] which amounts to the inclusion of an infinite number of channels. However, it has been shown[6] that Regge trajectories are bounded by an $s^{1/2}$ behavior at infinity in this model, in contrast to their linear behavior in the Veneziano model. It might therefore be desirable to look at alternative models.

Two such schemes have recently been proposed.[7] Each

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[1] G. Veneziano, Nuovo Cimento **57**, 190 (1968).
[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[3] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).
[4] P. D. B. Collins and R. C. Johnson, Phys. Rev. **182**, 1755 (1969).
[5] G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Letters **22**, 208 (1969); I. G. Halliday and L. M. Saunders, Nuovo Cimento **60A**, 115 (1969).
[6] C. I. Tan and J.-M. Wang, Phys. Rev. Letters **22**, 1152 (1969).
[7] L. A. P. Balázs, Phys. Rev. **176**, 1769 (1968).

is a simple relativistic generalization of a Schrödinger equation with an energy-dependent superposition of local Yukawa potentials.[8] By contrast, the Madelstam-Chew-Frautschi prescription corresponds to a nonlocal energy-independent potential in the nonrelativistic limit. In an S-matrix approach, there is really nothing to choose between the two, since they turn out to give the same analyticity structure. The differences arise only for more distant singularities (short-range effects) which cannot be calculated in present-day physics anyway. In the case of something like single-particle exchange, however, our potential turns out to give better convergence than the Mandelstam-Chew-Frautschi potential. One might thus invoke something like a "principle of maximal convergence" in choosing between them.

In Sec. 5 of Ref. 7, a prescription was given for the potential which would make the Bethe-Salpeter amplitude crossing symmetric. A similar thing can be done with the unitarity equation. Of course, we can always add to such a potential any other crossing-symmetric function. We shall see that such a function has to be nonsingular in the Mandelstam strip regions if unitarity is to be satisfied, however.

In the present paper, we concentrate on the unitarity equation, which was mentioned only briefly in Ref. 7. In Sec. II, it is set up by analogy with an on-shell dispersion formulation of nonrelativistic scattering with a local energy-dependent potential. It is argued that this gives an amplitude with an infinite number of subtractions in the case where the potential corresponds to the exchange of a particle with spin >0. A general expression is then given in Sec. III for a potential which will guarantee crossing symmetry.

As a first approximation to the crossing-symmetric equations, we might simply approximate the potential by a truncated crossed-channel absorptive part (a correction to this is discussed in Appendix B). In practice, this would be dominated by the lower resonances. The resulting unitarity equation then has Regge asymptotic behavior. In Sec. IV a method is given for solving this equation. In lowest order, this just leads to the determinantal approximation. Because our potential is not the same as the Mandelstam-Chew-Frautschi potential, however, this determinantal approximation differs in general from the one derived from the usual N/D equations.

The approximate version of Sec. IV gives a crossed-channel amplitude dominated by resonances at low energies and by Regge behavior at high energies. It therefore resembles the Veneziano model in at least these general features. A reasonable procedure might then be to take the input truncated absorptive part from the Veneziano model as a first guess. If we then solve the unitarity equation, we will not in general obtain the same Regge trajectory as in the Veneziano model. But if we vary the Veneziano parameters, we may hope to find a set for which this is approximately the case, at least at low energies, in the direct channel. The resulting amplitude would then be unitary and at the same time approximate the Veneziano amplitude.

The program we have just described is applied to $I=1$ $\pi\pi$ scattering in Sec. V. Experimental values are taken for the input Veneziano parameters and the unitarity equation is solved by the determinantal approximation of Sec. IV. The cutoff which truncates the absorptive part is fixed by the requirement that the output ρ resonance have the same mass as the input; since the pion mass is negligibly small in this problem, this parameter essentially serves to fix the energy scale of the problem. It is then found that the output resonances agree roughly with the input ones lying on the leading trajectory, so that we do indeed have approximate partial crossing. These results are not changed much by the introduction of a strong inelasticity factor, at least at lower energies.

The output resonances of Sec. V correspond to a rising Regge trajectory in the entire range of interest. In Appendix A it is shown that it actually rises to infinity. Another property which this trajectory possesses is that of approximate duality in the original Dolen-Horn-Schmid sense.[9] In fact, it is possible to turn the problem around and require the cutoff which truncates the absorptive part to be such that duality is satisfied exactly at each energy. This is considered in Appendix C.

II. RELATIVISTIC UNITARITY EQUATION

Let us begin by considering the nonrelativistic Schrödinger equation with an energy-dependent local potential

$$\nabla^2\psi+[q^2-W(r,s)]\psi=0, \tag{2.1}$$

where ψ is the wave function, q is the magnitude of the three-momentum, r is the radial distance, and s is the energy. We shall take the potential to be a superposition of Yukawa potentials

$$W(r,s)=\pi^{-1}\int_{t_0}^{\infty}dt'\,v(t',s)r^{-1}e^{-r\sqrt{t'}}. \tag{2.2}$$

For the time being, we also take $v(t,s)$ to be real everywhere. Suppose we now freeze s at some fixed value $s=\bar{s}$, within the potential only, in Eq. (2.1). The resulting problem is then equivalent to solving a Schrödinger equation with an energy-independent potential. Of course, the resulting amplitude is physical only when $s=\bar{s}$.

[8] Another example of such a generalization, at least in a perturbation theory context, was discussed by A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963). This equation, however, has certain spurious singularities, as was shown by R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966). This makes it unsuitable for our purposes.

[9] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters **19**, 402 (1967).

One way of solving a Schrödinger equation with an energy-independent superposition of Yukawa potentials is to use dispersion theory.[10] In its simplest form, we would just write a fixed-momentum-transfer dispersion relation for the physical amplitude A and combine this with unitarity. Thus, remembering that $\bar{s}$ is a fixed parameter, we have

$$A(s,t,\bar{s})=V(t,\bar{s})+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}A_s^{el}(s',t,\bar{s}), \quad (2.3)$$

where, from unitarity, the absorptive part $A_s^{el}=\mathrm{Im}A$ is given by

$$A_s^{el}(s,t,\bar{s})=\frac{\rho(s)}{4\pi}\int d\Omega\, A^*(s,t_1,\bar{s})A(s,t_2,\bar{s}), \quad (2.4)$$

and where $\rho(s)$ is the nonrelativistic phase-space factor which equals q; $-t$, $-t_1$, and $-t_2$ are the squares of the momentum transfers between the initial and final, the initial and intermediate, and the intermediate and final states, respectively; and V is the Born approximation, which has the form

$$V(t,\bar{s})=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}v(t',\bar{s}) \quad (2.5)$$

and is a constant for fixed t and $\bar{s}$. It thus plays the role of a subtraction constant in Eq. (2.3) and has to be added to the dispersion integral in (2.3) in order to guarantee the correct asymptotic behavior of A. Equations (2.3) and (2.4) now constitute a nonlinear integral equation which can, for example, be solved by iteration, starting with $A=V$ as a first approximation. Once we have solved the equation, we would evaluate the physical amplitude from

$$A(\bar{s},t)=A(\bar{s},t,\bar{s}), \quad (2.6)$$

since $A(s,t,\bar{s})$ is physically meaningful only when $s=\bar{s}$, as we have seen.

The above formalism can be readily generalized to the fully relativistic case. The only change is that A now becomes the invariant amplitude, s the square of the total c.m. energy, and ρ the relativistic phase-space factor. Thus, for the scattering of two spinless particles of mass μ, we have

$$\rho(s)=q/(q^2+\mu^2)^{1/2}. \quad (2.7)$$

Otherwise, Eqs. (2.3)–(2.5) remain the same, at least if we only have direct forces (it is straightforward to generalize the equations to include exchange forces). From Eq. (2.7) we see that we regain the nonrelativistic limit when $\mu\to\infty$.

At this point we should note that our equations are not in general the same as the ones written down by, say, Mandelstam[2] or Chew and Frautschi.[3] In the simple case we have been considering, these do indeed resemble Eqs. (2.3)–(2.5), but with the crucial difference that we replace the last $\bar{s}$ in Eq. (2.3) by s' and all the other $\bar{s}$ by s. We therefore no longer have a parametric dependence of the equations on the energy, so that we have an energy-independent potential in the nonrelativistic limit; it will, however, be nonlocal in general.

Both Eqs. (2.3) and (2.4) and the corresponding Mandelstam-Chew-Frautschi equations are ways of building up an amplitude from a potential, using only unitarity and analyticity. In an S-matrix theory, there is thus no way of deciding which is more correct. If $V(t,s)$ is simply calculated from the exchange of a particle of spin >0, so that it is a polynomial in s, our equations give more convergent integrals if we make an iterative expansion of Eqs. (2.3) and (2.4). Indeed, the corresponding Mandelstam-Chew-Frautschi integrals would diverge unless we introduce a cutoff. A "principle of maximal convergence" would, therefore, select our equations in this case.

When $V(t,s)$ is a polynomial in s, Eqs. (2.3) and (2.4) can be formally reduced to a fixed-t dispersion relation with an infinite number of subtractions. One merely uses the general relation

$$\int_{s_0}^{\infty}ds'\frac{s^nf(s')}{s'-s}=\sum_{i=1}^{n-1}c_is^i+\int_{s_0}^{\infty}ds'\frac{s'^nf(s')}{s'-s}. \quad (2.8)$$

In an iteration of Eqs. (2.3) and (2.4) a typical integral for $A(s,t)$ has the form of the left-hand side of Eq. (2.8). From Eq. (2.8) we see that it is an n-subtracted dispersion integral. Since n increases with the order of the iteration, we have an infinite number of subtractions if we go to all orders. It should be emphasized that this procedure is purely formal, however, since the integral on the right-hand side of Eq. (2.8) diverges in general.

We shall conclude this section by expressing the above equations in double-spectral language. We write

$$A(s,t,\bar{s})=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\frac{A_t(s,t',\bar{s})}{t'-t}. \quad (2.9)$$

From Eqs. (2.3)–(2.5), we then have

$$A_t(s,t,\bar{s})=v(t,\bar{s})+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho^{el}(s',t,\bar{s}), \quad (2.10)$$

where the double-spectral function is given by

$$\rho^{el}(s,t,\bar{s})=\frac{\rho(s)}{2\pi q^2}\int_{t_0}^{K>0}\int_{t_0}dt'dt''\times\frac{A_t^*(s,t',\bar{s})A_t(s,t'',\bar{s})}{K^{1/2}(q^2;t,t',t'')}, \quad (2.11)$$

with

$$K(q^2,t,t',t'')=t^2+t'^2+t''^2-2(tt'+tt''+t't'')-tt't''q^{-2}. \quad (2.12)$$

[10] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) 10, 62 (1960).

It is just the t discontinuity of A_s^{el}, so

$$A_s^{el}(s,t,\bar{s})=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\rho^{el}(s,t',\bar{s}). \tag{2.13}$$

The advantage of these equations is that it is possible to obtain A_t exactly at any finite value of t with only a finite number of iterations. The lowest iteration $A_t=v$ is exact for $t_0<t<4t_0$, while the nth iteration is exact for $t_0<t<(n+1)^2t_0$. If we expand these equations, we also see that Eq. (2.6) gives us an $A(s,t)$ which satisfies a Mandelstam representation.[11]

So far we have been taking $v(t,s)$ to be real and nonsingular for finite s. However, the above results are unaltered if we take a $V(t,s)$ which satisfies a Mandelstam representation. But the double-spectral function must then be zero in the elastic strip $s<s_I$, i.e., in the range of s where elastic unitarity holds. Otherwise $v(t,s)$ is complex and Eq. (2.3) does not give a unitary amplitude. For $s>s_I$ there is no reason why it should, so v would in general be complex. In this region the A_s^{el} in (2.3) and (2.4) is no longer the total absorptive part, since this also picks up a contribution from V. Similarly ρ^{el} in Eqs. (2.10) and (2.11) is only a part of the total double-spectral function.

III. CROSSING-SYMMETRIC FORM OF UNITARITY EQUATION

Up to now we have only been considering the case of scattering in a single channel. Let us now turn to a fictional situation in which we only have an s and a t channel (we will come to the more general case later). We will follow a generalization of a prescription which was given in Sec. 5 of Ref. 7. We first note that if we write equations corresponding to (2.3) and (2.6) in the t channel, we have

$$A(t,s,\bar{t})=V(s,\bar{t})+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}A_t^{el}(t',s,\bar{t}) \tag{3.1}$$

and

$$A(\bar{t},s)=A(\bar{t},s,\bar{t}), \tag{3.2}$$

with A_t^{el} calculated from unitarity as in Eq. (2.4). From crossing, $s_0=t_0=4\mu^2$.

Equations (2.3) and (3.1) suggest that a fully crossing-symmetric equation might have the form

$$A(s,t,\bar{s},\bar{t})=R(\bar{s},\bar{t})+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}A_s^{el}(s',t,\bar{s})+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{dt'}{t'-t}A_t^{el}(t',s,\bar{t}), \tag{3.3}$$

where

$$A(s,t,\bar{s},t)=A(s,t,\bar{s}), \tag{3.4}$$

$$A(s,t,s,\bar{t})=A(t,s,\bar{t}), \tag{3.5}$$

and $R(s,t)$ is a crossing-symmetric function which contains the contribution of all remaining effects. It is easy to see that Eqs. (3.3)–(3.5) lead to a crossing-symmetric amplitude $A(s,t)$. We also see that Eq. (2.3) is regained provided we take for the potential

$$V(t,s)=R(s,t)+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}A_t^{el}(t',s,t). \tag{3.6}$$

Similarly we can regain Eq. (3.1). This shows that one never actually needs the function $A(s,t,\bar{s},\bar{t})$ with both $s\neq\bar{s}$ and $t\neq\bar{t}$. One only needs either the function (3.4) or the function (3.5) in an actual calculation.

We have seen in Sec. II that $V(t,s)$ must satisfy a Mandelstam representation if we want the final amplitude to do likewise. We also saw that, if we want elastic unitarity to hold for $s<s_I$, the double-spectral function for V must be zero in that strip. Now from Eqs. (2.9) and (2.11), $\rho^{el}(s,t,\bar{s})$ is zero, not only outside the Mandelstam double-spectral regions but also for $t<4t_0$, so that $A_s^{el}(s,t,\bar{s})$ not only has only Mandelstam singularities but is also real for $t<4t_0$. By the same argument, in the t channel, $A_t^{el}(t,s,\bar{t})$ has only Mandelstam singularities and is real for $s<4s_0$, and hence for $s<s_I$, since we must have inelastic effects starting at $s=4s_0$, if not sooner. This, of course, means that the integral term in Eq. (3.6) satisfies a Mandelstam representation with a zero double-spectral function in the strip $s<s_I$. But we saw that this is also true of V. From Eq. (3.6) it must therefore be true of R. Since $R(s,t)$ is, in addition, crossing symmetric, its double-spectral function must be zero in both elastic strips $s<s_I$ and $t<t_I$.

Since Eq. (2.3) can be regained from Eqs. (3.3)–(3.5), so can the double-spectral function equations (2.9)–(2.12). A convenient expression for v can be obtained by combining Eqs. (3.6), (3.1), (3.2), and (2.5), which gives

$$v(t,s)=R_t(s,t)+[A_t(t,s)-V_t(s,t)]. \tag{3.7}$$

The t subscript means that we are to take the t discontinuity of the quantity in question. Now $V(s,t)$ is the t-channel potential, which, as we have seen, has a Mandelstam representation with a zero double-spectral function in the strip $t<t_I$. The same was true of $R(s,t)$. Thus $R_t(s,t)$ and $V_t(s,t)$ are zero in that strip and we have

$$v(t,s)=A_t(t,s), \quad t<t_I. \tag{3.8}$$

We can readily generalize the above equations to the more realistic case where we have all three double-spectral functions. To be more specific, let us consider $\pi\pi$ scattering. Then Eqs. (2.6), (2.9), (2.10), (2.12), (3.7), (3.8), and (2.5) become

$$A^I(s,t,u)=A^I(s,t,u,s), \tag{3.9}$$

$$A^I(s,t,u,\bar{s})=\frac{1}{\pi}\int_{t_0}^{\infty}dt'A_t^I(s,t',\bar{s})\left[\frac{1}{t'-t}+\frac{(-1)^I}{t'-u}\right], \tag{3.10}$$

[11] In the nonrelativistic limit, this was first shown to be true by J. M. Cornwall and M. Ruderman, Phys. Rev. 128, 1474 (1962).

$$A_t^I(s,t,\bar{s})=v^I(t,\bar{s})+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho_{el}{}^I(s',t,\bar{s}), \tag{3.11}$$

$$\rho_{el}{}^I(s,t,\bar{s})=\frac{\rho(s)}{\pi q^2}\int_{t_0}^{K>0}\int_{t_0}dt'dt''\times\frac{A_t{}^{I*}(s,t',\bar{s})A_t{}^I(s,t'',\bar{s})}{K^{1/2}(q^2;t,t';t'')}, \tag{3.12}$$

$$v^I(t,s)=R_t{}^I(s,t)+\sum_{I'}\beta_{II'}\{A_t{}^{I'}(t,s)-V_t{}^{I'}(s,t)\}, \tag{3.13}$$

$$v^I(t,s)=\sum_{I'}\beta_{II'}A_t{}^{I'}(t,s),\quad t<t_I \tag{3.14}$$

and

$$V^I(t,s)=\frac{1}{\pi}\int_{t_0}^{\infty}dt'\,v^I(t',s)\left[\frac{1}{t'-t}+\frac{(-1)^I}{t'-u}\right], \tag{3.15}$$

where $u=4\mu^2-s-t$, I is the isospin, $u_0=t_0=s_0$, and $\beta_{II'}$ is an element of the usual $\pi\pi$ crossing matrix

$$\beta_{II'}=\begin{pmatrix}\frac{1}{3} & 1 & 5/3\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix}.$$

As before, $R^I(s,t)$ is any fully crossing-symmetric function which has a Mandelstam representation with zero double-spectral functions in all the elastic strip regions. It presumably contains the contribution of all inelastic effects which have not been explicitly put into our equations.

It is also possible to generalize Eq. (3.3), thereby explicitly displaying the crossing symmetry. We write

$$\begin{aligned}A^I(s,t,u,\bar{s},\bar{t},\bar{u})&=R^I(\bar{s},\bar{t},\bar{u})\\&+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}[A_{sI}{}^{el}(s',t,\bar{s})+(-1)^IA_{sI}{}^{el}(s',u,\bar{s})]\\&+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\sum_{I'}\beta_{II'}[A_{tI'}{}^{el}(t',s,\bar{t})\\&+(-1)^{I'}A_{tI'}{}^{el}(t',u,\bar{t})]+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-u}\sum_{I'}\beta_{II'}(-1)^I\\&\times[A_{uI'}{}^{el}(u',s,\bar{u})+(-1)^{I'}A_{uI'}{}^{el}(u',t,\bar{u})],\end{aligned} \tag{3.16}$$

where the $A_s{}^{el}$ are given by Eq. (2.13) but the corresponding $\rho_{el}{}^I$ given by Eq. (3.12). Equation (3.4) is generalized to

$$A^I(s,t,u,\bar{s},t,u)=A^I(s,t,u,\bar{s}). \tag{3.17}$$

From Eqs. (3.16) and (3.17), together with Eq. (3.10), one can readily obtain Eqs. (3.11), (3.13), and (3.15).

Up to now we have been ignoring the possibility of divergences. These do not arise if the potential V is calculated from single-particle exchanges, but could arise in the above crossing-symmetric equations, particularly if we start making approximations. This could happen if $V(t,s)$ does not fall off rapidly enough with t, for example. A simple cutoff on t' and s' in Eq. (3.3) or on s' in Eq. (3.11) might not be sufficient in this case. A better procedure would be to put in a factor which makes Eq. (3.11) fall off rapidly with s, since its contribution to Eq. (3.13) will then fall off rapidly with t. One way of accomplishing this without destroying analyticity and unitarity is to insert a factor $g(s)h(s')$ into the integral of Eq. (3.11), where $g(s)$ falls off rapidly with large s and is analytic for $s<s_I$, while $h(s)=g^{-1}(s)$ for $s<s_I$ but does not grow too rapidly for large s. One possible choice for h is $h(s')=g^{-1}(s')\theta(s_c-s')+g^{-1}(s_c)\theta(s'-s_c)$, where $s_c\geq s_I$. A convenient choice for g is $g(s)=\exp[c(s_c-s)^{1/4}]$, which vanishes rapidly in all s directions and is nonsingular for $s<s_c$.

IV. APPROXIMATION SCHEME FOR UNITARITY EQUATION

The fully crossing-symmetric equations (3.9)–(3.14) are extremely difficult to solve directly. In practice we have to make an approximation for the potential, thereby destroying exact crossing. The simplest is to use Eq. (3.14), extending it to $t>t_I$ if necessary and putting in an adjustable cutoff $t=t_c$ to roughly take into account the effects of R_t and V_t. Thus

$$v^I(t,s)\simeq\sum_{I'}\beta_{II'}\,\mathrm{Re}A_t{}^{I'}(t,s)\theta(t_c-t). \tag{4.1}$$

We have to take the real part of $A_t{}^{I'}$ because it does in fact have an imaginary part for $t>t_I$. This imaginary part is canceled by R_t and V_t for $s<s_I$ in Eq. (3.13) since unitarity requires v to be real there, as we have seen. Somewhat more natural approximate prescriptions for $v^I(t,s)$, in which $\mathrm{Im}A_t$ is automatically taken care of, will be discussed in Appendix B. We shall see there that, if we start from A_t, we have to make corrections which introduce short-range repulsive terms into the potential. From our experience with the Schrödinger equation, we might then hope to have the calculated amplitude depend less sensitively on the cutoff t_c than it does if we simply use Eq. (4.1).

A cutoff of the type contained in (4.1) might at first sight appear rather crude. However, it should represent large-t effects in much the same way that a pole, for example, represents distant singularities in an effective-range approximation.[12] To see this more clearly, let us go back to the simple case given by Eqs. (2.3)–(2.5). From these equations we see that, if we are only interested in the amplitude in the physical region, we only need V for $t<0$. In practice we can get quite a good approximation for this, even if we distort v quite drastically for large t' in Eq. (2.5), as long as the distorted v resembles the original one in some average sense. This could easily be accomplished if t_c is adjusted accordingly.

[12] See, for instance, G. F. Chew, *S-Matrix Theory of Strong Interactions* (Benjamin, New York, 1961), Chap. 1.

Just as a more accurate effective-range approximation can always be constructed by adding extra poles, so here too we can presumably improve Eq. (4.1) by adding δ functions to it. This would correspond to adding poles to V. Alternatively we could add a δ function and its derivatives, say, at $t=t_c$. This amounts to adding a multipole expansion to V. Of course, this involves more parameters. In problems with spin, however, the potential may have to fall off more rapidly with t just to guarantee the existence of a solution. We could then give V, as given, say, by Eq. (2.5), a behavior t^{-n-1} simply by adding n multipoles and adjusting their residues so as to guarantee this behavior. This fixes the residues uniquely.

If we take a particular value of $\bar{s}$, we can directly apply a proof due to Mandelstam[13] to show that Eqs. (3.11) and (3.12) give Regge behavior for large t. This is actually needed to give meaning to the integral (3.10), which may diverge in the elementary sense, but can be defined by Regge continuation. It, of course, assumes that V falls off sufficiently rapidly for large t. A cutoff potential, such as the one given by Eq. (4.1), does in fact satisfy this requirement in the $\pi\pi$ problem.

To solve our equations we shall use a generalization of an iteration of the Noyes-Kowalski form of the Schrödinger equation.[14] Let us first go back to Eq. (2.1). This can be reexpressed as a Lippmann-Schwinger equation. In a given partial wave, it has the form

$$\Phi_l(s',s,\bar{s})=V_l(s',s,\bar{s}) + \frac{1}{\pi}\int_{s_0}^{\infty} ds''\,\frac{\rho(s'')}{s''-s}V_l(s',s'',\bar{s})\Phi_l(s'',s,\bar{s}), \quad (4.2)$$

where, as before, we are freezing s at $s=\bar{s}$ within the potential only, and where $\Phi_l(s,s,\bar{s})=A_l(s,\bar{s})$, the partial-wave projection of $A(s,t,\bar{s})$. If the potential is given by (2.2), then V_l is given by

$$V_l(s',s,\bar{s})=\frac{1}{\pi}\int_{t_0}^{\infty} dt'\,\frac{v(t',\bar{s})}{2q'q}Q_l((q'^2+q^2+t')/2q'q). \quad (4.3)$$

Equation (4.2) is a singular equation. To reduce it to nonsingular form, we write[14]

$$\Phi_l(s',s,\bar{s})=F_l(s',s,\bar{s})A_l(s,\bar{s}), \quad (4.4)$$

so that $F_l(s,s,\bar{s})=1$. From Eq. (4.2), we then have

$$A_l(s,\bar{s})=V_l(s,s,\bar{s})/D_l(s,\bar{s}), \quad (4.5)$$

with

$$D_l(s,\bar{s})=1-\frac{1}{\pi}\int_{s_0}^{\infty} ds''\,\frac{\rho(s'')}{s''-s}V_l(s,s'',\bar{s})F_l(s'',s,\bar{s}). \quad (4.6)$$

The function F_l is the solution of the nonsingular integral equation

$$F_l(s',s,\bar{s})=\frac{V_l(s',s,\bar{s})}{V_l(s,s,\bar{s})}+\frac{1}{\pi}\int_{s_0}^{\infty} ds''\,\frac{\rho(s'')}{s''-s} \times\left[V_l(s',s'',\bar{s})-\frac{V_l(s',s,\bar{s})V_l(s,s'',\bar{s})}{V_l(s,s,\bar{s})}\right]F_l(s'',s,\bar{s}), \quad (4.7)$$

which can, for example, be solved by iteration, dropping the integral as a first approximation. Note that Eq. (4.6) can also be written as

$$D_l(s,\bar{s})=h(s,\bar{s})-\frac{1}{\pi}\int_{s_0}^{\infty} ds''\,\frac{\rho(s'')}{s''-s}V_l(s'',s'',\bar{s}), \quad (4.8)$$

where

$$h(s,\bar{s})=1-\frac{1}{\pi}\int_{s_0}^{\infty} ds''\,\frac{\rho(s'')}{s''-s} \times[V_l(s,s'',\bar{s})F_l(s'',s,\bar{s})-V_l(s'',s'',\bar{s})], \quad (4.9)$$

which is nonsingular for $s>s_0$ and thus has only a left-hand cut.

An iterative expansion of Eq. (4.7) is equivalent to associating a parameter λ with V and expanding in powers of λ. From Eq. (4.9) this, in turn, is equivalent to expanding h:

$$h=1+\lambda h_1+\lambda^2 h_2+\cdots. \quad (4.10)$$

From Eqs. (4.7) and (4.9) we also see that h is nonsingular to any order for $s>s_0$. From Eqs. (4.5) and (4.8), this means that unitarity is not destroyed by a truncation of the series (4.10).

Now it is not necessary to obtain the expansion (4.10) from Eqs. (4.7) and (4.9). We could equally well expand A_l as given by Eqs. (4.5), (4.8), and (4.10), and compare it with the corresponding expansion of Eqs. (2.3) and (2.4) or of Eqs. (2.9)–(2.11). This enables us to evaluate h up to any order by purely on-shell techniques.

The above method can be readily generalized to the relativistic case. Here we no longer have an equivalent of Eqs. (4.6), (4.7), and (4.9), but we can continue using Eqs. (4.5), (4.8), and (4.10). The only change is that A is now the invariant amplitude, s the square of the total c.m. energy, and ρ the relativistic phase-space factor. The coefficients of Eq. (4.10) are required to be such that we reproduce, up to any given order, the amplitude obtained from Eqs. (3.9)–(3.12). Since these equations give a unitary amplitude in the elastic region, the function $h(s,\bar{s})$ must be nonsingular for $s_0<s<s_I$, as can be seen from Eq. (4.8). Since this is true for a continuum of possible values of λ, the coefficients in Eq. (4.10) must likewise be nonsingular for $s_0<s<s_I$. We therefore again obtain a unitary amplitude, even when the expansion (4.10) is truncated.

Since Eq. (4.8) gives a unitary amplitude, we might expect it to give rise to resonances or bound states.

[13] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).
[14] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965); K. L. Kowalski, *ibid.* **15**, 798 (1965).

Remembering that the physically interesting amplitude is $A_l(s,s)$, such a state will arise at $s=s_R$ if

$$\mathrm{Re}D_l(s_R,s_R)=0. \tag{4.11}$$

The corresponding residue is

$$4q_R{}^{2l}\Gamma_l=-\left[V_l(s,s,s)\Big/\left(\frac{d}{ds}D_l(s,s)\right)\right]_{s=s_R}, \tag{4.12}$$

where q_R is the value of q at $s=s_R$ and Γ_l is the usual reduced width.

From now on we shall make the lowest-order approximation for h, namely,

$$h(s,\bar{s})\simeq 1. \tag{4.13}$$

Equations (4.5) and (4.8) then reduce to something resembling the determinantal approximation. In general, however, it differs from the usual determinantal approximation to the N/D equations where, in addition to making the approximation (4.13), we would replace the last $\bar{s}$ in Eq. (4.8) by an s''. It is this difference which makes it possible for trajectories to rise to infinity in our scheme.

V. SIMPLE CALCULATIONS IN $I=1$ $\pi\pi$ SCATTERING

We shall now apply the approximation (4.13) to the $I=1$ state, taking Eq. (4.1) for v. This means that our input is the crossed-channel absorptive part. Now ultimately this must be determined by self-consistency, i.e., it must be consistent with the absorptive part calculated dynamically in the direct channel. We shall see *a posteriori* that such consistency is at least approximately attained if we take A_t from the Veneziano model, with experimental values for the trajectory and coefficient. It is possible to go even further and determine these parameters completely self-consistently so that a bootstrap solution does in fact exist.[15]

The Veneziano model gives an $I=1$ $\pi\pi$ amplitude[16]

$$B(s,t,u)=(-\bar{\beta})\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}-(t\to u), \tag{5.1}$$

where α is the Regge trajectory function. Now this amplitude actually has many of the same general properties as the one obtained by solving our unitarity equations, at least if we make the approximation (4.1). In particular they both have Regge behavior at large energies and satisfy the dispersion relation (3.10). It is therefore reasonable to hope that the two amplitudes can be made to be approximately equal to each other, at least for certain ranges of energy. One prerequisite is that the low-t crossed-channel absorptive parts be the same in both models. Assuming that secondary trajectories are not too important, the finite-energy sum rules implied by Eq. (3.10) might then hopefully give approximately the same Regge behavior for the high-t absorptive part as Eq. (5.1), which satisfies the same sum rules. Using Eq. (5.1), we therefore take

$$A_t{}^{I=1}=B_t\equiv-\pi\bar{\beta}\sum_{n=0}^{\infty}\frac{n-\alpha(s)}{\alpha'(R_n)}\begin{bmatrix}\alpha(s)+n-1\\ n\end{bmatrix}\times\delta(t-R_n), \tag{5.2}$$

where the R_n are the positions of the resonances, given by $\alpha(R_n)=n+1$. We next use Eq. (4.1) to evaluate v. The θ function will now mean taking a finite number $(N+1)$ of δ functions and retaining only a fraction r (≤ 1) of the highest one. In other words, we take

$$v(t,s)=-\pi\bar{\beta}\sum_{n=0}^{N}\frac{n-\alpha(s)}{\alpha'(R_n)}\begin{bmatrix}\alpha(s)+n-1\\ n\end{bmatrix}\times[1+(r-1)\delta_{n,N-1}]\delta(t-R_n). \tag{5.3}$$

It can be shown explicitly that most of the contribution to v comes from the exchange of the particles lying on the leading trajectory. In practice we make the usual linear approximation for $\alpha(s)$, although, ultimately, some inelasticity could be introduced by adding a cut to $\alpha(s)$ for $s>s_I$. This would make v complex in the inelastic region.

To simplify our equations further, we approximated v by a single δ function,

$$v(t,s)=\pi r(s)\delta(t-t_D), \tag{5.4}$$

with $r(s)$ and t_D adjusted so that the resulting Eq. (2.5) is the same as that given by Eq. (5.3) at $t=0$ and $t=-\infty$. The same effective-range argument can be used to justify this as was made in Sec. IV to justify the cutoff $t=t_c$. In addition, the extreme relativistic approximation was made for the phase-space factor, $\rho(s)\simeq 1$, so that the pion mass drops out of the problem. Finally, the Legendre function Q_l is approximated by its asymptotic form in Eq. (4.3). This is usually quite a good approximation, even for fairly low values of the argument of Q_l.

Table I gives the output s_R and Γ_l for the first five integer-l resonances generated by Eqs. (4.11) and (4.12).

TABLE I. Values of l, s_R, Γ_l, b/π, and T for the first five $I=1$ output resonances, assuming elastic unitarity for all s. The cutoff t_c was fixed so that s_R has the correct experimental value for $l=1$.

l	s_R	Γ_l	b/π	T
1	30	2.6×10^{-1}	1.5×10^{-2}	1.3×10^{-2}
2	84	3.9×10^{-3}	2.6×10^{-4}	2.7×10^{-4}
3	144	3.4×10^{-5}	2.4×10^{-6}	2.9×10^{-6}
4	208	1.7×10^{-7}	1.3×10^{-8}	2.0×10^{-8}
5	281	6.7×10^{-10}	4.6×10^{-11}	9.7×10^{-11}

[15] L. A. P. Balázs, Phys. Letters **29B**, 228 (1969). This calculation used Eq. (4.13) and was done in the vicinity of the ρ mass, which was not determined. Since the pion was taken to have a vanishingly small mass, however, the ρ mass merely serves to set the energy scale and is thus not really a parameter in the calculation.

[16] C. Lovelace, Phys. Letters **28B**, 264 (1968).

TABLE II. Values of l, s_R, Γ_l, and b/π for the first five $I=1$ Veneziano (input) leading-trajectory resonances. The Veneziano parameters were fixed so that the ρ width and the ρ and f^0 masses have the correct experimental values.

l	s_R	Γ_l	b/π
1	30	2.0×10^{-1}	1.2×10^{-2}
2	80	3.2×10^{-3}	2.4×10^{-4}
3	130	2.7×10^{-5}	2.4×10^{-6}
4	180	1.6×10^{-7}	1.6×10^{-8}
5	230	7.4×10^{-10}	8.0×10^{-11}

The cutoff is adjusted so the output ρ has the correct experimental position $s_R=30$; this corresponds to $N=5$ and $r=0.813$. The Veneziano parameters which come into Eq. (5.3) were fixed so that Eq. (5.1) gives the correct experimental ρ width and ρ and f^0 positions. These are listed in Table II, along with the remaining positions and widths of the first five resonances on the leading input trajectory. This was the main part of our crossed-channel input. Since these values agree approximately with the corresponding values in Table I, our model therefore has at least partial crossing and some resemblance to the Veneziano amplitude.

If we now plot s_R versus l for the resonances of Table I and interpolate with a polynomial, we obtain an implicit expansion for the Regge trajectory $\alpha(s)$. From this it is possible to evaluate $\alpha'(s)$ and so calculate the Regge residue functions

$$\frac{b}{\pi}=\frac{4}{\sqrt{\pi}}\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+1)}\Gamma_\alpha\frac{d\alpha}{ds} \tag{5.5}$$

at the resonance positions. These are listed in Table I. Table II gives the corresponding Veneziano values obtained by looking at the asymptotic behavior of Eq. (5.1), which gives

$$\frac{b}{\pi}=\frac{\alpha'^{\alpha}}{\Gamma(\alpha)}\bar{\beta}. \tag{5.6}$$

Once we have b and α for the leading output trajectory, we can check average duality by seeing how well the lowest-moment finite-energy sum rule[17] is satisfied. Now Eq. (3.10), together with Regge behavior, gives

$$\int_0^{N}d\bar{\nu}(A_t-A_t^{\text{Regge}})\bar{\nu}^n=0, \tag{5.7}$$

where $\bar{\nu}$ is the usual antisymmetric combination $\bar{\nu}=\frac{1}{2}(t-u)$. If we take for the Regge behavior

$$A_t^{\text{Regge}}(t,s)=b\bar{\nu}^{\alpha}, \tag{5.8}$$

we obtain, for $n=0$,

$$T(s)\equiv\frac{\alpha+1}{\pi N^{\alpha+1}}\int_{t_0}^{t_1}dt'\,A_t(t',s)=\frac{b}{\pi}, \tag{5.9}$$

where $t_1=N-\frac{1}{2}s+2\mu^2$. Of course, any amplitude with Regge asymptotic behavior will automatically satisfy Eq. (5.9) for sufficiently large t_1. Duality goes further and requires that it be satisfied even for comparatively small t_1, provided this is taken about midway between two t-channel input resonances.[9] Table I gives the values of $T(s)$ (with $t_1=255$) at the positions of the output resonances. We see that they are, in fact, roughly equal to the corresponding values of b/π, as required by Eq. (5.9).

From Table I we also see that the resonances lie on a trajectory which rises up to fairly large values of $\alpha(s)$.[18] In Appendix A we show that it actually rises to infinity. One input feature which we do not seem to have, however, is daughter trajectories. This is probably a defect of our approximation. Any simple generalization of a Schrödinger equation with an attractive potential whose strength increases sufficiently rapidly with energy can be expected to give higher resonances in a given partial wave; an indication that this might happen occurs in Finkelstein's calculation in Ref. 18. It will be necessary to do a more accurate calculation to see whether such daughter resonances do in fact develop in our formalism.

In the above calculation the cutoff t_c was taken to be a constant. It was then found that duality was approximately satisfied. We can turn this around and use duality to determine the cutoff at each energy. As discussed in Appendix C, such a procedure was essentially the one followed in Ref. 15, where self-consistent Veneziano parameters were actually obtained. This shows that bootstrap solutions are possible with our scheme, presumably because it satisfies the nonlinear constraint of unitarity.

A possible objection to the above calculation is that it is purely elastic, whereas inelastic effects are clearly important at higher energies. Such effects are discussed further in the following section, and a proper treatment of them would complicate the above calculations considerably. If we assume Regge asymptotic behavior, however, we can take them approximately into account by replacing Eq. (2.7) with a modified phase-space factor

$$\rho(s)=R(s)q/(q^2+\mu^2)^{1/2}. \tag{5.10}$$

In the asymptotic region, the factor $R=\sigma_{\text{tot}}/\sigma_{\text{el}}$ is then independent of l.[19] If it is calculated from Pomeranchuk

[17] Such sum rules were discovered independently by A. A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters **24B**, 181 (1967); K. Igi and S. Matsuda, Phys. Rev. Letters **18**, 625 (1967); R. Gatto, *ibid.* **18**, 803 (1967); L. A. P. Balázs and J. M. Cornwall, Phys. Rev. **160**, 1313 (1967); and R. Dolen, D. Horn, and C. Schmid, Ref. 9.

[18] This particular feature was also obtained by J. Finkelstein [Phys. Rev. **145**, 1185 (1966)] using only ρ exchange, and by S. H. Patil [*ibid.* **179**, 1405 (1969)] using only ρ, f^0, and g exchange. Both started from equations which do not have the correct Mandelstam analyticity. This is also true of the calculations of G. Tiktopulos, Phys. Letters **29B**, 185 (1969); U. Trivedi, Phys. Rev. **188**, 2241 (1969).

[19] See, for example, R. W. Childers and A. W. Martin, Phys. Rev. **182**, 1762 (1969).

trajectory exchange, it varies only slowly with energy. We shall take it to be actually energy independent and assume it persists down to the 2ρ threshold, where inelastic effects might be first expected to start becoming important. Assuming elastic unitarity below this threshold, we thus have

$$R(s)=1+c\theta(s-4m_\rho^2). \tag{5.11}$$

This model is almost certainly greatly oversimplified[20] but may at least be instructive. The constant c was calculated from the ratio of the total g-meson width as given by experiment (120 MeV) to the partial width as given in Table II.

If we now repeat our previous calculations, we obtain the results in Table III. Only the first two resonances were calculated since the crude form (5.11) fails above the second resonance because of the θ function, which actually gives an infinite number of resonances just below $s=4m_\rho^2$. A smoothed out expression would presumably remove this difficulty, but also would complicate the calculation. Note that the parameters of the output resonances are not too different from the corresponding ones in Table I. This does not, of course, mean that inelastic effects are unimportant. It just means that, at least below the inelastic threshold, they are roughly taken into account by simply taking a different cutoff t_c. It is interesting to note that the Table III cutoff $N=3$, $r=0.536$ would be just the sort we might expect if we calculated v from A_t, but with repulsive corrections for $t>4t_\rho$ strong enough to cancel out the effect of A_t. We shall see in Appendix B that repulsive corrections do in fact arise there. Hopefully they would lessen the dependence of our results on the cutoff.

Finally, it might be interesting to calculate the P-wave scattering length a_1. We obtain $a_1=0.023$ if we use elastic unitarity and $a_1=0.031$ if we take Eqs. (5.10) and (5.11). The corresponding Weinberg current-algebra value[21] is $a_1=(24\pi F_\pi)^{-2}=0.0295$.

VI. ALTERNATIVE CALCULATIONAL TECHNIQUES AND POSSIBLE IMPROVEMENTS

Suppose we again begin from the approximated potential given by Eq. (4.1). We have already seen that this gives Regge behavior for large t. We have also seen that, if we solve Eqs. (3.11) and (3.12) by iteration, starting from $A_t{}^I=v_t{}^I$, the nth iteration will be exact for $t_0<t<(n+1)^2t_0$. We can thus build $A_t{}^I$ exactly up to as large a value of t as we wish. Eventually we reach a point $t=t_1$ dominated by the leading Regge pole. At this point the most straightforward procedure for determining b and α would be simply to require that the value and derivative of A_t be the same as that given by Eq. (5.8). If we have more than one Regge trajectory, the asymptotic A_t would be a sum of terms, each of the form (5.8). The extra quantities could then be determined either by considering higher derivatives at $t=t_1$ or by matching at several values of t.

TABLE III. Values of l, s_R, Γ_l, b/π, and T (with $t_1=105$) for the first two $I=1$ output resonances, using Eqs. (5.10) and (5.11) for the phase-space factor. The cutoff was again fixed so that s_R has the correct experimental value for $l=1$.

l	s_R	Γ_l	b/π	T
1	30	2.8×10^{-1}	1.4×10^{-2}	1.3×10^{-2}
2	91	4.7×10^{-3}	2.9×10^{-4}	2.6×10^{-4}

One difficulty with the above scheme is that it requires A_t to be evaluated accurately up to very large values of t. A closely related scheme, for which this may not be necessary, is to insert an iterated A_t into the finite-energy sum rule (5.7). The Regge parameters can then be calculated by considering, say, $n=0$ and $n=1$, and using Eq. (5.8).

The above methods do not satisfy unitarity exactly. One which does is the Padé approach discussed already in Ref. 7. Another is the Schrödinger equivalent potential method.[22] Here one again takes an A_t obtained by iterating Eqs. (3.11) and (3.12) and calculates an equivalent Schrödinger potential by using Secs. 2 or 5 of Ref. 22. The latter procedure may also be applied with minor modifications to calculate a Logunov-Tavkhelidze potential.[8] Unfortunately, once we have a potential we also have to solve an integral or differential equation to obtain the amplitude. The same is true if we write a set of N/D equations for the amplitude $A_l(s,\bar{s})$, with the left-hand cut in s calculated from the iterated A_t; as before, $\bar{s}$ would be kept fixed in such a calculation, so that our equations would not be the same as the N/D equations of the usual dispersion theory.

Once we have calculated the absorptive part and double-spectral function, we can obtain an improved potential by crossing from Eqs. (3.13) and (3.15). The whole procedure can then be repeated any number of times and hopefully converges to a solution. Of course, such a scheme may lead to potentials which give divergences unless one puts in a cutoff of the type discussed in the last paragraph of Sec. III. If they do not, they almost certainly lead to Regge cuts.[23] The methods of the first two paragraphs of this section would then have to be modified since we would have to add the cut contribution to Eq. (5.8). The crudest procedure would be simply to take a model, such as that of Arnold,[24] for these cuts.

To go beyond Eqs. (3.9)–(3.15) we must include the effect of other channels. In some crude sense this is done by using Eq. (5.10). It could also be done by taking

[20] For a more detailed version, see L. A. P. Balázs, Phys. Rev. **132**, 867 (1963).

[21] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[22] L. A. P. Balázs, Phys. Rev. **137**, B1510 (1965).

[23] D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento **26**, 896 (1962); D. Amati, M. Cini, and A. Stanghellini, *ibid.* **30**, 193 (1963).

[24] R. C. Arnold, Phys. Rev. **153**, 1523 (1967).

some model for the crossing-symmetric amplitude $R^I(s,t)$, say, by calculating it from a finite number of box graphs. But the only correct procedure is to use multichannel unitarity, which adds on to Eq. (3.12) the contribution of other intermediate states. We then have to solve for several processes simultaneously, an extremely difficult problem in practice unless we make approximations such as using the determinantal approximation and taking Eq. (4.1) for the potential. Another problem is that double counting may occur. For example, if we consider the coupled $\pi\pi$-$\rho\rho$ system, the same box diagram for $\pi\pi \to \pi\pi$ (with two pions and two ρ's for the internal lines) will be generated from the inelastic part of unitarity in the direct channel and also be a part of the potential, since it is generated by the elastic part of unitarity in the crossed channel. Its contribution to the potential must therefore be explicitly removed. This can be done in a crossing-symmetric way and generally weakens the potential. The introduction of higher channels, which have the effect of introducing extra attraction, may thus be at least partially offset by the weakening of the potential required by the avoidance of double counting.

APPENDIX A: INFINITE RISE OF OUTPUT REGGE TRAJECTORY

We will show here that, at least for the determinantal approximation, the output Regge trajectory will rise to infinity if the potential contains the exchange of angular momentum $>5/4$. It is quite likely that this latter condition is different if we improve on the determinantal approximation, although it is probably safe to conjecture that the infinite rise of the trajectory will persist in all cases of interest.

Suppose for simplicity that we obtain V from the exchange of a single particle of mass m and spin l', so that

$$v(t,s)=gP_{l'}[1+(2s/m^2-4)]\delta(t-m^2), \tag{A1}$$

where g is a constant; it is trivial to extend our argument to the exchange of several particles. We have taken the external mass $\mu=1$. For $h=1$ and $s=\bar{s}$, the D function in Eq. (4.6) then contains the integral

$$I(\nu)=P\int_0^\infty \frac{d\nu'}{\nu'-\nu}\rho(s')F(\nu'), \tag{A2}$$

where $\nu=q^2=\frac{1}{4}s-1$ and

$$F(\nu')=(1/\nu')Q_l(1+(m^2/2\nu')). \tag{A3}$$

Since we shall be concerned mainly with large l, let us approximate Q_l by its asymptotic form,[25] so that for $2\nu'\gg m^2$,

$$F(\nu')\simeq(\tfrac{1}{2}\pi m)^{1/2}\frac{\exp[-(m/\sqrt{\nu'})(l+\frac{1}{2})]}{l^{1/2}\nu'^{5/4}}. \tag{A4}$$

[25] See, for example, W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea, New York, 1949), p. 23.

This rises monotonically with ν' up to a maximum and then falls off. The maximum occurs at

$$\nu_m=(4/25)m^2(l+\tfrac{1}{2})^2. \tag{A5}$$

However, we must have $\nu'\ll\nu_{\max}$ for Eq. (A4) to be valid.

Since $F(\nu')>0$ for all $\nu'>0$, Eq. (A2) gives

$$I(\nu_m/4M)>P\int_0^{\nu_m/2M}\frac{d\nu'}{\nu'-(\nu_m/4M)}\rho(s')F(\nu') +\int_{\nu_m/2M}^{\nu_m/M}d\nu'\frac{\rho(s')F(\nu')}{\nu'-(\nu_m/4M)}, \tag{A6}$$

where M is a fixed number $\gg1$, so that the asymptotic form (A4) is valid in the integrals of (A6). Since $F(\nu')$ is a monotonically rising function for $\nu'<\nu_{\max}$, the first integral must be >0. For the same reason, the second integral is bigger than the same integral with the numerator replaced by its value at the lower limit and the denominator replaced by its value at the upper limit. We therefore have

$$I(\nu_m/4M)>\tfrac{2}{3}F(\nu_m/2M)\rho((2\nu_m/M)+4). \tag{A7}$$

Since M is fixed, we can always take l big enough so that $\nu_m\gg4M_m^2$, where the asymptotic form of $P_{l'}$ can be used. From Eqs. (4.6) and (A7) we then have

$$1-\mathrm{Re}D>Kl^{2n-(5/2)}M^{(5/4)-n}e^{-5M}, \tag{A8}$$

where K is a fixed constant which depends on g, l', and m^2. Now since M and K are fixed, we can always find an l big enough so that $Kl^{2n-(5/2)}>M^{n-(5/4)}e^{5M}$, provided that $n>5/4$. From Eq. (A8) we then have

$$\mathrm{Re}D(\nu=\nu_m/4M)<0 \tag{A9}$$

for sufficiently large l.

Let us now turn to $\nu=0$. Here we use the fact that $Q_{l+1}(x)<Q_l(x)$ for any $x>1$. Since the only l dependence in $I(\nu)$, and hence in $1-D$, comes from the Q_l, this means that the magnitude of $1-D$ decreases with increasing l. Thus, if the magnitude <1 for some particular l, it will remain so for all higher l. But in our case, this condition is in fact met for $l=1$. We must therefore have $D>0$ for all $l\geq1$ at $\nu=0$. But this, when combined with Eq. (A9), means that, no matter how large l is, we always have a zero of ReD and hence a resonance. The Regge trajectory therefore rises to infinity.

APPENDIX B: CONSTRUCTION OF POTENTIAL FROM ABSORPTIVE PART

In Sec. IV, v was approximated by ReA_t up to $t=t_c$. One difficulty with this is that it is not analytic in s. One model which does not have this difficulty and still reduces to Eq. (3.14) would be

$$v^I(t,s)\simeq\sum_{I'}\beta_{II'}A_t{}^{el\cdot I'}(t,s,t)\theta(t_c-t), \tag{B1}$$

with A_t^{el} defined as in Eq. (2.4). This prescription is nonsingular for $s<s_I$ and thus does not introduce any spurious inelastic effects in the elastic region.

If, however, we wish to construct v from some given A_t, we must go to Eq. (3.11), but this time with the value at $s=\bar{s}$ subtracted out. Thus

$$A_t^I(s,t,\bar{s})=A_t^I(t,\bar{s}) + \frac{1}{\pi}\int_{s_0}^{\infty} ds'\,\rho_{el}^I(s',t,\bar{s})\left(\frac{1}{s'-s}-\frac{1}{s'-\bar{s}}\right). \quad \text{(B2)}$$

Starting from a given $A_t^I(t,\bar{s})$ we can construct ρ_{el}^I by iterating Eqs. (B2) and (3.12), with $A_t^I(s,t,\bar{s})=A_t^I(t,\bar{s})$ in lowest order. As usual, n iterations give an exact ρ_{el}^I up to $t=(n+1)^2t_0$. Once we have determined it up to $t=t_c$, we can obtain v from Eq. (3.11) evaluated at $s=\bar{s}$.

The above procedure requires an A_t^I which itself comes from a unitary model. If, for example, we only have the contribution of a small number of t-channel partial waves $\mathrm{Im}A_l(t)$, with $l\leq L$, we would again generate a complex v for $t>t_I$, with the attendant spurious inelasticity. This can be avoided by keeping only the contribution of the same number of partial waves in v and using Eq. (3.11) evaluated at $s=\bar{s}$. Then

$$v^I(t,s)=\sum_{l=0}^{L}(2l+1)P_l\left(1+\frac{2s}{t-4}\right) \times\left[\mathrm{Im}A_l(t)-\int_{-t+4}^{0}\frac{ds''}{t-4}P_l\left(1+\frac{2s''}{t-4}\right) \times\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s''}\rho_{el}^I(s',t,s'')\right]. \quad \text{(B3)}$$

Equations (B3), (3.11), and (3.12) now form a closed system of equations with the $\mathrm{Im}A_l(t)$ as input. As before, the nth iteration gives us a v up to $t=(n+1)^2t_0$.

APPENDIX C: SOLUTIONS CONSTRAINED BY AVERAGE DUALITY

In Sec. V, calculations were made in which the cutoff t_c was taken to be a constant, adjusted so that the output ρ have the same mass as the input. It was then found that duality was approximately satisfied by the solutions. We shall now turn this around and require the cutoff to be such that Eq. (5.9) is satisfied. We could do this at some particular energy (say, the mass of the ρ) and continue taking t_c to be a constant. But we will, instead, determine it separately at each energy. In practice, the absorptive part A_t in Eqs. (5.7) and (5.9) will be simply calculated from ρ, f^0, and g exchange in this case. We have already seen that this does not differ too much from using Eq. (5.2).

We shall restrict ourselves to the neighborhood of the ρ meson, both in l and in s. This means that we must consider unphysical as well as physical values of l. The integrals of the determinantal approximation are difficult to handle then and so we make exactly the same Pagels-type approximation as was used in Ref. 15. The potential is also approximated further as in Ref. 15, rather than the way it is in Eq. (5.4).

Suppose we take the experimental values of 765, 1250, and 1630 MeV for the ρ, f^0, and g masses. If we fix the separation point t_1 in Eq. (5.9) midway between the g and f^0 resonances, take the experimental value of 125 MeV for the input ρ width, and adjust the input f^0 width so that the output ρ mass has the correct experimental value, we obtain an input f^0 width of 130 MeV, an output ρ width of 172 MeV, and $\alpha'=0.0144$. If, instead, we set $t_1=t_g+\frac{1}{2}(t_g-t_f)$, take the experimental values of 125 and 140 MeV for the input ρ and f^0 widths, and adjust the input g width so that the output ρ mass has the correct value, we obtain an input g width of 58 MeV, an output ρ width of 161 MeV, and $\alpha'=0.0154$.

Instead of taking t_1 midway between two resonances, we could fix it by requiring Eq. (5.7) to be satisfied for $n=2$, as well as $n=0$. We can then repeat the above calculations. Thus, with t_1 between the f^0 and g mesons, we get an input f^0 width of 103 MeV, an output ρ width of 193 MeV, and $\alpha'=0.0129$. With t_1 above the g, we obtain an input g width of 38 MeV, an output ρ width of 171 MeV, and $\alpha'=0.0145$. These values are not too different from the ones calculated in the preceding paragraph.

Instead of taking given inputs and obtaining outputs which are only approximately consistent with them, we could try to determine our parameters self-consistently. In such a program it is better to use Eq. (5.2) in the sum rule (5.9), since we then have fewer parameters to determine. Now since the Veneziano model satisfies this sum rule automatically, at least to a very good approximation, we would get the same relation between β and α from the sum rule as we would from the Veneziano model directly. Thus our calculation reduces to the one in Ref. 15, where the latter relation was used and where self-consistent Veneziano parameters were actually obtained.