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Veneziano Model for Nucleon-Nucleon Scattering*†

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A Veneziano model is constructed for nucleon-nucleon scattering. Each parent trajectory in the model has no parity or isospin doublets, and its residues satisfy the factorization theorem and positivity conditions.

I. INTRODUCTION

A MODEL for scattering amplitudes that combines, in a natural way, crossing symmetry, narrow resonances, and Regge asymptotic behavior has been recently proposed by Veneziano.[1] So far this model has been discussed mainly in connection with reactions involving only bosons. In this paper a Veneziano model is constructed for the contribution to nucleon-nucleon scattering of three exchange-degenerate trajectories—the (ω,f^0), the (ρ,A_2), and the (π,B). The model contains nonleading terms, as was found to be necessary by Jacobs,[2] but the residue functions of each parent trajectory depend on only two arbitrary parameters.

Since "duality diagrams"[3] without exotic resonances cannot be drawn for nucleon-nucleon scattering, one might believe that it could not be described by the Veneziano model. This belief is not really justified, since Khuri[4] has shown that a large class of amplitudes exhibiting crossing symmetry, narrow resonances in two channels, and Regge asymptotic behavior can be expressed as a uniformly convergent sum of Veneziano-type terms. It is true, however, that the model cannot be extended to octet-octet scattering in an $SU(3)$-symmetric way. In particular, both the s and the u channels are exotic for the reaction $\Sigma^+\Sigma^-\rightarrow\Sigma^+\Sigma^-$, so it cannot be described by any form of dual resonance model. Consideration of the nucleon-nucleon problem nevertheless appears to be worthwhile.

Sections II and III of this paper are devoted to the necessary preliminaries. In Sec. II the helicity and invariant amplitudes for nucleon-nucleon scattering are reviewed. In Sec. III the Veneziano model is discussed briefly, and its partial-wave projection is calculated. The actual models for the (ω,f^0), the (ρ,A_2), and the (π,B) exchange-degenerate trajectories are constructed in Sec. IV. Since resonances occur only in the two identical nucleon-antinucleon channels, each trajectory can be and is treated separately. The Pauli principle is maintained exactly, and the three parent trajectories are free of parity and isospin doublets and have residues which satisfy the factorization theorem and positivity conditions. In the model for the (π,B) trajectory, a positive-parity, isospin-1 conspirator is also included. While it must be degenerate with the (π,B) trajectory, its residues vanish at $\alpha=0$, so that no low-mass positive-parity resonance is required. The conspirator can be removed by setting a certain parameter equal to zero.

The model is compared with experiment in Sec. V. Its contribution to the cross section at high energies and large angles is negligible compared to the experimental values, presumably indicating that only the Pomeranchuk or Regge cuts are important there.

II. KINEMATICS

Let the s channel, having initial momenta p_a and p_b and final momenta p_c and p_d, correspond to nucleon-nucleon scattering. Then the t and u channels both correspond to nucleon-antinucleon scattering. The Mandelstam variables are

$$s=(p_a+p_b)^2,\quad t=(p_a-p_c)^2,\quad u=(p_a-p_d)^2. \tag{1}$$

They are not independent, but satisfy

$$s+t+u=4m^2, \tag{2}$$

where m is the nucleon mass. It is useful to introduce also the three-momentum k_s and scattering angle θ_s in the center-of-mass frame of the s channel. Let

$$z_s=\cos\theta_s; \tag{3}$$

then

$$k_s^2=\tfrac{1}{4}s-m^2,\quad z_s=1+2t/(s-4m^2). \tag{4}$$

The corresponding quantities in the t and u channels are

$$\begin{aligned} k_t^2&=\tfrac{1}{4}t-m^2, & z_t&=1+2u/(t-4m^2),\\ k_u^2&=\tfrac{1}{4}u-m^2, & z_u&=1+2t/(u-4m^2). \end{aligned} \tag{5}$$

Since each nucleon's helicity can assume the values $\pm\frac{1}{2}$, there are for each isospin 16 possible s-channel helicity amplitudes[5,6] $f_{cd,ab}{}^s(s,t)$, the subscripts denoting the corresponding helicities. (The isospin label will be suppressed temporarily.) If for the single-particle states the covariant normalization

$$\langle \mathbf{p},\lambda\,|\,\mathbf{p}',\lambda'\rangle=(\mathbf{p}^0/m)(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'} \tag{6}$$

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‡ National Science Foundation Predoctoral Fellow.
[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] M. A. Jacobs, Phys. Rev. **184**, 1574 (1969).
[3] H. Harari, Phys. Rev. Letters **22**, 562 (1969).
[4] N. N. Khuri, Phys. Rev. **185**, 1876 (1969).
[5] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **1**, 404 (1959).
[6] K. Huang and S. Pinsky, Phys. Rev. **174**, 1915 (1968); **181**, 2154(E) (1969).

is used, then the helicity amplitudes are related to the unpolarized cross section by

$$\frac{d\sigma}{d\Omega}=\frac{m^4}{16\pi^2 s}\sum_{cd,ab}|f_{cd,ab}{}^s(s,t)|^2. \tag{7}$$

The helicity amplitudes have the partial-wave expansion

$$f_{cd,ab}{}^s(s,t)=\sum_{J=\lambda_m}^{\infty}(2J+1)\langle cd|F^J(s)|ab\rangle d_{\lambda\mu}{}^J(\theta_s),$$
$$\lambda=a-b,\quad \mu=c-d,\quad \lambda_m=\max(|\lambda|,|\mu|), \tag{8}$$

where $d_{\lambda\mu}{}^J(\theta_s)$ is the usual representation of a rotation about the y axis and $\langle cd|F^J(s)|ab\rangle$ is the s-channel partial-wave amplitude.

Invariance of the S matrix under parity, time reversal, and the interchange of identical particles implies that only five of the helicity amplitudes are independent. These five are conventionally taken to be[7]

$$\begin{aligned} f_1{}^s(s,t)&=f_{++,++}{}^s(s,t),\\ f_2{}^s(s,t)&=f_{++,--}{}^s(s,t),\\ f_3{}^s(s,t)&=f_{+-,+-}{}^s(s,t),\\ f_4{}^s(s,t)&=f_{+-,-+}{}^s(s,t),\\ f_5{}^s(s,t)&=f_{++,+-}{}^s(s,t).\end{aligned} \tag{9}$$

The others are then given by

$$\begin{aligned} &f_1{}^s=f_{--,--}{}^s,\\ &f_2{}^s=f_{--,++}{}^s,\\ &f_3{}^s=f_{-+,-+}{}^s,\\ &f_4{}^s=f_{-+,+-}{}^s,\\ &f_5{}^s=f_{--,+-}{}^s=f_{-+,--}{}^s=f_{-+,++}{}^s=-f_{--,-+}{}^s\\ &\quad=-f_{++,-+}{}^s=-f_{+-,++}{}^s=-f_{+-,--}{}^s,\end{aligned} \tag{10}$$

so that the unpolarized cross section is

$$\frac{d\sigma}{d\Omega}=\frac{m^4}{8\pi^2 s}[|f_1{}^s|^2+|f_2{}^s|^2+|f_3{}^s|^2+|f_4{}^s|^2+4|f_5{}^s|^2]. \tag{11}$$

The t-channel helicity amplitudes are denoted by $f_{cA,Db}{}^t(s,t)$, the capital letters indicating antinucleons. Their partial-wave expansion is

$$f_{cA,Db}{}^t(s,t)=\sum_{J=\lambda_m}^{\infty}(2J+1)\langle cA|G^J(t)|Db\rangle d_{\lambda\mu}{}^J(\theta_t),$$
$$\lambda=D-b,\quad \mu=c-A,\quad \lambda_m=\max(|\lambda|,|\mu|). \tag{12}$$

There are again five independent amplitudes $f_i{}^t(s,t)$, which are defined analogously to $f_i{}^s(s,t)$ in (9).

The helicity states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$ with definite angular momentum J appearing in (12) are not eigenstates of parity. It is useful to introduce linear combinations $|\lambda\pm\rangle$ of them, namely,[6,8]

$$\begin{aligned} |0+\rangle&=(|++\rangle+|--\rangle)/\sqrt{2},\\ |0-\rangle&=(|++\rangle-|--\rangle)/\sqrt{2},\\ |1+\rangle&=(|+-\rangle+|-+\rangle)/\sqrt{2},\\ |1-\rangle&=(|+-\rangle-|-+\rangle)/\sqrt{2}.\end{aligned} \tag{13}$$

TABLE I. $N\bar{N}$ helicity states.

State	Parity	G parity	Known trajectories $I=0$	Known trajectories $I=1$
$\|0+\rangle$	$+(-1)^J$	$+(-1)^{I+J}$	P,ω,f^0	ρ,A_2
$\|0-\rangle$	$-(-1)^J$	$+(-1)^{I+J}$	η	π,B
$\|1+\rangle$	$+(-1)^J$	$+(-1)^{I+J}$	P,ω,f^0	ρ,A_2
$\|1-\rangle$	$-(-1)^J$	$-(-1)^{I+J}$	$\cdots$	A_1

The good quantum numbers of these states and the known Regge trajectories coupled to them are listed in Table I. Since the only allowed off-diagonal transition is that between $|0+\rangle$ and $|1+\rangle$, one can define five new partial-wave amplitudes with good parity and G parity:

$$G_{\lambda\mu}{}^{J\pm}(t)=\langle\lambda\pm|G^J(t)|\mu\pm\rangle. \tag{14}$$

In terms of these,

$$\begin{aligned} \langle++|G^J|++\rangle&=\tfrac{1}{2}(G_{00}{}^{J+}+G_{00}{}^{J-}),\\ \langle++|G^J|--\rangle&=\tfrac{1}{2}(G_{00}{}^{J+}-G_{00}{}^{J-}),\\ \langle+-|G^J|+-\rangle&=\tfrac{1}{2}(G_{11}{}^{J+}+G_{11}{}^{J-}),\\ \langle+-|G^J|-+\rangle&=\tfrac{1}{2}(G_{11}{}^{J+}-G_{11}{}^{J-}),\\ \langle++|G^J|+-\rangle&=\tfrac{1}{2}G_{10}{}^{J+}.\end{aligned} \tag{15}$$

The parity-conserving helicity amplitudes $g_{iI}{}^t(s,t)$ are defined by their partial-wave expansions[6,8]:

$$\begin{aligned} g_{1I}{}^t(s,t)&=\sum_{J=0}^{\infty}(2J+1)G_{00,I}{}^{J+}(t)e_{00}{}^{J+}(z_t),\\ g_{2I}{}^t(s,t)&=\sum_{J=0}^{\infty}(2J+1)G_{00,I}{}^{J-}(t)e_{00}{}^{J+}(z_t),\\ g_{3I}{}^t(s,t)&=\sum_{J=1}^{\infty}(2J+1)[G_{11,I}{}^{J+}(t)e_{11}{}^{J+}(z_t)+G_{11,I}{}^{J-}(t)e_{11}{}^{J-}(z_t)],\\ g_{4I}{}^t(s,t)&=\sum_{J=1}^{\infty}(2J+1)[G_{11,I}{}^{J-}(t)e_{11}{}^{J+}(z_t)+G_{11,I}{}^{J+}(t)e_{11}{}^{J-}(z_t)],\\ g_{5I}{}^t(s,t)&=\sum_{J=1}^{\infty}(2J+1)G_{10,I}{}^{J+}(t)e_{10}{}^{J+}(z_t),\end{aligned} \tag{16}$$

where the isotropic spin label I has been restored. The functions $e_{\lambda\mu}{}^{J\pm}(z_t)$ are polynomials in z_t related to

[7] M. L. Goldberger, M. J. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[8] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

$d_{\lambda\mu}{}^J(\theta_t)$ by the equations

$$d_{\lambda\mu}{}^J(\theta_t)=(1+z_t)^{\frac{1}{2}|\lambda+\mu|}(1-z_t)^{\frac{1}{2}|\lambda-\mu|}\times[e_{\lambda\mu}{}^{J+}(z_t)+e_{\lambda\mu}{}^{J-}(z_t)],$$
$$e_{\lambda\mu}{}^{J\pm}(-z_t)=\pm(-1)^{J-\lambda_m}e_{\lambda\mu}{}^{J\pm}(z_t),\quad \lambda_m=\max(|\lambda|,|\mu|).\tag{17}$$

Explicitly,[8]

$$\begin{aligned}e_{00}{}^{J+}(z_t)&=P_J(z_t),\\ e_{11}{}^{J+}(z_t)&=[P_J'(z_t)+z_tP_J''(z_t)]/[J(J+1)],\\ e_{11}{}^{J-}(z_t)&=-P_J''(z_t)/[J(J+1)],\\ e_{10}{}^{J+}(z_t)&=-P_J'(z_t)/[J(J+1)]^{1/2},\end{aligned}\tag{18}$$

where $P_J(z)$ is the Legendre polynomial. Hence[6]

$$f_{iI}{}^t(s,t)=\sum_{j=1}^{5}\mathfrak{L}_{ij}(z_t)g_{jI}{}^t(s,t),\tag{19}$$

where

$$\mathfrak{L}(z_t)=\frac{1}{2}\begin{bmatrix}1&1&0&0&0\\1&-1&0&0&0\\0&0&(1+z_t)&(1+z_t)&0\\0&0&(1-z_t)&-(1-z_t)&0\\0&0&0&0&(1-z_t^2)^{1/2}\end{bmatrix}.\tag{20}$$

Similarly, $g_{iI}{}^s(s,t)$ with isospin I in the s channel is defined by

$$f_{iI}{}^s(s,t)=\sum_{j=1}^{5}\mathfrak{L}_{ij}(z_s)g_{jI}{}^s(s,t).\tag{21}$$

The inverse relations for (16) are[8]

$$\begin{aligned}G_{00,I}{}^{J+}(t)&=\frac{1}{2}\int_{-1}^{+1}dz_t\,g_{1I}{}^t(s,t)c_{00}{}^{J+}(z_t),\\ G_{00,I}{}^{J-}(t)&=\frac{1}{2}\int_{-1}^{+1}dz_t\,g_{2I}{}^t(s,t)c_{00}{}^{J+}(z_t),\\ G_{11,I}{}^{J+}(t)&=\frac{1}{2}\int_{-1}^{+1}dz_t[g_{3I}{}^t(s,t)c_{11}{}^{J+}(z_t)+g_{4I}{}^t(s,t)c_{11}{}^{J-}(z_t)],\\ G_{11,I}{}^{J-}(t)&=\frac{1}{2}\int_{-1}^{+1}dz_t[g_{4I}{}^t(s,t)c_{11}{}^{J+}(z_t)+g_{3I}{}^t(s,t)c_{11}{}^{J-}(z_t)],\\ G_{10,I}{}^{J+}(t)&=\frac{1}{2}\int_{-1}^{+1}dz_t\,g_{5I}{}^t(s,t)c_{10}{}^{J+}(z_t),\end{aligned}\tag{22}$$

where the functions $c_{\lambda\mu}{}^{J\pm}(z_t)$ are polynomials in z_t defined by

$$d_{\lambda\mu}{}^J(\theta_t)=(1+z_t)^{-\frac{1}{2}|\lambda+\mu|}(1-z_t)^{-\frac{1}{2}|\lambda-\mu|}\times[c_{\lambda\mu}{}^{J+}(z_t)+c_{\lambda\mu}{}^{J-}(z_t)],$$
$$c_{\lambda\mu}{}^{J\pm}(-z_t)=\pm(-1)^{J-\lambda_m}c_{\lambda\mu}{}^{J\pm}(z_t),\quad \lambda_m=\max(|\lambda|,|\mu|).\tag{23}$$

They can be expressed as linear combinations of the Legendre polynomials[8]:

$$\begin{aligned}c_{00}{}^{J+}(z_t)&=P_J(z_t),\\ c_{11}{}^{J+}(z_t)&=[(J+1)P_{J-1}(z_t)+JP_{J+1}(z_t)]/(2J+1),\\ c_{11}{}^{J-}(z_t)&=P_J(z_t),\\ c_{10}{}^{J+}(z_t)&=-[J(J+1)]^{1/2}[P_{J-1}(z_t)-P_{J+1}(z_t)]/(2J+1).\end{aligned}\tag{24}$$

Since the $e_{\lambda\mu}{}^{J\pm}(z_t)$ appearing in the partial-wave expansions (16) are polynomials, the only singularities of $g_{iI}{}^t(s,t)$ in z_t come from divergences of the series: They have no kinematic singularities in z_t. However, they still have kinematic singularities in t, being related to amplitudes $\bar{g}_{iI}{}^t(s,t)$ free of all kinematic singularities by[9]

$$\begin{aligned}g_{1I}{}^t(s,t)&=k_t^{-2}\bar{g}_{1I}{}^t(s,t),\\ g_{2I}{}^t(s,t)&=k_t^{-2}\bar{g}_{2I}{}^t(s,t),\\ g_{3I}{}^t(s,t)&=\bar{g}_{3I}{}^t(s,t),\\ g_{4I}{}^t(s,t)&=\bar{g}_{4I}{}^t(s,t),\\ g_{5I}{}^t(s,t)&=t^{1/2}\bar{g}_{5I}{}^t(s,t).\end{aligned}\tag{25}$$

Furthermore, the $g_{iI}{}^t(s,t)$ must satisfy certain kinematic constraints to cancel singularities in the crossing matrix. Of these constraints, the most important, which corresponds physically to the requirement that angular momentum be conserved in the forward direction in the s channel, is

$$g_{2I}{}^t(s,t)-z_tg_{3I}{}^t(s,t)-g_{4I}{}^t(s,t)=0\quad\text{at } t=0.\tag{26}$$

This relation is known in Regge phenomenology as the "conspiracy condition."[10]

Since the Veneziano model has only dynamical singularities, it is an appropriate representation not for the helicity amplitudes, but for the invariant amplitudes, which are free of kinematic singularities and constraints. The s-channel invariant amplitudes $F_{iI}{}^s(s,t)$ for isospin I are related to the Feynman amplitude by[7]

$$\mathfrak{M}_I{}^s=F_{1I}{}^s(s,t)(S-\tilde{S})+F_{2I}{}^s(s,t)(T+\tilde{T})+F_{3I}{}^s(s,t)(A-\tilde{A})+F_{4I}{}^s(s,t)(V+\tilde{V})+F_{5I}{}^s(s,t)(P-\tilde{P}),\tag{27}$$

where[11]

$$\begin{aligned}S&=\bar{u}(p_c)u(p_a)\bar{u}(p_d)u(p_b),\\ T&=\tfrac{1}{2}\bar{u}(p_c)\sigma^{\mu\nu}u(p_a)\bar{u}(p_d)\sigma_{\mu\nu}u(p_b),\\ A&=\bar{u}(p_c)\gamma^\mu\gamma_5u(p_a)\bar{u}(p_d)\gamma_\mu\gamma_5u(p_b),\\ V&=\bar{u}(p_c)\gamma^\mu u(p_a)\bar{u}(p_d)\gamma_\mu u(p_b),\\ P&=\bar{u}(p_c)i\gamma_5u(p_a)\bar{u}(p_d)i\gamma_5u(p_b),\end{aligned}\tag{28}$$

and $\tilde{S}$, $\tilde{T}$, $\tilde{A}$, $\tilde{V}$, $\tilde{P}$ are given the same expressions with p_c and p_d interchanged. The Pauli principle therefore

[9] L. L. Wang, Phys. Rev. **142**, 1187 (1966).
[10] R. J. N. Phillips, Nucl. Phys. **B2**, 657 (1967).
[11] The γ matrices satisfy $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}$, $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3$, $\sigma_{\mu\nu}=\frac{1}{2}i[\gamma_\mu,\gamma_\nu]$. This differs from the conventions of Ref. 7.

requires[7]

$$F_{iI}{}^s(s,u)=(-1)^{i+I}F_{iI}{}^s(s,t). \tag{29}$$

The sets of covariants S, T, A, V, P and $\tilde{S}, \tilde{T}, \tilde{A}, \tilde{V}, \tilde{P}$ are not independent, but are related by the Fierz matrix

$$\mathfrak{F}=\mathfrak{F}^{-1}=\tfrac{1}{4}\begin{bmatrix}1 & 1 & -1 & 1 & -1\\ 6 & -2 & 0 & 0 & -6\\ -4 & 0 & -2 & -2 & -4\\ 4 & 0 & -2 & -2 & 4\\ 1 & -1 & -1 & 1 & 1\end{bmatrix}. \tag{30}$$

Hence (27) can be written as

$$\mathfrak{M}_I{}^s=\sum_{i,j=1}^{5} F_{iI}{}^s(s,t)[\delta_{ij}+(-1)^i\mathfrak{F}_{ij}] \times\bar{u}(p_c)\Gamma^j u(p_a)\bar{u}(p_d)\Gamma^j u(p_b), \tag{31}$$

where

$$\Gamma^1=1, \quad \Gamma^2=(1/\sqrt{2})\sigma^{\mu\nu}, \quad \Gamma^3=\gamma^\mu\gamma_5, \quad \Gamma^4=\gamma^\mu, \quad \Gamma^5=i\gamma_5, \tag{32}$$

and repeated Lorentz indices are to be contracted. Since the t channel has initial momenta $-p_d$ and p_b and final momenta p_c and $-p_a$, with $p_c=p_a$ for forward scattering, the corresponding expansion for it is

$$\mathfrak{M}_I{}^t=\sum_{i,j=1}^{5} F_{iI}{}^t(s,t)[\delta_{ij}+(-1)^i\mathfrak{F}_{ij}] \times\bar{u}(-p_a)\Gamma^j u(-p_d)\bar{u}(p_c)\Gamma^j u(p_b). \tag{33}$$

Then crossing symmetry requires[12]

$$F_{iI}{}^t(s,t)=\sum_{J=0}^{1}\sum_{j=1}^{5} K_{IJ}\Gamma_{ij}F_{jJ}{}^s(s,t), \tag{34}$$

where

$$\Gamma_{ij}=(\Gamma^{-1})_{ij}=(-1)^{i+1}\mathfrak{F}_{ji} =\tfrac{1}{4}\begin{bmatrix}1 & 6 & -4 & 4 & -1\\ -1 & 2 & 0 & 0 & 1\\ -1 & 0 & -2 & -2 & -1\\ -1 & 0 & 2 & 2 & -1\\ -1 & -6 & -4 & 4 & 1\end{bmatrix} \tag{35}$$

and

$$K=K^{-1}=\frac{1}{2}\begin{bmatrix}-1 & -3\\ -1 & 1\end{bmatrix}. \tag{36}$$

The invariant amplitudes do not have simple partial-wave expansions or definite quantum numbers, so it is necessary to relate them to the parity-conserving helicity amplitudes. This can be done[7] by explicitly evaluating the covariants in (33) for spinors of the appropriate helicities. The result is

$$g_{iI}{}^t(s,t)=\sum_{j=1}^{5} Q_{ij}(t,z_t)F_{jI}{}^t(s,t), \tag{37}$$

where

$$Q(t,z_t)=\frac{1}{m^2}\begin{bmatrix}-k_t{}^2 & 2E_t{}^2z_t & 4k_t{}^2 & 2m^2z_t & k_t{}^2\\ (E_t{}^2+m^2) & -2k_t{}^2z_t & (4E_t{}^2+2m^2) & 0 & -k_t{}^2\\ 0 & 2m^2 & 0 & 2E_t{}^2 & 0\\ -k_t{}^2 & 0 & -2k_t{}^2 & 0 & -k_t{}^2\\ 0 & -2mE_t & 0 & -2mE_t & 0\end{bmatrix}, \tag{38}$$

$$E_t{}^2=\tfrac{1}{4}t.$$

Also, since the s and t channels are treated equivalently,

$$g_{iI}{}^s(s,t)=\sum_{j=1}^{5} Q_{ij}(s,z_s)F_{jI}{}^t(s,t), \tag{39}$$

where $g_{iI}{}^s(s,t)$ is defined by (21).

III. VENEZIANO MODEL

To construct a Veneziano model for the invariant amplitudes of nucleon-nucleon scattering, it is necessary to generalize the original model to the class of functions

$$V_c{}^{ab}(\alpha(t),\alpha(u))=\frac{\Gamma(a-\alpha(t))\Gamma(b-\alpha(u))}{\Gamma(c-\alpha(t)-\alpha(u))}, \quad \max(a,b)\leqslant c\leqslant a+b. \tag{40}$$

Here a, b, and c are non-negative integers, $\Gamma(z)$ is the gamma function,[13] and

$$\alpha(t)=\alpha_0+\alpha' t \tag{41}$$

is the linear Regge trajectory, which is assumed to be the same in both the t and the u channels. Therefore $V_c{}^{ab}(\alpha(t),\alpha(u))$ is a meromorphic function of s and t whose only singularities are simple poles at

$$\alpha(t)=a,\ a+1,\ a+2,\ \ldots,$$
$$\alpha(u)=b,\ b+1,\ b+2,\ \ldots. \tag{42}$$

The condition $\max(a,b)\leqslant c\leqslant a+b$ in (40) ensures that there are no simultaneous poles in t and u and that the residue of the pole at $\alpha(t)=n$ is a polynomial in z_t of degree not greater than n. It is shown in Appendix A that, as $\alpha(t)\to n$,

$$V_c{}^{ab}(\alpha(t),\alpha(u))\sim\sum_{l=0}^{n+b-c}(2l+1)R_{nl}{}^{abc}P_l(z_t)\frac{1}{n-\alpha(t)}, \tag{43}$$

[12] The derivation of this is essentially identical to that given in Ref. 7 for the s-u crossing matrices.

[13] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge U. P., Cambridge, England, 1965).

where

$$R_{nl}{}^{abc}=\frac{1}{2}\int_{-1}^{+1} dz_t\, P_l(z_t) \times \mathrm{Res}\{V_c{}^{ab}(\alpha(t),\alpha(u)),\alpha(t)=n\}\,, \quad (44)$$

so that the pole actually corresponds to a superposition of resonances with several spins. In particular,

$$R_{nn}{}^{abb}=\frac{(-1)^a}{(n-a)!}\frac{(n!)^2}{(2n+1)!}(n-\alpha_0-4m^2\alpha')^n\,,$$

$$R_{n,n-1}{}^{abb}=\frac{1}{2}\frac{(-1)^a}{(n-a)!}\frac{n[(n-1)!]^2}{(2n-1)!} \times(3\alpha_0+4m^2\alpha'-2b+1) \times(n-\alpha_0-4m^2\alpha')^{n-1}\,, \quad (45)$$

$$R_{n,n-1}{}^{a,b,b+1}=-\frac{(-1)^a}{(n-a)!}\frac{[(n-1)!]^2}{(2n-1)!}(n-\alpha_0-4m^2\alpha')^{n-1}\,,$$

$$R_{nl}{}^{abc}=0\,,\quad l>n+b-c\,.$$

The asymptotic behavior of $V_c{}^{ab}(\alpha(t),\alpha(u))$ can be obtained by using the Stirling approximation[13] for $\Gamma(z)$. As $u\to\infty$ with $|\arg u|>0$ and t fixed,

$$V_c{}^{ab}(\alpha(t),\alpha(u))\sim\Gamma(a-\alpha(t))(-\alpha'u)^{\alpha(t)+b-c} \times\left[1+\frac{1}{2\alpha'u}(\alpha't+\alpha_0+b-c) \times(\alpha't+3\alpha_0-b-c+1)+\cdots\right], \quad (46)$$

which result, like the resonance spectrum, corresponds to an infinite family of Regge poles:

$$\alpha_k(t)=\alpha(t)-k\,,\quad k=0,\,1,\,2,\,\ldots\,. \quad (47)$$

As $u\to\infty$ with $0<|\arg u|<\pi$ and s fixed, the amplitude decreases exponentially, there being no Regge poles in the s channel. Finally, as $s\to\infty$ with $|\arg s|<\pi$ and z_s fixed,

$$V_c{}^{ab}(\alpha(t),\alpha(u)) \sim(2\pi)^{1/2}(1-z_s)^{a-\alpha_0-\frac{1}{2}}(1+z_s)^{b-\alpha_0-\frac{1}{2}}(2)^{-c+2\alpha_0+\frac{1}{2}} \times(2\alpha'k_s{}^2)^{a+b-c-\frac{1}{2}}\exp\{-2\alpha'k_s{}^2[\ln 4 -(1+z_s)\ln(1+z_s)-(1-z_s)\ln(1-z_s)]\}\,. \quad (48)$$

For physical z_s the square bracket in the exponent is always positive and is given approximately by[1]

$$\ln 4-(1+z_s)\ln(1+z_s)-(1-z_s)\ln(1-z_s) \approx\ln 4\sin\theta_s\,. \quad (49)$$

Hence the gross behavior of the large-angle cross section depends only on $k_s{}^2\sin\theta_s$.

The finite-energy sum rule[14] for the amplitude

$$A(s,t)=V_c{}^{ab}(\alpha(t),\alpha(u))+V_c{}^{ba}(\alpha(t),\alpha(u)) \quad (50)$$

is

$$\int_{u_0}^{u} du'(u')^m\,\mathrm{Im}A(s',t)=-\frac{\Gamma(1-\alpha(t))\sin\pi(\alpha(t)+b-c)(u)^{m+1}(\alpha'u)^{\alpha(t)}}{\alpha(t)+b-c+m+1}\,,\quad b>a,$$
$$=-\frac{2\Gamma(1-\alpha(t))\sin\pi(\alpha(t)+b-c)(u)^{m+1}(\alpha'u)^{\alpha(t)}}{\alpha(t)+b-c+m+1}\,,\quad b=a, \quad (51)$$

where $\mathrm{Im}A(s',t)$ is a sequence of δ functions. The original Veneziano model has the remarkable property that it satisfies this relation approximately even for small u.[1] The functions (50) share this property for $b=a$, but not for $b>a$, as is illustrated in Fig. 1. Of course, (51) must be satisfied in all cases for large u.

The Veneziano model constructed in Sec. IV for nucleon-nucleon scattering contains terms like (50) with $b>a$, so it will satisfy the finite-energy sum rules only asymptotically. It is very difficult to tell whether or not this is true of the physical amplitudes. To determine uniquely the imaginary parts of these amplitudes, it is necessary to measure not only the cross section and polarization, but also various spin correlation functions. Even if this were done, there still would remain the problem of determining the amplitude in the large unphysical region between $t=4m_\pi{}^2$ and $t=4m^2$.

[14] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

IV. VENEZIANO MODEL FOR NUCLEON-NUCLEON SCATTERING

The s channel, corresponding to nucleon-nucleon scattering, contains no known resonances except the deuteron, which will be ignored. The identical t and u channels, corresponding to nucleon-antinucleon scattering, both contain meson resonances which are assumed to lie on straight, exchange-degenerate Regge trajectories $\alpha(t)$. This assumption of exchange degeneracy, and hence of a real amplitude in the s channel, is necessary because the Veneziano model has no imaginary part in a channel without resonances. Unless exotic resonances are assumed to exist, therefore, the Veneziano model cannot be used for the Pomeranchuk trajectory.[15]

Since the invariant amplitudes $F_{iI}{}^s(s,t)$ in (27) are free of kinematic singularities and constraints, it is

[15] H. Harari, Phys. Rev. Letters **20**, 1395 (1968).

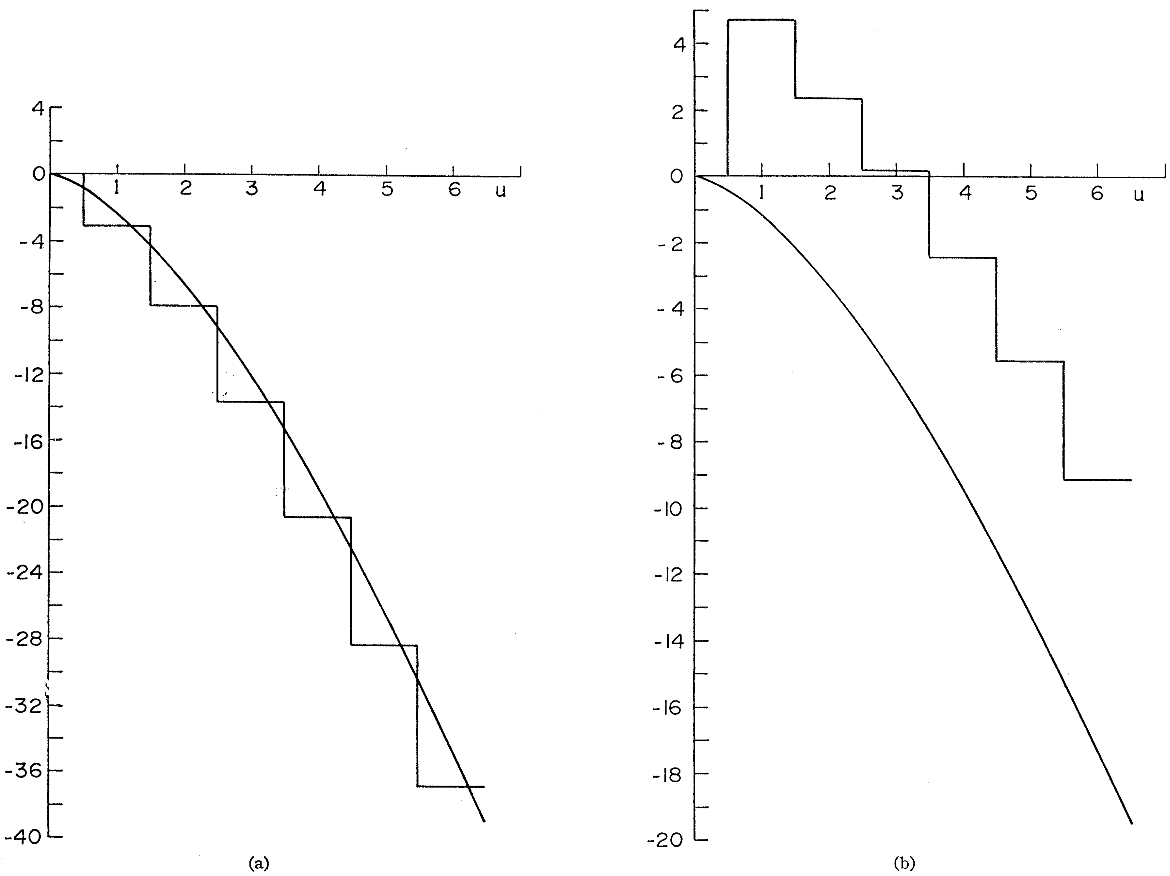

FIG. 1. Comparison of the left-hand side (smooth curves) and right-hand side (broken curves) of (51) for $t=0$ and $\alpha(t)=\frac{1}{2}+t$. In (a), $A(s,t)=2V_1^{11}(\alpha(t),\alpha(u))$. In (b), $A(s,t)=V_3^{13}(\alpha(t),\alpha(u))+V_3^{31}(\alpha(t),\alpha(u))$.

appropriate to write them as a sum of terms of the form $V_c^{ab}(\alpha(t),\alpha(u))$. It assumed here, as has already been done in Sec. III, that the same trajectory α appears in both the t and the u channel of any given term; the full model is then obtained by adding the separate models for each trajectory. This assumption, which greatly simplifies the calculation, is, of course, not possible for some other scattering processes.

For a given Regge trajectory $\alpha(t)$, the most general Veneziano model consistent with the Pauli principle (29) is

$$F_{iI}{}^s(s,t)=\sum_{abc}\beta_{iI}{}^{abc}[V_c{}^{ab}(\alpha(t),\alpha(u))+(-1)^{i+I}V_c{}^{ba}(\alpha(t),\alpha(u))], \quad (52)$$

$$\alpha(t)=\alpha_0+\alpha' t,$$

where it is to be understood that

$$\beta_{iI}{}^{abc}=0 \quad \text{unless } a\leqslant b\leqslant c\leqslant a+b. \quad (53)$$

In general, the pole of such a model at $\alpha(t)=n$, $n=a$, $a+1$, $a+2,\ldots$, appears in all of the partial-wave amplitudes $G_{\lambda\mu,I}{}^{J\pm}(t)$ with $J\leqslant n+1$. The problem is to choose the coefficients $\beta_{iI}{}^{abc}$ so that the resonances on the ancestor trajectory (those with $J=n+1$) are eliminated and so that the resonances on the parent trajectory (those with $J=n$) have definite parity and isospin and have residues which satisfy the factorization theorem and positivity conditions. This can be done by choosing an appropriate set of terms in (52) and calculating for them the residues $G_{\lambda\mu,I}{}^{\pm}(n,J)$ defined by

$$G_{\lambda\mu,I}{}^{J\pm}(t)\sim\frac{G_{\lambda\mu,I}{}^{\pm}(n,J)}{n-\alpha(t)}, \quad \alpha(t)\to n, \quad (54)$$

where $G_{\lambda\mu,I}{}^{J\pm}(t)$ are the t-channel partial-wave amplitudes (22). The problem is then reduced to forcing $G_{\lambda\mu,I}{}^{\pm}(n,n+1)$ to vanish and $G_{\lambda\mu,I}{}^{\pm}(n,n)$ to satisfy certain constraints. Since the initial model (52) satisfies the Pauli principle, there is no need to consider the u-channel resonances separately. Furthermore, the asymptotic behavior and the resonance structure of the Veneziano model are correlated by the finite energy sum rules (51), and this is sufficient to ensure that all of the amplitudes have the Regge behavior appropriate to the given parent trajectory.[16]

[16] This correlation is sometimes referred to as "duality."

In what follows, the above procedure is carried out for three exchange-degenerate pairs of trajectories—the (ω,f^0), the (ρ,A_2), and the (π,B).[17] For simplicity these pairs will be called the ω, the ρ, and the π trajectories, respectively.

ω Trajectory

The residue functions $\beta_{\lambda\mu,I}{}^{\pm}(t)$ for the Regge pole at $J=\alpha(t)$ are defined by

$$\beta_{\lambda\mu,I}{}^{\pm}(t)=\lim_{J\to\alpha(t)}[J-\alpha(t)]G_{\lambda\mu,I}{}^{J\pm}(t). \qquad (55)$$

For any choice of terms in (52) they have the form

$$\beta_{\lambda\mu,I}{}^{\pm}(t)=[1/\Gamma(\alpha(t)+\tfrac{3}{2})](\alpha' k_t^2)^{\alpha(t)-1}\gamma_{\lambda\mu,I}{}^{\pm}(t), \qquad (56)$$

where the $\gamma_{\lambda\mu,I}{}^{\pm}(t)$ are polynomials. If the model for the ω trajectory is to contain a pole corresponding to the ω resonance itself, it must include in (52) terms of the form

$$\beta_{iI}{}^{1bb}[V_b{}^{1b}+(-1)^{i+I}V_b{}^{b1}] \qquad (57)$$

for some values of b. The degrees of the polynomials $\gamma_{\lambda\mu,I}{}^{\pm}(t)$ corresponding to these terms are independent of b. Now the only other terms which can contribute to the parent trajectory are

$$\begin{aligned}&\beta_{iI}{}^{a,b,b}[V_b{}^{ab}+(-1)^{i+I}V_b{}^{ba}], \quad a\geqslant 2,\\ &\beta_{iI}{}^{a,b,b+1}[V_{b+1}{}^{ab}+(-1)^{i+I}V_{b+1}{}^{ba}], \quad a\geqslant 1.\end{aligned} \qquad (58)$$

Of these, all except

$$\begin{aligned}&\beta_{iI}{}^{1,b,b+1}[V_{b+1}{}^{1b}+(-1)^{i+I}V_{b+1}{}^{b1}],\\ &\beta_{iI}{}^{2,b,b+1}[V_{b+1}{}^{2b}+(-1)^{i+I}V_{b+1}{}^{b2}]\end{aligned} \qquad (59)$$

give polynomials $\gamma_{\lambda\mu,I}{}^{\pm}(t)$ of higher degree than those coming from (57). Hence the simplest possible residue function for the parent trajectory is obtained by choosing to include in the model just those terms in (57) and (59). In the absence of any better criterion, this choice is made herein.

The calculation for these terms of the residues $G_{\lambda\mu,I}{}^{\pm}(n,n+1)$ and $G_{\lambda\mu,I}{}^{\pm}(n,n)$ defined by (54) is carried out in Appendix B. It is shown there that the absence of ancestors,

$$G_{\lambda\mu,I}{}^{\pm}(n,n+1)=0 \quad \text{for all } n \qquad (60)$$

implies

$$\begin{aligned}&\sum_{b\geqslant 1}{}'(\beta_{1I}{}^{1bb}-2\beta_{2I}{}^{1bb}-\beta_{5I}{}^{1bb})=0,\\ &\sum_{b\geqslant 1}{}'(\beta_{1I}{}^{1bb}-2\beta_{4I}{}^{1bb}+\beta_{5I}{}^{1bb})=0,\\ &\beta_{3I}{}^{1bb}=0,\end{aligned} \qquad (61)$$

where

$$\sum_{b\geqslant a}{}'\beta_{iI}{}^{abc}=(-1)^{i+I}\beta_{iI}{}^{aac}+\sum_{b=a}^{\infty}\beta_{iI}{}^{abc}. \qquad (62)$$

[17] Exchange degeneracy in pion-pion scattering requires in addition that the ρ and f^0 trajectories be degenerate. Of course this does not force their residues in nucleon-nucleon scattering to be equal, and in fact the $I=1$ residues are much smaller than the $I=0$ ones.

The residues $G_{\lambda\mu,I}{}^{\pm}(n,n)$ of the parent trajectory then depend only on certain combinations of the $\beta_{iI}{}^{abc}$, namely,

$$\begin{aligned}
A_{1I}&=\sum_{b\geqslant 1}{}'\beta_{1I}{}^{1bb},\\
A_{2I}&=\sum_{b\geqslant 1}{}'b(\beta_{1I}{}^{1bb}-2\beta_{2I}{}^{1bb}-\beta_{5I}{}^{1bb}),\\
A_{4I}&=\sum_{b\geqslant 1}{}'b(\beta_{1I}{}^{1bb}-2\beta_{4I}{}^{1bb}+\beta_{5I}{}^{1bb}),\\
A_{5I}&=\sum_{b\geqslant 1}{}'\beta_{5I}{}^{1bb},\\
B_{2I}&=\sum_{b\geqslant 1}{}'(\beta_{1I}{}^{1,b,b+1}-2\beta_{2I}{}^{1,b,b+1}-\beta_{5I}{}^{1,b,b+1}),\\
B_{3I}&=\sum_{b\geqslant 1}{}'\beta_{3I}{}^{1,b,b+1},\\
B_{4I}&=\sum_{b\geqslant 1}{}'(\beta_{1I}{}^{1,b,b+1}-2\beta_{4I}{}^{1,b,b+1}+\beta_{5I}{}^{1,b,b+1}),\\
C_{2\lambda}&=\sum_{b\geqslant 2}{}'(\beta_{1I}{}^{2,b,b+1}-2\beta_{2I}{}^{2,b,b+1}-\beta_{5I}{}^{2,b,b+1})\\
&\qquad+(-1)^I(-\beta_{1I}{}^{122}-2\beta_{2I}{}^{122}+\beta_{5I}{}^{122}),\\
C_{3I}&=\sum_{b\geqslant 2}{}'\beta_{3I}{}^{2,b,b+1},\\
C_{4I}&=\sum_{b\geqslant 2}{}'(\beta_{1I}{}^{2,b,b+1}-2\beta_{4I}{}^{2,b,b+1}+\beta_{5I}{}^{2,b,b+1})\\
&\qquad+(-1)^I(-\beta_{1I}{}^{122}-2\beta_{4I}{}^{122}-\beta_{5I}{}^{122}).
\end{aligned} \qquad (63)$$

Actually, C_{2I} and C_{4I} appear only in the combinations

$$A_{2I}-C_{2I}, \quad B_{2I}+C_{2I}, \quad A_{4I}-C_{4I}, \quad B_{4I}+C_{4I}, \qquad (64)$$

so there are eight free parameters for each isospin.

These parameters are constrained by the conditions imposed on the leading trajectory. From Table I, the absence of parity and isospin doublets on the parent trajectory requires that

$$G_{00,0}{}^{-}(n,n)=G_{11,0}{}^{-}(n,n)=G_{\lambda\mu,1}{}^{\pm}(n,n)=0. \qquad (65)$$

The factorization theorem implies that

$$G_{00,0}{}^{+}(n,n)G_{11,0}{}^{+}(n,n)=[G_{10,0}{}^{+}(n,n)]^2. \qquad (66)$$

Finally, since

$$(n-\alpha_0-4m^2\alpha')^{n-1}=(4\alpha' k_t^2)^{\alpha(t)-1}|_{\alpha(t)=n}, \qquad (67)$$

reality of the coupling constants demands

$$\begin{aligned}&G_{00,0}{}^{+}(n,n)/(n-\alpha_0-4m^7\alpha')^{n-1}\geqslant 0,\\ &G_{11,0}{}^{+}(n,n)/(n-\alpha_0-4m^7\alpha')^{n-1}\geqslant 0.\end{aligned} \qquad (68)$$

In Appendix B it is shown that there is a nontrivial solution to these conditions having two free parameters, which are chosen to be A_{11} and A_{51}. Then

$$\begin{aligned}
&A_{10}=A_{11},\\
&A_{20}-C_{20}=A_{21}-C_{21}=-4A_{51},\\
&A_{40}-C_{40}=A_{41}-C_{41}=0,\\
&A_{50}=A_{51},\\
&B_{20}+C_{20}=B_{21}+C_{21}=4\alpha_0 A_{51},\\
&B_{30}=B_{31}=-2m^2\alpha' A_{11}A_{51}/(A_{11}+A_{51}),\\
&B_{40}+C_{40}=B_{41}+C_{41}=12m^2\alpha' A_{11}A_{51}/(A_{11}+A_{51}),\\
&C_{30}=C_{31}=0,\\
&A_{11}+A_{51}<0.
\end{aligned} \qquad (69)$$

Hence, a Veneziano model for the ω trajectory satisfying all of the desired conditions is given by

$$F_{iI}^{s}(s,t)=\sum_{b=1}^{\infty}\beta_{iI}^{1bb}[V_b^{1b}(\alpha(t),\alpha(u))+(-1)^{i+I}V_b^{b1}(\alpha(t),\alpha(u))]+\sum_{b=1}^{\infty}\beta_{iI}^{1,b,b+1}[V_{b+1}^{1b}(\alpha(t),\alpha(u))+(-1)^{i+I}V_{b+1}^{b1}(\alpha(t),\alpha(u))]+\sum_{b=2}^{\infty}\beta_{iI}^{2,b,b+1}[V_{b+1}^{2b}(\alpha(t),\alpha(u))+(-1)^{i+I}V_{b+1}^{b2}(\alpha(t),\alpha(u))], \quad (70)$$

where the coefficients β_{iI}^{abc} are chosen to satisfy (61) and (69). From Appendix B, the nonzero Regge residue functions for the parent trajectory of this model are

$$\beta_{00,0}^{+}(t)=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}(-A_{11}-A_{51})\times[\alpha(t)]\left(\alpha't-4m^2\alpha'\frac{A_{11}}{A_{11}+A_{51}}\right)^2,$$

$$\beta_{11,0}^{+}(t)=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}(-A_{11}-A_{51})\times[\alpha(t)+1](\alpha't)(4m^2\alpha')\left(\frac{A_{51}}{A_{11}+A_{51}}\right)^2, \quad (71)$$

$$\beta_{10,0}^{+}(t)=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}(-A_{11}-A_{51})\times\{\alpha(t)[\alpha(t)+1]\}^{1/2}(\alpha't)^{1/2}(4m^2\alpha')^{1/2}\times\left(\frac{A_{51}}{A_{11}+A_{51}}\right)\left(\alpha't-4m^2\alpha'\frac{A_{11}}{A_{11}+A_{51}}\right).$$

The square roots appearing in $\beta_{10,0}^{+}$ in (71) are just the kinematic singularities of $G_{10,0}^{J+}(t)$ arising from (24) and (25). However, the factors of $\alpha(t)$ in $\beta_{00,0}^{+}(t)$ and of $\alpha't[\alpha(t)+1]$ in $\beta_{11,0}^{+}(t)$ are dynamical predictions of the model. They imply[8] that the ω trajectory chooses sense at $\alpha(t)=-1$ and nonsense at $\alpha(t)=0$; the compensating trajectory at $\alpha(t)=0$ is of course just the first daughter trajectory.

ρ Trajectory

While the ρ trajectory has isospin 1, it couples to the same spin states as the ω, so the two cases are virtually identical. The solution analogous to (69) is

$$\begin{aligned}
&-\tfrac{1}{3}A_{10}=A_{11},\\
&-\tfrac{1}{3}(A_{20}-C_{20})=A_{21}-C_{21}=-4A_{51},\\
&-\tfrac{1}{3}(A_{40}-C_{40})=A_{41}-C_{41}=0,\\
&-\tfrac{1}{3}A_{50}=A_{51},\\
&-\tfrac{1}{3}(B_{20}+C_{20})=B_{21}+C_{21}=4\alpha_0A_{51},\\
&-\tfrac{1}{3}B_{30}=B_{31}=-2m^2\alpha'A_{11}A_{51}/(A_{11}+A_{51}),\\
&-\tfrac{1}{3}(B_{40}+C_{40})=B_{41}+C_{41}=12m^2\alpha'A_{11}A_{51}/(A_{11}+A_{51}),\\
&-\tfrac{1}{3}C_{30}=C_{31}=0,\\
&A_{11}+A_{51}>0.
\end{aligned} \quad (72)$$

The nonzero residues for the parent trajectory are

$$\beta_{00,1}^{+}(t)=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}(A_{11}+A_{51})\times[\alpha(t)]\left(\alpha't-4m^2\alpha'\frac{A_{11}}{A_{11}+A_{51}}\right)^2,$$

$$\beta_{11,1}^{+}(t)=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}(A_{11}+A_{51})\times[\alpha(t)+1](\alpha't)(4m^2\alpha')\left(\frac{A_{51}}{A_{11}+A_{51}}\right)^2, \quad (73)$$

$$\beta_{10,1}^{+}(t)=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}(A_{11}+A_{51})\times\{\alpha(t)[\alpha(t)+1]\}^{1/2}(\alpha't)^{1/2}(4m^2\alpha')^{1/2}\times\left(\frac{A_{51}}{A_{11}+A_{51}}\right)\left(\alpha't-4m^2\alpha'\frac{A_{11}}{A_{11}+A_{51}}\right).$$

Of course, the values of A_{11} and A_{51} here are independent of those for the ω trajectory.

π Trajectory and Conspiracy

Since the π trajectory is lower than the ω and the ρ, it can ordinarily be ignored. It is important only for proton-neutron charge exchange scattering, the cross section for which has a sharp forward peak with a width on the order of the pion mass. By itself, the π trajectory cannot explain such a peak: It couples only to the amplitude $g_{21}^{t}(s,t)$, and its contribution must therefore vanish at $t=0$ to satisfy the kinematic constraint (26). Suppose, however, that there is a natural-parity isospin-1 trajectory which is coupled to $g_{31}^{t}(s,t)$ and is degenerate with the π at $t=0$. Then both it and the π can have nonzero contributions at $t=0$ if these contributions just cancel each other in (26). This possibility[10] is known as conspiracy, and the other trajectory is called the conspirator.

It is often assumed that the conspirator has a very small slope so that it does not produce a low-mass positive-parity meson. In the framework of the Veneziano model, this is not possible, for if the t and u channels of the same term contain trajectories of different slopes, then the amplitude grows exponentially with t for certain angles in the t channel. If the π and the conspirator trajectories appear in different terms, then the pion residue is required to vanish at $t=0$. Hence the conspirator must be degenerate with the π, and its residue must be forced to vanish at $\alpha(t)=0$.

Reasoning analogous to that used for the ω trajectory leads to the inclusion in the model of the terms

$$\beta_{iI}^{0bb}[V_b^{0b}+(-1)^{i+I}V_b^{b0}],\qquad \beta_{iI}^{1,b,b+1}[V_{b+1}^{1b}+(-1)^{i+I}V_{b+1}^{b1}]. \quad (74)$$

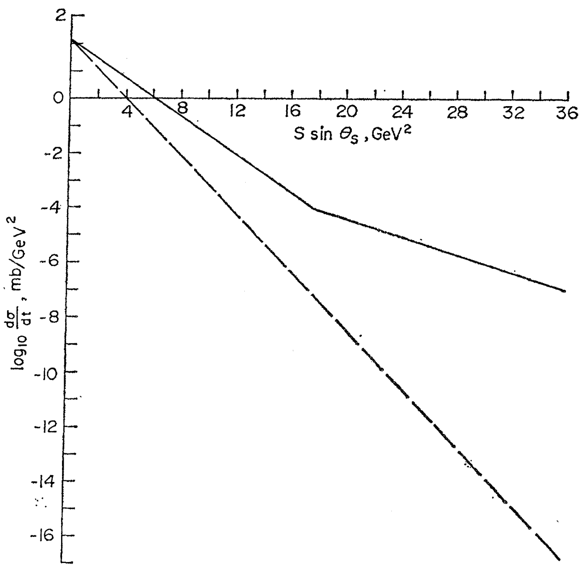

FIG. 2. The solid curve is the phenomenological formula (82) for the proton-proton cross section. The dashed curve is the approximate prediction (81) of the Veneziano model for $\alpha'=0.87$ GeV^{-2} as required by PCAC [C. Lovelace, Phys. Letters **28B**, 264 (1969)].

{There can be no terms of the form

$$\beta_{iI}{}^{0,b,b+1}[V_{b+1}{}^{0b}+(-1)^{i+I}V_{b+1}{}^{b0}], \tag{75}$$

since these have simultaneous poles in t and u.} The conditions required to eliminate the ancestors are

$$\begin{aligned}&\sum_{b\geqslant 0}{}'(\beta_{1I}{}^{0bb}-2\beta_{2I}{}^{0bb}-\beta_{5I}{}^{0bb})=0,\\&\sum_{b\geqslant 0}{}'(\beta_{1I}{}^{0bb}-2\beta_{4I}{}^{0bb}+\beta_{5I}{}^{0bb})=0,\\&\beta_{3I}{}^{0bb}=0,\end{aligned} \tag{76}$$

where the $\sum'$ notation is defined by (62). The residues $G_{\lambda\mu,I}{}^{\pm}(n,n)$ of the parent trajectory then depend only on

$$\begin{aligned}D_{1I}&=\sum_{b\geqslant 0}{}'\beta_{1I}{}^{0bb},\\D_{2I}&=\sum_{b\geqslant 0}{}'b(\beta_{1I}{}^{0bb}-2\beta_{2I}{}^{0bb}-\beta_{5I}{}^{0bb}),\\D_{4I}&=\sum_{b\geqslant 0}{}'b(\beta_{1I}{}^{0bb}-2\beta_{4I}{}^{0bb}+\beta_{5I}{}^{0bb}),\\D_{5I}&=\sum_{b\geqslant 0}{}'\beta_{5I}{}^{0bb},\\E_{2I}&=\sum_{b\geqslant 1}{}'(\beta_{1I}{}^{1,b,b+1}-2\beta_{2I}{}^{1,b,b+1}-\beta_{5I}{}^{1,b,b+1})\\&\quad+(-1)^I(-\beta_{1I}{}^{011}-2\beta_{2I}{}^{011}+\beta_{5I}{}^{011}),\\E_{3I}&=\sum_{b\geqslant 1}{}'\beta_{3I}{}^{1,b,b+1},\\E_{4I}&=\sum_{b\geqslant 1}{}'(\beta_{1I}{}^{1,b,b+1}-2\beta_{4I}{}^{1,b,b+1}+\beta_{5I}{}^{1,b,b+1})\\&\quad+(-1)^I(-\beta_{1I}{}^{011}-2\beta_{4I}{}^{011}-\beta_{5I}{}^{011}).\end{aligned} \tag{77}$$

The trajectory $\alpha(t)$ is to be interpreted as a π trajectory degenerate with a natural parity, isospin-1 conspirator trajectory; the latter must produce no resonance at $\alpha(t)=0$. By Table I this requires that

$$\begin{aligned}G_{\lambda\mu,0}{}^{\pm}(n,n)&=0,\\G_{11,1}{}^{-}(n,n)&=0,\\G_{\lambda\mu,1}{}^{+}(0,0)&=0,\\G_{00,1}{}^{+}(n,n)G_{11,1}{}^{+}(n,n)&=[G_{10,1}{}^{+}(n,n)]^2.\end{aligned} \tag{78}$$

Hence,

$$\begin{aligned}D_{1I}=D_{2I}=D_{4I}=E_{3I}=E_{4I}&=0,\\-\tfrac{1}{3}D_{50}&=D_{51},\\-\tfrac{1}{3}E_{20}&=E_{21}.\end{aligned} \tag{79}$$

If (76) and (79) are satisfied, then the nonzero residues of the parent Regge trajectory are

$$\begin{aligned}\beta_{00,1}{}^{+}(t)&=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t{}^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}(\alpha't)[\alpha(t)](E_{21}),\\\beta_{00,1}{}^{-}(t)&=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t{}^2)^{\alpha(t)}}{\Gamma(\alpha(t)+\frac{3}{2})}\\&\quad\times[\alpha't(4D_{51})+\alpha(t)(-E_{21})],\\\beta_{11,1}{}^{+}(t)&=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t{}^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}\\&\quad\times[\alpha(t)+1](4m^2\alpha')(E_{21}),\\\beta_{10,1}{}^{+}(t)&=\frac{(\pi)^{1/2}}{4m^2\alpha'}\frac{(\alpha'k_t{}^2)^{\alpha(t)-1}}{\Gamma(\alpha(t)+\frac{3}{2})}\{\alpha(t)[\alpha(t)+1]\}^{1/2}\\&\quad\times(\alpha't)^{1/2}(4m^2\alpha')^{1/2}(E_{21}).\end{aligned} \tag{80}$$

Setting $E_{21}=0$ makes the natural-parity residues vanish and gives a model for an evasive pion trajectory.

The residue $\beta_{11,1}{}^{+}(t)$ in (80) does not vanish at the nonsense point $\alpha(t)=0$. However, since it is proportional to E_{21}, (77) implies that it receives contributions only from terms of the form $V_{b+1}{}^{1b}(\alpha(t),\alpha(u))$, which are regular at $\alpha(t)=0$. Hence the apparent pole in $G_{11,1}{}^{J+}(t)$ at $\alpha(t)=0$ is absent, being removed of course by the compensating trajectory mechanism.[8]

V. COMPARISON WITH EXPERIMENT

The Veneziano model is of particular interest in connection with high-energy large-angle scattering, for which the Pauli principle is expected to be important. According to (45) and (46), its contribution to the cross section in this region has the approximate form

$$d\sigma/dt=A\exp(-\ln4\alpha's\sin\theta_s), \tag{81}$$

independent of the details of the model. In the same region the experimental proton-proton cross section can be represented by the phenomenological formula[18]

$$d\sigma/dt = B\exp(-s\sin\theta_s/g)\,, \tag{82}$$

where for $s\sin\theta_s < 16.0$ GeV^2,

$$\begin{aligned} g &= (1.24\pm0.01)\ \text{GeV}^2\,,\\ B &= (134.6\pm11.7)\ \text{mb GeV}^{-2}\,, \end{aligned} \tag{83}$$

and for $s\sin\theta_s > 20.0$ GeV^2,

$$\begin{aligned} g &= (2.77\pm0.02)\ \text{GeV}^{-2}\,,\\ B &= (56.4\pm3.4)\ \mu\text{b GeV}^{-2}. \end{aligned} \tag{84}$$

The model fails to reproduce this break in the experimental cross section. Furthermore, comparison of (81) and (82) shows that the values of the trajectory slope needed to obtain agreement on either side of the break are

$$\begin{aligned} \alpha' &= 0.58\ \text{GeV}^{-2} \quad \text{for } s\sin\theta_s < 16.0\ \text{GeV}^2,\\ \alpha' &= 0.26\ \text{GeV}^{-2} \quad \text{for } s\sin\theta_s > 20.0\ \text{GeV}^2. \end{aligned} \tag{85}$$

Since these values are both significantly less than the slope of an ordinary Regge trajectory, the contribution of the Veneziano model to the cross section is negligible for large values of $s\sin\theta_s$, as is shown in Fig. 2. The only parts of the amplitude which are important there are those which have been omitted from the model, such as the contributions of the Pomeranchuk trajectory and of cuts. This is supported by the fact that the large-angle data have been successfully fitted by a phenomenological model including just these contributions.[6]

A detailed phenomenological fit to forward nucleon-nucleon scattering has been made including the contributions of the Pomeranchuk, the ω, and the f^0 trajectories.[19] The compensation method used in this fit differs from that predicted by the Veneziano model (71) for the ω and f^0 trajectories, but this difference may not be significant. A more serious problem is that the phenomenological fit requires that the ω residues change sign at $t\approx -0.2$ GeV^2, whereas no change of sign occurs in (71). Perhaps this could be corrected by including more terms in the model, but then the same zero would have to occur in the f^0 residue function, and there is no evidence for it in pion-nucleon scattering.

The models for the (ρ,A_2) and for the (π,B) with a conspirator both contribute to proton-neutron charge-exchange scattering at small angles. Since ordinary Regge-pole theory, even with conspiracy, does not explain the measured cross section,[20] the Veneziano model cannot do so either. It is probably necessary to include the contributions of Regge cuts.

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APPENDIX A

The behavior of the function $V_c^{ab}(\alpha(t),\alpha(u))$ defined by (40) as $\alpha(t)\to n$, $n=a, a+1, \ldots$, is given by

$$V_c^{ab}(\alpha(t),\alpha(u)) \sim \frac{(-1)^{n-a}}{(n-a)!} r_{n+b-c}[c-n-\alpha(u)] \frac{1}{n-\alpha(t)}\,, \tag{A1}$$

where $r_n(x)$ is the Pochhammer polynomial,

$$\begin{aligned} r_n(x) &= \Gamma(x+n)/\Gamma(x)\\ &= x(x+1)\cdots(x+n-1)\,. \end{aligned} \tag{A2}$$

From[13]

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z\,, \tag{A3}$$

it follows that

$$r_n(-x) = (-1)^n r_n(x-n+1)\,. \tag{A4}$$

This, together with (5) and (41), implies that, as $\alpha(t)\to n$,

$$V_c^{ab}(\alpha(t),\alpha(u)) \sim \frac{(-1)^{a+b+c}}{(n-a)!} r_{n+b-c}(\alpha_0 - 2k_t^2\alpha' + 2k_t^2\alpha' z_t - b + 1) \times \frac{1}{n-\alpha(t)}\,, \tag{A5}$$

so that the residue of the pole at $\alpha(t)=n$ is a polynomial in z_t of degree $n+b-c$. Hence R_{nl}^{abc} in (43) is given by

$$R_{nl}^{abc} = \frac{1}{2}\frac{(-1)^{a+b+c}}{(n-a)!}\int_{-1}^{+1} dz_t \times r_{n+b-c}(\alpha_0 - 2k_t^2\alpha' + 2k_t^2\alpha' z_t - b+1) P_l(z_t)\,. \tag{A6}$$

This integral can be evaluated by noting that

$$r_n(x) = \sum_{k=0}^{n} \rho_{nk} x^{n-k}\,, \tag{A7}$$

where the coefficients ρ_{nk} are given by

$$\begin{aligned} \rho_{n0} &= 1\,,\\ \rho_{nn} &= 0\,, \quad n>0\,,\\ \rho_{nk} &= \sum_{i_1<i_2<\cdots<i_k=1}^{n-1} i_1 i_2\cdots i_k\,, \quad k\neq 0, n\,, \end{aligned} \tag{A8}$$

and satisfy

$$\rho_{nk} = \rho_{n-1,k} + (n-1)\rho_{n-1,k-1}\,. \tag{A9}$$

[20] K. Huang and I. J. Muzinich, Phys. Rev. **164**, 1726 (1967).

Then, by (A7) and the binomial theorem,

$$R_{nl}{}^{abc}=\frac{1}{2}\frac{(-1)^{a+b+c}}{(n-a)!}\sum_{i=0}^{n+b-c}\rho_{n+b-c,i}\sum_{j=0}^{n+b-c-i}\binom{n+b-c-i}{j}\times(\alpha_0-2k_t{}^2\alpha'-b-1)^{n+b-c-i-j}(2k_t{}^2\alpha')^j\times\int_{-1}^{+1}dz_t\,z_t{}^jP_l(z_t)\,,\quad\text{(A10)}$$

$$k_t{}^2=(n-\alpha_0)/4\alpha'-m^2.$$

But[13]

$$\int_{-1}^{+1}dz\,z^jP_l(z)=\frac{2^{l+1}j!(\frac{1}{2}j+\frac{1}{2}l)!}{(\frac{1}{2}j-\frac{1}{2}l)!(j+l+1)!}\,,\quad j\geqslant l,\ j-l\text{ even}$$

$$=0\,,\quad\text{otherwise.}\qquad\text{(A11)}$$

From (A9)–(A11), it is easy to obtain (42).

APPENDIX B

For the general Veneziano model (52) the t-channel parity-conserving helicity amplitudes $g_{iI}{}^t(s,t)$ are obtained by using (34) and (37). They are

$$\begin{aligned}
g_{1I'}{}^t(s,t)=\frac{1}{4m^2}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\{&[V_c{}^{ab}+(-1)^IV_c{}^{ba}][E_t{}^2(-12\beta_{2I}{}^{abc}-8\beta_{4I}{}^{abc})+m^2(12\beta_{2I}{}^{abc}+8\beta_{4I}{}^{abc})\\
&+E_t{}^2z_t(4\beta_{2I}{}^{abc})+m^2z_t(4\beta_{4I}{}^{abc})]\\
&+[V_c{}^{ab}-(-1)^IV_c{}^{ba}][E_t{}^2(-6\beta_{1I}{}^{abc}-8\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})+m^2(6\beta_{1I}{}^{abc}+8\beta_{3I}{}^{abc}+2\beta_{5I}{}^{abc})\\
&+E_t{}^2z_t(-2\beta_{1I}{}^{abc}+2\beta_{5I}{}^{abc})+m^2z_t(-2\beta_{1I}{}^{abc}+4\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})]\}\,,\\
g_{2I'}{}^t(s,t)=\frac{1}{4m^2}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\{&[V_c{}^{ab}+(-1)^IV_c{}^{ba}][E_t{}^2(12\beta_{2I}{}^{abc}-8\beta_{4I}{}^{abc})+m^2(4\beta_{4I}{}^{abc})\\
&+E_t{}^2z_t(-4\beta_{2I}{}^{abc})+m^2z_t(4\beta_{2I}{}^{abc})]\\
&+[V_c{}^{ab}-(-1)^IV_c{}^{ba}][E_t{}^2(-2\beta_{1I}{}^{abc}-8\beta_{3I}{}^{abc}-6\beta_{5I}{}^{abc})+m^2(-2\beta_{1I}{}^{abc}-12\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})\\
&+E_t{}^2z_t(2\beta_{1I}{}^{abc}-2\beta_{5I}{}^{abc})+m^2z_t(-2\beta_{1I}{}^{abc}+2\beta_{5I}{}^{abc})]\}\,,\qquad\text{(B1)}\\
g_{3I'}{}^t(s,t)=\frac{1}{4m^2}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\{&[V_c{}^{ab}+(-1)^IV_c{}^{ba}][E_t{}^2(4\beta_{4I}{}^{abc})+m^2(4\beta_{2I}{}^{abc})]\\
&+[V_c{}^{ab}-(-1)^IV_c{}^{ba}][E_t{}^2(-2\beta_{1I}{}^{abc}+4\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})+m^2(-2\beta_{1I}{}^{abc}+2\beta_{5I}{}^{abc})]\}\,,\\
g_{4I'}{}^t(s,t)=\frac{1}{4m^2}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\{&[V_c{}^{ab}+(-1)^IV_c{}^{ba}][E_t{}^2(-4\beta_{4I}{}^{abc})+m^2(4\beta_{4I}{}^{abc})]\\
&+[V_c{}^{ab}-(-1)^IV_c{}^{ba}][E_t{}^2(2\beta_{1I}{}^{abc}+12\beta_{3I}{}^{abc}+2\beta_{5I}{}^{abc})+m^2(-2\beta_{1I}{}^{abc}-12\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})]\}\,,\\
g_{5I'}{}^t(s,t)=\frac{1}{4m^2}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\{&[V_c{}^{ab}+(-1)^IV_c{}^{ba}][mE_t(-4\beta_{2I}{}^{abc}-4\beta_{4I}{}^{abc})]\\
&+[V_c{}^{ab}-(-1)^IV_c{}^{ba}][mE_t(4\beta_{1I}{}^{abc}-4\beta_{3I}{}^{abc})]\}\,,
\end{aligned}$$

where

$$V_c{}^{ab}=V_c{}^{ab}(\alpha(t),\alpha(u))\,.\qquad\text{(B2)}$$

The partial-wave amplitudes $G_{\lambda\mu,I}{}^{J\pm}(t)$ are then given by (22), where the $c_{\lambda\mu}{}^{J\pm}(z_t)$ are linear combinations of Legendre polynomials. The residues $G_{\lambda\mu,I}{}^{\pm}(n,J)$ in (54) are therefore expressible in terms of the $R_{nl}{}^{abc}$ defined in (44) and evaluated in Appendix A. The extra factors of z_t in (B1) are handled by using[13]

$$z_tP_l(z_t)=\frac{l+1}{2l+1}P_{l+1}(z_t)+\frac{l}{2l+1}P_{l-1}(z_t)\,.\qquad\text{(B3)}$$

Hence

$$G_{00,I'}{}^{+}(n,J)=\frac{1}{16m^2\alpha'}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\Bigg\{[R_{nJ}{}^{abc}+(-1)^I R_{nJ}{}^{bac}]$$
$$\times[(n-\alpha_0)(-12\beta_{2I}{}^{abc}-8\beta_{4I}{}^{abc})+4m^2\alpha'(12\beta_{2I}{}^{abc}+8\beta_{4I}{}^{abc})]$$
$$+\left[\frac{J+1}{2J+1}[R_{n,J+1}{}^{abc}+(-1)^I R_{n,J+1}{}^{bac}]+\frac{J}{2J+1}[R_{n,J-1}{}^{abc}+(-1)^I R_{n,J-1}{}^{bac}]\right]$$
$$\times[(n-\alpha_0)(4\beta_{2I}{}^{abc})+4m^2\alpha'(4\beta_{4I}{}^{abc})]+[R_{nJ}{}^{abc}-(-1)^I R_{nJ}{}^{bac}]$$
$$\times[(n-\alpha_0)(-6\beta_{1I}{}^{abc}-8\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})+4m^2\alpha'(6\beta_{1I}{}^{abc}+8\beta_{3I}{}^{abc}+2\beta_{5I}{}^{abc})]$$
$$+\left[\frac{J+1}{2J+1}[R_{n,J+1}{}^{abc}-(-1)^I R_{n,J+1}{}^{bac}]+\frac{J}{2J+1}[R_{n,J-1}{}^{abc}-(-1)^I R_{n,J-1}{}^{bac}]\right]$$
$$\times[(n-\alpha_0)(-2\beta_{1I}{}^{abc}+2\beta_{5I}{}^{abc})+4m^2\alpha'(-2\beta_{1I}{}^{abc}+4\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})]\Bigg\}\,,$$

$$G_{00,I'}{}^{-}(n,J)=\frac{1}{16m^2\alpha'}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\Bigg\{[R_{nJ}{}^{abc}+(-1)^I R_{nJ}{}^{bac}][(n-\alpha_0)(12\beta_{2I}{}^{abc}-8\beta_{4I}{}^{abc})+4m^2\alpha'(4\beta_{4I}{}^{abc})]$$
$$+\left[\frac{J+1}{2J+1}[R_{n,J+1}{}^{abc}+(-1)^I R_{n,J+1}{}^{bac}]+\frac{J}{2J+1}[R_{n,J-1}{}^{abc}+(-1)^I R_{n,J-1}{}^{bac}]\right]$$
$$\times[(n-\alpha_0)(-4\beta_{2I}{}^{abc})+4m^2\alpha'(4\beta_{2I}{}^{abc})]+[R_{nJ}{}^{abc}-(-1)^I R_{nJ}{}^{bac}]$$
$$\times[(n-\alpha_0)(-2\beta_{1I}{}^{abc}-8\beta_{3I}{}^{abc}-6\beta_{5I}{}^{abc})+4m^2\alpha'(-2\beta_{1I}{}^{abc}-12\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})]$$
$$+\left[\frac{J+1}{2J+1}[R_{n,J+1}{}^{abc}-(-1)^I R_{n,J+1}{}^{bac}]+\frac{J}{2J+1}[R_{n,J-1}{}^{abc}-(-1)^I R_{n,J-1}{}^{abc}]\right]$$
$$\times[(n-\alpha_0)(2\beta_{1I}{}^{abc}-2\beta_{5I}{}^{abc})+4m^2\alpha'(-2\beta_{1I}{}^{abc}+2\beta_{5I}{}^{abc})]\Bigg\}\,,$$

$$G_{11,I'}{}^{+}(n,J)=\frac{1}{16m^2\alpha'}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\Bigg\{\left[\frac{J+1}{2J+1}[R_{n,J-1}{}^{abc}+(-1)^I R_{n,J-1}{}^{bac}]\right.$$
$$\left.+\frac{J}{2J+1}[R_{n,J+1}{}^{abc}+(-1)^I R_{n,J+1}{}^{bac}]\right][(n-\alpha_0)(4\beta_{4I}{}^{abc})+4m^2\alpha'(4\beta_{2I}{}^{abc})] \tag{B4}$$
$$+\left[\frac{J+1}{2J+1}[R_{n,J-1}{}^{abc}-(-1)^I R_{n,J-1}{}^{bac}]+\frac{J}{2J+1}[R_{n,J+1}{}^{abc}-(-1)^I R_{n,J+1}{}^{bac}]\right]$$
$$\times[(n-\alpha_0)(-2\beta_{1I}{}^{abc}+4\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})+4m^2\alpha'(-2\beta_{1I}{}^{abc}+2\beta_{5I}{}^{abc})]$$
$$+[R_{nJ}{}^{abc}+(-1)^I R_{nJ}{}^{bac}][(n-\alpha_0)(-4\beta_{4I}{}^{abc})+4m^2\alpha'(4\beta_{4I}{}^{abc})]+[R_{nJ}{}^{nbc}-(-1)^I R_{nJ}{}^{bac}]$$
$$\times[(n-\alpha_0)(2\beta_{1I}{}^{abc}+12\beta_{3I}{}^{abc}+2\beta_{5I}{}^{abc})+4m^2\alpha'(-2\beta_{1I}{}^{abc}-12\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})]\Bigg\}\,,$$

$$G_{11,I'}^{-}(n,J)=\frac{1}{16m^2\alpha'}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\Bigg\{\bigg[\frac{J+1}{2J+1}[R_{n,J-1}{}^{abc}+(-1)^I R_{n,J-1}{}^{bac}]$$

$$+\frac{J}{2J+1}[R_{n,J+1}{}^{abc}+(-1)^I R_{n,J+1}{}^{bac}]\bigg][(n-\alpha_0)(-4\beta_{4I}{}^{abc})+4m^2\alpha'(4\beta_{4I}{}^{abc})]$$

$$+\bigg[\frac{J+1}{2J+1}[R_{n,J-1}{}^{abc}-(-1)^I R_{n,J-1}{}^{bac}]+\frac{J}{2J+1}[R_{n,J+1}{}^{abc}-(-1)^I R_{n,J+1}{}^{bac}]\bigg]$$

$$\times[(n-\alpha_0)(2\beta_{1I}{}^{abc}+12\beta_{3I}{}^{abc}+2\beta_{5I}{}^{abc})+4m^2\alpha'(-2\beta_{1I}{}^{abc}-12\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})]$$

$$+[R_{nJ}{}^{abc}+(-1)^I R_{nJ}{}^{bac}][(n-\alpha_0)(4\beta_{4I}{}^{abc})+4m^2\alpha'(4\beta_{2I}{}^{abc})]+[R_{nJ}{}^{abc}-(-1)^I R_{nJ}{}^{bac}]$$

$$\times[(n-\alpha_0)(-2\beta_{1I}{}^{abc}+4\beta_{3I}{}^{abc}-2\beta_{5I}{}^{abc})+4m^2\alpha'(-2\beta_{1I}{}^{abc}+2\beta_{5I}{}^{abc})]\Bigg\}\,,$$

$$G_{10,I'}^{+}(n,J)=\frac{1}{4m\sqrt{(\alpha')}}\sum_{I=0}^{1}K_{I'I}\sum_{abc}\Bigg\{\frac{[J(J+1)]^{1/2}}{2J+1}\{[R_{n,J+1}{}^{abc}+(-1)^I R_{n,J+1}{}^{bac}]$$

$$-[R_{n,J-1}{}^{abc}-(-1)^I R_{n,J-1}{}^{bac}]\}[(n-\alpha_0)^{1/2}(-2\beta_{2I}{}^{abc}-2\beta_{4I}{}^{abc})]$$

$$+\frac{[J(J+1)]^{1/2}}{2J+1}\{[R_{n,J+1}{}^{abc}-(-1)^I R_{n,J+1}{}^{bac}]-[R_{n,J-1}{}^{abc}-(-1)^I R_{n,J-1}{}^{bac}]\}$$

$$\times[(n-\alpha_0)^{1/2}(2\beta_{1I}{}^{abc}-2\beta_{3I}{}^{abc})]\Bigg\}\,.$$

To proceed further it is necessary to make a particular choice of terms, as was done in (57) and (59) for the ω trajectory. For these terms, it follows from (45) and (B4) that

$$G_{\lambda\mu,I}{}^{\pm}(n,J)=0 \quad \text{for all } J\geqslant n+2\,. \tag{B5}$$

The only nonzero contributions to $G_{\lambda\mu,I}{}^{\pm}(n,\, n+1)$ come from the $R_{n,J-1}{}^{1bb}$ terms in (B4) and from the cross terms $R_{n,J-1}{}^{b1b}$ with $b=1$; the latter are automatically included if the $\sum'_{b\geqslant1}$ notation (62) is used. Since from (45)

$$R_{nn}{}^{1bb}=R_{nn}{}^{111} \quad \text{for all } b\,, \tag{B6}$$

these contributions are

$$G_{00,I'}^{+}(n,\,n+1)=\frac{1}{16m^2\alpha'}R_{nn}{}^{111}\frac{n+1}{2n+3}\sum_{I=0}^{1}K_{I'I}\sum_{b\geqslant1}{}'\,[(n-\alpha_0)(4\beta_{2I}{}^{1bb}-2\beta_{1I}{}^{1bb}+2\beta_{5I}{}^{1bb})$$
$$+4m^2\alpha'(4\beta_{4I}{}^{1bb}-2\beta_{1I}{}^{1bb}+4\beta_{3I}{}^{1bb}-2\beta_{5I}{}^{1bb})],$$

$$G_{00,I'}^{-}(n,\,n+1)=\frac{1}{16m^2\alpha'}R_{nn}{}^{111}\frac{n+1}{2n+3}\sum_{I=0}^{1}K_{I'I}\sum_{b\geqslant1}{}'\,[(n-\alpha_0)(-4\beta_{2I}{}^{1bb}+2\beta_{1I}{}^{1bb}-2\beta_{5I}{}^{1bb})$$
$$+4m^2\alpha'(4\beta_{2I}{}^{1bb}-2\beta_{1I}{}^{1bb}+2\beta_{5I}{}^{1bb})],$$

$$G_{11,I'}^{+}(n,\,n+1)=\frac{1}{16m^2\alpha'}R_{nn}{}^{111}\frac{n+2}{2n+3}\sum_{I=0}^{1}K_{I'I}\sum_{b\geqslant1}{}'\,[(n-\alpha_0)(4\beta_{4I}{}^{1bb}-2\beta_{1I}{}^{1bb}+4\beta_{3I}{}^{1bb}-2\beta_{5I}{}^{1bb}) \tag{B7}$$
$$+4m^2\alpha'(4\beta_{2I}{}^{1bb}-2\beta_{1I}{}^{1bb}+2\beta_{5I}{}^{1bb})],$$

$$G_{11,I'}^{-}(n,\,n+1)=\frac{1}{16m^2\alpha'}R_{nn}{}^{111}\frac{n+2}{2n+3}\sum_{I=0}^{1}K_{I'I}\sum_{b\geqslant1}{}'\,[(n-\alpha_0)(-4\beta_{4I}{}^{1bb}+2\beta_{1I}{}^{1bb}+12\beta_{3I}{}^{1bb}-2\beta_{5I}{}^{1bb})$$
$$+4m^2\alpha'(4\beta_{4I}{}^{1bb}-2\beta_{1I}{}^{1bb}-12\beta_{3I}{}^{1bb}-2\beta_{5I}{}^{1bb})],$$

$$G_{10,I'}^{+}(n,\,n+1)=\frac{1}{4m(\alpha')^{1/2}}R_{nn}{}^{111}\frac{[(n+1)(n+2)]^{1/2}}{2n+3}\sum_{I=0}^{1}K_{I'I}\sum_{b\geqslant1}{}'\,[(n-\alpha_0)^{1/2}(-2\beta_{2I}{}^{1bb}-2\beta_{4I}{}^{1bb}+2\beta_{1I}{}^{1bb}-2\beta_{3I}{}^{1bb})].$$

If there are to be no ancestors, these contributions must vanish, implying that

$$\sum_{b\geqslant 1}{}' (\beta_{1I}{}^{1bb}-2\beta_{2I}{}^{1bb}-\beta_{5I}{}^{1bb})=0\,,\quad \sum_{b\geqslant 1}{}' (\beta_{1I}{}^{1bb}-2\beta_{4I}{}^{1bb}+\beta_{5I}{}^{1bb})=0\,,\quad \sum_{b\geqslant 1}{}' \beta_{3I}{}^{1bb}=0\,. \tag{B8}$$

The last condition just reflects the fact that $V_b{}^{1b}$ grows more rapidly with u than does $F_{31}{}^{s}(s,t)$. Henceforth it will be assumed that

$$\beta_{3I}{}^{1bb}=0\,, \tag{B9}$$

in agreement with (61).

For the parent trajectory residues $G_{\lambda\mu,I}{}^{\pm}(n,n)$, the only new cross terms $R_{nl}{}^{bac}$ in (B4) which contribute are the $R_{n,J-1}{}^{212}$ ones; these are explicitly taken into account in C_{20} and C_{40} in (63). Substituting (45) in (B4) and using (61) and (63) yields

$$\begin{aligned}
G_{00,I'}{}^{+}(n,n)={}&\frac{1}{16m^2\alpha'}\frac{n(n!)}{(2n+1)!}(n-\alpha_0-4m^2\alpha')^{n-1}\sum_{I=0}^{1}K_{I'I}\{n^2[16A_{1I}-4A_{2I}+4C_{2I}]\\
&+n[-4B_{2I}+4C_{2I}+\alpha_0(-32A_{1I}+4A_{2I}-4C_{2I})+4m^2\alpha'(-32A_{1I}-4A_{4I}-8C_{3I}+4C_{4I})]\\
&+[(\alpha_0+4m^2\alpha')^2(16A_{1I})+\alpha_0(4B_{2I}+4C_{2I})+4m^2\alpha'(8B_{3I}-4B_{4I}+8C_{3I}-4C_{4I})]\}\,,\\
G_{00,I'}{}^{-}(n,n)={}&\frac{1}{16m^2\alpha'}\frac{n(n!)}{(2n+1)!}(n-\alpha_0-4m^2\alpha')^{n}\sum_{I=0}^{1}K_{I'I}\\
&\times\{n[4A_{2I}+16A_{5I}-4C_{2I}]+[4B_{2I}+4C_{2I}+\alpha_0(-16A_{5I})]\}\,,\\
G_{11,I'}{}^{+}(n,n)={}&\frac{1}{16m^2\alpha'}\frac{(n+1)!}{(2n+1)!}(n-\alpha_0-4m^2\alpha')^{n-1}\sum_{I=0}^{1}K_{I'I}\{n^2[-4A_{4I}-8C_{3I}+4C_{4I}]\\
&+n[8B_{3I}-4B_{4I}+8C_{3I}-4C_{4I}+\alpha_0(4A_{4I}+8C_{3I}-4C_{4I})+4m^2\alpha'(-4A_{2I}+4C_{2I})]\\
&+[\alpha_0(-8B_{3I}+4B_{4I}-8C_{3I}+4C_{4I})+4m^2\alpha'(-4B_{2I}-4C_{2I})]\}\,,\\
G_{11,I'}{}^{-}(n,n)={}&\frac{1}{16m^2\alpha'}\frac{(n+1)!}{(2n+1)!}(n-\alpha_0-4m^2\alpha')^{n}\sum_{I=0}^{1}K_{I'I}\\
&\times\{n[4A_{4I}-24C_{3I}-4C_{4I}]+[24B_{3I}+4B_{4I}+24C_{3I}+4C_{4I}]\}\,,\\
G_{10,I'}{}^{+}(n,n)={}&\frac{1}{16m^2\alpha'}\frac{[n(n+1)]^{1/2}n!}{(2n+1)!}(n-\alpha_0-4m^2\alpha')^{n-1}(4m^2\alpha')^{1/2}(n-\alpha_0)^{1/2}\sum_{I=0}^{1}K_{I'I}\\
&\times\{n[-4A_{2I}-4A_{4I}+4C_{2I}-8C_{3I}+4C_{4I}]+[-4B_{2I}+8B_{3I}-4B_{4I}-4C_{2I}+8C_{3I}-4C_{4I}]\}\,.
\end{aligned} \tag{B10}$$

The satisfaction of the conditions (65), (67), and (68) is now reduced to an algebraic problem, the solution of which is given in (69). The residues (70) are obtained by substituting (69) into (B10) and replacing n by $\alpha(t)$.