Linearly Rising Trajectories in an Infinite-Component Field Theory*

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(Received 4 May 1970)

We analyze a particular set of infinite-component wave equations from a non-group-theoretical point of view for the purpose of constructing a field theory. Using difference-equation techniques, we are able to study equations with much more general mass spectra than group theory would allow. We solve one equation exactly (corresponding to an equation previously solved group theoretically) and go on to analyze a model of the ρ trajectory which is asymptotically linear. As expected, a ghost appears in the theory, but we are able to modify the equation in a simple way and cause the ghost to disappear.

I. INTRODUCTION

MODELS of strong interactions involving infinitely many particles have been considered in various forms in the past few years. The existence of large numbers of particles in nature suggests that it may be profitable to consider models that have an infinite number of particles in the zeroth-order approximation instead of just a few. Various approaches have included dual theories and their generalizations,[1] field theories,[2] wave equations,[3] and attempts at realizing current algebra on an infinite set of one-particle states.[4] A field theory in which the mass spectrum is determined by a wave equation has been constructed in a previous paper,[5] and the present paper is concerned with further development of this approach. Our departure from previous work is to investigate techniques to solve wave equations that do not have simple group-theoretic solutions.

It is a valid question to ask why a wave equation is necessary at all since it is not essential in the construction of the field theory.[6] Our reason to use it is that the particles which enter the field theory appear to be bound states, whose dynamics are in some sense governed by the wave equation. The questions which arise naturally from our approach are: (a) What is the correspondence between the properties of the wave equation and the properties of its spectrum; (b) what dynamical system, if any, is the equation, and hence the field, describing? This paper is principally concerned with the first of these questions. An attempt to answer the second will be delayed to a later paper. Clearly, the nature of the spectrum will be a strong guide in any attempt to answer the dynamical question.

Our starting point is to consider a boson field $\varphi(x)$ which transforms under the infinitely reducible representation of the homogeneous Lorentz group:

$$R=\sum_{k=0}^{\infty}\oplus(\tfrac{1}{2}k,\tfrac{1}{2}k)\,. \tag{1.1}$$

Here $(\frac{1}{2}k,\frac{1}{2}k)$ denotes the usual $(k+1)^2$-dimensional irreducible representation. It is possible to construct Lorentz tensor operators on this representation. Introducing a Lorentz 4-vector L_μ and a scalar M, we form a first-order wave equation for the field:

$$(\partial_\mu L^\mu - M)\varphi(x)=0\,, \tag{1.2}$$

where L_μ and M are infinite-dimensional matrices in the space of R.

The mass spectrum allowed by (1.2) is most readily analyzed by first Fourier-transforming the equation

$$(ip_\mu L^\mu - M)\tilde{\varphi}(p)=0 \tag{1.3}$$

and then going to the rest frame

$$(ip_0L^0 - M)\tilde{\varphi}(p_0)=0\,. \tag{1.4}$$

The masses are then the values of p_0 for which (1.4) has a nontrivial solution.

Given the eigenvalues p_0 and the c-number eigenvectors of this equation, we can construct the second-quantized field by the prescription given in Ref. 5. We shall not discuss this aspect further here except to emphasize that the procedure is completely covariant, and once the field is constructed all the machinery of relativistic quantum field theory is at our disposal. In particular, one can write Lorentz-invariant couplings, calculate propagators and vertices, and derive expressions for the S matrix.

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* Supported in part by the U. S. Office of Naval Research and in part by the National Science Foundation.
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[3] There are a very large number of proposed wave equations that have appeared in the literature in the past few years. Some of these combine wave equations and field theory; others are wave-function theories; still others concentrate on deriving results from the standpoint of dynamical groups. We refer to a representative sample of such theories; others may be traced through references cited therein. Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppls. **37-38**, 368 (1966); Phys. Rev. **160**, 1171 (1967); C. Fronsdal, *ibid.* **156**, 1653 (1967); **156**, 1665 (1967); **171**, 1811 (1968); A. O. Barut, Dennis Corrigan, and Hagen Kleinert, *ibid.* **167**, 1527 (1968); R. Casalbuoni, R. Gatto, and G. Longhi, Nuovo Cimento Letters **2**, 159 (1969); H. Leutwyler, Phys. Letters **31B**, 214 (1970).
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Up to this point, L_μ and M are to some degree arbitrary, since only their transformation properties have been specified. We must further specify the reduced matrix elements of these operators in order to completely define the wave equation. Traditionally these are chosen so that L_μ and M are generators of some group, leading to an equation that is easily solved. In this paper we undertake a more general investigation of infinite-component wave equations, in which we forsake group theory for a direct calculation of the infinite-dimensional determinant whose roots give the allowed mass spectrum. Our techniques are not limited to choices of reduced matrix elements which allow group-theoretical solutions, and hence we can study equations with much more general mass spectra.

The "natural" group-theoretical choice of reduced matrix elements leads to a rising mass spectrum with a hydrogenlike accumulation point,[5,7] which we believe is undesirable in a model of strong interactions. We exhibit a model for which no simple group-theoretical solution is known that has asymptotically linearly rising trajectories.

In Sec. II we write down the most general first-order wave equation in the context of our representation (1.1). We connect the matrix equation to a difference equation in Sec. III, and discuss its general properties. The difference equation is the basis for the investigations of this paper. In Sec. IV we derive an expression for the behavior of the leading trajectory function $\alpha(s)$ for large s. In Sec. V we solve the difference equation for the case that was previously solved group theoretically. Finally, in Sec. VI we treat a case which has asymptotically linear trajectories.

II. WAVE EQUATION

Let us focus our attention on the equation

$$(\partial_\mu L^\mu - M)\varphi(x)=0\,, \tag{1.2}$$

and write the most general form consistent with the transformation properties of the field $\varphi(x)$. We choose $\varphi(x)$ to transform under an infinitely reducible representation of the Lorentz group. To be specific, we let the column index on φ be given by the triplet $(kj\sigma)$, with $j\leq k$ and $-j\leq\sigma\leq j$, and under Lorentz transformation we demand

$$U(\Lambda,a)\,\varphi_{kj\sigma}(x)\,U^{-1}(\Lambda,a)=\sum_{j'\sigma'} D_{j\sigma,j'\sigma'}{}^{(\frac{1}{2}k,\frac{1}{2}k)}(\Lambda^{-1})\,\varphi_{kj'\sigma'}(\Lambda x+a)\,. \tag{2.1}$$

Here $D_{j\sigma,j'\sigma'}{}^{(\frac{1}{2}k,\frac{1}{2}k)}(\Lambda)$ is the matrix representative of the homogeneous Lorentz transformation Λ in the $(\frac{1}{2}k,\frac{1}{2}k)$ finite-dimensional representation.

Since L_μ and M have definite transformation properties, their form is restricted by their commutation relations with the generators $\mathbf{J}$ and $\mathbf{K}$ of the Lorentz group. M must commute with the generators, while L_μ satisfies the relations

$$[J_i,L_j]=i\epsilon_{ijk}L_k\,,\quad [J_i,L_0]=0\,,\qquad [K_i,L_j]=-i\delta_{ij}L_0\,,\quad [K_i,L_0]=-iL_i\,, \tag{2.2}$$

where $\mathbf{J}$ and $\mathbf{K}$ satisfy the usual commutation rules

$$[J_i,J_j]=i\epsilon_{ijk}J_k\,,\quad [J_i,K_j]=i\epsilon_{ijk}K_k\,,\qquad [K_i,K_j]=-i\epsilon_{ijk}J_k\,. \tag{2.3}$$

By Schur's lemma, M must be of the form

$$M_{kj\sigma,k'j'\sigma'}=m(k)\delta_{kk'}\delta_{jj'}\delta_{\sigma\sigma'}\,. \tag{2.4}$$

It follows from the Wigner-Eckart theorem that the form of L_0 must be[5]

$$(L_0)_{kj\sigma,k'j'\sigma'}=(a_{kj}\delta_{k,k'+1}-a_{k'j}\delta_{k',k+1})\delta_{jj'}\delta_{\sigma\sigma'}\,, \tag{2.5}$$

where

$$a_{kj}=[(k-j)(k+j+1)]^{1/2}r(k)\,. \tag{2.6}$$

(We have made the additional requirement that L_0 be antisymmetric; a slightly more general form is possible if we relax this requirement, but this will not affect the generality of our wave equation.) Here $r(k)$ and $m(k)$ are arbitrary functions of k; they are the reduced matrix elements of L_μ and M referred to in the Introduction. For convenience, we shall take L_0 to be anti-Hermitian and M to be Hermitian, i.e., $r(k)$ and $m(k)$ must be real.

The technique for deriving (2.5) and (2.6) from the commutation rules has been outlined in Appendix A of Ref. 5. The model which was solved group theoretically there corresponds to the choice $r(k)=\frac{1}{2}$, and $m(k)=\alpha(k+1)+\beta$, where α and β are constants.

Let us derive a condition under which the eigenvalues p_0 will all be real. Assuming that none of the $m(k)$ vanish, we can rewrite (1.4) as

$$M^{1/2}[M^{-1/2}(iL_0)M^{-1/2}-1/p_0]M^{1/2}\tilde{\varphi}=0\,. \tag{2.7}$$

Thus p_0 will be real if $B\equiv M^{-1/2}(iL_0)M^{-1/2}$ is a Hermitian matrix. Calculating the elements of B using (2.4) and (2.5), we find

$$B_{kj\sigma,k'j'\sigma'}=i\left[\frac{a_{kj}}{[m(k)m(k-1)]^{1/2}}\delta_{k,k'+1}-\frac{a_{k'j}}{[m(k')m(k'-1)]^{1/2}}\delta_{k',k+1}\right]\delta_{jj'}\delta_{\sigma\sigma'}\,, \tag{2.8}$$

and, therefore, from (2.6), our condition is that $r(k)/[m(k)m(k-1)]^{1/2}$ must be real.

We shall see in Sec. VI how the violation of this condition results in the appearance of ghosts.

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III. WAVE EQUATION AS DIFFERENCE EQUATION

In this section we derive a second-order difference equation, which, together with boundary conditions, is equivalent to the matrix equation (1.4). We solve the equation asymptotically and classify the solutions according to the normalization properties of the eigenvectors.

First let us note that (1.4) is diagonal in j and σ. We define a new matrix $\bar{L}$ by

$$(L_0)_{kj\sigma,k'j'\sigma'}=(\bar{L}_0^{(j)})_{kk'}\delta_{jj'}\delta_{\sigma\sigma'}\,, \tag{3.1}$$

where now

$$(\bar{L}_0^{(j)})_{kk'}=a_{kj}\delta_{kk'+1}-a_{k'j}\delta_{k'k+1}\,. \tag{3.2}$$

Equation (1.4) becomes

$$(\mathfrak{M}^{(j)})_{kk'}\varphi_{k'}^{(j)}(p_0)=0\,, \tag{3.3}$$

where

$$(\mathfrak{M}^{(j)})_{kk'}=-ip_0(\bar{L}_0^{(j)})_{kk'}+m(k)\delta_{kk'}\,. \tag{3.4}$$

Now if $\mathfrak{M}$ were a finite matrix the values of p_0 would be determined by simply setting det $\mathfrak{M}=0$. In the infinite-dimensional case, we can define a sequence of functions $D_n(p_0)$ as the determinants of truncated matrices. Thus

$$D_1=\det m(j)=m(j)\,,$$

$$D_2=\det\begin{pmatrix} m(j) & ip_0a_{j+1,j} \\ -ip_0a_{j+1,j} & m(j+1)\end{pmatrix},\ \text{etc.} \tag{3.5}$$

By simply expanding along the $(n+1)$st column of D_{n+1}, we arrive at the following recursion formula:

$$D_{n+1}(p_0)=m(n+j)D_n(p_0)-p_0^2a_{n+j,j}^2D_{n-1}(p_0)\,. \tag{3.6}$$

Since this is a second-order difference equation, we must specify two boundary conditions. Rather than use the explicit forms given in (3.5), it is convenient to specify $D_{-1}(p_0)<\infty$ and $D_0(p_0)=1$. To simplify notation, we write everything in terms of the index $n=k-j$, $n=0, 1, 2, \ldots$. We put $a_{kj}^2=n(n+\lambda)r_n^2$, where $\lambda\equiv 2j+1$, and let $m(k)\equiv m_n$. Furthermore, we write $p_0^2=x$ and note that (2.6) is a function of x only, so that our solutions will occur in pairs $\pm p_0$. In the context of infinite-component field theory, this means that we shall have both particles and antiparticles, as in the Dirac case. With these changes, (3.6) is

$$D_{n+1}(x)=m_nD_n(x)-xn(n+\lambda)r_n^2D_{n-1}(x)\,. \tag{3.6'}$$

Next we show that although two arbitrary functions m_n and r_n^2 appear in (3.6′), the problem is determined by a single function only. We set

$$D_n=\Big(\prod_{l=0}^{n-1}m_l\Big)\frac{E_{n+1}}{2^n} \tag{3.7}$$

(and $D_0=E_1$), which yields

$$\begin{aligned}E_{n+2}&=2E_{n+1}-4xn(n+\lambda)(r_n^2/m_nm_{n-1})E_n\\ &\equiv 2E_{n+1}-xn(n+\lambda)G_nE_n\,.\end{aligned} \tag{3.8}$$

Recall from Sec. II that G_n must be positive if only solutions for positive x are to occur, and notice that G_n depends on n only in the combination $(n+j)$.

We are interested in those values of x which cause $D_n(x)\to 0$ for large n. It might seem that to solve explicitly for each $D_n(x)$ is therefore obtaining a lot of unnecessary information. However, we can show that there is in fact a simple relationship between $D_n(x)$ and the actual components of the eigenvectors φ_n. To see this, we write (3.3) as

$$m_n\varphi_n-ip_0a_n\varphi_{n-1}+ip_0a_{n+1}\varphi_{n+1}=0\,, \tag{3.9}$$

where we have denoted $a_{n+j,j}$ simply by a_n. Let

$$\varphi_n=\Big(\frac{i}{p_0}\Big)^n\Big(\prod_{l=1}^{n}a_l\Big)^{-1}\chi_n\quad(\text{and }\chi_0=\varphi_0)\,. \tag{3.10}$$

Then

$$\Big(\frac{i}{p_0}\Big)\frac{m_n}{a_n}\chi_n-ip_0a_n\chi_{n-1}+\Big(\frac{i}{p_0}\Big)^2ip_0\frac{a_{n+1}}{a_{n+1}a_n}\chi_{n+1}=0\,, \tag{3.11}$$

or, multiplying by $-ip_0a_n$, we have

$$m_n\chi_n-p_0^2a_n^2\chi_{n-1}-\chi_{n+1}=0\,, \tag{3.12}$$

which is exactly Eq. (3.6).

Since φ_n has the same boundary conditions as D_n (except that we leave the choice of φ_0 arbitrary), we deduce that

$$\chi_n=D_n\varphi_0 \tag{3.13}$$

or

$$\varphi_n=\Big(\frac{i}{p_0}\Big)^n\Big(\prod_{l=1}^{n}a_l\Big)^{-1}D_n\varphi_0\,, \tag{3.14}$$

$n=1, 2, \ldots$. We have assumed in this derivation that $a_n\neq 0$ for $n>0$, i.e., that $r_n\neq 0$ for $n>0$.

It will be useful at this point to study the asymptotic solutions to (3.8) in order to classify the types of spectra we may expect as functions of the asymptotic behavior of G_n. We shall show that when $G_n\to\sigma n^{\tau-2}$, the solutions fall into three classes: (1) For $\tau>0$, we find that the only acceptable solutions, in the sense that $\sum_n\varphi_n^*\varphi_n$ diverges no worse than a δ function, are a continuum of "scattering" solutions for $0\leq x<\infty$. (2) For $\tau=0$, we find that there is an ionization point $x=1/\sigma$, below which there can exist normalizable solutions (i.e., "bound states") and above which there is a continuum as in case (1). (3) For $\tau<0$, we find that only bound-state solutions exist.

Before substantiating these statements case by case, let us comment on the normalization properties of the solutions to (3.3). Since

$$p_0(i\bar{L}_0)\varphi=M\varphi\,, \tag{3.15}$$

where $i\bar{L}_0$ and M are assumed Hermitian, we deduce that

$$[p_0^{(1)}-p_0^{(2)}]\varphi^{*(1)}(i\bar{L}_0)\varphi^{(2)}=0. \qquad (3.16)$$

Thus the natural metric with respect to which we shall examine the normalization of our solutions is

$$(\varphi^{(1)},\varphi^{(2)})=\varphi^{*(1)}(i\bar{L}_0)\varphi^{(2)}. \qquad (3.17)$$

In considering the norm (φ,φ) (which is not, however, necessarily positive definite), it will be more convenient to consider the equivalent expression $\varphi^*M\varphi$. We wish to know, therefore, the behavior of the quantity

$$\varphi_{n+1}{}^*m_{n+1}\varphi_{n+1}/\varphi_n{}^*m_n\varphi_n\equiv Z \qquad (3.18)$$

as a function of the ratio

$$R=E_{n+1}/E_n. \qquad (3.19)$$

Using (3.14) and (3.7), we have

$$\frac{\varphi_{n+1}}{\varphi_n}=\left(\frac{i}{p_0}\right)\frac{1}{a_{n+1}}\frac{D_{n+1}}{D_n}=\left(\frac{i}{p_0}\right)\left(\frac{m_n}{a_{n+1}}\right)\frac{1}{2}\frac{E_{n+2}}{E_{n+1}}, \qquad (3.20)$$

and, therefore, for large n,

$$Z=\frac{1}{p_0{}^2}\left(\frac{m_{n+1}m_n}{4r_{n+1}{}^2}\right)\frac{|R|^2}{n(n+\lambda)}$$
$$=|R|^2/xn^2G=|R|^2/x\sigma n^\tau. \qquad (3.21)$$

By the ratio test, then, the series $\sum_n \varphi_n{}^*m_n\varphi_n$ will converge for $|R|^2<x\sigma n^\tau$, will diverge for $|R|^2>x\sigma n^\tau$, and the case $|R|^2=x\sigma n^\tau$ is ambiguous. Keeping only leading powers in n, (3.8) reads

$$E_{n+2}=2E_{n+1}-x\sigma n^\tau E_n. \qquad (3.22)$$

There are two possible ways to satisfy this.

(i) Two of the terms in (3.22) have the same asymptotic behavior and the third dies relative to them; or
(ii) all three terms have the same asymptotic behavior.

We now examine (3.22) in the three cases $\tau>0$, $\tau=0$, and $\tau<0$.

(1) $\tau>0$. The reader can check that only possibility (i) exists in this case, and that in fact we must have

$$E_{n+2}=-x\sigma n^\tau E_n, \qquad (3.23)$$

so that, to leading order in n,

$$E_n=c(-x\sigma)^{n/2}[\Gamma(n+\alpha)]^{\tau/2}, \qquad (3.24)$$

where c and α are arbitrary constants. Thus we find that $|R|^2=x\sigma n^\tau$, which is the ambiguous case. In analogy to the Schrödinger equation, for example, we interpret this as the existence of a continuum of solutions where the eigenvectors have δ-function normalization.

Notice that according to (3.24) and the assumed positivity of x and σ, the E_n are alternately real and imaginary. To obtain purely real solutions (we know, after all, from the boundary conditions that the E_n must be real), we can make use of the fact that (3.22) has real coefficients to construct the solutions

$$E_n{}^{(1)}=E_n+E_n{}^* \quad\text{and}\quad E_n{}^{(2)}=i(E_n-E_n{}^*).$$

(2) $\tau=0$. Here only possibility (ii) exists, and Eq. (3.22) becomes

$$E_{n+2}=2E_{n+1}-x\sigma E_n, \qquad (3.25)$$

with solution

$$E_n=cq^n, \qquad (3.26)$$

$$q=1\pm(1-x\sigma)^{1/2}. \qquad (3.27)$$

Therefore, we have $R=q$ in this case. For $x>1/\sigma$, the square root is pure imaginary, so that $|R|^2=x\sigma$, and thus we have the ambiguous situation again, i.e., a continuum of masses for $x>1/\sigma$.

For $x<1/\sigma$, $|R|^2=2-x\sigma\pm2\ (1-x\sigma)^{1/2}$. Therefore, φ will be normalizable if $\pm(1-x\sigma)^{1/2}$ is less than $x\sigma-1$. Since $x\sigma-1<0$, we see that the minus sign yields a normalizable solution, while the plus sign yields a non-normalizable one. This is again analogous to a Schrödinger equation in the regime where bound states exist: For each energy, there is one solution which is asymptotically growing, and another which is asymptotically dying. We make the interpretation, then, that bound states may exist when $\tau=0$ and $x<1/\sigma$. Incidentally, this analysis of the $\tau=0$ case was confirmed in Ref. 5, where bound states were indeed found below an ionization point α, with a continuum for $x>\alpha$.

(3) $\tau<0$. Here the possibility exists to set $E_{n+2}=2E_{n+1}$ asymptotically, that is,

$$E_n=c(2^n). \qquad (3.28)$$

However, this means $R=2$, and $2>x\sigma n^\tau$ for $\tau<0$ and n sufficiently large. Thus the solution obtained in this way is not normalizable. The other possibility is to set

$$2E_{n+1}-x\sigma n^\tau E_n=0, \qquad (3.29)$$

that is,

$$E_n=c(\tfrac{1}{2}x\sigma)^n[\Gamma(n+\alpha)]^\tau \qquad (3.30)$$

for large n. Then $|R|^2=(\frac{1}{2}x\sigma)^2n^{2\tau}$, which can be made smaller than $x\sigma n^\tau$ for any given x if n is chosen large enough. Hence we expect bound states to occur for indefinitely large x. This behavior will be explicitly demonstrated in Sec. VI.

Before proceeding to discuss particular cases of Eq. (3.8) in greater detail, we remark that once we have reduced our problem to the solution of a difference equation, the angular momentum j appears only as a parameter. The equation can be discussed and in some cases solved exactly for nonintegral values of j; we shall find, in fact, that we can extrapolate smoothly between integers to generate x as a function of the continuous variable j. This we take to be the Regge continuation.

IV. ASYMPTOTIC BEHAVIOR OF LEADING TRAJECTORY

In Sec. III we established conditions on the function G_n [$\equiv G(n+j)$], i.e., on the reduced matrix elements, for there to be a discrete spectrum. In this section we go one step further and derive an expression for the asymptotic behavior of the leading trajectory for the cases in which the discrete spectrum exists for $0\leq x\leq\infty$ (i.e., the $\tau<0$ case of Sec. III). This will guide us in choosing G_n to get linear (or nearly linear) trajectories. Given that

$$G_n\sim(n+j)^{\tau-2}[1+O(1/(n+j))], \tag{4.1}$$

for large n, we show that $x\sim j^{-\tau}$ for large j on the leading trajectory or in Regge language $\alpha(s)\sim s^{-1/\tau}$. Linear trajectories require $\tau=-1$. The hydrogenlike spectrum arises in the limiting case $\tau\to 0^-$.

Consider the difference equation for E_n, Eq. (3.8),

$$E_{n+2}-2E_{n+1}+xn(n+\lambda)G_nE_n=0, \tag{4.2}$$

with the boundary conditions $E_0<\infty$, $E_1=1$. We introduce the forward difference operator

$$\Delta f_n=f_{n+1}-f_n. \tag{4.3}$$

Then Eq. (4.2) becomes

$$[\Delta^2-1+xn(n+\lambda)G_n]E_n=0. \tag{4.2$'$}$$

For j large, the function G_n is needed only for large values of its argument since n always starts at zero. Hence we substitute its asymptotic form

$$G_n=(n+j)^{\tau-2}. \tag{4.4}$$

The constant in front is chosen to be unity since any constant can be absorbed in x. We restrict ourselves to functions G that have an expansion implied by Eq. (4.1).

A. Conversion to Differential Equation

If we forget for the moment that Δ^2 is a difference operator and consider it to be a derivative operator D^2 [$\equiv(d/dn)^2$], this equation looks like a Schrödinger equation, where n plays the role of a relative coordinate, and we can therefore analyze it using familiar techniques.

We must first establish a criterion under which we can replace Δ by D. These operators are related by the expansion

$$(\Delta+1)=e^D \tag{4.5}$$

when acting on an infinitely differentiable function. Hence

$$\Delta^2=D^2+\frac{2^3-2}{3!}D^3+\frac{2^4-2}{4!}D^4+\cdots. \tag{4.6}$$

We will first ignore D^3 and the higher derivatives and find the ground state (i.e., the leading trajectory). We then justify this approximation in Sec. IV C by showing that the higher derivatives are small relative to D^2 for large j. Hence for large j we can drop the higher derivatives to get the leading j behavior.

B. Approximate Solution of Differential Equation

We wish to find the ground state of the differential equation

$$\begin{aligned}&[-d^2/dn^2+1+xH(n)]E(n)=0,\\ &H(n)=-n(n+2j)/(n+j)^{2-\tau}.\end{aligned} \tag{4.7}$$

This is Eq. (4.2′) with two modifications: $\Delta^2\to D^2$, $n(n+\lambda)G_n\to$ its asymptotic form. We cannot solve this equation exactly but can solve it for large j because it reduces to the harmonic-oscillator equation in this limit.

The function $H(n)$ is zero for $n=0$, and $n=\infty$. We need it in the region $n\geq 0$ and here it is a smooth function with a single dip at $n=n_0$, where

$$n_0=j\left[\left(\frac{\tau-2}{\tau}\right)^{1/2}-1\right]. \tag{4.8}$$

Expanding $H(n)$ about n_0 we get

$$\begin{aligned}&H(n)=H(n_0)+\tfrac{1}{2}(n-n_0)^2H''(n_0)+\cdots,\\ &H(n_0)=-j^\tau\left(\frac{-2}{\tau}\right)\left(\frac{\tau}{\tau-2}\right)^{(2-\tau)/2},\\ &H''(n_0)=j^{\tau-2}(-2\tau)\left(\frac{\tau}{\tau-2}\right)^{(2-\tau)/2}.\end{aligned} \tag{4.9}$$

Higher terms in the expansion fall successively faster with j. We may therefore keep only the first two terms in the limit of large j.

Our equation has the form

$$[-d^2/dn^2-A+B(n-n_0)^2]E(n)=0. \tag{4.10}$$

The ground-state eigenfunction and the eigenvalue condition are

$$\begin{aligned}&E(n)=\exp[-\tfrac{1}{2}A(n-n_0)^2],\\ &B=A^2.\end{aligned} \tag{4.11}$$

Hence our final result is

$$x=j^{-\tau}(-\tfrac{1}{2}\tau)\left(\frac{\tau-2}{\tau}\right)^{(2-\tau)/2}\left[1+O\left(\frac{1}{j}\right)\right]. \tag{4.12}$$

C. Justification for Using $\Delta^2\approx D^2$

Starting with the relation

$$\Delta^2=(e^D-1)^2, \tag{4.13}$$

we have approximated the right-hand side with $\Delta^2\approx D^2$. This was necessary in order to obtain a second-order differential equation. Now that we have a solution we show that the higher derivatives are vanishingly small for large j.

Let us examine the expression

$$Q=\frac{(\Delta^2-D^2)\exp[-\frac{1}{2}A(n-n_0)^2]}{D^2\exp[-\frac{1}{2}A(n-n_0)^2]}=\frac{\exp[-2A(n-n_0+1)]-2\exp[-\frac{1}{2}A(2n-2n_0+1)]+1}{-A+A^2(n-n_0)^2}-1. \quad (4.14)$$

As j gets large, $A\sim 1/j$ but n takes on all values. However, let us examine Q in the neighborhood of the dip in $H(n)$, i.e., where the solution is significant. The important region is then $\frac{1}{2}A(n-n_0)^2\sim 1$. Hence $A(n-n_0)\sim A^{1/2}\sim 1/j^{1/2}$. We therefore may expand the exponential and find $Q\sim 1/j^{1/2}$ and thus justify the dropping of higher derivatives for large j.

D. Summary

We have presented a simple nonrigorous calculation for the trajectory function obtaining

$$x=\text{const}\times j^{-\tau}[1+O(1/j)]. \quad (4.15)$$

We have dropped terms at many stages which contribute to order $1/j$ but believe to have a correct expression for the power behavior and the coefficient of the power. For the case of linear trajectories, i.e., $\tau=-1$, we get

$$x=\tfrac{3}{2}\sqrt{3}\,j. \quad (4.16)$$

(The scaling of x is arbitrary and so this is not a constraint on the slope of physical trajectories. We quote this number for comparison with a calculation in Sec. VI.)

Without examining the correction terms we do not know where the asymptotic behavior sets in. In Sec. VI we solve an equation numerically that should yield asymptotic linear trajectories by the arguments of this section. We find complete agreement with this calculation and also the surprising fact that our formula can work very well even for small j.

V. EXACT SOLUTION OF DIFFERENCE EQUATION

In this section we solve (3.6′) exactly in the case that has already been treated group theoretically in Ref. 5. We do this for two reasons: first, to bolster our confidence in the validity of our approach; and second, to bring out certain additional properties of typical solutions to our equation.

As pointed out in Sec. II, the case of interest is given by $r(k)=\frac{1}{2}$, $m(k)=\alpha(k+1)+\beta$ in (3.6′). Redefining $m_n=an+b$, with $a=\alpha$ and $b=\alpha(j+1)+\beta$, we have

$$D_{n+1}(x)=(an+b)D_n(x)-\tfrac{1}{4}xn(n+\lambda)D_{n-1}(x). \quad (5.1)$$

Rather than define E_n as we did in Sec. III, we shall define a slightly different function E_n' by

$$D_n=\Gamma(n)E_n', \quad (5.2)$$

which gives, in (5.1),

$$n(n-1)E_{n+1}'=(an+b)(n-1)E_n' -\tfrac{1}{4}xn(n+\lambda)E_{n-1}'. \quad (5.3)$$

Following the approach of Sec. III, we look for the asymptotic solution of (5.3), i.e., for the solution of

$$E_{n+1}'-aE_n'+\tfrac{1}{4}xE_{n-1}'=0. \quad (5.4)$$

E_n' is easily determined to be

$$E_n'=q^n \quad (n\text{ large}), \quad (5.5)$$

with

$$q=\tfrac{1}{2}a[1\pm(1-x/a^2)^{1/2}]. \quad (5.6)$$

As in Sec. III, only the minus sign in (5.6) will lead to normalizable solutions, so we make that choice. We extract the asymptotic behavior (5.5) by letting

$$E_n'=q^n f_{n+1}, \quad (5.7)$$

which yields

$$n(n-1)q^2f_{n+2}-(an+b)(n-1)qf_{n+1} +\tfrac{1}{4}xn(n+\lambda)f_n=0. \quad (5.8)$$

Following the techniques of Sec. IV, we introduce the first and second difference operators by

$$f_{n+1}=\Delta f_n+f_n,\qquad f_{n+2}=\Delta^2 f_n+2\Delta f_n+f_n. \quad (5.9)$$

Using $x=4q(a-q)$ and the definitions (5.9), we find

$$n(n-1)q\Delta^2 f_n+[(2q-a)n-b](n-1)\Delta f_n +\{[(a-q)(\lambda+1)-b]n+b\}f_n=0. \quad (5.10)$$

In order to have f_n regular for $n\geq 0$, and to preserve the asymptotic behavior (5.5), we want a solution to (5.10) which is a polynomial in n:

$$f_n=\sum_{l=u}^{L} b_l n^l.$$

Actually, it will be more convenient to make an equivalent expansion not in powers of n, but in a series of modified Pochhammer polynomials:

$$f_n=\sum_l c_l\{n\}_l, \quad (5.11)$$

where

$$\{n\}_l=(n+l-2)!/(n-2)!. \quad (5.12)$$

The $\{n\}_l$ have a number of useful properties which we now list:

$$(n-1)\Delta\{n\}_l=l\{n\}_l,\qquad n(n-1)\Delta^2\{n\}_l=l(l-1)\{n\}_l, \quad (5.13)$$

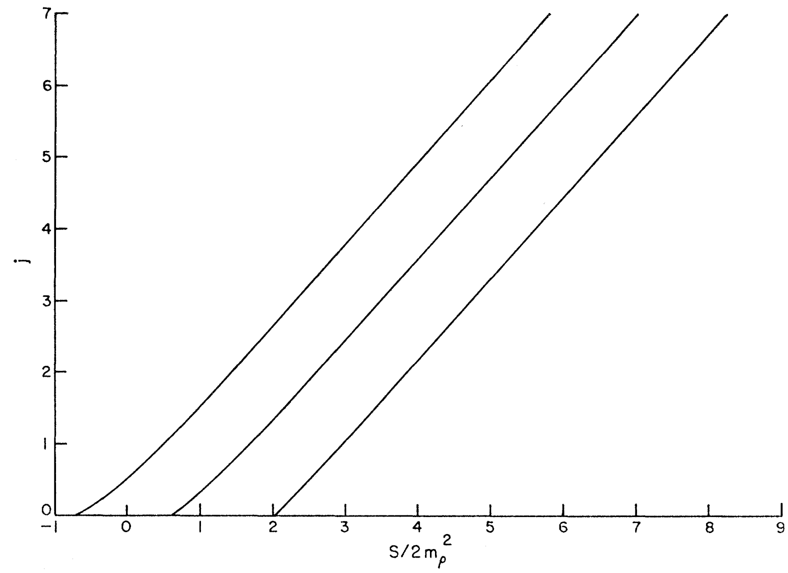

FIG. 1. The ρ trajectory and first two daughters calculated by putting $a=-\frac{3}{2}$, $b=-\frac{1}{2}$, $c=1$ in Eq. (6.1).

and

$$n\{n\}_l=\{n\}_{l+1}-(l-1)\{n\}_l.$$

These properties (5.13) allow us to solve for the eigenvalues x and the expansion coefficients c_l by introducing a differential equation in a variable r, whose solution is

$$f(r)=\sum_l c_l r^l, \tag{5.14}$$

where the c_l in (5.14) are the same as those in (5.11). Using (5.13), one can check that if we make the replacements

$$(n-1)\Delta\to rd/dr,$$
$$n(n-1)\Delta^2\to r^2d^2/dr^2, \tag{5.15}$$

and

$$n\to r-rd/dr+1,$$

then the differential operators on the right perform the same operations on r^l that the difference operators on the left do on $\{n\}_l$. Hence the differential equation we wish to study is

$$qr^2d^2f/dr^2+[(2q-a)(r-rd/dr+1)-b]rdf/dr$$
$$+[(a-q)(\lambda+1)-b](r-rd/dr+1)f+bf=0. \tag{5.16}$$

First we make the substitution $f(r)=r^{\lambda+1}g(r)$ to obtain

$$(a-q)rg''+[(2q-a)r+(a-q)(\lambda+1)]g'$$
$$+[q(\lambda+1)-b]g=0. \tag{5.17}$$

Finally, we change variable to

$$y=[(a-2q)/(a-q)]r, \tag{5.18}$$

which gives

$$yg''+[(\lambda+1)-y]g'$$
$$+(a-2q)^{-1}[(\lambda+1)q-b]g=0. \tag{5.19}$$

This is the equation for Laguerre polynoimals, provided we set

$$[(\lambda+1)q-b]/(a-2q)=M, \tag{5.20}$$

where $M=0, 1, 2, \ldots$. Defining $N=M+j+1$, we have

$$q=\tfrac{1}{2}(\alpha+\beta/N). \tag{5.21}$$

Comparing (5.21) and (5.6), and remembering that we chose the minus sign in (5.6) to obtain a normalizable solution, we see that we can have bound states only if

$$\beta/\alpha<0.$$

This is a result which we derived from the properties of of the group $O(2,1)$ in Ref. 5. From (5.21) we have

$$x=4q(a-q)=\alpha^2-\beta^2/N^2, \tag{5.22}$$

with $N=j+1, j+2, \ldots$.

This is the result found in Ref. 5. The solutions to our difference equation (5.8) are

$$f_n^{(N)}=\sum_{l=2(j+1)}^{N+j+1}c_l\{n\}_l, \tag{5.23}$$

corresponding to eigenvalues (5.22). The coefficients c_l are those in the expansion

$$r^{\lambda+1}L_{N-j-1}^{(\lambda)}\left(\frac{a-2q}{a-q}r\right)=\sum_{l=2(j+1)}^{N+j+1}c_l r^l. \tag{5.24}$$

We have thus accomplished our objectives of reproducing the discrete spectrum of Ref. 5, using techniques reminiscent of two-body potential theory in nonrelativistic quantum mechanics. The solution is related to a Laguerre polynomial, as the hydrogenlike spectrum (5.22) might suggest. This is not exactly the Schrödinger problem, of course, both because of the shift α^2, and

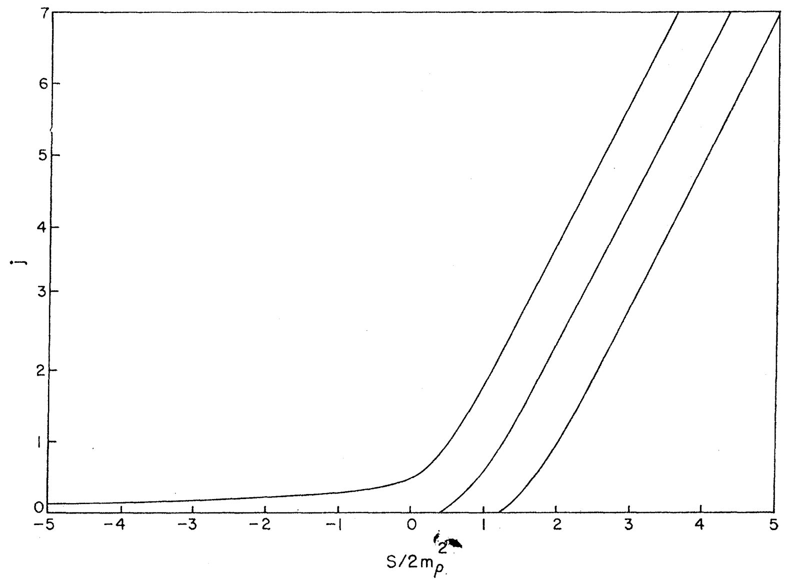

FIG. 2. The ρ trajectory and first two daughters incorporating the modification (6.3) with $d=0$, $e=-1$.

the fact that x is the energy squared, not the energy. Furthermore, as we stressed earlier, the solution f_n is related to the components of an eigenvector in a fully relativistic problem.

VI. NUMERICAL SOLUTION OF DIFFERENCE EQUATION WITH NEARLY LINEAR TRAJECTORIES

Difference equations are tailor-made for a computer, and it is tempting to abandon an analytic treatment of a difference equation that we cannot solve exactly. It is trivial to find the spectrum by truncating the equation and searching for zeros in x. However, since we are using the wave equation in order to construct a field we would strongly prefer to have exact expressions for the eigenvalues and eigenvectors.

The numerical calculation in this section serves two purposes. First, it is a check on the calculation of Sec. IV and furthermore shows that the spectrum can be well represented by its asymptotic form (4.16) over a wide range in j. Second, it can be used to establish the exact positions of low-lying states. We believe that in the construction of Feynman graphs and form factors, the approximate analytic expression for the spectrum and the asymptotic behavior of eigenvectors (Sec. III) will be much more useful information.

We consider the simplest equation we can write down that should give asymptotically linear trajectories:

$$[\Delta^2-1+xn(n+\lambda)G_n]E_n=0\,,$$
$$G_n=\frac{1}{[(n+j+a)\,(n+j+b)\,(n+j+c)]}, \tag{6.1}$$

where a, b, and c are arbitrary j-independent parameters. Attempts were made to solve this equation exactly. It is possible to solve for isolated values of j but we have not been able to find an analytic solution for general j. If we choose $b=a-1$ and $j=c-1$, one can easily verify that the gound-state solution is

$$E_n^{(0)}=(\tfrac{1}{2}x)^n[\Gamma(n+a+c-2)]^{-1}\,,\qquad x=4(c+a-1)\,. \tag{6.2}$$

Further excited-state eigenfunctions are of the form $E_n^{(0)}$ times polynomials in n.

We solved Eq. (6.1) numerically for arbitrary j for various choices of the parameters and verified that the trajectories were asymptotically linear and gave the slope predicted by Eq. (4.16). In presenting our results, we have in mind constructing a model of the ρ trajectory and must choose appropriate values of a, b, and c. The choice of these constants can be made somewhat systematic. There exists a solution for $x=0$ if $n(n+\lambda)G_n$ blows up for a particular value of n. [This corresponds to a matrix element $m(k)$ vanishing in the wave equation (3.3).] We can make it blow up at $n=1$ and $j=\frac{1}{2}$ (the intercept of the ρ) by choosing $a=-\frac{3}{2}$. We then choose $b=-\frac{1}{2}$ so that there exists an exact solution of our equation for some value of j. Finally, we choose $c=1$ and this allows us to solve $j=0$ exactly.

The results are shown in Fig. 1. The leading trajectory has an intercept $j=\frac{1}{2}$. We choose to scale x so that the leading trajectory passes through $j=1$ at $s=0.5$ corresponding to the mass of the ρ. We wish to stress again that the near linearity of the trajectories for small j was an unexpected result.

We note that there is a ghost state in the spectrum; i.e., $s<0$ for $j=0$. This was not unexpected since G_n is not positive definite and hence ghost states are allowed (see Sec. II).

We can alter G_n so that it is positive semidefinite. Consider the form

$$G_n^{(\text{new})}=G_n^{(\text{old})}\left(\frac{n+j+e}{n+j+d}\right). \quad (6.3)$$

This choice should change neither the asymptotic linearity nor the slope. We choose $e=-1$ so that the numerator factor vanishes for $n=1$ and $j=0$, i.e., at the point where $G_n^{(\text{old})}$ is negative. The constant d was arbitrarily chosen to be zero.

These results are shown in Fig. 2. The ghost disappeared because the leading trajectory was found to approach $j=0$ asymptotically. The asymptotic slope of the trajectories differ from Fig. 1 since a different scaling passes the leading trajectory through the ρ mass.

ACKNOWLEDGMENT

We thank Kurt Gottfried for valuable discussions in the early part of this work.

PHYSICAL REVIEW D VOLUME 2, NUMBER 4 15 AUGUST 1970

Crossing Regge Trajectories and Pole-Cut Relationships*

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(Received 4 May 1970)

The general features of typical pole-cut relationships with crossing Regge-pole trajectories are considered. The possible shapes of the resulting physical pole trajectories are described.

I. INTRODUCTION

In a recent paper,[1] we discussed the question of possible left-hand branch lines of Regge-pole trajectories. These branch lines are of interest in connection with diffraction scattering,[1–5] and possibly also for other high-energy properties. Since, *a priori*, one may perhaps think that there are other possibilities, we have pointed out that Regge trajectories can have such branch lines only as a consequence of the crossover of two (or more) *pole* trajectories. The relevant constraint is the condition that these branch points of the trajectory $\alpha(s)$ are not inherited by the continued partial-wave amplitude $F(s,\lambda)$.

From the phenomenological point of view, we may not want to have two trajectories which correspond to different branches of the same analytic function. It was therefore the main point of Ref. 1 to show that one can use fixed or moving branch points in the complex λ plane of $F(s,\lambda)$ in order to remove one of the two crossing Regge trajectories into a secondary sheet with respect to these λ branch lines.[6–8] It is the purpose of this paper to explore the general features of the resulting pole-cut relationships.

Crossing Regge poles and corresponding pole-cut relationships are possible structures in the complex λ plane which may well play an important role in phenomenological calculations, and which may give more concise parametrizations than poles and cuts separately. There is no proof at present that such structures are necessary within the framework of dispersion theory, but there are indications from potential theory,[9,10] relativistic perturbation theory,[11] and certain iteration schemes[12] that they may be relevant.

Suppose we have two Regge trajectories $\alpha_1(s)$ and $\alpha_2(s)$ which are pole surfaces of the continued partial-wave amplitude $F(s,\lambda)$. Then this amplitude has the meromorphic terms

$$F(s,\lambda)=\frac{\beta_1(s)}{\lambda-\alpha_1(s)}+\frac{\beta_2(s)}{\lambda-\alpha_2(s)}+\cdots. \quad (1)$$

* Supported in part by the U. S. Atomic Energy Commission.
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