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Feynman Rules for an $O(4)$ Family with No Ghosts

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When a resonance is observed in an experiment, it is usually assumed that the resonance has a definite spin J. Should, however, an $O(4)$ family of Regge trajectories hit the physical region, we could expect to see a family of resonances of different spins roughly degenerate in energy. For such a family of resonances, there should presumably be a master field which contains all the spins in the family and which interacts as a unit with other hadrons. In this paper we discuss the general tensor construction of such multispin fields. We believe that the splitting of the A_2 meson is a manifestation of a master field and, in a following paper, study the phenomenology of the $O(4)$ resonance system.

I. INTRODUCTION

THE $O(4)$ classification of Regge trajectories has long been accepted at the unphysical point $s=0$.[1–3] While the dynamics of Regge trajectories is not known, it is a popular belief that trajectories are parallel and linear so that when the parent trajectory passes through, say, an integer, the daughter trajectories of an $O(4)$ family also pass through integers below the parent spin. If daughter trajectories stay parallel, then the daughter trajectories could become physical at the same energy as the parent.[4]

When such a family of resonances appears, it becomes interesting to ask if the family of mesons, say, can be described in terms of one master field instead of a phenomenological field for each spin that is present in the family. The resonances being physical realizations of an $O(4)$ family of trajectories, one would expect the master field to be a field belonging to $O(4)$, or more accurately, $O(3,1)$ in the physical region.

The analytic continuation implied in going from $O(4)$ to $O(3,1)$ cures two defects of an $O(4)$ field theory: (i) the problem of nonpositive residues at an $O(4)$ pole (i.e., ghost states), and (ii) for $M\neq 0$ the problem of parity doubling within the multiplet.

The $O(3,1)$ multispin field theory that we discuss is interesting also from another point of view. The success as well as popularity of the Veneziano amplitude[5] has led to efforts at a deeper understanding of the dynamics of the Veneziano model. It appears that an $O(4)$ dual quark model leads to the Veneziano amplitude,[6–9] with all levels of the model having $M=0$.

Within the context of the dual quark model, the production of an excited state of the system is not forbidden. Since there are ghosts among the excited states, the quark model, in principle, can lead to production amplitudes for ghosts.

The point of view we take is that the success of the quark model suggests a dynamical reason for an $O(4)$ degeneracy of energy levels, but that the ghosts are due to an improper use of $O(4)$. At the present level of phenomenology, we are simply asking the following question: If such a family of resonances has been physically produced (no ghosts), what can be said simply about the system?

Phenomenologically, granted that a family of resonances has been produced, it is then a problem of how to describe the decay characteristics of the family. For the case of a particle of definite spin J, $O(3)$ tensors can be used very effectively to describe the angular distributions of the decay products.[10] In the present case, $O(4)$ tensors appropriately continued should be

[1] M. Toller, Nuovo Cimento **54A**, 295 (1968).
[2] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).
[3] G. Domokos, Phys. Rev. **159**, 1387 (1967).
[4] For an interesting discussion of this in connection with the Gell-Mann–Zweig model, see H. Harari, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).
[5] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[6] S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).
[7] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969).
[8] Y. Nambu, in Proceedings of the International Conference on Symmetries and Quark Models, Wayne State University, 1969 (unpublished).
[9] L. Susskind, Phys. Rev. Letters **23**, 545 (1969); Phys. Rev. D **1**, 1182 (1970).
[10] See the comprehensive article by C. Zemach, Phys. Rev. **140**, B97 (1965).

used. Since $O(3,1)$ tensors[11] are less well known than angular momentum tensors, we shall construct them in this paper.

This paper is organized as follows. In Sec. II we review briefly the spinor formalism for the description of particles with spin. This serves to define our notation as well. In Sec. III we describe a formal construction of $O(3,1)$ tensor fields which could be used for an $O(4)$ family of trajectories becoming physical. The propagators for the tensors are written down in Sec. IV. A compact form for the fully contracted propagator is given in terms of the Gegenbauer functions $C_n{}^1$, for $M=0$, and represents a particularly simple analytic continuation away from $s=0$. While our construction remains equally as simple for the $M\neq 0$ case as for the case $M=0$, the propagator at $p_\mu=0$ does not belong to one irreducible representation of the $O(4)$; it is reducible. This is because of our requirement that there be no parity doubling in the family of resonances that we deal with.

We believe that the A_2-meson multiplet is an example of the master fields we have constructed, belonging to $(n=2,\ M=0)$. In the following paper[12] we discuss the phenomenology of the A_2-meson system in terms of our master field.

II. REVIEW

It is helpful to review briefly the usual construction of the field for a particle of spin J.[13] It will serve to introduce our notations as well as set the stage for our generalization of the procedure.

We begin by recalling the canonical annihilation operators for particles with definite spin J, which under Lorentz transformations behave as $[p_\mu \equiv (\mathbf{p}, i\omega)]$

$$U(\Lambda)a(\mathbf{p};\,j,\sigma)U^\dagger(\Lambda) =\left(\frac{\omega'}{\omega}\right)^{1/2}\sum_{\sigma'} D_{\sigma\sigma'}{}^{(j)}(R_W{}^{-1}(\Lambda,p))a(\mathbf{p}';\,j,\sigma'),\quad (2.1)$$

where

$$p_\mu{}'=\Lambda_{\mu\nu}p_\nu$$

and σ is the component of spin along the z axis. $R_W(\Lambda,p)$ is the Wigner rotation

$$R_W(\Lambda,p)=L^{-1}(\Lambda p)\Lambda L(p) \quad (2.2)$$

and $L(p)$ is the boost

$$|\mathbf{p};\,j,\sigma\rangle=(m/\omega)^{1/2}U(L(p))|0;\,j,\sigma\rangle. \quad (2.3)$$

$D_{\sigma\sigma'}{}^{(j)}$ is the usual unitary representation of the rotation group.

To construct fields which transform locally under the Lorentz group, it is necessary to use explicitly representations of the full Lorentz group rather than the little group. For a finite-component field, nonunitary representations are used. For our purposes we shall use only $(j,0)$ and $(0,j)$ representations ($\mathbf{p}=\hat{n}m\sinh\theta$):

$$D_{\sigma\sigma'}{}^{(j,0)}(L(p))=[\exp(-\theta\hat{n}\cdot\mathbf{J})]_{\sigma\sigma'}\equiv D_{\sigma\sigma'}{}^{(j)}(\mathbf{p}), \quad (2.4)$$

$$D_{\sigma\sigma'}{}^{(0,j)}(L(p))=[\exp(\theta\hat{n}\cdot\mathbf{J})]_{\sigma\sigma'}\equiv \bar{D}_{\sigma\sigma'}{}^{(j)}(\mathbf{p}). \quad (2.5)$$

For arbitrary Lorentz transformations, the relation

$$D^{(j)}(\Lambda)=[\bar{D}^{(j)}(\Lambda^{-1})]^\dagger \quad (2.6)$$

holds. There is a charge conjugation matrix C which acting on a $(j,0)$ representation changes it to a $(0,j)$ representation and vice versa, viz.,

$$CD(\Lambda)C^{-1}=[\bar{D}(\Lambda)]^*. \quad (2.7)$$

The matrix C satisfies the well-known properties

$$C^\dagger C=1\,,\quad C^*C=(-)^{2j}. \quad (2.8)$$

The usual construction of fields proceeds to introduce spinors by considering the following linear combination of the canonical operators:

$$\mathcal{A}(\mathbf{p};\,j,\alpha)\equiv(\omega/m)^{1/2}\sum_\sigma D_{\alpha\sigma}{}^{(j)}(\mathbf{p})a(\mathbf{p};\,j,\sigma), \quad (2.9)$$

such that under Lorentz transformations they transform simply as

$$U(\Lambda)\mathcal{A}(\mathbf{p};\,j,\alpha)U^\dagger(\Lambda)=\sum_\beta D_{\alpha\beta}{}^{(j)}(\Lambda^{-1})\mathcal{A}(\mathbf{p}';\,j,\beta). \quad (2.10)$$

The field $\varphi_\alpha{}^{(j)}(x)$ is just the Fourier transform of (2.9) and the appropriate antiparticle creation operator. A notable feature of this construction is that under parity this field goes over into another field $\chi_\alpha{}^{(j)}(x)$ which transforms differently under Λ, viz., as the $(0,j)$ representation. For, under parity,

$$\mathcal{P}\mathcal{A}(\mathbf{p};\,j,\alpha)\mathcal{P}^{-1} =\eta(\omega/m)^{1/2}\sum_\sigma \bar{D}_{\alpha\sigma}{}^{(j)}(-\mathbf{p})a(-\mathbf{p};\,j,\sigma) \quad (2.11)$$

$$\equiv\eta\mathcal{A}(-\mathbf{p};\,j,\dot{\alpha}), \quad (2.12)$$

and under Λ, we have

$$U(\Lambda)\mathcal{A}(\mathbf{p};\,j,\dot{\alpha})U^\dagger(\Lambda)=\sum_\beta \bar{D}_{\alpha\beta}{}^{(j)}(\Lambda^{-1})\mathcal{A}(\mathbf{p}';\,j,\beta). \quad (2.13)$$

Since basically there is no parity doubling in the theory, the new fields $\chi_\alpha{}^{(j)}(x)$ are not independent of $\varphi_\alpha{}^{(j)}(x)$. The dependence between $\chi_\alpha{}^{(j)}(x)$ and $\varphi_\alpha{}^{(j)}(x)$ has been studied by Weinberg.[13] In his $2(2j+1)$-component field formalism, the dependence between φ_α and χ_α is in fact a "generalized" Dirac equation.

An alternative procedure for construction of fields is to use the tensorial basis. (For half-integer spins, the tensorial basis is added on to a basic Dirac field.) The

[11] We mean by an $O(3,1)$ tensor the field whose propagator at $p_\mu=0$ becomes an $O(4)$ propagator.

[12] N. P. Chang and C. A. Nelson, following paper, Phys. Rev. D 2, 966 (1970).

[13] S. Weinberg, Phys. Rev. **133**, B1318 (1964); **181**, 1893 (1969).

advantage of the tensorial construction is that under parity the tensors do not "double up." The construction of tensors from the basic canonical annihilation operators is easy and well known for a particle of spin j.[14] It is not so well known for an $O(4)$ system of particles with different spins.[15]

The situation may arise, if Regge trajectories form $O(4)$ families, that an $O(4)$ family with (n,M) quantum numbers becomes physical at a given energy where we should expect the system of particles to have spins from $n=A+B$ to minimum spin $M=|A-B|$. Such a system of particles will have to be described by a master field belonging to the (A,B) representation of the $O(3,1)$ group. It is the construction of such a master field that we shall discuss in Sec. III.

To complete our review, we note the quantization condition for the particles

$$[a(\mathbf{p};j,\sigma),a^*(\mathbf{p}';j,\sigma')]_\pm=\delta(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'}, \quad (2.14)$$

where the $\pm$ refers to the Fermi and Bose statistics, respectively. In terms of the spinor operators, the quantization condition becomes

$$[\mathfrak{a}(\mathbf{p};j,\alpha),\mathfrak{a}^*(\mathbf{p}';j,\beta)]_\pm =(\omega/m)\delta(\mathbf{p}-\mathbf{p}')\Pi_{\alpha\beta}{}^{(j)}(p)/(m^2)^j, \quad (2.15)$$

$$[\mathfrak{a}(\mathbf{p};j,\alpha),\mathfrak{a}^*(\mathbf{p}';j,\dot\beta)]_\pm=(\omega/m)\delta(\mathbf{p}-\mathbf{p}')\delta_{\alpha\beta}, \quad (2.16)$$

where

$$m^{-2j}\Pi_{\alpha\beta}{}^{(j)}(p)=[\exp(-2\theta\hat{n}\cdot\mathbf{J})]_{\alpha\beta} \quad (2.17)$$

$$\equiv(-)^{2j}(t_{\mu_1\mu_2\cdots\mu_{2j}})_{\alpha\beta}p_{\mu_1}p_{\mu_2}\cdots p_{\mu_{2j}}. \quad (2.18)$$

The matrices $(t_{\mu_1\mu_2\cdots})_{\alpha\beta}$ satisfy the following important properties (see Weinberg)[13]: (i) symmetry in $\mu_1, \mu_2, \ldots, \mu_{2j}$; (ii) traceless in any pair of μ indices; (iii) as matrices they satisfy

$$D^{(j)}(\Lambda)t_{\mu_1\cdots\mu_{2j}}D^{(j)}(\Lambda)^\dagger=\Lambda_{\mu_1\nu_1}{}^{-1}\cdots\Lambda_{\mu_{2j}\nu_{2j}}{}^{-1}t_{\nu_1\cdots\nu_{2j}}. \quad (2.19)$$

The matrices $\Pi^{(j)}$ are tabulated for $j\leq 3$ in Ref. 13, from which the matrices $t_{\mu_1\mu_2\cdots}$ can be directly calculated.

III. n=INTEGER TENSOR FIELDS

Let us imagine that we have a system of particles degenerate in mass but each having a different spin, with the spin content

$$S=n,\, n-1,\, \ldots,\, M. \quad (3.1)$$

For purposes of our discussion we may either suppose that these are the spin states of an $O(4)$ quark model or that these particles are an $O(4)$ family of Regge trajectories becoming physical. In either case we assume these states to be physical (no ghosts) and that they can be described by the annihilation operators

$$a(\mathbf{p};A,a;B,b), \quad (3.2)$$

where we set $n=A+B$, $M=|A-B|$. For definiteness, we take A to be greater than B.

Under Λ, the operators transform as

$$U(\Lambda)a(\mathbf{p};A,a;B,b)U^\dagger(\Lambda) =(\omega'/\omega)^{1/2}\sum_{a',b'}D_{aa'}{}^{(A)}(R_W{}^{-1})D_{bb'}{}^{(B)}(R_W{}^{-1}) \times a(\mathbf{p}';A,a';B,b'). \quad (3.3)$$

Since the transformation matrices are but rotation matrices, the operators of definite spin S can be projected from the operators (3.2) covariantly,

$$a(\mathbf{p};S,m)=\sum_{a,b}(-)^{A-B+m}\begin{pmatrix}A & B & S\\ a & b & -m\end{pmatrix}(2S+1)^{1/2} \times a(\mathbf{p};A,a;B,b), \quad (3.4)$$

where the $3j$ symbol is as defined in Edmonds[16] with the property

$$\sum_{a',b'}\begin{pmatrix}A & B & S\\ a' & b' & -m\end{pmatrix}D_{a'a}{}^{(A)}(R)D_{b'b}{}^{(B)}(R) =\sum_{m'}D_{mm'}{}^{(S)}(R)(-)^{m'-m}\begin{pmatrix}A & B & S\\ a & b & -m'\end{pmatrix}. \quad (3.5)$$

In the usual spinorial way of constructing fields, the procedure would be to form the operators

$$\mathfrak{a}(\mathbf{p};A,\alpha;B,\dot\beta) =(\omega/m)^{1/2}\sum_{a,b}D_{\alpha a}{}^{(A)}(\mathbf{p})\bar{D}_{\beta b}{}^{(B)}(\mathbf{p})a(\mathbf{p};A,a;B,b), \quad (3.6)$$

which transform under Λ like the (A,B) representation. The Fourier transform of these spinor operators, together with the appropriate antiparticle operators, then is the $\varphi^{(A,B)}$ field. Under parity, the fields go over into a new field $\varphi^{(B,A)}(x)$ which, however, is not independent of the original $\varphi^{(A,B)}$ field. This complicates somewhat the construction of the propagator for this field. As noted before, the tensorial construction does not suffer from this complication.

Consider the following pseudotensor combination of the operators (3.2):

$$\mathfrak{a}_{\mu_1\cdots\mu_n}(\mathbf{p})\equiv(\omega/m)^{1/2}\sum_{S=M}^{n}\sum_{a,b,a',b',m}(Ct_{\mu_1\cdots\mu_n})_{ab} \times\begin{pmatrix}\tfrac12 n & \tfrac12 n & S\\ a & b & m\end{pmatrix}\begin{pmatrix}A & B & S\\ a' & b' & m\end{pmatrix}(2S+1) \times a(\mathbf{p};Aa';Bb'). \quad (3.7)$$

[14] See, e.g., M. D. Scadron, Phys. Rev. **165**, 1640 (1968).

[15] The $O(4)$ propagators at $p_\mu=0$ have been constructed by H. F. Jones, Nuovo Cimento **59A**, 81 (1969).

[16] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957).

Under Λ, these operators transform as follows:

$$U(\Lambda)\mathcal{A}_{\mu_1\cdots\mu_n}(\mathbf{p})U^\dagger(\Lambda)=(\mathcal{R}^{-1})_{\mu_1\nu_1}\cdots(\mathcal{R}^{-1})_{\mu_n\nu_n}\mathcal{A}_{\nu_1\cdots\nu_n}(\mathbf{p}'), \quad (3.8)$$

where

$$\mathcal{R}_{\mu\nu}=[L^{-1}(\Lambda p)\Lambda L(p)]_{\mu\nu} \quad (3.9)$$

is the 4×4 matrix representation of the Wigner rotation. The ordinary tensor operators can be obtained from (3.7) by simply contracting the pseudotensors with the 4×4 matrix representation of the boost:

$$A_{\mu_1\cdots\mu_n}(\mathbf{p})=[L(p)]_{\mu_1\nu_1}\cdots[L(p)]_{\mu_n\nu_n}\mathcal{A}_{\nu_1\cdots\nu_n}(\mathbf{p}). \quad (3.10)$$

The matrices $[L(p)]_{\mu\nu}$ satisfy the simple properties

$$[L(p)]_{\mu\nu}[L(p)]_{\mu\lambda}=\delta_{\nu\lambda}, \quad p_\mu[L(p)]_{\mu\nu}=im\delta_{\nu 4}. \quad (3.11)$$

The pseudotensor components are equal to the tensor components in the rest frame. We use the pseudotensors merely as a convenience, although formally they look very much like $O(4)$ tensors. (They are not for $M\neq 0$; see below.)

By construction, the tensors $A_{\mu_1\mu_2\cdots\mu_n}(\mathbf{p})$ have the following expected tensor properties: (i) symmetry in all μ's; (ii) traceless for $n>1$,

$$\delta_{\mu_1\mu_2}A_{\mu_1\mu_2\mu_3\cdots\mu_n}(\mathbf{p})=0; \quad (3.12)$$

(iii) for $M=A-B\neq 0$,

$$(p_{\mu_1}\cdots p_{\mu_{n'}})A_{\mu_1\mu_2\cdots\mu_n}(\mathbf{p})=0, \quad n'=2B+1. \quad (3.13)$$

Properties (i) and (ii) are obvious, while (iii) needs to be shown. In terms of the pseudotensors, the property (iii) reads

$$\mathcal{A}_{4\cdots 4\mu_{n'+1}\cdots\mu_n}(\mathbf{p})=0, \quad n'=2B+1, M\geq 1. \quad (3.14)$$

Because of condition (3.14), the pseudotensor $A_{\mu_1\cdots\mu_n}(\mathbf{p})$ is not an $O(4)$ tensor.

Condition (3.14) follows immediately from the observation that the operator $A_{4\cdots 4\mu_{n'+1}\cdots\mu_n}(\mathbf{p})$ describes spins from $n-n'$, $n-n'-1$, down to 0, i.e., spins $=M-1$, $M-2$, ..., 0. But by construction those spins are absent from (3.7), and thus, those operators have to vanish.

Next we discuss the parity transformation. Our construction is general enough to allow for any set of intrinsic parities among the system of particles. The two sets of intrinsic parities that would be simplest from the tensor point of view are (i) when all the members of the multiple have the same spin and (ii) when the parities alternate from one spin to the next lower spin. The first case is trivial.

In the second case, the parity transformation law is

$$\mathcal{P}A_{\mu_1\cdots\mu_n}(\mathbf{p})\mathcal{P}^{-1}=\eta g_{\mu_1\nu_1}\cdots g_{\mu_n\nu_n}A_{\nu_1\cdots\nu_n}(-\mathbf{p}), \quad (3.15)$$

where $g_{lm}=\delta_{lm}=-g_{44}$, $g_{l4}=g_{4l}=0$. The parity transformation law for the pseudotensors is the same as (3.15), while in terms of the original operators (3.2), the law is

$$\mathcal{P}a(\mathbf{p};A,a;B,b)\mathcal{P}^{-1}=\eta\sum_{a',b',S,m}\begin{pmatrix}A & B & S\\ a & b & m\end{pmatrix}\begin{pmatrix}B & A & S\\ b' & a' & m\end{pmatrix}(2S+1)\times a(-\mathbf{p};A,a';B,b'). \quad (3.16)$$

Because of (3.16), the spinorial construction of fields in the case of an alternating sequence of parities is quite cumbersome.

Before going to Sec. IV, a remark on the particular choice of construction in (3.7) is perhaps in order. It is of course a trivial affair to write down general tensor fields which contain various spins and which satisfy the contraints (3.12) and (3.13). It is, however, not clear what combinations of nth-rank tensors formed out of varying spin-tensor fields would give rise to a propagator which when analytically continued to the point $s=0$ would correspond to an $O(4)$ propagator. The construction (3.7) has that property. It is in this sense that we call (3.7) an $O(3,1)$ tensor versus the $O(3)$ tensors for particles of definite spin even though the latter are also tensors with Lorentz indices.

IV. n=INTEGER PROPAGATORS

We shall take

$$[a(\mathbf{p};A,a;B,b),a^*(\mathbf{p}';A,a';B,b')]=\delta(\mathbf{p}-\mathbf{p}')\delta_{aa'}\delta_{bb'} \quad (4.1)$$

to be the basic rule for quantization for our system of particles. This rule of quantization guarantees that there are no ghosts present in the multiplet.

From (4.1) the commutation rule for the pseudotensors becomes, formally,

$$[\mathcal{A}_{\mu_1\cdots\mu_n}(\mathbf{p}),\mathcal{A}_{\nu_1\cdots\nu_n}{}^*(\mathbf{p}')]=(\omega/m)\delta(\mathbf{p}-\mathbf{p}')\Delta_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_n}{}^{(n,M)}, \quad (4.2)$$

where

$$\Delta_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_n}{}^{(n,M)}=\sum_{S=M}^{n}\sum_{a',b',a,b}(Ct_{\mu_1\cdots\mu_n})_{ab}\times\begin{pmatrix}\frac{1}{2}n & \frac{1}{2}n & S\\ a & b & m\end{pmatrix}(2S+1)\begin{pmatrix}\frac{1}{2}n & \frac{1}{2}n & S\\ a' & b' & m\end{pmatrix}\times(t_{\nu_1\cdots\nu_n}{}^\dagger C)_{b'a'}. \quad (4.3)$$

For $M=0$, the commutator reduces to

$$[\mathcal{A}_{\mu_1\cdots\mu_n}(\mathbf{p}),\mathcal{A}_{\nu_1\cdots\nu_n}{}^*(\mathbf{p}')]=(\omega/m)\delta(\mathbf{p}-\mathbf{p}')\sum_a(t_{\mu_1\cdots\mu_n}t_{\nu_1\cdots\nu_n}{}^\dagger)_{aa}. \quad (4.4)$$

For a self-conjugate system of particles, a local field can be constructed ($A_\mu^\star \equiv g_{\mu\nu}A_\nu^*$):

$$A_{\mu_1\cdots\mu_n}(x) = (2\pi)^{-3/2}(\tfrac{1}{2}m)^{1/2} \times \int \frac{d^3p}{\omega}[A_{\mu_1\cdots\mu_n}(\mathbf{p})e^{ip\cdot x} + A_{\mu_1\cdots\mu_n}^\star(\mathbf{p})e^{-ip\cdot x}]. \quad (4.5)$$

The propagator for this field is, in momentum space,

$$\frac{1}{(2\pi)^4 i}\frac{S_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_n}{}^{(n,M)}(p;\,m^2)}{p^2+m^2-i\epsilon}, \quad (4.6)$$

where

$$S_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_n}{}^{(n,M)} = L_{\mu_1\lambda_1}\cdots L_{\mu_n\lambda_n}(Lg)_{\rho_1\nu_1}\cdots(Lg)_{\rho_n\nu_n} \times \Delta_{\lambda_1\cdots\lambda_n,\rho_1\cdots\rho_n}{}^{(n,M)}, \quad (4.7)$$

with

$$L_{\mu\nu} \equiv [L(p)]_{\mu\nu}.$$

It is important to note that the $\Delta^{(n,M)}$ are a set of Kronecker δ's and for the $M=0$ case are in fact the $O(4)$ propagators that are known. For our derivation it has been convenient to introduce the pseudotensors for precisely the reason that their propagators look like $O(4)$ propagators.

The contraction with the boost matrices can be done by the following set of rules for replacing the Kronecker δ's in $\Delta^{(n,M)}$.

Replace

$$\begin{aligned} \delta_{l_il_j} &\text{ by } \delta_{\lambda_i\lambda_j}+p_{\lambda_i}p_{\lambda_j}/m^2, \\ \delta_{\lambda_i4}\delta_{\lambda_j4} &\text{ by } -p_{\lambda_i}p_{\lambda_j}/m^2, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \delta_{l_ir_j} &\text{ by } \delta_{\lambda_i\rho_j}+p_{\lambda_i}p_{\rho_j}/m^2, \\ \delta_{\lambda_i4}\delta_{\rho_j4} &\text{ by } +p_{\lambda_i}p_{\rho_j}/m^2, \end{aligned} \quad (4.9)$$

and for the Kronecker δ's in $\rho_i\rho_j$, use the same rules as in (4.8). These rules follow easily from an explicit representation of the boost matrix

$$L_{\mu4} = -ip_\mu/m, \quad L_{ij} = \delta_{ij} + p_ip_j/[m(\omega+m)]. \quad (4.10)$$

The simplicity of the rules (4.8) and (4.9), as well as the propagator (4.6), can best be appreciated by considering the fully contracted propagator

$$D^{(n,M)}(P,Q;\,p;\,m^2) = P_{\mu_1}\cdots P_{\mu_n}S_{\mu_1\cdots\mu_n,\nu_1\cdots\nu_n}{}^{(n,M)}Q_{\nu_1}\cdots Q_{\nu_n} \equiv P^n\!:S^{(n,M)}\!:Q^n. \quad (4.11)$$

The full propagator can be obtained by differentiation.

Consider first the case $M=0$. Then $\Delta^{(n,0)}$ is a set of four-dimensional Kronecker δ's and the fully contracted propagator in that case is[17,18]

$$P^n\!:\Delta^{(n,0)}\!:Q^n = (PQ)^n 2^{-n} C_n{}^1(P\cdot Q/PQ), \quad (4.12)$$

where

$$PQ \equiv (P^2Q^2)^{1/2}. \quad (4.13)$$

By the rules (4.8) and (4.9), we find immediately

$$D^{(n,0)}(P,Q;\,p;\,m^2) = (PQ)^n 2^{-n} C_n{}^1(z), \quad (4.14)$$

$$z = (P\cdot Q + 2P\cdot pQ\cdot p/m^2)/PQ. \quad (4.15)$$

$C_n{}^1$ is the Gegenbauer function,[19] also known as the Tschebyscheff polynomials.

An important property of (4.14) and (4.15) is that while by construction the propagator describes a family of spins $n, n-1, \ldots, 0$ on the mass shell $p^2=-m^2$, it at the same time gives a very easy continuation off the mass shell to the point $p_\mu=0$ where the propagator is an $O(4)$ $M=0$ propagator.

The spin content of the propagator (4.14) can be exhibited in terms of the corresponding contracted propagator for a particle of definite spin J:

$$\Delta_J(P,Q;\,p;\,m^2) = (pq)^J \mathcal{P}_J(x), \quad (4.16)$$

$$pq \equiv \left[\left(P^2 + \frac{(P\cdot p)^2}{m^2}\right)\left(Q^2+\frac{(Q\cdot p)^2}{m^2}\right)\right]^{1/2}, \quad (4.17)$$

$$x = (P\cdot Q + P\cdot pQ\cdot p/m^2)/(pq),$$

$$\mathcal{P}_J(x) \equiv [2^J(J!)^2/(2J)!]P_J(x) \xrightarrow[x\to\infty]{} x^J + O(x^{J-2}). \quad (4.18)$$

By the addition theorem for Gegenbauer functions,[20] we find

$$D^{(n,0)}(P,Q;\,p;\,m^2) = \sum_{J=0}^{n} \Delta_J(P,Q;\,p;\,m^2)(PQ)^{n-J} \times \mathcal{C}_{n-J}{}^{J+1}\!\left(\frac{P\cdot p}{m(-P^2)^{1/2}}\right)\mathcal{C}_{n-J}{}^{J+1}\!\left(\frac{Q\cdot p}{m(-Q^2)^{1/2}}\right) \times [c_{n,J}]^2, \quad (4.19)$$

with

$$[c_{n,J}]^2 = 2^{n-J}\frac{(2J+1)!}{(n+J+1)!(J!)^2}\frac{(n!)^2}{(n-J)!} \quad (4.20)$$

and

$$\mathcal{C}_{n-J}{}^{J+1}(x) \equiv 2^{J-n}[(n-J)!J/n!] \times C_{n-J}{}^{J+1}(x) \xrightarrow[x\to\infty]{} x^n + O(x^{n-2}). \quad (4.21)$$

The meaning of the relation (4.19) will be made clear when we consider the construction of these $O(3,1)$ tensors from a more general point of view. As is well known, there are many nth-rank tensors one can write that have spin content $n, n-1, \ldots, M$. To each spin-J tensor we can attribute an arbitrary weight $c_{n,J}$. The full propagator of such a collection of spins will depend on the coefficients $[c_{n,J}]^2$. Not all such coefficients will

[17] R. Delbourgo, K. Koller, and R. M. Williams, J. Math. Phys. **10**, 957 (1969).

[18] H. J. Jones (Ref. 15). See also Y. Iwasaki, Phys. Rev. **173**, 1608 (1968).

[19] Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. II, Sec. 10.9.

[20] Reference 19, Vol. I, Eq. (3.15.19).

have the property that at $p_\mu=0$ the propagator be identified with $O(4)$ propagators. The coefficients (4.20) will. Of course, to make this remark meaningful we would have to agree on a normalization for the tensors.

Following the notation of Zemach,[10] let

$$T^{(n)}(p\cdots pH^{(l)})$$

be an $O(4)$ tensor in the sense of (3.12) and (3.13), where $H^{(l)}$ is the usual spin-l tensor:

$H_{\mu_1\cdots\mu_l}{}^{(l)}(p)$:

(i) symmetric and traceless in any pair of μ's,

(ii) $p_{\mu_1}H_{\mu_1\mu_2\cdots\mu_l}{}^{(l)}(p)=0$. (4.22)

Then our normalization is $[P\equiv(-P^2)^{1/2}]$

$$P^n\colon T^{(n)}(p\cdots pH^{(l)}) = P^l\colon H^{(l)}(P)^{n-l}\mathcal{C}_{n-l}{}^{l+1}(P\cdot p/mP). \quad (4.23)$$

A few examples will make this transparent:

$n=2,\ l=1$:

$$T_{\mu\nu}{}^{(2)}(pH^{(1)})=(1/2m)(p_\mu H_\nu{}^{(1)}+p_\nu H_\mu{}^{(1)}); \quad (4.24)$$

$n=3,\ l=2$:

$$T_{\mu\nu\lambda}{}^{(3)}(pH^{(2)}) =(1/3m)(p_\mu H_{\nu\lambda}{}^{(2)}+p_\nu H_{\mu\lambda}{}^{(2)}+p_\lambda H_{\mu\nu}{}^{(2)}); \quad (4.25)$$

$n=4,\ l=2$:

$$T_{\mu\nu\lambda\rho}{}^{(4)}(ppH^{(2)}) =(1/6m^2)(p_\mu p_\nu H_{\lambda\rho}{}^{(2)}+\text{permutations}) -(p^2/48m^2)(\delta_{\mu\nu}H_{\lambda\rho}{}^{(2)}+\text{permutations}). \quad (4.26)$$

Thus, if we now display the spin content of (3.10) by writing

$$A_{\mu_1\cdots\mu_n}(\mathbf{p})=H_{\mu_1\cdots\mu_n}{}^{(n)}(\mathbf{p})+c_{n,n-1}T_{\mu_1\cdots\mu_n}{}^{(n)}(pH^{(n-1)}) +c_{n,n-2}T_{\mu_1\cdots\mu_n}{}^{(n)}(ppH^{(n-2)})+\cdots, \quad (4.27)$$

then the fully contracted propagator due to (4.27) would be precisely that given by (4.19).

So far we have considered the case $M=0$. The generalization to $M\neq0$ is easy. The only change is in the spin sum in (4.19),

$$D^{(n,M)}(P,Q;p;m^2)=\sum_{J=M}^{n}\Delta_J(P,Q;p;m^2)(PQ)^{n-J} \times\mathcal{C}_{n-J}{}^{J+1}\left(\frac{P\cdot p}{mP}\right)\mathcal{C}_{n-J}{}^{J+1}\left(\frac{Q\cdot p}{mQ}\right)[c_{n,J}]^2, \quad (4.28)$$

with $c_{n,J}$ as given by (4.20).

For $M\neq0$, this propagator when continued to $p_\mu=0$ does not become the corresponding $O(4)$ $M\neq0$ propagator. This is not surprising since it is known that the $O(4)$ $M\neq0$ propagators contain parity doubling, and here, by construction, we do not have parity doublets in the family.

What actually happens is that instead of one $(n,\ M\neq0)$ family of $O(4)$ trajectories making their appearance, there are several $O(4)$ families, the parent $(n,0)$ trajectory with daughters $(0,0)$, $(1,0)$, $\ldots$, $(M-1,\ 0)$ which remove the spins 0, 1, 2, $\ldots$, $M-1$ on shell. This will become clear in an example in Sec. V.

For $M\neq0$, therefore, our $O(3,1)$ tensor does not belong at $p_\mu=0$ to an irreducible representation of $O(4)$; it is reducible under $O(4)$.

V. EXAMPLES OF TENSOR FIELDS

It is useful to write down a few examples of the $O(3,1)$ tensors that we have constructed. In the following we display the spin content by using $\varphi(x)$, $V_\mu(x)$, and $T_{\mu\nu}(x)$ as symbols for spin-0, -1, and -2 fields, respectively. The propagators in momentum space are written down in the uncontracted form for easy reference.

$n=1,\ M=1$:

$$A_\mu{}^{(1,1)}(x)=V_\mu(x), \quad (5.1)$$

$$S_{\mu\nu}{}^{(1,1)}(p)=\delta_{\mu\nu}+p_\mu p_\nu/m^2\equiv D_{\mu\nu}(p); \quad (5.2)$$

$n=1,\ M=0$:

$$A_\mu{}^{(1,0)}(x)=V_\mu(x)+(1/m)\partial_\mu\varphi(x), \quad (5.3)$$

$$S_{\mu\nu}{}^{(1,0)}(p)=\delta_{\mu\nu}+2p_\mu p_\nu/m^2; \quad (5.4)$$

$n=2,\ M=2$:

$$A_{\mu\nu}{}^{(2,2)}(x)=T_{\mu\nu}(x), \quad (5.5)$$

$$S_{\mu\nu,\alpha\beta}{}^{(2,2)}(p)=\tfrac{1}{2}(D_{\mu\alpha}D_{\nu\beta}+D_{\nu\alpha}D_{\mu\beta})-\tfrac{1}{3}D_{\mu\nu}D_{\alpha\beta}; \quad (5.6)$$

$n=2,\ M=1$:

$$A_{\mu\nu}{}^{(2,1)}(x)=T_{\mu\nu}(x)+(1/\sqrt{2}m) \times(\partial_\mu V_\nu(x)+\partial_\nu V_\mu(x)), \quad (5.7)$$

$$S_{\mu\nu,\alpha\beta}{}^{(2,1)}(p)=\tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\nu\alpha}\delta_{\mu\beta})-\tfrac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta} +(1/m^2)(p_\mu p_\alpha\delta_{\nu\beta}+p_\nu p_\alpha\delta_{\mu\beta} +p_\mu p_\beta\delta_{\nu\beta}+p_\nu p_\beta\delta_{\mu\alpha}) -(1/3m^2)(p_\mu p_\nu\delta_{\alpha\beta}+p_\alpha p_\beta\delta_{\mu\nu}) +(8/3)p_\mu p_\nu p_\alpha p_\beta/m^4 \quad (5.8)$$

$$=S_{\mu\nu,\alpha\beta}{}^{(2,2)}(p)+(1/2m^2)(p_\mu p_\alpha D_{\nu\beta} +p_\nu p_\alpha D_{\mu\beta}+p_\nu p_\beta D_{\mu\alpha}+p_\mu p_\beta D_{\nu\alpha}); \quad (5.9)$$

$n=2,\ M=0$:

$$A_{\mu\nu}{}^{(2,0)}(x)=T_{\mu\nu}(x) +(1/\sqrt{2}m)(\partial_\mu V_\nu(x)+\partial_\nu V_\mu(x)) +(2/\sqrt{3})(\partial_\mu\partial_\nu-\tfrac{1}{4}\delta_{\mu\nu}\partial^2)\varphi(x), \quad (5.10)$$

$$S_{\mu\nu,\alpha\beta}{}^{(2,0)}(p)=\tfrac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\nu\alpha}\delta_{\mu\beta})-\tfrac{1}{4}\delta_{\mu\nu}\delta_{\alpha\beta} +(p_\mu p_\alpha\delta_{\nu\beta}+p_\nu p_\alpha\delta_{\mu\beta}+p_\nu p_\beta\delta_{\mu\alpha} +p_\mu p_\beta\delta_{\nu\alpha})/m^2+4p_\mu p_\nu p_\alpha p_\beta/m^4 \quad (5.11)$$

$$=S_{\mu\nu,\alpha\beta}{}^{(2,1)}(p)+\tfrac{4}{3}(p_\mu p_\nu+\tfrac{1}{4}\delta_{\mu\nu}m^2) \times(p_\alpha p_\beta+\tfrac{1}{4}m^2\delta_{\alpha\beta})/m^4. \quad (5.12)$$

The fully contracted propagator for the $n=2$, $M=0$ case reads

$$D^{(2,0)}(P,Q;p;m^2)=(P\cdot Q+2P\cdot pQ\cdot p/m^2)^2-\tfrac{1}{4}P^2Q^2, \quad (5.13)$$

while the fully contracted propagator for the $n=2$, $M=1$ case is the difference between two $(n,0)$ propagators,

$$D^{(2,1)}(P,Q;p;m^2)=D^{(2,0)}(P,Q;p;m^2)-(4/3m^4)[(P\cdot p)^2+\tfrac{1}{4}P^2m^2]\times[(Q\cdot p)^2+\tfrac{1}{4}Q^2m^2]D^{(0,0)} \quad (5.14)$$

$$=D^{(2,0)}(P,Q;p;m^2)-\tfrac{4}{3}\mathcal{C}_2{}^1(P\cdot p/mP)\times\mathcal{C}_2{}^1(Q\cdot p/mQ)D^{(0,0)}(P,Q;p;m^2). \quad (5.15)$$

Note that for the $M\neq 0$ case there are daughter n trajectories which remain ghosts on the mass shell and serve to "remove" spin 0, 1, ..., M from the family, on the mass shell.

VI. FIELD LAGRANGIAN

The fields $A_{\mu_1\cdots\mu_n}(x)$ satisfy very simple field equations, viz.,

$$(\partial^2-m^2)A_{\mu_1\cdots\mu_n}(x)=j_{\mu_1\cdots\mu_n}(x). \quad (6.1)$$

$j_{\mu_1\cdots\mu_n}(x)$ is the source current for the field. For $M=0$, there are *no further* differential equations that the fields must satisfy. The quantum field theory is therefore very simple and free of worries about interactions causing spontaneous breakdown of the subsidiary conditions[21]:

$$\mathcal{L}=-\tfrac{1}{2}\partial_\nu A_{\mu_1\cdots\mu_n}{}^\star\partial_\nu A_{\mu_1\cdots\mu_n}-\tfrac{1}{2}m^2A_{\mu_1\cdots\mu_n}{}^\star A_{\mu_1\cdots\mu_n}+\mathcal{L}_{\text{int}}. \quad (6.2)$$

Typical examples of $\mathcal{L}_{\text{int}}$ will illustrate the simplicity of this field theory compared with most phenomenological field theories of resonances. Suppose we deal with the A_2-meson multiplet and assume it to be an (2,0) multiplet. Then the minimal coupling between the A_2 system and $\gamma\rho$, for instance, would be uniquely given by

$$gA_{\mu\lambda}{}^\star(x)F_{\mu\nu}(x)G_{\lambda\nu}(x)+\text{H.c.},\quad F_{\mu\nu}\equiv\partial_\nu A_\mu-\partial_\mu A_\nu,\quad G_{\lambda\nu}\equiv\partial_\lambda\rho_\mu-\partial_\mu A_\lambda, \quad (6.3)$$

A_μ, ρ_μ being the photon and ρ-meson fields, respectively.

[21] There are many treatments of field theory for higher spins. See Ref. 12, and W. K. Tung, Phys. Rev. **156**, 1385 (1967).

This coupling is reminiscent of $SU(12)_{\mathcal{L}}$ coupling[22,23] for it fixes the relative couplings of each of the mesons in the multiplet 2^+, 1^-, and 0^+. A phenomenological field theory of resonance would have assigned a spin tensor to each particle and the relative coupling between 2^+, 1^-, and 0^+ would have been arbitrary.

It is this simplifying aspect of what we call "$O(4)$" field theory that we explore with regard to the A_2-meson system in a following paper.[12]

Of course, there will be occasions where even the minimal coupling between the master field and the decay-product fields is not unique. (This happens in the $A_2\to\pi\rho$ decays.) But in no case will using the master field introduce more arbitrary coupling constants than in a phenomenological theory.

For the case $M\neq 0$, the field $A_{\mu_1\cdots\mu_n}(x)$ will satisfy an additional equation

$$\partial_{\mu_1}\cdots\partial_{\mu_{n'}}A_{\mu_1\cdots\mu_{n'}\mu_{n'+1}\cdots\mu_n}(x)=0,\quad n'=n-M+1. \quad (6.4)$$

This kind of subsidiary condition has appeared many times before in quantum field theories of arbitrary spin.[21] As a Feynman rule, this subsidiary condition has been taken into account in the construction of the free propagator. Higher-order self-energy corrections to the propagator may not, in general, preserve the condition, but we have not studied this in detail.

Finally, we remark on the comparison between this field theory without ghosts and a field theory with ghosts. The propagators for our $A_{\mu_1\cdots\mu_n}(x)$ fields diverge as badly as before in higher-spin field theories. It is thus an unrenormalizable theory. On the other hand, if we had let the field $A_{\mu_1\cdots\mu_n}$ have ghosts, the propagators would behave like $(1/p)^2$ as $p_\mu\to\infty$. Such a field theory, formally, is convergent. The choice between the two may be a matter of taste; on the phenomenological level at which we shall be using the theory, however, it would appear to be best to use physical (no ghosts) propagators for the $O(4)$ family produced in a reaction.

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[22] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); A. Pais, *ibid.* **13**, 175 (1964); F. Gürsey, A. Pais, and L. A. Radicati, *ibid.* **13**, 299 (1964); B. Sakita, *ibid.* **13**, 643 (1964); Phys. Rev. **136**, B1759 (1964); A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. (London) **A284**, 146 (1965).

[23] Other examples of multispin fields can be found in the work by J. Schwinger, Phys. Rev. **140**, B158 (1965). See also S. J. Chang, Phys. Rev. **178**, 2421 (1969).