CONCLUSIONS

We interpret these results to mean that while the Regge model has certain distinct advantages, such as its simplicity and small number of parameters, the quality of the fits obtained is not substantially superior to that of other models.[2] In order to fit finer details of the data, it will be necessary to destroy some of the simplicity of the model. There is some hope that a clever use of a dual amplitude in this reaction will provide a better insight to the dynamics. There certainly are resonances present in the data at this energy, and a complete description of the data cannot be made without taking them into account. In the meantime, multiperipheralism seems to be an excellent approximation.

There is some controversy over whether or not it is necessary to Reggeize the pion exchange in a multiperipheral process. This study has not demonstrated any such necessity, nor has it shown the Reggeization to be bad. It seems at this point to be largely a matter of taste. Clearly, one-pion exchange in one form or another is here to stay, since reasonably good descriptions of a wide range of data are obtained from such models with a minimum of embellishment. For simple descriptions of new data, the Regge form does have the advantage of a simple parametrization.

ACKNOWLEDGMENTS

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Double-Scattering Models and Chew-Low Extrapolations*

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A formalism is presented by which absorptive corrections to single-particle and Regge-pole exchange amplitudes in quasi-two-body inelastic processes can be expressed as a highly convergent power series in t. This representation does not require a partial-wave decomposition of the scattering amplitudes and can be written down by inspection in terms of the single-particle or Regge-pole helicity amplitudes. The method is used to calculate the effects of absorption in the determination, through Chew-Low extrapolation techniques, of $\pi\pi$ phase shifts when the dipion effective mass is in the ρ region. These calculations show quantitatively the importance of absorption in the choice of extrapolation procedures.

I. INTRODUCTION

THE consideration of absorptive effects in quasi-two-body inelastic processes has received renewed theoretical attention lately with the application of absorptive corrections to Regge-pole exchange.[1–6] The absorption-modified Regge-pole (i.e., Regge-cut) model and the already successful absorption-modified one-pion-exchange model[7] seem together to form a useful phenomenonological framework through which a wealth of experimental data can be described.[1–8]

Unfortunately these double-scattering models, achieving considerable success in describing experimental results, have remained largely a tool of theorists and have not been exploited fully by experimentalists in extracting information from data. A basic reason for this is that calculations of double-scattering effects have been both cumbersome and slow. The original approach to absorption-model calculations,[7] based on the impact-parameter representation, requires a numerical integration for every value of four-momentum transfer for which the scattering amplitude is calculated. A more recent, and perhaps more "exact," approach[9]

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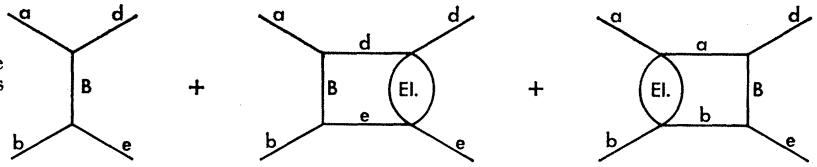

FIG. 1. Diagrams approximating the Sopkovich formula. The intermediate states are on the mass shell.

necessitates a partial-wave decomposition of the single-particle or Regge-pole exchange amplitude as well as a partial-wave decomposition of the initial- and final-state elastic scattering amplitudes. These partial-wave amplitudes are always tedious to extract if spin is taken into account and often require numerical integration. Further, the region of small four-momentum transfer squared, t, where double-scattering models are most reliable, is dominated by high partial waves. Consequently, accurate calculations require summing large numbers of terms (typically 100 or more[10]) in the partial-wave expansion of the double-scattering modified amplitudes. The unwieldy character of these calculations has long been an unattractive feature of the absorption model and inhibits a widespread application of double-scattering models to detailed analysis of data.

In this paper a different representation of the double-scattering corrections to single-particle or Regge-pole exchange amplitudes is derived. It is based on an expansion in t, as befits a peripheral model, rather than on a partial-wave expansion. The representation enjoys the following properties: (a) It is applicable to all those quasi-two-body inelastic processes in which the driving term is any t-channel single-particle exchange amplitude or any Regge-pole amplitude belonging to a general class described below; (b) the double-scattering corrections can be written as an expansion in t directly from the single-particle or Regge-pole exchange amplitude *by inspection* with *no* partial-wave decomposition of any amplitude required; (c) the expansion is highly covergent in t and thus yields fast numerical results on a computer; and (d) no new approximation is made in addition to those standard to the absorption model.[7,9,11]

In Sec. II we develop the double-scattering corrections to a driving amplitude of very general form including, as special cases, single-particle and Regge-pole exchange amplitudes. The application of this general formulation yields in Sec. III the double-scattering corrections to single-particle exchange and Regge-pole exchange. Also in this section we apply the method to calculate the effects of absorption in the determination, through Chew-Low extrapolation techniques,[12] of $\pi\pi$ phase shifts when the dipion effective mass is in the ρ region. A discussion of our results and suggestions for future applications are contained in Sec. IV.

II. THEORY

Consider the quasi-two-body inelastic process $a+b\to d+e$. Our normalization of the helicity amplitudes[13] $\langle\theta;\delta\epsilon|M|0;\alpha\beta\rangle$ is chosen such that the c.m. differential cross section for this reaction with unpolarized particles is given by

$$\frac{d\sigma}{d\Omega}=\frac{1}{(2S_a+1)(2S_b+1)}\sum_{\alpha\beta\delta\epsilon}|\langle\theta;\delta\epsilon|M|0;\alpha\beta\rangle|^2, \quad (2.1)$$

where θ is the c.m. scattering angle, S_j is the spin of particle j, and α, β, δ, ϵ are the helicities of particles a, b, d, e, respectively. We approximate[6,14,15] the Sopkovich formula[16] by the contributions to the reaction arising from the diagrams shown in Fig. 1, which can be written in operator form

$$M=B+i\Lambda(T_FB+BT_I), \quad (2.2)$$

in which B is the driving (single-particle, Regge-pole exchange) term and T_FB, BT_I are the double-scattering (absorption, Regge-cut) corrections. T_I and T_F are the elastic scattering transition operators associated with the initial and final states, and Λ is the "coherent inelastic factor" introduced in Ref. 14 which phenomenologically takes into account the contribution of intermediate states other than the (mass-shell) states a, b, d, e to the double-scattering correction.

In the c.m. system, Eq. (2.2) can be evaluated in the linear momentum representation or, equivalently, in the angular momentum representation. In terms of the former,

$$\begin{aligned}\langle\theta;\delta\epsilon|M|0;\alpha\beta\rangle=&\langle\theta;\delta\epsilon|B|0;\alpha\beta\rangle\\&+\frac{i\Lambda p}{4\pi}\Big[\sum_{\alpha'\beta'}\int d\Omega'(\langle\theta;\delta\epsilon|T_F|\Omega';\alpha'\beta'\rangle\\&\times\langle\Omega';\alpha'\beta'|B|0;\alpha\beta\rangle+\langle\theta;\delta\epsilon|B|\Omega';\alpha'\beta'\rangle\\&\times\langle\Omega';\alpha'\beta'|T_I|0;\alpha\beta\rangle)\Big], \quad (2.3)\end{aligned}$$

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where for simplicity we have introduced high-energy kinematics in which the c.m. momenta of the initial and final states are equal to p. The amplitudes appearing in (2.3) are normalized in accordance with Eq. (2.1). In the angular momentum representation, Eq. (2.2) takes the form

$$M_{\delta\epsilon\alpha\beta}{}^{J}=B_{\delta\epsilon\alpha\beta}{}^{J}+i\Lambda\sum_{\alpha'\beta'}(T_{F\delta\epsilon\alpha'\beta'}{}^{J}B_{\alpha'\beta'\alpha\beta}{}^{J}+B_{\delta\epsilon\alpha'\beta'}{}^{J}T_{I\alpha'\beta'\alpha\beta}{}^{J}). \quad (2.4)$$

The amplitudes of Eq. (2.3) are related to those of (2.4) as follows:

$$\langle\theta;\delta\epsilon|A|0;\alpha\beta\rangle=(1/p)\sum_{J}(2J+1)A_{\delta\epsilon\alpha\beta}{}^{J}d_{\lambda\nu}{}^{J}(\theta), \quad (2.5)$$

where $\lambda=\alpha-\beta$, $\nu=\delta-\epsilon$, and $d_{\lambda\nu}{}^{J}(\theta)$ are the reduced rotation functions.

It should be noted that Eq. (2.4) is equivalent to the familiar absorption-model prescription[7,9] if the initial- and final-state elastic scatterings are taken as equal. For high-energy peripheral processes, however, the use of Eq. (2.4) in calculations is extremely unwieldy, requiring an often cumbersome partial-wave decomposition of the amplitudes B, T_I, and T_F, as well as large numbers of partial waves (typically 100 or more[10]), in order to get accurate results in the region of small four-momentum transfer squared t. It will be shown for these processes that the scattering amplitude $\langle\theta;\delta\epsilon|M|0;\alpha\beta\rangle$ is much more conveniently calculable from an expansion (to be derived below) in t, than from a partial-wave expansion.

Consider the amplitude $\langle\theta;\delta\epsilon|B|\Omega';\alpha'\beta'\rangle$. The states $|\theta'\phi';\alpha'\beta'\rangle$ and $|\theta;\delta\epsilon\rangle$ can be written in terms of angular momentum states with the aid of the relation

$$|\theta\phi;\alpha\beta\rangle=\sum_{JM}\left(\frac{2J+1}{4\pi}\right)^{1/2}D_{M\lambda}{}^{J}(\phi,\theta,-\phi)|JM;\alpha\beta\rangle, \quad (2.6)$$

where $\lambda=\alpha-\beta$ and $D_{M\lambda}{}^{J}$ is the Wigner rotation function. Upon performing this expansion and using the addition theorem for the rotation functions, one can easily show that

$$\langle\theta;\delta\epsilon|B|\theta'\phi';\alpha'\beta'\rangle=\langle\theta''\phi'';\delta\epsilon|B|0;\alpha'\beta'\rangle e^{-i\lambda'(\phi''-\psi'')}, \quad (2.7)$$

where $\lambda'=\alpha'-\beta'$ and

$$\begin{aligned}\cos\theta''&=\cos\theta'\cos\theta+\sin\theta\sin\theta'\cos\phi',\\ \cos\theta'&=\cos\theta''\cos\theta+\sin\theta\sin\theta''\cos\phi'',\\ \sin\phi''\sin\theta''&=-\sin\phi'\sin\theta',\\ \psi''&=\phi''+\tfrac{1}{2}\sin\theta\sin\theta'\sin\phi'+O(tt'/p^4),\end{aligned} \quad (2.8)$$

where $t=-2p^2(1-\cos\theta)$ and $t'=-2p^2(1-\cos\theta')$ in the approximation of high-energy kinematics.

Thus the third term of (2.3), defined to be $\delta M_{\delta\epsilon\alpha\beta}{}^{I}$, can be written

$$\delta M_{\delta\epsilon\alpha\beta}{}^{I}=\frac{i\Lambda p}{4\pi}\sum_{\alpha'\beta'}\int d\Omega''\times\{\exp[-i(\nu-\lambda')\phi''-i(\lambda'-\lambda)\phi'-i\lambda'(\phi''-\psi'')]\times\langle\theta'';\delta\epsilon|B|0;\alpha'\beta'\rangle\langle\theta';\alpha'\beta'|T_I|0;\alpha\beta\rangle\}, \quad (2.9)$$

where $\nu=\delta-\epsilon$, $\lambda'=\alpha'-\beta'$, $\lambda=\alpha-\beta$, and in which we have factored out the azimuthal dependence of the amplitudes under the integral and changed the integration variable from Ω' to Ω''. We make the usual assumption that the elastic scattering amplitude is primarily helicity nonflip and can be written

$$\langle\theta';\alpha'\beta'|T_I|0;\alpha\beta\rangle=N\delta_{\alpha\alpha'}\delta_{\beta\beta'}e^{\frac{1}{2}At'}, \quad (2.10)$$

where $t'=-2p^2(1-\cos\theta')$; N is complex and taken independent of t'. From the optical theorem, $N=ip\sigma\times(1-i\rho)/4\pi$, where σ is the total initial-state cross section, and ρ is the ratio of the real to imaginary parts of the elastic scattering amplitude. In what follows, it will be seen that representing the elastic scattering amplitude as a sum of terms of the form (2.10) with arbitrary phases presents no further complications.

If the term of order tt'/p^4 in (2.8) is neglected, the integral over ϕ'' in (2.9) can be performed, resulting in

$$\delta M_{\delta\epsilon\alpha\beta}{}^{I}=-\frac{\Lambda\sigma p^2(1-i\rho)}{8\pi}\left(1-\frac{\lambda(\nu-\lambda)}{2Ap^2}\right)\times\int d\cos\theta''\{\langle\theta'';\delta\epsilon|B|0;\alpha\beta\rangle I_n(Ap^2\sin\theta\sin\theta'')\times\exp[-Ap^2(1-\cos\theta\cos\theta'')]\}, \quad (2.11)$$

where I_n is the modified Bessel function of the first kind and $n=|\lambda-\nu|$. The other double-scattering correction $\delta M_{\delta\epsilon\alpha\beta}{}^{F}$ can be calculated in the same way. For simplicity of presentation, however, we take the final-state and initial-state elastic scatterings equal, in which case

$$\left(1-\frac{\lambda(\nu-\lambda)}{2Ap^2}\right)^{-1}\delta M_{\delta\epsilon\alpha\beta}{}^{I}=\left(1+\frac{\nu(\nu-\lambda)}{2Ap^2}\right)^{-1}\delta M_{\delta\epsilon\alpha\beta}{}^{F}.$$

Equation (2.3) can now be rewritten in terms of four-momentum transfer squared and reads, with an obvious simplifying change of notation,

$$M_{\delta\epsilon\alpha\beta}=B_{\delta\epsilon\alpha\beta}-\frac{\Lambda\sigma(1-i\rho)}{8\pi}\left(1+\frac{n^2}{4Ap^2}\right)\int_{-4p^2}^{0}dt''\times\{B_{\delta\epsilon\alpha\beta}(t')\exp[\tfrac{1}{2}A(t+t''+\tfrac{1}{2}p^{-2}tt'')]\times I_n[A(-t-\tfrac{1}{4}p^{-2}t^2)^{1/2}(-t''-\tfrac{1}{4}p^{-2}t''^2)^{1/2}]\}. \quad (2.12)$$

This formula can be computed by numerical integration for an arbitrary driving term $B_{\delta\epsilon\alpha\beta}$. However, as we have indicated previously, for $|t|$ small in magnitude

compared to $4p^2$, Eq. (2.12) can be expanded in a simple, highly convergent power series in t, provided that the driving term is taken to be one of two currently popular models—single-particle exchange or Regge-pole exchange.

High-Energy Small-Angle Approximation

In the small-$|t|$, high-energy region where $|t|\ll 4p^2$ and $|4p^2|\to\infty$, the double-scattering term of Eq. (2.12) takes the form

$$\delta M_{\delta\epsilon\alpha\beta}=-\frac{\Lambda\sigma}{8\pi}(1-i\rho)\int_{-\infty}^{0}dt' B_{\delta\epsilon\alpha\beta}(t')e^{\frac{1}{2}A(t+t')}\times I_n(A(tt')^{1/2}),\quad (2.13)$$

a result derived previously by use of the "impact-parameter representation"[17]; however, Eq. (2.12) makes clear the approximations inherent to the "impact-parameter representation," and could be used to calculate corrections to this approximation.

We assume, and confirm later, that the driving term $B_{\delta\epsilon\alpha\beta}$ is of the form[18]

$$B_{\delta\epsilon\alpha\beta}=(-t)^{n/2}\beta(t)P_{\delta\epsilon\alpha\beta}(t)e^{\frac{1}{2}Bt},\quad (2.14)$$

where $P_{\delta\epsilon\alpha\beta}$ is a polynomial in t and $\beta(t)$ is a function which has a Laplace transform representation,

$$\beta(t)=\int_0^\infty F(x)e^{xt}dx,$$

in the physical region of four-momentum transfer. The integral over t' in Eq. (2.13) can then be performed, giving

$$\delta M_{\delta\epsilon\alpha\beta}=-\frac{\Lambda\sigma(1-i\rho)}{8\pi}e^{\frac{1}{2}At}(-t)^{n/2}A^nP_{\delta\epsilon\alpha\beta}(\partial/\partial x)\times\int_0^{C^{-1}}du\,F(z)u^{n-1}e^{-\frac{1}{2}A^2tu},\quad (2.15)$$

where $z=\frac{1}{2}(1/u-C)$, $x=\frac{1}{2}C$, and $C=A+B$. Expressing $F(z)$ in terms of the inverse Laplace transform, expanding the exponential under the integral of (2.15), and integrating term by term brings (2.15) into the form

$$\delta M_{\delta\epsilon\alpha\beta}=-\frac{\Lambda\sigma(1-i\rho)}{8\pi}e^{\frac{1}{2}At}(-t)^{n/2}A^n\times\sum_{m=0}^{\infty}\frac{D^m}{m!}\left[P_{\delta\epsilon\alpha\beta}(\partial/\partial x)\frac{K_{n+m+1}(x)}{C^{n+m}}\right],\quad (2.16)$$

[17] See, for example, Ref. 6.

[18] We only exhibit the t dependence of this amplitude explicitly.

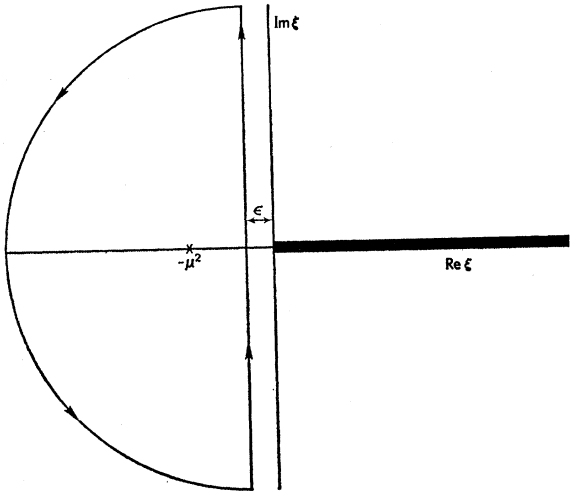

FIG. 2. Integration contour for evaluating K_n for single-particle exchange.

where $D=-\frac{1}{2}A^2t$ and

$$K_n(x)=\frac{1}{2\pi i}\int_{-i\infty-\epsilon}^{i\infty-\epsilon}d\xi\,e^{-\xi x}\beta(-\xi)E_n(-\xi x),\quad (2.17)$$

in which E_n is the exponential integral function, the properties of which are discussed in the Appendix. Equation (2.16) is the basic result of this paper.[19] For the special cases of t-channel single-particle and Regge-pole exchange, it will be shown in Sec. III that (2.17) can be evaluated explicitly without difficulty.

III. APPLICATIONS

Single-Particle Exchange

In the high-energy, small-$|t|$ region, the single-particle exchange amplitude (t-channel exchange) is of the form[20]

$$B_{\delta\epsilon\alpha\beta}=[(-t)^{n/2}/(\mu^2-t)]P_{\delta\epsilon\alpha\beta}(s,t),\quad (3.1)$$

where μ is the exchanged particle mass, $n=|\lambda-\nu|$, and $P_{\delta\epsilon\alpha\beta}$ is a polynomial in t. Consistent with the derivation of Eq. (2.16), we have neglected terms of order $t/4p^2$ in writing (3.1). In order to evaluate (2.16) with this driving term, one sets $\beta(t)=(\mu^2-t)^{-1}$ and $C=A$. From the asymptotic properties of $E_n(z)$ given in the Appendix, it is easy to show that the integral (2.17) can be continued as shown in Fig. 2 on the left-hand infinite semicircle in the complex ξ plane. The only contribution to the integral arises from the propagator pole, giving

$$K_n=\frac{1}{2\pi i}\int_{-i\infty-\epsilon}^{i\infty-\epsilon}d\xi\frac{e^{-\frac{1}{2}\xi A}}{\mu^2+\xi}E_n(-\frac{1}{2}\xi A)=e^{\frac{1}{2}\mu^2A}E_n(\tfrac{1}{2}\mu^2A).\quad (3.2)$$

[19] At relatively low energies, where $t_{\min}(\equiv t$ at $\cos\theta=1)$ is not negligible, t and t' should be replaced by $t-t_{\min}$ and $t'-t_{\min}$, respectively, in Eqs. (2.13)–(2.16).

[20] L. Durand III and Y. T. Chiu, Phys. Rev. **139**, B646 (1965).

Thus,

$$M_{\delta\epsilon\alpha\beta}=(-t)^{n/2}\left\{\frac{P_{\delta\epsilon\alpha\beta}}{\mu^2-t}-\frac{\Lambda\sigma(1-i\rho)}{8\pi}e^{\frac{1}{4}At}A^n \times\left[\sum_{m=0}^{\infty}\frac{D^m}{m!2^{n+m}}P_{\delta\epsilon\alpha\beta}(s,\partial/\partial x)F_{n+m+1}(x)\right]\right\}, \quad (3.3)$$

where $x=\frac{1}{2}A$ and $F_n(x)=e^{\mu^2 x}E_n(\mu^2 x)/x^{n-1}$. From Eq. (3.3), one can write down in a form particularly suitable for numerical calculations the general absorption-modified single-particle exchange amplitude by *inspection*. We illustrate the use of (3.3) and its good convergence properties by a specific example.

$\pi+p\to\rho+N$; Chew-Low Extrapolation

In the high-energy limit, assuming π exchange, only three independent helicity amplitudes $B_{\delta\epsilon\beta}$ survive as driving terms for the reaction $\pi+p\to\rho+N$:

$$B_{++-}=N_B t/(\mu^2-t), \quad (3.4)$$

$$B_{0+-}=-\frac{N_B(-t)^{1/2}(M_\rho+M_\rho^{-1}t)}{\sqrt{2}(\mu^2-t)}, \quad (3.5)$$

$$B_{-+-}=N_B(-t)/(\mu^2-t), \quad (3.6)$$

where μ is the pion mass, M_ρ is the ρ mass, s is the c.m. energy squared, and N_B is a common factor appropriate to the normalization (2.1), which depends on c.m. energy and coupling constants. The absorption-modified amplitudes for this reaction can be read off from Eq. (3.3) by inspection.

Consider, for example, the amplitude M_{0+-} with net helicity flip $n=1$. From (3.5) and (3.3), one finds immediately that

$$M_{0+-}=-\frac{N_B(-t)^{1/2}}{\sqrt{2}}\left[\frac{M_\rho+M_\rho^{-1}t}{\mu^2-t}-\frac{\Lambda\sigma(1-i\rho)}{8\pi}e^{\frac{1}{4}At}A \times\sum_{m=0}^{\infty}\frac{D^m}{m!2^{m+1}}(M_\rho+M_\rho^{-1}\partial/\partial x)F_{m+2}(x)\right]. \quad (3.7)$$

The other amplitudes can be found in the same way. Since the representation of the absorption-modified amplitude given in (3.7) is highly covergent in the nonphysical as well as physical region of t, it provides a convenient basis for taking absorptive effects into account in a modified Chew-Low extrapolation.[12]

As a simple, semiquantitative example, we consider the reaction $\pi+p\to(\pi\pi)+N$ when the effective mass $M_{\pi\pi}$ of the dipion system is in the ρ region where the $I=1$, P-wave $\pi\pi$ phase shift dominates. If other phase shifts can be neglected in this region, the dipion angular distribution, integrated over azimuthal angle, is given in terms of spin-density matrices in the Gottfried-Jackson (GJ) frame by[21]

$$W(\theta^*)=\tfrac{3}{2}\rho_{11}+\tfrac{3}{2}(\rho_{00}-\rho_{11})\cos^2\theta^* \equiv a_0+a_2\cos^2\theta^*. \quad (3.8)$$

In order to extract information about the $\pi\pi$ phase shifts, experimentalists often extrapolate to the pion pole what, in this dipion mass region, is equivalent to[22–24]

$$A_l(t)\equiv -a_l(\mu^2-t)^2\frac{1}{t}\frac{d\sigma}{d\Omega}(\pi p\to\rho N). \quad (3.9)$$

At the pion pole, keeping only the dominant P-wave $\pi\pi$ interaction, one should have (a) $A_0(\mu^2)=0$, and (b) $A_2(\mu^2)$ proportional to the P-wave $\pi\pi$ cross section,[22–24] which should reach the unitarity limit when $M_{\pi\pi}=M_\rho$. Any believable extrapolation procedure should give (a) and (b) before the sensitive question of measuring other phase shifts is considered.

With the absorption-modified amplitudes (3.7) we can calculate $A_l(t)$ in the physical region as well as the nonphysical region of t and test whether current parametrizations[22–24] of these quantities are compatible with absorptive effects. To do this, we recall that the density-matrix elements in the helicity (H) frame are given in terms of the amplitudes (3.7) by

$$\hat{\rho}_{mm'}=N\sum_{\epsilon\beta}M_{m\epsilon\beta}M_{m'\epsilon\beta}^*, \quad (3.10)$$

and in the GJ frame by[7]

$$\rho_{mm'}=\sum_{\mu\mu'}d_{m\mu}^1(\psi)\hat{\rho}_{\mu\mu'}d_{m'\mu'}^1(\psi), \quad (3.11)$$

where N is chosen such that $\mathrm{Tr}\hat{\rho}=1$. In the high-energy, small-$|t|$ approximation,

$$\sin\psi=\frac{2(-t)^{1/2}}{M_\rho(1-M_\rho^{-2}t)}, \quad \cos\psi=\frac{1+M_\rho^{-2}t}{1-M_\rho^{-2}t}. \quad (3.12)$$

and

In Fig. 3 we show $A_2(t)$ calculated using the absorption-modified amplitudes (3.7) with Λ taking a range of values. $\Lambda=0$ corresponds to no absorption, and $\Lambda=1.61$ corresponds, in the language of the absorption model, to "total absorption of the lowest partial wave."[25] The striking feature of this figure is the appearance of a

[21] K. Gottfried and J. D. Jackson, Nuovo Cimento **33**, 309 (1964).

[22] S. Marateck *et al.*, Phys. Rev. Letters **21**, 1613 (1968).

[23] J. P. Baton, G. Laurens, and J. Reignier, Phys. Letters **25B**, 419 (1967).

[24] V. Hagopian *et al.*, in Proceedings of the Conference on $\pi\pi$ and $K\pi$ Interactions, Argonne National Laboratory, 1969, p. 149 (unpublished).

[25] The lowest partial wave is completely absorbed if $\Lambda=4\pi A/\sigma$. In this paper we take $A=8$ GeV^{-2}, $\sigma=25$ mb, and $\rho=0$ since the calculations are highly insensitive to ρ.

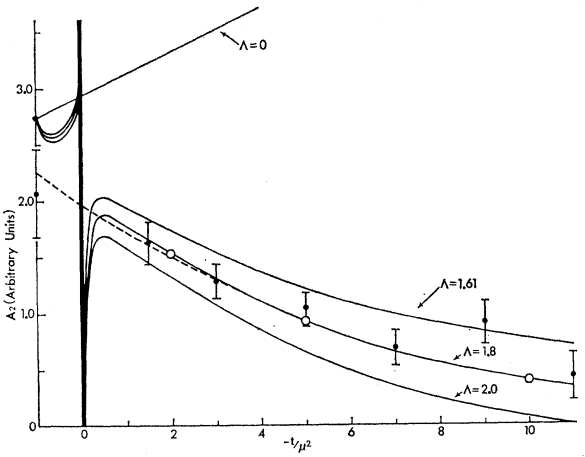

FIG. 3. Calculation of $A_2(t)$ showing dependence on Λ. The data are from Ref. 24.

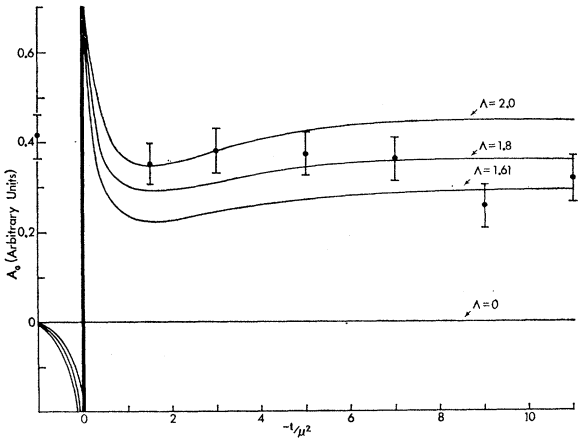

FIG. 4. Calculation of $A_0(t)$ showing dependence on Λ. The data are from Ref. 24.

pole in $A_2(t)$ at $t=0$ which arises from absorption. The presence of this pole invalidates the use of any polynomial representation (in t) of $A_2(t)$ as a basis for extrapolating from the physical region to the pion pole. For comparison we also exhibit in Fig. 3 recent experimental measurements of $A_2(t)$ corresponding to the reaction $\pi^- p \to \pi^-\pi^+ n$ with $740 \leq M_{\pi\pi} \leq 780$ MeV together with a quadratic extrapolation of the data to $t=\mu^2$.[24] Although, in this semiquantitative calculation, we have not attempted detailed fitting of our model to the data, one observes that $\Lambda=1.8$ appears to represent the data fairly well.[26] In order to estimate the error incurred from a quadratic extrapolation of $A_2(t)$, we have fitted a quadratic to the calculated $A_2(t)$, with $\Lambda=1.8$, through the points $-t=2$, 5, and 10 μ^2. Extrapolation of this polynomial to the pion pole gives rise to an error of 15% in $A_2(\mu^2)$, as shown by the dashed curve in Fig. 3. This error would be reflected in a $\pi\pi$ cross section 15% too small in this dipion mass region and which would thus fall below the unitarity limit. This prediction should be compared to the experimental extrapolation of Fig. 3 which leads to a P-wave $\pi\pi$ cross section 20% below the unitarity limit.[22,24]

Figure 4 shows $A_0(t)$ calculated with the same range of Λ's, together with corresponding experimental data and a linear extrapolation of the data to the pion pole.[24] Here the pitfalls of extrapolating over a pole with a polynomial are even more dramatic. Although $A_0(t)$ should not extrapolate precisely to zero in the presence of an $I=0$, or $I=2$ S-wave $\pi\pi$ interaction,[22–24] it is clear from Fig. 4 that the absorption model predicts that $A_0(t)$ is made up largely of P-wave "leakage" coming from absorptive depolarization effects and that a simple polynomial extrapolation of $A_0(t)$ from the physical region to $t=\mu^2$ will vastly overestimate the contribution of the S-wave $\pi\pi$ interaction at the pion pole. This fact has been observed by experimentalists.[24] In addition to $A_0(t)$ and $A_2(t)$, density-matrix elements in either the GJ or H reference frame generally develop, because of absorption, poles in the unphysical region between $t=\mu^2$ and $t=0$ and cannot be represented by polynomials in t for purposes of extrapolation.[27] The existence of this pole structure in $A_2(t)$, $A_0(t)$, and density-matrix elements is essentially a model-independent effect which does not depend on the details of the absorption model and which will arise in any model which destroys the delicate relationship (evasion) at $t=0$ among the helicity amplitudes in the single-particle-exchange mechanism.[28] The predictions, given in Figs. 3 and 4, of the quantitative *effect* in extrapolations of the poles in $A_0(t)$ and $A_2(t)$ at $t=0$ are, of course, dependent on the validity of the absorption model.

Using the amplitudes (3.7) it is equally simple to find those physical quantities which can be represented by a finite polynomial in the region $\mu^2 \geq t \geq -10\mu^2$ when $M_{\pi\pi} \approx M_\rho$. We list a few:

$$\text{(a)} \qquad (\mu^2-t)^2 \frac{d\sigma}{d\Omega}(\pi p \to \rho N),$$

$$\text{(b)} \qquad \rho_{ii}(\mu^2-t)^2 \frac{d\sigma}{d\Omega}(\pi p \to \rho N),$$

where ρ_{ii} is any diagonal density-matrix element evaluated in either the GJ or H reference frame. In Figs. 5 and 6 are shown $\frac{3}{2}\rho_{11}(\mu^2-t)^2 d\sigma/d\Omega$ and $\frac{3}{2}(\rho_{00}-\rho_{11})$

[26] The relative normalization of the curves of Figs. 3–6 was fixed by normalizing the calculated $A_2(t)$, with $\Lambda=1.8$, of Fig. 3 to the measured results of Ref. 24. This value of Λ also fits density-matrix elements well (independent of normalization) and agrees favorably with the values of Λ found in Ref. 6 and Ref. 14.

[27] For discussion of attempts at polynomial extrapolations of density-matrix elements, see J. P. Baton and G. Laurens, Phys. Letters **26B**, 471 (1968); Phys. Rev. **176**, 1574 (1968); and Ref. 24.

[28] See, for example, G. Kane and M. Ross, Phys. Rev. **177**, 2353 (1969); C. D. Froggatt and D. Morgan, *ibid.* **187**, 2044 (1969); and P. K. Williams, Phys. Rev. D **1**, 1312 (1970).

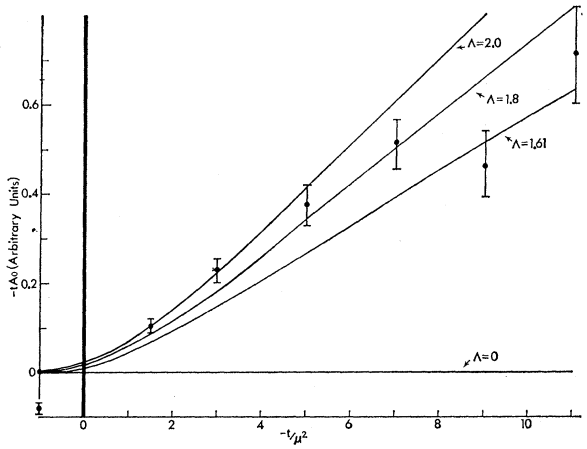

FIG. 5. Calculation of $-tA_0(t)$ showing dependence on Λ. The data are reconstructed from Ref. 24.

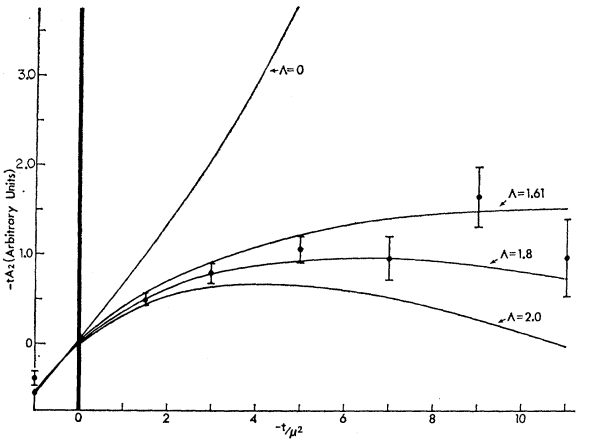

FIG. 6. Calculation of $-tA_2(t)$ showing dependence on Λ. The data are reconstructed from Ref. 24.

$\times(\mu^2-t)^2 d\sigma/d\Omega$, where the density-matrix elements are evaluated in the GJ frame and are again calculated for a range of values of Λ in order to demonstrate the dependence of these objects on absorption strength. The data points and extrapolated points are reconstructed from the results of Ref. 24.

From these figures, it is clear that the main structure exhibited in the calculated curves lies in the t region $|t|<2\mu^2$, where reliable data are predicted to be of critical importance for a model-independent extrapolation. Since most experimental data lie outside of this region, however, a model-dependent extrapolation may be required with the present data. For this purpose, the absorption-modified amplitudes (3.3) are well suited: They can be written down in easily calculable form by inspection; there is only one parameter Λ (in addition to phase shifts); and the representation is highly convergent in t, which means that the amplitudes can be readily continued to the unphysical region of t and evaluated rapidly on a computer. To stress this last point, we take terms only up to t^2 in the expansion (3.7). This approximation results in density-matrix elements accurate to within 8% (0.8%) and $d\sigma/d\Omega$ accurate to within 1% (0.1%) compared to the "exact" $[t^{\max}=t^{35}]$ calculation at $-t=10\mu^2$ ($5\mu^2$).

Regge-Pole Exchange

It has been proposed that the s-channel helicity amplitudes corresponding to the exchange (in the t channel) of a bosonic Regge trajectory $\alpha(t)$, with signature τ, parity η, G parity g, and isospin I, is of the form[29]

$$B_{\delta\epsilon\alpha\beta}=\frac{(-t)^{n/2}\gamma_{\delta\epsilon\alpha\beta}(1+\tau e^{-i\pi\alpha})}{(\sqrt{s})\sin\pi\alpha}\left(\frac{s}{s_0}\right)^{\alpha}, \quad (3.13)$$

where $n=|\lambda-\nu|$, s_0 is the scale factor, and the residue $\gamma_{\delta\epsilon\alpha\beta}$ is a regular function of t. In evaluating the double-scattering correction (2.16) to Regge-pole exchange, it is convenient to consider trajectories of even and odd signature separately.

Even Signature

If $\tau=+$ and $\alpha(t)=\alpha_0+\alpha' t$, Eq. (3.13) can be written

$$B_{\delta\epsilon\alpha\beta}=(-t)^{n/2}(e^G/\sqrt{s})\beta_{\delta\epsilon\alpha\beta}(t)e^{\frac{1}{2}Bt}, \quad (3.14)$$

where

$$G=-\tfrac{1}{2}i\pi\alpha_0+\alpha_0\ln(s/s_0), \quad (3.15)$$

$$B=2\alpha'\ln(s/s_0)-i\pi\alpha', \quad (3.16)$$

and

$$\beta_{\delta\epsilon\alpha\beta}=\frac{\gamma_{\delta\epsilon\alpha\beta}(t)}{\sin\frac{1}{2}\pi\alpha}. \quad (3.17)$$

For linear trajectories, $\beta_{\delta\epsilon\alpha\beta}$ is a meromorphic function of t and has the representation

$$\beta_{\delta\epsilon\alpha\beta}=\sum_{\kappa}\frac{Q_{\delta\epsilon\alpha\beta}{}^{\kappa}(t)}{\mu_\kappa{}^2-t}+H(t). \quad (3.18)$$

The sum in (3.18) runs over the particles on the Regge trajectory, μ_κ is the mass of the κth recurrence, $Q_{\delta\epsilon\alpha\beta}{}^{\kappa}$ is a polynomial in t, and $H(t)$ is an entire function.

We assume that the entire function $H(t)$ is a polynomial or, in the small-$|t|$ region where the double-scattering formula (2.16) is valid, can be approximated by a polynomial in t. With this weak restriction,

$$\beta_{\delta\epsilon\alpha\beta}(t)=\sum_{\kappa}\frac{P_{\delta\epsilon\alpha\beta}{}^{\kappa}(t)}{\mu_\kappa{}^2-t}, \quad (3.19)$$

where again $P_{\delta\epsilon\alpha\beta}{}^{\kappa}(t)$ is a polynomial.[30] It is a simple matter now to extract from (2.16) the double-scat-

[29] G. Cohen-Tannoudji, Ph. Salin, and A. Morel, Nuovo Cimento **55A**, 412 (1968).

[30] This is a generalization of the Regge amplitudes used in Ref. 6.

tering-corrected Regge-pole exchange amplitude:

$$M_{\delta\epsilon\alpha\beta}=B_{\delta\epsilon\alpha\beta}-\frac{\Lambda\sigma(1-i\rho)}{8\pi\sqrt{s}}(-t)^{n/2}e^{(\frac{1}{2}At+G)}A^n \times\sum_{m=0}^{\infty}\frac{D^m R_{n+m+1}{}^{+}}{m!2^{n+m}}, \quad (3.20)$$

in which $x=\frac{1}{2}(A+B)$, $n=|\lambda-\nu|$, $D=-\frac{1}{2}A^2t$, and

$$R_n{}^{+}=\sum_{\kappa}P_{\delta\epsilon\alpha\beta}{}^{\kappa}(\partial/\partial x)\frac{e^{\mu_\kappa{}^2x}E_n(\mu_\kappa{}^2x)}{x^{n-1}}. \quad (3.21)$$

The validity of (3.21) is established, as for the case of single-particle exchange, by closing the contour of (2.17) along the left-hand infinite semicircle of the ξ plane and noting from the asymptotic forms of the exponential integral function (see the Appendix) that only the poles associated with the particles lying on the Regge trajectory contribute to the integral.

Odd Signature

If $\tau=-$, the double-scattering corrected Regge-pole amplitude can be written as (3.20) except that

$$R_n{}^{-}=i\sum_{\kappa}P_{\delta\epsilon\alpha\beta}{}^{\kappa}(\partial/\partial x)\frac{e^{\mu_\kappa{}^2x}E_n(\mu_\kappa{}^2x)}{x^{n-1}}. \quad (3.22)$$

Using Eq. (3.20) and the analogous expression for negative signature, one can write down by *inspection* the double-scattering-corrected Regge-pole exchange amplitude in a form convenient for numerical calculations.

IV. DISCUSSION

We have derived a representation for the double-scattering corrections to single-particle and Regge-pole exchange in quasi-two-body inelastic processes which can be written down in easily calculable form by inspection. No partial-wave decomposition of any amplitude is required. Besides simplifying the calculations necessary in the usual *comparison* of double-scattering models with data, this formulation, yielding fast numerical results, facilitates substantially the inclusion of double-scattering effects in the detailed *analysis* of data.

As an example of such an application, we have considered the effects of absorption in Chew-Low extrapolations to determine $\pi\pi$ phase shifts, employing the absorption model (a) as a guide to finding, in the experimentally inaccessible region $-\mu^2\leq -t\leq\mu^2$, the analytic structure of the quantities often extrapolated by experimentalists and (b) as a model to predict the detailed t dependence of these quantities in the region $-\mu^2\leq -t\leq 11\mu^2$. These semiquantitative calculations indicate that reliable data in the region $|t|<0.04$ GeV^2 will be of critical importance for a model-independent extrapolation to succeed. The lack of data in this region may require at the present time, however, the use of the absorption model not just as a guide, but as an extrapolation *model* in the determination of $\pi\pi$ phase shifts. For such a purpose the representation of the absorption model given in this paper, easily continuable to the unphysical region, is well suited. This formulation also provides a convenient and fast method for taking into account double-scattering corrections in ascertaining spin-parity quantum numbers of peripherally produced resonances.

ACKNOWLEDGMENTS

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APPENDIX

For completeness we summarize the properties[31] of the exponential integral functions relevant to this paper.

Definition and Series Representation

$$E_n(z)=\int_1^{\infty}\frac{e^{-zt}}{t^n}dt \quad (n=0, 1, 2, \ldots; \mathrm{Re}z>0)$$

$$=\frac{(-z)^{n-1}}{(n-1)!}[-\ln z+\psi(n)] - \sum_{m=0,\ m\neq n-1}^{\infty}\frac{(-z)^m}{(m-n+1)m!} \quad (|\arg z|<\pi),$$

where

$$\psi(1)=-\gamma, \quad \psi(n)=-\gamma+\sum_{m=1}^{n-1}\frac{1}{m} \quad (n>1),$$

and $\gamma=0.5772156649\ldots$.

Derivatives and Asymptotic Expansion

$$dE_n(z)/dz=-E_{n-1}(z) \quad (n=1, 2, 3, \ldots),$$

$$E_n(z)\sim(e^{-z}/z)(1-n/z+\cdots) \quad (|\arg z|<\tfrac{3}{2}\pi).$$

[31] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), Chap. 5, p. 227.