three independent parameters d, f, and s, defined in Sec. IV. Then we obtain a constraint for the d, f, and s couplings. By using the GO mass formulas which can also be derived from the same commutator $[\dot{V}_{K^0},A_{K^0}]=0$, we obtain

$$\sqrt{3}d[2(\Sigma'^2+\Lambda'^2-n'^2-\Xi'^2)-(\Sigma^2-\Lambda^2)] \\ +2f[(n'^2-\Xi'^2)-(n^2-\Xi^2)] \\ +2\sqrt{3}\sin\theta s(\Lambda'^2-Y'^2)=0. \quad \text{(D1)}$$

The equation

$$\lim_{|\mathbf{q}|\to\infty} \langle n(\mathbf{q})|[\dot{V}_{K^0},A_{K^0}]|\Xi'(\mathbf{q})\rangle=0,$$

also leads to the same constraint. If both the B and B' are octets, Eq. (D1) takes a form

$$\sqrt{3}d[(\Sigma'^2-\Lambda'^2)-(\Sigma^2-\Lambda^2)] \\ +2f[(n'^2-\Xi'^2)-(n^2-\Xi^2)]=0. \quad \text{(D2)}$$

As an example consider the case $B'=(\frac{1}{2})^-$ and $B=(\frac{1}{2})^+$; Eq. (D1) gives a constraint for the physical couplings:

$$\sqrt{3}(\Sigma'-\Lambda)^{-1}[2(\Sigma'^2+\Lambda'^2-n'^2-\Xi'^2)-(\Sigma^2-\Lambda^2)]g_{\Sigma'^+\Lambda\pi} \\ +2(\Sigma'-\Sigma)^{-1}[(n'^2-\Xi'^2)-(n^2-\Xi^2)]g_{\Sigma'^+\Sigma^0\pi} \\ +2\sqrt{3}\sin\theta(Y-\Sigma)^{-1}(\Lambda'^2-Y'^2)g_{Y'\Sigma^-\pi}=0. \quad \text{(D3)}$$

Since this constraint is sensitive to the errors in the mass values (because it involves mass differences), it may not be very useful at present.

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Broken Chiral Symmetry and the η Meson*

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The scheme of broken chiral symmetry given by Gell-Mann, Oakes, and Renner is generalized to include the η meson with proper consideration of the octet-singlet mixing. It is shown that at least one of the η and the η' must violate the partially conserved axial-vector current (PCAC) condition (or Adler's condition). This is not simply because they are much heavier than the pion. The analysis of the decays $\eta\to 3\pi$ and $\eta\to 2\gamma$, which are known for large deviations from the simple SU_3 predictions, indicates that PCAC for the η is violated rather severely.

I. INTRODUCTION AND SUMMARY

An investigation has been made by Gell-Mann, Oakes, and Renner[1] of the consequence of the Hamiltonian

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}', \quad (1.1)$$

where $\mathcal{H}_0$ is invariant under $SU_3\times SU_3$, while $\mathcal{H}'$ takes a simple form

$$\mathcal{H}'=-u_0-cu_8 \quad (1.2)$$

with $c\approx-\sqrt{2}$. This Hamiltonian leads to a very satisfactory understanding of the mesons as far as pions and K mesons are concerned. It turns out, however, that one needs something extra beyond the straightforward extension of the scheme if one tries to include the η meson in the consideration. The argument is given briefly in Sec. II, together with a proposal of a possible way out of the difficulties. Although the same scheme has also been proposed recently by Lee,[2] we exploit its consequence with a different emphasis. A natural and interesting conclusion is that there is not as strong a reason for assuming partial conservation of axial-vector current (PCAC) for the η meson as that for the pion and K meson. By PCAC we mean here that the amplitude in the soft-meson limit can be calculated on the basis of the equal-time commutator involving the corresponding axial-vector charge. In this sense, its violation also leads to the violation of Adler's condition.[2a]

One may argue that this is not surprising simply because the η is much heavier than the pion. This argument is, however, not always correct. Note, for example, that there is so far no reason to prevent PCAC from holding for the K meson which is as heavy as the η.[3] It is also interesting to note that Adler's condition can be satisfied rather naturally for the pion as well as for the K meson in the Veneziano-type meson-meson scattering amplitudes, while the same is not true for the η,[4–7] indicating that PCAC does not hold for the η. The last two sections are devoted to

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provide another indication against the validity of PCAC for the η.

In Sec. III we present some of the formulas necessary for the following calculations. We propose a divergence equation for the η which allows a violation of PCAC. The formalism is applied in Sec. IV to the decays $\eta \to 3\pi$ and $\eta \to 2\gamma$ which have been known for remarkable deviation from the simple SU_3 predictions. The decay $\eta \to 3\pi$ is calculated on the basis of the "tadpole term" $-du_3$ added to the Hamiltonian (1.1).[8,9] We also use the π^0 pole-dominance model which is strongly supported by the excellent fit of the predicted slope to the observed spectrum of $\eta \to \pi^+\pi^-\pi^0$.[4,10,11]

Here one notices an important question about the soft-meson technique. It has been pointed out that the soft-meson calculation gives only half of the meson mass. One may expect that the other half comes from the "kinetic-energy part," as suggested by the free meson field model.[1] If this is the case, it is rather unlikely that the same additional term contributes to the π^0-η junction which is off-diagonal, unlike the masses. In this model, which will be referred to as the "half-mass model" since the π^0-η junction is related to half of the masses through the SU_3 coefficients, the soft-meson limit will give the full amplitude for the transition π^0-η. On the other hand, one may also expect that the physical π^0-η junction is twice as large as its soft-meson limit simply because the same is true for the masses. This "full-mass model" corresponds to the point of view which has been accepted commonly in the literature.[12] Although the latter model tends to narrow the discrepancy between theory and experiment, we consider both models in view of the theoretical uncertainties on this point.

The decay $\eta \to 2\gamma$ is calculated from the anomalous term in the divergence equation.[13,14] The decay $\pi^0 \to 2\gamma$ calculated in this method agrees with the experiment if the quarks are integrally charged, or if the quarks are fractionally charged and the anomalous term is modified by the factor 3 because of the possible higher-order correction.[15] In accordance with the recent discussion by Mathur, Okubo, and Rao,[16] we investigate three models: the Maki-Hara model[17] (MH) and the Han-Nambu model[18] (HN), both of which give integral quark charges, and the Gell-Mann–Zweig model[19] (GZ) with the multiplicative factor 3 in front of the anomalous term.

For all these models we examine two different kinds of fits, which are characterized as follows. (i) The octet-singlet mixing angle in the η field could be large. (ii) The mixing angle is small, but the magnitude of the tadpole term could be different from the usual estimate.

In all of these fits, reasonable agreements to the experiment are obtained only if an appreciable violation of PCAC for the η is assumed. No attempt is made to select or eliminate any of the models without further improvement in the approximation which will be discussed elsewhere.

II. DIFFICULTIES WITH η

First one may simply assume $\varphi_\eta \propto D_8$ ($D_i \equiv \partial_\mu \mathfrak{F}_{\mu i}{}^5$) where a possible small mixing with the unitary singlet component has been neglected. The Hamiltonian (1.1), (1.2) with $c=-\sqrt{2}$ gives

$$D_8 = 2(v_8 \cos\theta_0 - v_0 \sin\theta_0)\,, \tag{2.1}$$

where $\theta_0 = \sin^{-1}\frac{1}{3}\sqrt{3} = 35.3°$. There is a large admixture of the unitary singlet component. Moreover, the particular angle θ_0 suggests that the η is made up of the λ quark and its antiparticle, alone, like the $\varphi(1020)$ or $f^*(1515)$. This seems to contradict the present experimental data.[20] A small deviation of c from $-\sqrt{2}$ will not improve the situation.

Next, one may expect that this difficulty will be avoided if one takes into account the mixing between octet and singlet components. Then $F_0{}^5$ comes into play. The simplest commutation relations will be

$$\begin{aligned} [F_0{}^5, u_j] &= -i d_{0jk} v_k = -i(\sqrt{\tfrac{2}{3}}) v_j\,, \\ [F_0{}^5, v_j] &= i(\sqrt{\tfrac{2}{3}}) u_j\,. \end{aligned} \tag{2.2}$$

One may assume tentatively that $\mathcal{H}_0$ is invariant under $U_3 \times U_3$, i.e.,

$$[F_0{}^5, \mathcal{H}_0] = 0\,. \tag{2.3}$$

The total Hamiltonian is then invariant under $U_2 \times U_2$ because $\mathcal{H}'$ is invariant under $SU_2 \times SU_2$ for $c=-\sqrt{2}$. Consequently, one would have an isoscalar massless

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[11] If one assumes the usual nontadpole interaction, the dominance of the π^0 pole term cannot be justified because a large cancellation must occur between this term and the other terms to meet the soft-pion conditions, as have been discussed in detail by the present author [Y. Fujii, Nucl. Phys. **B7**, 601 (1968)]. The soft-pion conditions from the tadpole interaction, on the other hand, can be satisfied by the π^0 pole term alone if the σ term in the $\pi\pi$ scattering amplitude belongs to the representation $(3,3^*)+(3^*,3)$. The possible large violation of PCAC for the η, as first proposed in the above-mentioned paper, is again considered in the present paper in a new context.

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[19] M. Gell-Mann, Phys. Letters **8**, 214 (1964).

[20] See, e.g., H. Harari, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 195.

pseudoscalar meson.[21] A small deviation of c from $-\sqrt{2}$ will predict the η which still should be as light as the pion.

For these reasons we give up the assumption (2.3) but still keep (2.2). We break up $\mathcal{H}_0$ into two parts:

$$\mathcal{H}_0=\mathcal{H}_{00}+\mathcal{H}_{01},$$

where $\mathcal{H}_{00}$ is invariant under $U_3\times U_3$, while for $\mathcal{H}_{01}$ we assume

$$[F_i{}^5,\mathcal{H}_{01}]=i\delta_{i0}\mathcal{P}\quad(i=0,\ldots,8), \tag{2.4}$$

where $\mathcal{P}(x)$ is a unitary singlet pseudoscalar density. In order that all the relations in (2.4) be mutually compatible, one must have[22]

$$[F_i{}^5,\mathcal{P}]=0\quad(i=1,\ldots,8). \tag{2.5}$$

An important consequence is that $\mathcal{P}$ cannot be proportional to v_0, since $[F_i{}^5,v_0]=i(\sqrt{\tfrac{2}{3}})u_i\neq 0$. It is not easy to give a simple representation of $\mathcal{P}$ and $\mathcal{H}_{01}$ in terms of quarks. According to (2.4), one obtains (for $c=-\sqrt{2}$)

$$D_0=-\tfrac{1}{2}\sqrt{2}D_8+\mathcal{P}. \tag{2.6}$$

Let us now discuss the mixing problem of the η and the η', which are supposed to be two linear combinations of φ_8 ($I=Y=0$ member of the octet) and φ_0 (the unitary singlet). If one assumes PCAC for both η and η', one would have a set of the equations

$$\begin{aligned}\varphi_\eta&=a_{11}\varphi_8+a_{12}\varphi_0=b_{11}D_8+b_{12}D_0,\\ \varphi_{\eta'}&=a_{21}\varphi_8+a_{22}\varphi_0=b_{21}D_8+b_{22}D_0.\end{aligned} \tag{2.7}$$

Using (2.1) and (2.6), and decomposing both sides of (2.7) into the octet and singlet parts, one obtains the following four equations:

$$\begin{aligned}a_{\alpha1}\varphi_8&=2(b_{\alpha1}-b_{\alpha2}/\sqrt{2})\cos\theta_0 v_8,\\ a_{\alpha2}\varphi_0&=-2(b_{\alpha1}-b_{\alpha2}/\sqrt{2})\sin\theta_0 v_0+b_{\alpha2}\mathcal{P}\quad(\alpha=1,2).\end{aligned} \tag{2.8}$$

From the last two equations one obtains

$$\frac{a_{12}}{a_{22}}=\frac{-2(b_{11}-b_{12}/\sqrt{2})\sin\theta_0 v_0+b_{12}\mathcal{P}}{-2(b_{21}-b_{22}/\sqrt{2})\sin\theta_0 v_0+b_{22}\mathcal{P}}. \tag{2.9}$$

Here one recalls that $\mathcal{P}$ is not proportional to v_0. The right-hand side of (2.9) can be a constant only if the b's satisfy the condition

$$\frac{b_{11}-b_{12}/\sqrt{2}}{b_{12}}=\frac{b_{21}-b_{22}/\sqrt{2}}{b_{22}}$$

or[23]

$$\det b=0.$$

Using this relation, one further obtains

$$\det a=0.$$

It is, therefore, impossible to have two fields both of which satisfy the PCAC condition.[24]

The origin of this unusual conclusion can be traced back to the fact that we have three pseudoscalar densities (v_8, v_0, and $\mathcal{P}$), but still have only two divergences (D_8 and D_0). One may start from the three fields instead of the two (φ_8 and φ_0) to obtain three observed pseudoscalar mesons η, η', and probably the $E(1420)$. Still it is obviously impossible for all of these three fields to obey PCAC at the same time simply because threre are only two axial-vector currents available. It is not immediately clear which of these two or three fields violate PCAC. It is even possible that none of these obeys PCAC. In the following we show some indication that the η does not obey the PCAC condition.

III. FORMALISM

In order to provide a basis of calculation, we assume

$$\begin{aligned}-\langle\varphi_i|u_j|\varphi_k\rangle&=\alpha\delta_{j0}\delta_{ik}+\alpha'\delta_{i0}\delta_{j0}\delta_{k0}+\beta d_{ijk}\\ &\quad+(\sqrt{\tfrac{2}{3}})\tfrac{1}{2}\beta'(\delta_{i0}\delta_{jk}+\delta_{ij}\delta_{k0})\quad(i,j,k=0,\ldots,8).\end{aligned} \tag{3.1}$$

The terms with α' and β' have been added to the formula (3.6) of Ref. 1.[25]

For $i=1,\ldots,7$, one can proceed in the same way as in Ref. 1 and get the results

$$\begin{aligned}&f_\pi=f_K=f,\\ &\langle 0|v_j|\varphi_k\rangle=\delta_{jk}\beta/2f\quad(j=1,\ldots,8;\,k=0,\ldots,8),\\ &\langle 0|v_0|\varphi_k\rangle=\delta_{k0}\hat{\beta}/2f\quad(k=0,\ldots,8),\end{aligned} \tag{3.2}$$

where

$$\hat{\beta}=\beta+\tfrac{1}{2}\beta'.$$

Here and throughout the paper we simplify the formulas by choosing

$$m_\pi{}^2=0,$$

so that

$$c=-\sqrt{2},\quad\alpha=0,\quad\beta=(\sqrt{\tfrac{2}{3}})m_K{}^2.$$

If one allows the violation of PCAC, one may add to the right-hand side of the first equation of (2.7) the terms which cannot be reduced to D_8 or D_0. The simplest candidate of such terms will be v_8, v_0, or $\mathcal{P}$. One can, however, use (2.1) to express v_8, for example, in terms of v_0 and D_8. Also (2.6) can be used to eliminate $\mathcal{P}$. We thus assume the form

$$(m_\eta{}^2/2f_\eta)\varphi_\eta=D_8\cos\theta-D_0\sin\theta+gv_0, \tag{3.3}$$

[21] The axial-vector current defined by $\mathfrak{F}_{N\mu}{}^5=\mathfrak{F}_{0\mu}{}^5\cos\theta_0+\mathfrak{F}_{8\mu}{}^5\times\sin\theta_0$ is conserved for $c=-\sqrt{2}$. It is easy to see that this current is made up of nucleon quarks only.

[22] The same scheme has been proposed recently by Y. Y. Lee (Ref. 2).

[23] One obtains c det $b=0$ for arbitrary c.

[24] Mathur, Okubo, and Rao (Ref. 16) assume the PCAC equations of the form (2.7) [their Eqs. (35a) and (35b)] which are not justified in our context.

[25] Equation (3.1) is supposed to be valid even for large SU_3 violation, as long as the pion or the K meson is considered to belong to a pure octet. Possible admixture of the states other than the original octet with the same I and Y will be neglected to a good approximation, since there are no known pseudoscalar mesons belonging to other SU_3 multiplets and the contribution from the continuum states is usually negligibly small.

where f_η, θ, and g are to be determined later. We consider that PCAC is violated largely if the parameter g turns out to be of order unity. We still require that the physical amplitude can be well approximated by the amplitude evaluated at the vanishing η momentum (the smoothness condition). Then we have the relation

$$\langle\eta|u_j|\varphi_k\rangle \cong im_\eta^2 \int d^4x\, T\langle 0|\varphi_\eta(x)u_j(0)|\varphi_k\rangle\,, \quad (3.4)$$

into which (3.3) is to be substituted. The first two terms on the right-hand side of (3.3) give

$$-2if_\eta\langle 0|[F_8{}^5\cos\theta - F_0{}^5\sin\theta,\, u_j(0)]|\varphi_k\rangle\,, \quad (3.5)$$

while the last term gives

$$\lim_{p\to 0} 2if_\eta g \int d^4x\, e^{-ipx} T\langle 0|v_0(x)u_j(0)|\varphi_k\rangle\,. \quad (3.6)$$

We write a dispersion relation for (3.6) as a function of p^2 and keep only the one pseudoscalar-meson state inserted in the intermediate state. We thus obtain

$$2f_\eta g \sum_X (1/2m_X{}^2)\langle 0|v_0|X\rangle\langle X|u_j|\varphi_k\rangle$$

for (3.6). The calculation may be further simplified by keeping only the η meson as X.

We also assume[26]

$$\varphi_\eta = z(\varphi_8\cos\varphi - \varphi_0\sin\varphi)\,, \quad (3.7)$$

where the angle φ could be completely independent of the other angle θ introduced in (3.3). The right-hand side of (3.4) is then expressed in terms of z, φ, α, α', β, and $\hat{\beta}$ by using (3.1). Comparing both sides of (3.4), one obtains the following set of equations:

$$\hat{\beta}/\beta = \tan\theta/\tan\varphi\,, \quad (3.8a)$$

$$\alpha'/\beta = (\sqrt{\tfrac{3}{2}})(\tan\theta/\tan\varphi)^2\,, \quad (3.8b)$$

$$\frac{f_\eta}{f} = z\frac{\cos\varphi}{\cos\theta}\left[1 - \tfrac{1}{4}(\sqrt{\tfrac{3}{2}})g\frac{z^2}{x}\frac{\sin\theta\cos^2\varphi}{\cos^2\theta}\right]^{-1}, \quad (3.8c)$$

where

$$x = m_\eta{}^2/m_8{}^2 = \tfrac{1}{2}(\sqrt{\tfrac{3}{2}})m_\eta{}^2/\beta\,.$$

We next consider the matrix element of (3.3) between the vacuum and $|\varphi_8\rangle$ to obtain

$$f_\eta/f = xz(\sqrt{\tfrac{2}{3}})\cos\varphi/\cos(\theta_0 - \theta)\,. \quad (3.8d)$$

Comparing this with (3.8c), we obtain

$$g = 4(\sqrt{\tfrac{2}{3}})z^{-2}\frac{\cos^2\theta}{\sin\theta\cos^2\varphi}(x - 1 - \tfrac{1}{2}\sqrt{2}\tan\theta)\,. \quad (3.8e)$$

[26] The "third" field may enter in the right-hand side of (3.7). We simply hope that its contribution, if any, can be neglected for the η which is the lightest of the particles with the same quantum numbers.

In the same way we calculate the matrix element of (3.3) between the vacuum and $|\varphi_0\rangle$ and obtain

$$(\gamma/\beta)\sin\theta\sin\varphi = -\sqrt{2}\cos(\theta_0-\theta)\cos\varphi(\tan\theta - \sqrt{2}\tan^2\varphi) - (4/\sqrt{3})z^{-2}\sin\theta/\cos\varphi\,, \quad (3.8f)$$

where

$$\gamma/2f = \langle 0|\mathcal{P}|\varphi_0\rangle\,.$$

If PCAC is assumed for the η (i.e., $g=0$), Eq. (3.8e) reduces to

$$\tan\theta = \sqrt{2}(x-1)\,,$$

which shows that θ whould be very small because x must be taken almost equal to 1 in the present approximation. (Note that $x=1$ means that the η mass satisfies the Gell-Mann–Okubo mass formula.) In the following, we choose $x=1$ to simplify the formulas. Equation (3.8e) now becomes[27]

$$g = -(4/\sqrt{3})z^{-2}\cos\theta/\cos^2\varphi\,. \quad (3.8g)$$

There is another type of equation which involves the η mass. Again in the soft-η limit we have

$$m_\eta{}^2 = \langle\eta|\mathcal{H}_{01}|\eta\rangle + \langle\eta|\mathcal{H}'|\eta\rangle\,. \quad (3.9)$$

The first term can be calculated in the same way as above, giving the result

$$\langle\eta|\mathcal{H}_{01}|\eta\rangle = z^2\gamma\tan\theta\sin\varphi\cos\varphi\,. \quad (3.10)$$

The second term of (3.9) is calculated in terms of $\langle\varphi_j|\mathcal{H}'|\varphi_k\rangle$ with $j,k=0,8$. Again using (3.1), together with the relations (3.8), one obtains

$$\langle\eta|\mathcal{H}'|\eta\rangle = (\sqrt{6})\beta z^2\frac{\cos^2(\theta_0-\theta)}{\cos^2\theta}\cos^2\varphi\,. \quad (3.11)$$

Substituting (3.10) and (3.11) into (3.9), one obtains another equation for γ/β in terms of x, z, φ, and θ, which is equivalent to (3.8f) if

$$z^2 = \frac{1+\sqrt{2}\tan\theta}{1+(\tan\theta)/\sqrt{2}}\,. \quad (3.12)$$

One finds that the range of θ corresponding to $-\sqrt{2} < \tan\theta < -\tfrac{1}{2}\sqrt{2}$ should be excluded. Note also that z is different from unity for $\theta \neq 0$. This means that the mixing in the η is not a simple mass-mixing type in general. Therefore, the angle φ may (or may not) be larger than the usual estimate ($\lesssim \pm 10°$), in spite of a small deviation of the η mass from the Gell-Mann–Okubo mass formula. In the following section we try two different types of analysis by assuming that φ is

[27] Equation (3.8g) shows that g does not vanish for $\theta=0$. One notices, however, that in our formulas g is always multiplied by $\hat{\beta}\sim\beta\tan\theta$. For $\theta=0$, therefore, there is no contribution from the g term anyway. One also finds that g vanishes for $\theta=\pm\tfrac{1}{2}\pi$. The factor $\hat{\beta}$ is then infinitely large, so that the product $g\hat{\beta}$ remains finite.

either small as usually expected or could be much larger.[28]

IV. APPLICATION TO $\eta \to 3\pi$ AND $\eta \to 2\gamma$

The decay $\eta \to 3\pi$ is caused by the electromagnetic interaction with $|\Delta I|=1$, which is the sum of the usual second order term and the "tadpole term." The nontadpole contribution is considered to be negligibly small because it vanishes at the limit of the soft π^0.[11,29] The effect of the tadpole term will be represented by a term

$$\mathcal{H}''=-du_3 \tag{4.1}$$

which is added to the Hamiltonian (1.1). The constant d can be estimated either by simply comparing the π-K mass difference and the K^+-K^0 mass difference (and using $c\approx-\sqrt{2}$), or by directly computing (half of) the K^+-K^0 mass difference using PCAC. Both methods give the same result,

$$d_K=-(2.0\pm0.2)\times10^{-2}\,. \tag{4.2}$$

The estimate on the basis of the U-spin invariance is about 30% larger than (4.2).[30] This value gives a reasonably good over-all fit to the electromagnetic mass splittings of the hadrons including the baryons (taking the nontadpole contribution into account).[31] Although this size of d has been used frequently in the literature, we take a slightly more relaxed point of view, and introduce the ratio r,

$$r=d_\eta/d_K\,,$$

where d_η is the effective d coefficient responsible for the decay $\eta\to3\pi$. We expect that r is within a reasonable range, probably between 0.5 and 2.

Assuming the complete dominance of the tadpole term and the conventional PCAC for the η with $d_\eta=d_K$, one obtains the π^0-η junction

$$\langle\pi^0|\mathcal{H}''|\eta\rangle=-d_K\tfrac{1}{4}\sqrt{2}m_\eta^2\equiv\mathfrak{M}_0. \tag{4.3}$$

The full amplitude for $\eta\to\pi^+\pi^-\pi^0$ is now calculated under the assumption of the π^0 pole dominance;

$$M(s_3)=\sqrt{2}d_Kf^2(s_3-m_\pi^2)\,, \tag{4.4}$$

where $s_3=-(p_\eta-p_{\pi^0})^2$, and the result from the current algebra has been used for the part $\pi^0\to\pi^+\pi^-\pi^0$.[32] This amplitude gives an excellent agreement to the observed slope,[4,10] giving a strong support to the assumption of π^0 pole dominance.[33] If, however, one accepts the estimate (4.2), the average amplitude $M_{\rm av}$ for $s_3=\frac{1}{3}m_\eta^2+m_\pi^2$ is calculated to be

$$M_{\rm av}=-0.170 \tag{4.5}$$

for the full-mass model in which the SU_3 symmetry correlates the π^0-η junction with the full masses of the pseudoscalar mesons (see Sec. I), while the result is half of (4.5) for the half-mass model. These predictions are too small compared with the experimental result

$$|M_{\rm av}{}^{\rm expt}|=0.37\pm0.06\,. \tag{4.6}$$

The value (4.5) corresponds to $\Gamma(\eta\to\pi^+\pi^-\pi^0)=124$ eV, while (4.6) corresponds to 610 eV.

In the soft-π^0 limit, our general formulas give

$$\begin{aligned}\mathfrak{M}&=-2fd_\eta(1/\sqrt{3})\langle0|v_8-\sqrt{2}v_0|\eta\rangle\\&=-d_\eta z(\beta/\sqrt{3})\ \cos\varphi(1-\sqrt{2}\ \tan\theta)\,,\end{aligned} \tag{4.7}$$

where use was made of (3.2), (3.7), and (3.8a). The same result is also obtained in the soft-η limit by the method described in Sec. III. The conventional result (4.3) corresponds to $\theta=\varphi=0$, and $r=1$. We define the ratio

$$R_{3\pi}=\mathfrak{M}/\mathfrak{M}_0=rz\cos\varphi(1-\sqrt{2}\ \tan\theta)\,. \tag{4.8}$$

Comparing (4.5) and (4.6), we expect to get

$$|R_{3\pi}{}^{\rm expt}|=2.2\pm0.7 \tag{4.9}$$

for the full-mass model. The number should be doubled for the half-mass model. If one chooses $\theta=\varphi=0$ in the latter model, one should have $r=4.4\pm1.5$, which we consider to be far beyond an acceptable range. In (4.9), the error includes the experimental error ($\sim$12%) as well as the theoretical uncertainties (tentatively assumed $\sim$20%).

We next discuss another controversial decay, $\eta\to2\gamma$. Although some of the theoretical predictions give much better agreement to the experiment than the simple SU_3 prediction,[6,34] it seems worthwhile to see if the violation of PCAC for the η is really responsible for the expected large deviation from the SU_3 symmetry.

We use Adler's argument on the anomalous term in the PCAC equation.[13] In the case of $\pi^0\to2\gamma$, the relevant equation is given by

$$D_3=(m_\pi^2/2f)\varphi_{\pi^0}+\xi_3(\alpha/16\pi)\epsilon_{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}\,, \tag{4.10}$$

where $F_{\mu\nu}$ is the electromagnetic field strength. The second term gives a good agreement to the observed rate for $\xi_3=1$. This choice corresponds to the quark model with the integral charges, or that with the fractional charges with the multiplicative factor 3 in the anomalous term. We examine the consequence of the three different quark models: the MH model,[17]

[28] The authors of Ref. 16 also conclude that the mixing mechanism would be much more complicated than the simple mass or the current mixing.

[29] D. G. Sutherland, Phys. Letters **23**, 384 (1966).

[30] S. Okubo and B. Sakita, Phys. Rev. Letters **11**, 50 (1963).

[31] S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964).

[32] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[33] The decay amplitude for $\eta\to3\pi^0$ can also be calculated in the same approximation to give $M=-4f^2\langle\pi^0|\mathcal{H}''|\eta\rangle$. It is interesting to note that the same result is obtained in the limit of two soft π^0's without using the one-π^0 pole dominance; $\lim M=(-2if)^2\times\langle\pi^0|[F_3{}^5,[F_3{}^5,\mathcal{H}'']]|\eta\rangle=-4f^2\langle\pi^0|\mathcal{H}''|\eta\rangle$. This fact seems to give another support to the π^0 pole-dominance model. In this scheme we always predict $\Gamma(\eta\to3\pi^0)/\Gamma(\eta\to\pi^+\pi^-\pi^0)\simeq1.6$, which is consistent with the recent experimental result $1.50_{-0.29}{}^{+0.15}$ [C. Baglin *et al.*, Phys. Letters **29B**, 445 (1969)].

[34] L. H. Chan, L. Clavelli, and R. Torgerson, Phys. Rev. **185**, 1754 (1969).

the HN model,[18] and the GZ model[19] with the above modification. The coefficient ξ_8 corresponding to D_8 turns out to be $\xi_8=\frac{1}{3}\sqrt{3}$ independent of the models. On the other hand, ξ_0 is given by $\sqrt{\frac{2}{3}}$ for MH, $4\sqrt{\frac{2}{3}}$ for HN, and $2\sqrt{\frac{2}{3}}$ for GZ. Then the coefficient ξ_η of the anomalous term added to the right-hand side of (3.3) is simply given by

$$\xi_\eta=\xi_8\cos\theta-\xi_0\sin\theta\,. \tag{4.11}$$

The conventional result corresponds to $\theta=0$, agreeing with the usual SU_3 limit. The ratio $R_{2\gamma}$ of the amplitude to the conventional one is then given by

$$R_{2\gamma}=\sqrt{3}(f_\eta/f)\xi_\eta\,. \tag{4.12}$$

By using (3.8c) with $x=1$, one finds that this is again a simple function of θ and φ:

$$R_{2\gamma}=\sqrt{2}z\cos\varphi\;\xi_\eta/\cos(\theta_0-\theta)\,, \tag{4.13}$$

which should be compared with

$$|R_{2\gamma}{}^{\text{expt}}|=2.47\pm0.25\,, \tag{4.14}$$

corresponding to the observed value

$$\Gamma(\eta\to2\gamma)=(1.0\pm0.2)\text{ keV}\,.$$

We try to find two different types of the solution in fitting the rates of $\eta\to3\pi$ and $\eta\to2\gamma$. In solutions I (full mass) and I′ (half mass) we set $r=1$, but we allow a large mixing angle φ. We first note that from (4.8) and (4.13) the ratio

$$\mathcal{R}=R_{3\pi}/R_{2\gamma} \tag{4.15}$$

is independent of φ. We determine θ by comparing (4.15) with the experimental result

$$|\mathcal{R}^{\text{expt}}|=0.89\pm0.23\,. \tag{4.16}$$

Further, using (4.13), we determine φ.

The solutions are, however, not unique. After removing some solutions which lead to unreasonably large $\hat{\beta}$, α', g, or z^{-1}, we obtain several solutions as shown in Table I. Angle θ is large in any of the solutions and the parameter g is certainly different from zero. Angle φ

TABLE I. Solutions I and I′ with $r=1$, for the full-mass and the half-mass models, respectively. The sign in the subscript corresponds to the unknown sign of $\mathcal{R}$.

Model and solution	θ (deg)	$\lvert\varphi\rvert$ (deg)	z	g
MH I_-	-70 ± 2	75 ± 5	1.78 ± 0.08	$-3.6_{-4.2}^{+1.5}$
MH I_-'	-76 ± 3	66 ± 7	$1.60_{-0.03}^{+0.06}$	$-1.3_{-1.3}^{+0.6}$
HN I_+	77 ± 3	71 ± 6	1.32 ± 0.02	$-2.9_{-1.8}^{+0.6}$
GZ I_+	62_{-2}^{+5}	0_{-0}^{+35}	1.26 ± 0.02	$-0.70_{-0.16}^{+0}$
GZ I_+'	76_{-5}^{+4}	46_{-19}^{+10}	1.32 ± 0.03	-0.66 ± 0.09

TABLE II. Solutions II and II′ with $\varphi\approx0$, for the full-mass and the half-mass models, respectively. The sign in the subscript corresponds to the unknown sign of $R_{2\gamma}$.

Model and solution	θ (deg)	r	z	g
GZ II_+	-25_{-8}^{+5}	$1.8_{-0.6}^{+1.8}$	$0.71_{-0.32}^{+0.10}$	$-4.2_{-8.9}^{+0.8}$
GZ II_-	65 ± 4	$0.9_{-0.4}^{+0.7}$	1.27 ± 0.02	-0.63 ± 0.10
GZ II_-'	65 ± 4	$1.8_{-0.8}^{+1.4}$	1.27 ± 0.02	-0.63 ± 0.10
HN II_+	-13.5 ± 2.0	$1.8_{-0.6}^{+0.7}$	0.89 ± 0.02	-2.8 ± 0.1

in the solution GZ I_+ could be negligibly small, yet the factor z is considerably different from unity.

The large angle φ, as suggested here, may not be excluded from the present experimental data on the η production (see, however, Ref. 20). Also, a complication may occur if the process under consideration is dominated by the $A_{2}\pi\eta$ coupling, since it is not yet clear whether there are two A_2 mesons and how they couple to the other particles.[7] It should also be emphasized that the mixing angle of the η' could be quite different from that of the η. It is even possible that the third pseudoscalar field plays an important role in the η'. For these reasons it is rather unlikely that angle φ is related to the decay modes of η' in any simple manner, without additional assumptions on the η'.

In the second alternative solutions II and II′, we assume a small φ as suggested by the usual mass mixing model, but allowing $r\neq1$. We set $\varphi=0$ except in the denominators of (3.8). We first determine θ from $R_{2\gamma}$, and then determine r from $R_{3\pi}$. In addition to the unreasonable solutions as described above, we also exclude the solutions which lead to $|r|\ll0.5$ or $\gg2$. Reasonable solutions are then selected as shown in Table II. The ratio r lies well within an acceptable range, and the choice $g=0$ (i.e., PCAC for the η) is definitely ruled out. The solution GZ II_- is found to be almost equivalent to GZ I_+. For the solution HN II_+, θ is rather small. Consequently, z is rather close to unity, so that the mixing will be very close to the simple mass mixing for this solution.

Summarizing all the solutions so far obtained, we conclude that a large amount of violation of PCAC for the η is strongly suggested from the decays $\eta\to3\pi$ and $\eta\to2\gamma$. A further improvement of the approximations will be needed, however, to select or eliminate any of the assumed models. The experimental determination of the mixing angle φ is also of great importance.

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