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Dispersion Relations and Asymptotic Behavior of the Veneziano Partial-Wave Amplitude in the Complex s Plane*

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The asymptotic behavior of the Veneziano partial-wave $I=1$ amplitude $V_l(s)$ for $\pi\pi$ scattering is studied in the complex s plane for physical l values. The ρ-f^0 exchange-degenerate trajectory is of the form $\alpha(s)=as+b$. For $b<1$ and $3b+4am_\pi^2\geq 1$, it is shown that, asymptotically, $V_l(s)\sim o(s^{b-1})$. Under the same conditions, the resonance partial widths for fixed l have the property $\Gamma s_R\sim o(s_R^{b-3/2})$. The discontinuity of $V_l(s)$ across the left-hand cut oscillates, and if $b<1$, then, asymptotically, disc $V_l(s)\sim o(s^{-2b-4am_\pi^2})$. In the case $-2am_\pi^2<b<1$, disc $V_l(s)\to 0$ as $s\to-\infty$ and $V_l(s)\to 0$ as $|s|\to\infty$ and $V_l(s)$ can be written in the form of unsubtracted partial-wave dispersion relations, i.e., as an integral along the left-hand cut plus the sum of an infinite number of poles along the right-hand real axis. Thus for the particular case of the ρ-trajectory ($b\approx\frac{1}{2}$, $a\approx 1$ BeV^{-2}), an unsubtracted dispersion relation can be written.

I. INTRODUCTION

A SIMPLE representation for the scattering amplitude which meets the requirements of Regge asymptotic behavior and crossing symmetry and which exhibits zero-width resonance poles has been introduced by Veneziano.[1]

In this paper, we study the asymptotic properties of the $I=1$ continued Veneziano-Lovelace[2] $\pi\pi$ partial-wave amplitude in the complex s plane. Although the analysis is limited to $\pi\pi$ scattering, the methods used should be applied easily to other Veneziano-type amplitudes.

In Sec. II the formalism is developed. We assume that the exchange degenerate ρ-f^0 trajectory is linear and given by $\alpha(s)=as+b$. The analysis is carried out for $b<1$, since $b>1$ violates the Froissart-Gribov bound and $b=1$ corresponds to the Pomeranchuk trajectory. We first consider the discontinuity across the left-hand cut. We show that disc $V_l(s)\sim o(s^{-2b-4am_\pi^2})$ as $s\to-\infty$.[3] In the special case of the ρ meson where $b\simeq\frac{1}{2}$ and $a\simeq 1$ BeV^{-2}, disc $V_l(s)$ goes to zero faster than $1/s$ along the left-hand cut. If $2b+4am_\pi^2>0$, disc $V_l(s)\to 0$ as $s\to-\infty$.

In Sec. III we examine the asymptotic behavior of the partial-wave amplitude in the complex s plane. Our results are conveniently expressed in terms of a parameter x, which is defined to be the minimum of $1-b$ and $2b+4am_\pi^2$. For $b<1$ and δ any positive number, it is shown that (i) $V_l(s)\to 0$ as $|s|\to\infty$ provided that $\mathrm{Im}s\neq 0$ and $2b+4am_\pi^2>0$; (ii) $V_l(s)\sim o(s^{b-1})$ if $3b+4am_\pi^2\geq 1$ or if $|\mathrm{Im}s/\ln s|\to\infty$ as $|s|\to\infty$; (iii) $s^{x-\delta}V_l(s)\to 0$ as $|s|\to\infty$ if $\mathrm{Im}s\neq 0$. When $b=\frac{1}{2}$ and $a=1$ BeV^{-2}, this reduces to $V_l(s)\sim o(1/\sqrt{s})$ as $|s|\to\infty$. Part of the proof of this theorem is given in Appendix C. It is interesting to note that the condition $3b+4am_\pi^2\geq 1$ is just the result found by Shapiro and Yellin[4] in order to guarantee positivity of the resonance widths of the first daughter trajectory.

In Sec. IV the asymptotic behavior of $\pi\pi$ partial widths is studied. For fixed l we find that $\Gamma s_R\sim o(s_R^{-x-1/2})$, where the parameter x has been specified above. When $3b+4am_\pi^2\geq 1$, a more concise formula is $\Gamma s_R\sim o(s_R^{b-3/2})$.

We show that partial-wave dispersion relations can be obtained for the $I=1$ Veneziano amplitude when $b<1$. It will be necessary to make subtractions if $2b+4am_\pi^2<0$. This result disagrees with Drago and Matsuda,[5] who suggested without proof that partial-wave dispersion relations could not be used, and also with Sivers and Yellin.[5] For $I=0$ or $I=2$, the presence of a $V(t,u)$ term in the amplitude causes disc $V_l(s)$ to diverge exponentially as $s\to-\infty$ and partial-wave dispersion relations are not valid.

We propose a method for obtaining a unitary $\pi\pi$ scattering amplitude from the Veneziano model.

II. DISCONTINUITY ACROSS LEFT-HAND CUT

We define the amplitude $V(s,t)$

$$V(s,t)=-\gamma\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}\tag{1}$$

and write for the isospin-1 $\pi\pi$ scattering amplitude

$$A^1(s,t,u)=V(s,t)-V(s,u).\tag{2}$$

Here $\alpha(s)$ is the ρ-f^0 exchange-degenerate trajectory and is assumed to have a linear form:

$$\alpha(s)=as+b,\quad a>0.\tag{3}$$

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[1] G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2] C. Lovelace, Phys. Letters 28B, 264 (1968).

[3] The asymptotic relation $f(x)\sim o(x^a)$ means that, for any positive number δ, $f(x)x^{-a-\delta}\to 0$ as $x\to\infty$. For example, if $f(x)\to x^a/\ln x$ as $x\to\infty$, then $f(x)\sim o(x^a)$.

[4] J. Shapiro and J. Yellin, LRL Report No. 18500 (unpublished).

[5] F. Drago and S. Matsuda, Phys. Rev. 181, 2095 (1969); D. Sivers and J. Yellin, Ann. Phys. (N. Y.) 55, 107 (1969).

The partial-wave projection of $A^1(s,t,u)$ for physical l values can be written[6]

$$V_l(s)=\gamma\frac{\alpha(s)}{aq^2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)}Q_l\left(1+\frac{n+1-b}{2aq^2}\right),\quad l>\mathrm{Re}\,\alpha(s). \tag{4}$$

For $b<1$, this expression is an analytic function of s with a left-hand cut (LHC) starting at $s=s_L=4m_\pi^2+(b-1)/a$ and a series of branch points on the cut at $s_L-(n-1)/a$ for $n=1, 2, \ldots$.

The discontinuity across the cut is given by

$$\mathrm{disc}\ V_l(s)=\tfrac{1}{2}\pi\gamma\frac{\alpha(s)}{aq^2}\sum_{n=0}^{p}\frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)}\times P_l\left(1+\frac{n+1-b}{2aq^2}\right),\quad s\le s_L \tag{5}$$

where p is the largest integer less than or equal to $b-1-4aq^2$. Because p is a step function of s, disc $V_l(s)$ may be discontinuous at the branch points of Eq. (4). For all other values of s on the left-hand cut, disc $V_l(s)$ and its derivatives are defined and continuous. A typical graph of disc $V_l(s)$ is shown in Fig. 1.

Our primary objective is to place an asymptotic bound on the behavior of disc $V_l(s)$ as $s\to-\infty$. This is conveniently done by rewriting (5) in terms of an infinite series. In Appendix A we show that

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)}n^k=0\quad\text{if}\quad\alpha(s)<-1-k \tag{6}$$

for all non-negative integers k. Since the Legendre function $P_l(1+(n+1-b)/2aq^2)$ can be expanded in powers of $1+(1-b)/2aq^2$ and $n/2aq^2$, (5) becomes

$$\mathrm{disc}\ V_l(s)=-\tfrac{1}{2}\pi\gamma\frac{\alpha(s)}{aq^2}\sum_{n=p+1}^{\infty}\frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)}\times P_l\left(1+\frac{n+1-b}{2aq^2}\right),\quad s\le s_L \tag{7}$$

which converges (absolutely) for $\alpha(s)<-1-l$. Using the relation $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$, this equation may be rewritten in the form

$$\mathrm{disc}\ V_l(s)=\tfrac{1}{2}\gamma\frac{\alpha(s)}{aq^2}\sin\pi\alpha(s)\Gamma(-\alpha(s))\times\sum_{n=p+1}^{\infty}\frac{\Gamma(n+\alpha(s)+1)}{\Gamma(n+1)}P_l\left(1+\frac{n+1-b}{2aq^2}\right),\quad s\le s_L. \tag{8}$$

[6] D. I. Fivel and P. K. Mitter, Phys. Rev. **183**, 1240 (1969).

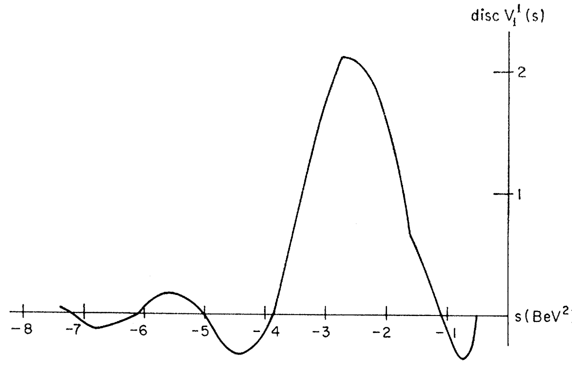

FIG. 1. Discontinuity across the left-hand cut of $V_1^1(s)$. In this graph $b=\frac{1}{2}$, $a=1$ BeV^{-2}, $m_\pi=0$, and $\gamma=0.5$.

From this expression, we see that, in general, disc $V_l(s)$ is an oscillating function of s with an infinite number of zeros on the real s axis. For $2b+4am_\pi^2>0$ and $\alpha(s)\le-2-l$, there are zeros at $\alpha(s)=-2-l$, $-3-l$, $-4-l$, $\ldots$, and these are unique. In addition, there will be a finite number of zeros for $\alpha(s_L)>\alpha(s)>-2-l$. When $2b+4am_\pi^2<0$, the position of zeros of disc $V_l(s)$ is not obviously determined from Eq. (8), since the zeros of $\sin\pi\alpha$ are cancelled by $\Gamma(n+\alpha(s)+1)$.

Expanding $P_l(1+(n+1-b)/2aq^2)$ in powers of $n/2aq^2$, we observe that the asymptotic behavior of (8) as $s\to-\infty$ is controlled by terms of the form

$$\sin\pi\alpha(s)\Gamma(-\alpha(s))\sum_{n=p+1}^{\infty}\frac{\Gamma(n+\alpha(s)+1)}{\Gamma(n+1)}\left(\frac{n}{s}\right)^k,\quad\text{with }0\le k\le l. \tag{9}$$

From the definition of p, there exists a number ζ, $0\le\zeta<1$, such that $as=b+4am_\pi^2-1-p-\zeta$. Rewriting (9) in terms of p and ζ, we obtain

$$M(p)\sin\pi(2b+4am_\pi^2+1-\zeta)\frac{\Gamma(1-2b-4am_\pi^2+p+\zeta)}{p^k}\times\sum_{n=p+1}^{\infty}\frac{\Gamma(n+2b+4am_\pi^2-p-\zeta)}{\Gamma(n+1)}n^k,\quad 0\le k\le l \tag{10}$$

where $M(p)$ is bounded as $p\to+\infty$. An analysis in Appendix B shows that Eq. (10) is bounded by $p^{-(1+2b+4am_\pi^2-\zeta)}$ as $p\to+\infty$. Therefore, to leading order in p, disc $V_l(s)$ is also bounded by $p^{-(1+2b+4am_\pi^2-\zeta)}$. We may conclude that for any positive real number δ

$$s^{2b+4am_\pi^2-\delta}\ \mathrm{disc}\ V_l(s)|_{\mathrm{LHC}}\to0\quad\text{as}\quad s\to-\infty. \tag{11}$$

In the case $2b+4am_\pi^2>0$, disc $V_l(s)\to0$ as $s\to-\infty$.

III. ASYMPTOTIC BEHAVIOR OF PARTIAL-WAVE AMPLITUDE

In order to study the asymptotic properties of $V_l(s)$, we construct a formula[5] from Eq. (4) which is defined in the entire complex s plane:

$$V_l(s) = -\gamma\frac{(aq^2)^l}{l!}(-1)^l\Gamma(1-\alpha(s))\int_{-1}^{1}dt(1-t^2)^l \times\frac{\partial^l}{\partial\epsilon^l}\frac{\Gamma(\epsilon+2q^2a(t+1))}{\Gamma(-\alpha(s)+\epsilon+2q^2a(t+1))}\bigg|_{\epsilon=1-b}. \quad (12)$$

For $b<1$ this formula has poles on the positive real s axis and has a LHC. The properties of the LHC and corresponding branch points have already been discussed on the basis of Eq. (4). $V_l(s)$ is a holomorphic function in any domain not containing the real s axis. In studying the asymptotic properties of $V_l(s)$, we will assume $b<1$ and $\mathrm{Im}s\neq 0$ as $|s|\to\infty$.

In (12) we eliminate aq^2 in terms of α and make the substitution $t=2r-1$ on the integral. Then defining a constant $c=b+4am_\pi^2$, we may write the amplitude as a function of α as follows:

$$V_l(\alpha) = -\frac{2\gamma(-1)^l}{l!}\int_0^1\frac{\partial^l K_l(\alpha,r,\epsilon)}{\partial\epsilon^l}\bigg|_{\epsilon=1-b}dr, \quad (13)$$

where

$$K_l(\alpha,r,\epsilon) = r^l(1-r)^l(\alpha-c)^l\frac{\Gamma(1-\alpha)\Gamma(\epsilon-rc+r\alpha)}{\Gamma(\epsilon-rc-(1-r)\alpha)}. \quad (14)$$

Our study of $V_l(\alpha)$ as $|\alpha|\to\infty$ will be based on the asymptotic properties of the integrand of (13). For an arbitrary complex constant a, we use the standard formula

$$\ln\Gamma(z+a)\to(z+a-\tfrac{1}{2})\ln z - z+\tfrac{1}{2}\ln 2\pi+O(1/z), \quad |\arg z|<\pi \quad (15)$$

to expand the Γ functions of (14). Regardless of the magnitude of α, there will always be regions near $r=0$ and $r=1$ in the integral of (13) for which $|r\alpha|$ and $|(1-r)\alpha|$ are small or zero. Hence the expansion formula (15) will not be applicable to all three Γ functions of (14) near $r=0$ or $r=1$. It is therefore convenient to divide the region of integration into three parts: a region near $r=0$, a region where $|r\alpha|$ and $|(1-r)\alpha|\to\infty$ as $|\alpha|\to\infty$, and a region near $r=1$. We choose a number η with the property $0<\eta<1$ and write

$$\int_0^1\frac{\partial^l K_l}{\partial\epsilon^l}dr = \int_0^{|\alpha|^{-\eta}}\frac{\partial^l K_l}{\partial\epsilon^l}+\int_{|\alpha|^{-\eta}}^{1-|\alpha|^{-\eta}}\frac{\partial^l K_l}{\partial\epsilon^l}dr +\int_{1-|\alpha|^{-\eta}}^{1}\frac{\partial^l K_l}{\partial\epsilon^l}dr. \quad (16)$$

Consider the integral of $\partial^l K_l/\partial\epsilon^l$ from $|\alpha|^{-\eta}$ to $1-|\alpha|^{-\eta}$. For all r in the region of integration, $|r\alpha|$ and $|(1-r)\alpha|\to\infty$ as $|\alpha|\to\infty$ and

$$\ln\left|\frac{\partial^l K_l}{\partial\epsilon^l}\right|\to\mathrm{Re}\alpha[r\ln r+(1-r)\ln(1-r)]-r\pi|\mathrm{Im}\alpha| \quad (17)$$

to leading order in α. When $\mathrm{Re}\alpha$ is bounded from below, (17) shows that $\partial^l K_l/\partial\epsilon^l$ decreases exponentially to zero as $|\alpha|\to\infty$. In this case the second integral of (16) must go to zero exponentially as $|\alpha|\to\infty$.

When $\mathrm{Re}\alpha\to-\infty$ as $|s|\to\infty$, $\partial^l K_l/\partial\epsilon^l$ diverges exponentially provided $\mathrm{Im}\alpha$ does not go to infinity too fast. Then for infinitely many r in the region of integration of (13), the integrand diverges as $\mathrm{Re}\alpha\to-\infty$. However, because there is cancellation of positive and negative values of the integrand, the integral itself may not blow up.

Our study of the asymptotic properties of $V_l(s)$ will be divided into two parts depending on whether $\mathrm{Re}\alpha\to-\infty$. We begin by assuming that $\mathrm{Re}\alpha$ is bounded from below. From (17) it follows that

$$\int_{|\alpha|^{-\eta}}^{1-|\alpha|^{-\eta}}\frac{\partial^l K_l}{\partial\epsilon^l}dr\to 0\text{ (exponentially) as }|\alpha|\to\infty \quad \text{if }\mathrm{Re}\alpha\not\to-\infty. \quad (18)$$

We consider the integral of $\partial^l K_l/\partial\epsilon^l$ with respect to r from $r=0$ to $r=|\alpha|^{-\eta}$. Any point r in the region of integration must approach zero as $|\alpha|\to\infty$. The quantity $|\alpha r|$ can approach infinity or remain bounded as $|\alpha|\to\infty$ and the asymptotic behavior of $\partial^l K_l/\partial\epsilon^l$ to leading order in α is given by

$$\ln\left|\frac{\partial^l K_l}{\partial\epsilon^l}\right|\to\mathrm{Re}\alpha[r\ln r+(1-r)\ln(1-r)]-r\pi|\mathrm{Im}\alpha| +(1-\epsilon)\ln|\alpha| \quad \text{if }|r\alpha|\to\infty \quad (19)$$

$$\to l\ln|r\alpha|+(1-\epsilon-r\,\mathrm{Re}\alpha)\ln|\alpha| \quad \text{if }|r\alpha|\text{ is bounded.} \quad (20)$$

Because $\mathrm{Re}\alpha$ is bounded from below as $|\alpha|\to\infty$, the most divergent behavior possible for $|\partial^l K_l/\partial\epsilon^l|$ for all r in the region $0\leq r\leq|\alpha|^{-\eta}$ is $G(\alpha)|\alpha|^{1-\epsilon}$, where $G(\alpha)$ has the property $\ln|G(\alpha)|/\ln|\alpha|\to 0$ as $|\alpha|\to\infty$. Thus we can write

$$\left|\int_0^{|\alpha|^{-\eta}}\frac{\partial^l K_l}{\partial\epsilon^l}dr\right|\leq\int_0^{|\alpha|^{-\eta}}\left|\frac{\partial^l K_l}{\partial\epsilon^l}\right|dr\leq|G(\alpha)|\,|\alpha|^{b-\eta}. \quad (21)$$

For any real number $\delta>0$, we may choose an η such that $1-\delta<\eta<1$, and therefore from (21)

$$|\alpha|^{1-b-\delta}\int_0^{|\alpha|^{-\eta}}\frac{\partial^l K_l}{\partial\epsilon^l}dr\to 0 \quad \text{as }|\alpha|\to\infty\quad\text{if }\mathrm{Re}\alpha\not\to-\infty. \quad (22)$$

For r in the region $1-|\alpha|^{-\eta}\leq r\leq 1$, $|(1-r)\alpha|$ will either approach infinity or remain bounded as $|\alpha|\to\infty$. When $|(1-r)\alpha|\to\infty$ as $|\alpha|\to\infty$, Eq. (17) may be used to show that $\partial^l K_l/\partial\epsilon^l$ approaches zero exponentially if $\mathrm{Re}\alpha\not\to-\infty$. When $|(1-r)\alpha|$ is bounded as

$|\alpha|\to\infty$, we obtain

$$\ln\left|\frac{\partial^l K_l}{\partial\epsilon^l}\right|\to l\ln|(1-r)\alpha| +[\epsilon-c-(1-r)\ \mathrm{Re}\alpha]\ln|\alpha|-\pi|\ \mathrm{Im}\alpha| \quad (23)$$

to leading order in α. Then provided $\mathrm{Re}\alpha\not\to-\infty$, the most divergent behavior of $\partial^l K_l/\partial\epsilon^l$ as $|\alpha|\to\infty$ is $\bar{G}(\alpha)|\alpha|^{\epsilon-c}$, where $\ln|\bar{G}(\alpha)|/\ln|\alpha|\to 0$ as $|\alpha|\to\infty$. If $|\mathrm{Im}\alpha/\ln\alpha|\to\infty$ as $|\alpha|\to\infty$, $\partial^l K_l/\partial\epsilon^l$ goes to zero regardless of the value of $\epsilon-c$. Yet when $\mathrm{Im}\alpha$ is bounded as $|\alpha|\to\infty$, $\partial^l K_l/\partial\epsilon^l$ can blow up if $\epsilon-c>0$.

To obtain an asymptotic bound on the integral of $\partial^l K_l/\partial\epsilon^l$ from $r=1-|\alpha|^{-\eta}$ to $r=1$, we write

$$\int_{1-|\alpha|^{-\eta}}^{1}\left|\frac{\partial^l K_l}{\partial\epsilon^l}\right|dr\leq|\bar{G}(\alpha)|\,|\alpha|^{\epsilon-c-\eta},\quad \mathrm{Re}\alpha\not\to-\infty\,. \quad (24)$$

For any $\delta>0$, η can be chosen such that $1-\delta<\eta<1$ and we obtain

$$|\alpha|^{2b+4am_\pi^2-\eta}\int_{1-|\alpha|^{-\eta}}^{1}\frac{\partial^l K_l}{\partial\epsilon^l}dr\to 0 \quad (25)$$

as $|\alpha|\to\infty$ if $\mathrm{Re}\alpha\not\to-\infty$. The integral of (25) will approach zero as $|\alpha|\to\infty$ even when $2b+4am_\pi^2<0$ provided that $|\mathrm{Im}\alpha/\ln\alpha|\to\infty$ as $|\alpha|\to\infty$.

The asymptotic behavior of $V_l(\alpha)$ can now be determined. We define a number x to be the minimum of $1-b$ and $2b+4am_\pi^2$. Combining (13), (16), (18), (22), and (25), for $b<1$ we have shown that for any number $\delta>0$

$$s^{x-\delta}V_l(s)\to 0\,, \quad (26)$$

$$s^{1-b-\delta}V_l(s)\to 0 \quad \text{if} \quad |\mathrm{Im}s/\ln s|\to\infty \quad (27)$$

as $|s|\to\infty$ if $\mathrm{Re}s\not\to-\infty$ and $\mathrm{Im}s\neq 0$. In Appendix C we show that this theorem is exactly true even when $\mathrm{Re}s\to-\infty$ as $|s|\to\infty$.

IV. PARTIAL-WAVE DISPERSION RELATIONS

In this section we assume for convenience that $-2am_\pi^2<b<1$. When this condition is satisfied, $V_l(s)\to 0$ as $|s|\to\infty$ and disc $V_l(s)\to 0$ as $s\to-\infty$. We perform an integral of $V_l(s')/(s'-s)$ over the contour shown in Fig. 2. The result is an unsubtracted partial-wave dispersion relation

$$V_l(s)=\sum_{n=1}^{\infty}\frac{\beta_n^{(l)}}{s-s_n}+\frac{1}{\pi}\int_{-\infty}^{s_L}\frac{\mathrm{disc}\ V_l(s')ds'}{s-s'}, \quad (28)$$

where $s_n=(n-b)/a$ and $\beta_n^{(l)}$ is the residue of $V_l(s)$ at $s=s_n$. When the ρ-f^0 trajectory is purely real and $b<1$, the residues $\beta_n^{(l)}$ are zero whenever $l>n$. Hence a more concise representation for $V_l(s)$ can be written

$$V_l(s)=\sum_{n=l}^{\infty}\frac{\beta_n^{(l)}}{s-s_n}+\frac{1}{\pi}\int_{-\infty}^{s_L}\frac{\mathrm{disc}\ V_l(s')ds'}{s'-s}. \quad (29)$$

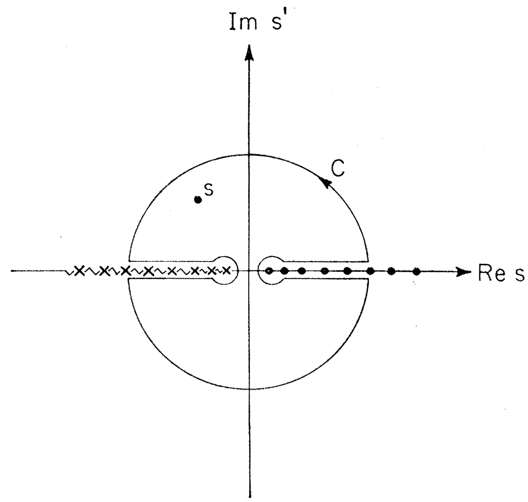

FIG. 2. Contour used to obtain Veneziano partial-wave dispersion relations.

Since $V_l(s)$ and the integral in (29) are well defined for all s, if $\mathrm{Im}s\neq 0$, we conclude that the sum in (29) converges for all s provided that $\mathrm{Im}s\neq 0$.

It can be shown directly that the sum in (29) converges. To do this we determine the asymptotic behavior of the residues. From the definition of $\beta_n^{(l)}$ and Eqs. (13) and (14), we obtain

$$\beta_n^{(l)}=\frac{2\gamma(-1)^{n+l+1}}{l!a\Gamma(n)}(n-c)^l\int_0^1\frac{\partial^l}{\partial\epsilon^l}r^l(1-r)^l \times\frac{\Gamma(\epsilon-rc+rn)}{\Gamma(\epsilon-rc-(1-r)n)}dr. \quad (30)$$

The behavior of this equation as $n\to\infty$ follows easily by analogy to the analysis of Eqs. (13) and (14). We define a parameter x to be the minimum of $1-b$ and $2b+4am_\pi^2$. Then for any $\delta>0$

$$n^{x-\delta}\beta_n^{(l)}\to 0 \quad \text{as} \quad n\to\infty \quad \text{with } l \text{ fixed}. \quad (31)$$

Since $1/(s-s_n)$ behaves as $1/n$ in the limit as $n\to\infty$, the sum in (29) converges if $x>0$. But $x>0$ is equivalent to $-2am_\pi^2<b<1$.

The partial widths of $\pi\pi$ resonances in the zero-width approximation are related to $\beta_n^{(l)}$ by the formula

$$\Gamma_n^{(l)}=-\beta_n^{(l)}/\sqrt{s_n} \quad \text{as} \quad n\to\infty\,. \quad (32)$$

Combining (31) and (32), for fixed l the result is

$$n^{1/2+x-\delta}\Gamma_n^{(l)}\to 0 \quad \text{as} \quad n\to\infty\,.$$

Therefore, for the standard ρ-f^0 trajectory, which has $a\simeq 1$ BeV^{-2} and $b\simeq\frac{1}{2}$, the partial widths predicted by the Veneziano model must go to zero as fast as $1/s$ as $s\to\infty$.

The outstanding problem of the Veneziano model is its failure to satisfy unitarity. We suggest a method of unitarizing the Veneziano amplitude which uses the

N/D equations.[7] The N/D equations are derived in the standard way from a unitary amplitude with a right- and left-hand cut. Then the Veneziano amplitude may be used to obtain an input discontinuity across the LHC. From our results in Sec. II, disc $V_l(s)$ is an oscillating function which decreases to zero as $s \to -\infty$. Hence a unitary solution to the N/D equations can be obtained and we may look for a bootstrapped ρ meson in the output. In this way, one can determine how closely the Veneziano LHC approximates the true LHC for $\pi\pi$ scattering.

ACKNOWLEDGMENT

We wish to thank Professor Richard J. Eden for discussions on this subject.

APPENDIX A

In this Appendix we prove the result stated in Sec. II, namely,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)} n^k = 0 \quad \text{if} \quad \alpha(s) < -1-k. \tag{A1}$$

The proof is by induction. For $k=0$ we observe that

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)} = (1-x)^{-1-\alpha(s)}\Big|_{x=1} = 0 \quad \text{if} \quad \alpha(s) < -1. \tag{A2}$$

Now assume that (A1) is true for $k=0, 1, 2, \ldots, l$. Since there exist constants $c_i(l)$ such that

$$n^{l+1} = n(n-1)\cdots(n-l) + \sum_{i=1}^{l} c_i n^i, \tag{A3}$$

we can write

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)} n^{l+1} = \sum_{n=l+1}^{\infty} \frac{\Gamma(n+\alpha(s)+1)}{\Gamma(n-l)\Gamma(\alpha(s)+1)}. \tag{A4}$$

The latter sum is equal to

$$\frac{\Gamma(l+\alpha(s)+2)}{\Gamma(\alpha(s)+1)} (1-x)^{-l-\alpha(s)-2}\Big|_{x=1} = 0 \quad \text{if} \quad \alpha(s) < -l-2. \tag{A5}$$

This completes the proof.

APPENDIX B

We prove that Eq. (10) is bounded by $p^{-(1+2b+4am_\pi^2-\zeta)}$ as $p \to +\infty$. We do this by finding the asymptotic behavior of the sum in Eq. (10). When $1+2b+4am_\pi^2-\zeta \neq 0, -1, -2, \ldots$, the sum can be rewritten

$$\frac{\Gamma(1+2b+4am_\pi^2-\zeta)}{p!} \left[(p+1)^{k-1} + \sum_{n=1}^{\infty} \frac{(n+2b+4am_\pi^2-\zeta)\cdots(2+2b+4am_\pi^2-\zeta)(1+2b+4am_\pi^2-\zeta)}{(p+n)\cdots(p+2)(p+1)} (p+n+1)^{k-1} \right]. \tag{B1}$$

We choose an integer N with the property

$$N \geq |1+2b+4am_\pi^2| \quad \text{and} \quad N \geq |2b+4am_\pi^2|.$$

Then the magnitude of (B1) is bounded by

$$\frac{(p+1)^{k-1}}{p!} + \frac{1}{(N-1)!} \sum_{n=1}^{\infty} \frac{(n+p+1)^{k-1}}{(n+N)(n+N+1)\cdots(n+p)} \tag{B2}$$

to leading order in p. In the case where $k=0$, the sum in (B2) can be performed[8] and (B2) is equal to

$$[p!(p-N+1)]^{-1}.$$

For $k \geq 1$, $(n+p+1)^{k-1}$ may be expanded by the formula

$$(n+p+1)^{k-1} = \sum_{r=0}^{k-1} \binom{k-1}{r} (n+N)^r (p-N+1)^{k-1-r}. \tag{B3}$$

This leads to a new upper bound for (B2):

$$\frac{(p+1)^{k-1}}{p!} + \frac{1}{(N-1)!} \sum_{r=0}^{k-1} \binom{k-1}{r} (p-N+1)^{k-1-r} \times \sum_{n=1}^{\infty} \frac{1}{(n+N+r)\cdots(n+p+1)(n+p)}. \tag{B4}$$

The sum over n is calculated[8] with the result that (B4) behaves as $p^{k-1}/p!$ to leading order in p.

When $2b+4am_\pi^2+1-\zeta$ is a nonpositive integer, $-m$, which occurs only if $2b+4am_\pi^2<0$, the first $m+1$ terms of the sum in Eq. (10) have poles. The poles are multiplied by the zeros of the sine function and the series itself may be terminated after $m+1$ terms. The first term dominates asymptotically and behaves as $p^{k-1}/p!$.

We have shown that the product of the sine function and sum in Eq. (10) is bounded by $p^{k-1}/p!$ as $p \to +\infty$. It follows easily that Eq. (10) is bounded by $p^{-(1+2b+4am_\pi^2-\zeta)}$.

[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (English translation) (Academic, New York, 1965).

APPENDIX C

In this Appendix we prove the result stated in Sec. III in the case where $\mathrm{Re}\alpha\rightarrow-\infty$. Expression (12) for $V_l(\alpha)$ may be replaced by another integral representation which allows us to study the asymptotic behavior of $V_l(\alpha)$ as $\mathrm{Re}\alpha\rightarrow-\infty$. Fixing ϵ and α, we define a function $g(z)$ by

$$g(z)=(1-z^2)^l\frac{\partial^l}{\partial\epsilon^l}\frac{\Gamma(\epsilon-\frac{1}{2}c+\frac{1}{2}\alpha+\frac{1}{2}z(\alpha-c))}{\Gamma(\epsilon-\frac{1}{2}c-\frac{1}{2}\alpha+\frac{1}{2}z(\alpha-c))}\bigg|_{\epsilon=1-b},\quad\text{(C1)}$$

and from (12) note that

$$V_l(\alpha)=-\gamma\frac{(\alpha-c)^l}{4^l l!}(-1)^l\Gamma(1-\alpha)\int_{-1}^{1}g(t)dt.\quad\text{(C2)}$$

We compute the integral of $g(z)$ in the complex z plane over the closed contour shown in Fig. 3. $g(z)$ is a meromorphic function and has no poles inside or on the contour of integration, provided that $b<1$ and $\mathrm{Im}\alpha<0$. In what follows we assume that $\mathrm{Im}\alpha<0$, in which case the integral of $g(z)$ vanishes. Later the results are extended to the case where $\mathrm{Im}\alpha>0$.

As the height R of C_3 approaches $+\infty$ (see Fig. 3), the integral of $g(z)$ along C_3 approaches zero, provided that $\mathrm{Re}\alpha<-l$. The integrals along C_1 and C_2 converge absolutely if $\mathrm{Re}\alpha<-1-l$. Since these inequalities are satisfied in the limit as $\mathrm{Re}\alpha\rightarrow-\infty$, we may write

$$\int_{-1}^{1}g(t)dt=-\int_{C_1}g(z)dz-\int_{C_2}g(z)dz\quad\text{as}\quad R\rightarrow+\infty\,.\quad\text{(C3)}$$

A new representation of $V_l(\alpha)$ is obtained from (C2) and (C3):

$$V_l(\alpha)=\frac{-\gamma(-1)^l}{l!4^l}\left[\int_0^\infty\frac{\partial^l I_l(\alpha,y,\epsilon)}{\partial\epsilon^l}dy-\int_0^\infty\frac{\partial^l\bar{I}_l(\alpha,y,\epsilon)}{\partial\epsilon^l}dy\right],\quad\text{(C4)}$$

with

$$I_l(\alpha,y,\epsilon)=iy^l(y+2i)^l(\alpha-c)^l\times\frac{\Gamma(1-\alpha)\Gamma(\epsilon+\frac{1}{2}iy(\alpha-c))}{\Gamma(\epsilon-c+(\frac{1}{2}iy-1)(\alpha-c))},\quad\text{(C5)}$$

$$\bar{I}_l(\alpha,y,\epsilon)=iy^l(y-2i)^l(\alpha-c)^l\times\frac{\Gamma(1-\alpha)\Gamma(\epsilon+(\frac{1}{2}iy+1)(\alpha-c))}{\Gamma(\epsilon-c+\frac{1}{2}iy(\alpha-c))},\quad\text{(C6)}$$

if $\mathrm{Re}\alpha<-1-l$ and $\mathrm{Im}\alpha<0$.

The asymptotic behavior of $V_l(\alpha)$ as $\mathrm{Re}\alpha\rightarrow-\infty$ can be determined by an examination of the integrands of (C4) in the limit as $|\alpha|\rightarrow\infty$. Since the asymptotic behavior of these integrands will depend on the value of y, it is convenient to divide each of the two integrals in (C4) into three parts:

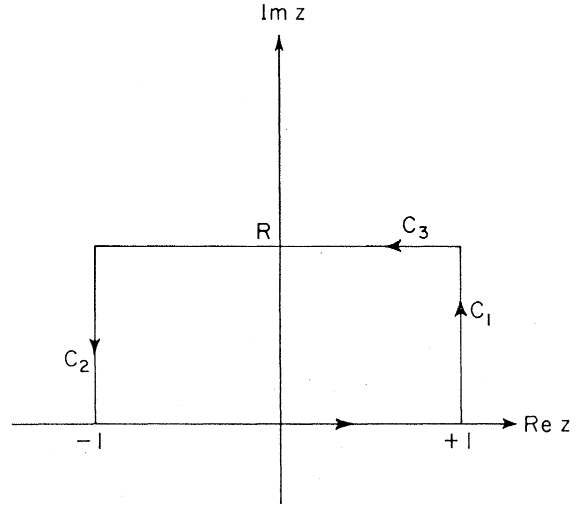

FIG. 3. Contour used in the analysis of $V_l(s)$ as $\mathrm{Re}s\rightarrow-\infty$.

$$\int_0^\infty\frac{\partial^l I_l}{\partial\epsilon^l}dy=\int_0^{|\alpha|^{-\lambda}}\frac{\partial^l I_l}{\partial\epsilon^l}dy+\int_{|\alpha|^{-\lambda}}^{|\alpha|}\frac{\partial^l I_l}{\partial\epsilon^l}dy+\int_{|\alpha|}^{\infty}\frac{\partial^l I_l}{\partial\epsilon^l}dy,\quad\text{(C7)}$$

where λ is an arbitrary parameter which satisfies $0<\lambda<1$. The same relation can be written for $\bar{I}_l$.

We will now place asymptotic bounds on each of the integrals appearing in (C7) in the limit as $\mathrm{Re}\alpha\rightarrow-\infty$. When y is in the region $|\alpha|^{-\lambda}\leq y\leq|\alpha|$, the asymptotic behavior of $\ln|\partial^l I_l/\partial\epsilon^l|$ to leading order is given by

$$\ln\left|\frac{\partial^l I_l}{\partial\epsilon^l}\right|\rightarrow\mathrm{Re}\alpha[\tfrac{1}{2}y\arg(1+2i/y)+\tfrac{1}{2}\ln(1+\tfrac{1}{4}y^2)]+\mathrm{Im}\alpha[\tfrac{1}{4}y\ln(1+4/y^2)-\arg(1-\tfrac{1}{2}iy)]\quad\text{as}\quad\mathrm{Re}\alpha\rightarrow-\infty\,.\quad\text{(C8)}$$

The argument functions of (C8) are restricted to values less than π by (15). When $\mathrm{Im}\alpha<0$ and $\mathrm{Re}\alpha\rightarrow-\infty$, this expression for $\ln|\partial^l I_l/\partial\epsilon^l|$ approaches $-\infty$ as $|\alpha|\rightarrow\infty$. More precisely, there exist positive numbers M and $\bar{M}$ and a function $H(\alpha)$ such that for all y in the interval $|\alpha|^{-\lambda}\leq y\leq|\alpha|$

$$\left|\frac{\partial^l I_l}{\partial\epsilon^l}\right|\leq H(\alpha)\exp(M\,\mathrm{Re}\alpha|\alpha|^{-\lambda}+\bar{M}\,\mathrm{Im}\alpha|\alpha|^{-\lambda}),\quad\text{(C9)}$$

where $\ln|H(\alpha)|/(M\,\mathrm{Re}\alpha|\alpha|^{-\lambda}+\bar{M}\,\mathrm{Im}\alpha|\alpha|^{-\lambda})\rightarrow0$ as $|\alpha|\rightarrow\infty$. Therefore,

$$\left|\int_{|\alpha|^{-\lambda}}^{|\alpha|}\frac{\partial^l I_l}{\partial\epsilon^l}dy\right|\leq H(\alpha)\exp(M\,\mathrm{Re}\alpha|\alpha|^{-\lambda}+\bar{M}\,\mathrm{Im}\alpha|\alpha|^{-\lambda})\times[|\alpha|-|\alpha|^{-\lambda}].\quad\text{(C10)}$$

We conclude that the integral of $\partial^l I_l/\partial\epsilon^l$ from $|\alpha|^{-\lambda}$ to $|\alpha|$ decreases exponentially to zero as $|\alpha|\to\infty$ when $\mathrm{Re}\alpha\to-\infty$ and $\mathrm{Im}\alpha<0$.

For $y\geq|\alpha|$, the asymptotic behavior of $\partial^l I_l/\partial\epsilon^l$ to leading order is given by

$$\ln\left|\frac{\partial^l I_l}{\partial\epsilon^l}\right|\to(\mathrm{Re}\alpha+l)\ln y+\tfrac{1}{2}\pi\,\mathrm{Im}\alpha. \quad (C11)$$

We may use this equation to write

$$\left|\int_{|\alpha|}^{\infty}\frac{\partial^l I_l}{\partial\epsilon^l}dy\right|\leq\bar{H}(\alpha)e^{(\pi/2)\mathrm{Im}\alpha}\int_{|\alpha|}^{\infty}y^{\mathrm{Re}\alpha+l}dy, \quad (C12)$$

where

$$\ln|\bar{H}(\alpha)|/[\tfrac{1}{2}\pi\,\mathrm{Im}\alpha+(\mathrm{Re}\alpha+l)\ln y]\to0 \quad\text{as}\quad |\alpha|\to\infty.$$

For $\mathrm{Im}\alpha<0$ and $\mathrm{Re}\alpha\to-\infty$, this integral goes to zero exponentially.

Consider the integral of $\partial^l I_l/\partial\epsilon^l$ from 0 to $|\alpha|^{-\lambda}$. For y in this range, $|\alpha y|$ can approach infinity or remain bounded as $|\alpha|\to\infty$. The asymptotic behavior of $\ln|\partial^l I_l/\partial\epsilon^l|$ when $|\alpha y|\to\infty$ is given by the right-hand side of (C8) plus the quantity $(1-\epsilon)\ln|\alpha|$. When αy remains bounded as $|\alpha|\to\infty$, we obtain

$$\left|\frac{\partial^l I_l}{\partial\epsilon^l}\right|\to|\alpha y|^l|\alpha|^{1-\epsilon+y/2\,\mathrm{Im}\alpha} \quad\text{for}\quad \mathrm{Im}\alpha<0. \quad (C13)$$

We conclude that the most divergent behavior of $\partial^l I_l/\partial\epsilon^l$ for y in the region $0\leq y\leq|\alpha|^{-\lambda}$ is $|\alpha|^b$. Hence for any constant $\delta>0$

$$|\alpha|^{1-b-\delta}\int_0^{|\alpha|^{-\lambda}}\frac{\partial^l I_l}{\partial\epsilon^l}dy\to0. \quad (C14)$$

In the limit as $\mathrm{Re}\alpha\to-\infty$ with $\mathrm{Im}\alpha<0$, we have shown that for any number $\delta>0$

$$\alpha^{1-b-\delta}\int_0^{\infty}\frac{\partial^l I_l(\alpha,y,\epsilon)}{\partial\epsilon^l}dy\to0. \quad (C15)$$

We now determine the conditions under which the integral of $\partial^l\bar{I}_l/\partial\epsilon^l$ goes to zero as $|\alpha|\to\infty$. For y in the region $|\alpha|^{-\lambda}\leq y\leq|\alpha|$, the expansion of $\ln|\partial^l\bar{I}_l/\partial\epsilon^l|$ to leading order in α becomes

$$\ln\left|\frac{\partial^l\bar{I}_l}{\partial\epsilon^l}\right|\to\mathrm{Re}\alpha[\tfrac{1}{2}y\arg(1+2i/y)+\tfrac{1}{2}\ln(1+\tfrac{1}{4}y^2)]$$
$$+\mathrm{Im}\alpha[-\arg(-1-\tfrac{1}{2}iy)-\tfrac{1}{4}y\ln(1+4/y^2)] \quad (C16)$$

as $|\alpha|\to\infty$ if $\mathrm{Im}\alpha<0$ and $\mathrm{Re}\alpha\to-\infty$. If $y\geq|\alpha|$, it follows that

$$\ln\left|\frac{\partial^l\bar{I}_l}{\partial\epsilon^l}\right|\to(\mathrm{Re}\alpha+l)\ln y+\tfrac{1}{2}\pi\,\mathrm{Im}\alpha \quad (C17)$$

as $\mathrm{Re}\alpha\to-\infty$ with $\mathrm{Im}\alpha<0$. Arguments similar to those used to prove (C10) and (C12) were zero as $|\alpha|\to\infty$ can now be used to show that

$$\int_{|\alpha|^{-\lambda}}^{\infty}\frac{\partial^l\bar{I}_l}{\partial\epsilon^l}dy\to0\ \text{(exponentially)}, \quad (C18)$$

where $\mathrm{Re}\alpha\to-\infty$ and $\mathrm{Im}\alpha<0$.

For $0\leq y\leq|\alpha|^{-\lambda}$, $|\alpha y|$ approaches infinity or remains bounded as $|\alpha|\to\infty$. Formula (C16) plus $(\epsilon-c)\ln|\alpha|$ gives the asymptotic behavior of $\partial^l\bar{I}_l/\partial\epsilon^l$ as $|\alpha|\to\infty$ and $|\alpha y|\to\infty$. When $|\alpha y|$ is bounded as $|\alpha|\to\infty$, we obtain to leading order in α

$$\ln\left|\frac{\partial^l\bar{I}_l}{\partial\epsilon^l}\right|\to(\epsilon-c)\ln|\alpha|$$
$$+\mathrm{Im}\alpha(\pi-\tfrac{1}{2}y\ln|\alpha|)+l\ln|\alpha y|. \quad (C19)$$

If $|\mathrm{Im}\alpha/\ln\alpha|\to\infty$ as $|\alpha|\to\infty$ with $|\alpha y|$ bounded, $\partial^l\bar{I}_l/\partial\epsilon^l$ approaches zero exponentially. In general, the most divergent behavior of $\partial^l\bar{I}_l/\partial\epsilon_l$ is $|\alpha|^{\epsilon-c}$. These results and Eq. (C18) can be combined to show that as $\mathrm{Re}\alpha\to-\infty$ with $\mathrm{Im}\alpha<0$

$$\int_0^{\infty}\frac{\partial^l\bar{I}_l}{\partial\epsilon^l}dy\to0\ \text{(exponentially)} \quad (C20)$$

if $|\mathrm{Im}\alpha/\ln\alpha|\to\infty$, and

$$\alpha^{2b+4am_\pi^2-\delta}\int_0^{\infty}\frac{\partial^l\bar{I}_l}{\partial\epsilon^l}dy\to0, \quad (C21)$$

where δ is any positive number.

Based on Eqs. (C4), (C15), (C20), and (C21), we have extended the result stated in Sec. III to the case of $\mathrm{Re}\alpha\to-\infty$ and $\mathrm{Im}\alpha<0$.

The representation (12) has the property $V_l(\alpha)=V_l^*(\alpha^*)$. This means that $|V_l(\alpha)|$ and $|V_l(\alpha^*)|$ have the same properties as $|\alpha|\to\infty$ and the conclusions obtained in this Appendix apply equally well in the case where $\mathrm{Im}\alpha>0$.

Note added in proof. The formula which we have derived for $\pi\pi$ partial widths was also obtained by Sivers and Yellin.[5] The asymptotic and oscillatory behavior of disc $V_l(s)$ has been studied independently by Atkinson with conclusions similar to our own [D. Atkinson (private communication)].