PHYSICAL REVIEW D VOLUME 2, NUMBER 4 15 AUGUST 1970

Scale Invariance, Conformal Invariance, and the High-Energy Behavior of Scattering Amplitudes

DAVID J. GROSS
CERN Geneva, Switzerland
and
Palmer Physical Laboratory,
Princeton University, Princeton, New Jersey 07540

AND

J. WESS
CERN Geneva, Switzerland
and
University of Karlsruhe, Karlsruhe, Germany

(Received 13 October 1969)

The constraints placed by scale invariance upon the asymptotic behavior of scattering amplitudes, in theories with no dimensional coupling constants, are discussed. It is proved that for a wide class of Lagrangian theories, which include all renormalizable interactions except for ϕ^3 coupling, scale invariance implies invariance under conformal transformations. The equations that scattering amplitudes should satisfy in theories where the breaking of scale invariance is due solely to nonzero masses are derived, under the assumption of large energies compared to the masses. These equations are derived by two methods, first by direct scale and conformal transformations of the Green's functions and second by considering the low-energy theorem for the emission of the divergence of the currents which generate scale and conformal invariance. These divergences are essentially given by the trace of the symmetric energy-momentum tensor. A low-energy theorem is proved for the emission of an energy-momentum tensor (or graviton) from an arbitrary process up to quadratic terms in the graviton momenta. The equations of scale and conformal invariance are applied to the scattering of scalars off spinors, and photons. It is argued that scale invariance leads to asymptotic behavior that is governed by fixed cuts and not by Regge poles in the angular momentum. Although the strong interactions seem to be manifestly non-scale invariant, scale and conformal invariances may prove useful in discussing asymptotic behavior in quantum electrodynamics, in model field theories, and in high-energy inelastic electroproduction.

I. INTRODUCTION

A SCALE transformation is one that scales all coordinates according to

$$S(\lambda):\quad x_\mu \to \lambda x_\mu . \tag{1.1}$$

At the same time all momenta are subjected to the inverse scaling law

$$S(\lambda):\quad p_\mu \to (1/\lambda) p_\mu . \tag{1.2}$$

However, physical constants are left unchanged by this transformation. In our units $\hbar = c = 1$, and they remain unchanged under a scale transformation. Furthermore, all masses are unchanged.

For the world to be invariant under scale transformations, it is clearly necessary that all particles have vanishing masses[1] [since $S(\lambda):\ p^2 \to (1/\lambda)p^2$], and that there be no dependence of physical amplitudes upon dimensional constants. A Lagrangian field theory will be scale invariant if it contains only massless particles and dimensionless coupling constants. Quantum electrodynamics would therefore be scale invariant if the electron mass were zero.

The dynamics of the strongly interacting hadrons is manifestly non-scale invariant. Aside from the obvious fact that there exists an abundance of massive particles, the scattering amplitudes reveal many dimensional constants. These include the widths of diffraction peaks (mass^{-2}) and the slopes of Regge trajectories (mass^{-2}).

What is the purpose therefore of considering these transformations if nature so blatantly violates scale invariance? One can certainly not hope to treat scale invariance as an exact symmetry.[2] However, when all relevant energies are very large, one might hope that it would be possible to neglect all masses and thus recover scale invariance. Consider, for example, quantum electrodynamics. At very large energies and momentum transfers (compared to the electron mass) the scattering amplitudes should exhibit the features of the scale-invariant theory that would result if the electron were massless.

Accordingly, in this paper, we study scale invariance as an asymptotic (high-energy) symmetry of scattering amplitudes, in theories where the breaking of scale invariance is due only to the existence of particles with nonvanishing masses (i.e., no dimensional coupling

[1] J. Wess, Nuovo Cimento **18**, 1086 (1960). G. Mack and A. Salam (Ref. 6) have suggested that the scale current might be conserved even in a theory with massive particles, if the invariance is broken by a degenerate vacuum with only a σ particle being massless.

[2] Many authors have discussed the physical meaning and the possible applications of scale and conformal invariance. A partial list of literature can be found in F. Gürsey, Nuovo Cimento **3**, 98 (1956); T. Fulton, R. Rohrlich, and L. W. Hen, Rev. Mod. Phys. **34**, 442 (1962); H. A. Kastrup, Ann. Phys. (N. Y.) **7**, 388 (1962); H. A. Kastrup, in *Particles, Currents, Symmetries*, edited by P. Urban (Springer-Verlag, New York, 1968) pp. 407–468; K. Wilson, Phys. Rev. **179**, 1499 (1969).

constants). These include all renormalizable field theories, except for $\lambda\phi^3$ coupling.

Maxwell's equations and the quantum electrodynamics of massless particles are invariant under conformal transformations in addition to scale transformations. These are defined as a four-parameter group that transforms coordinates according to

$$C(\alpha):\quad x_\mu \to \frac{x_\mu+\alpha_\mu x^2}{1+2\alpha\cdot x+\alpha^2 x^2}. \tag{1.3}$$

These conformal and scale transformations form with the Poincaré transformations a 15-dimensional group. There is no reason, *a priori*, why a theory which is scale invariant should necessarily be conformally invariant. However, we shall prove that this is the case for essentially all Lagrangian field theories of interest; i.e., scale invariance implies conformal invariance. We therefore shall study the restrictions that conformal and scale invariance impose on the asymptotic behavior of scattering amplitudes.

At first sight it would seem that scale transformations necessarily take one off the mass shell $[p^2\to(1/\lambda)p^2]$, and that therefore one cannot obtain any useful information regarding on-mass-shell amplitudes. That this is not the case will be evident from the way in which we derive the consequences of scale and conformal invariance. It is well known that if

$$Q=\int d^3x\, J_0(x,t)$$

generates a symmetry transformation $(\partial_t Q=0)$, then the consequences of this symmetry can be derived from the low-energy theorem on the emission of the divergence of the conserved current J_μ. In our case we shall show that the divergence of the currents that generate scale and conformal transformations are related to the trace of the symmetric energy-momentum tensor. Using the conservation of this tensor, one can derive a low-energy theorem for the emission of a soft-energy momentum tensor (or a graviton which couples to this tensor) from any physical process up to terms of order k^2 in the momenta (k) of the graviton. As in the case of soft-photon emission, the low-energy theorem is expressed in terms of the on-mass-shell S matrix for the original process and does not involve off-mass-shell derivatives. Upon taking the trace of this low-energy theorem, we shall derive the equations for the on-mass-shell S matrix that embody scale [coefficient of the $(k_\mu)^0$ term] and conformal (the coefficients of the k_μ terms) invariance.

The equations we thereby derive are quite strong. In fact, the propagator of a scale-invariant theory is uniquely determined to be the free propagator. However, the propagator cannot be trivial, even in the case of a massless scale-invariant theory. The resolution of this paradox lies in the fact that owing to infrared divergences the limit of zero masses does not strictly exist. Alternatively, one can say that even for zero masses there exist a scale—the infrared cutoff λ. This scale appears, to any finite order in perturbation theory, in logarithmic terms [e.g., $\ln(s/\lambda)$] which violate scale invariance. Accordingly, in theories with finite masses, but which are otherwise scale invariant, there will usually appear logarithmic violations of asymptotic scale invariance, e.g., of the form $\ln(s/M^2)$. Therefore, all our results concerning the asymptotic behavior of amplitudes in such theories are only true up to logarithmic terms of this nature. In other words, we can determine the power behavior of the amplitudes, but we have no control over the logarithms. Furthermore, we must make the assumption, which seems to be valid in all specific examples we have investigated, that when one works to all orders in perturbation theory, the logarithmic terms do not build up to powers.

Finally, to what use shall we put these equations? One application is quantum electrodynamics, where we know that the theory is scale invariant except for the finite electron mass, and one can therefore partially determine the asymptotic behavior, up to logarithmic terms, of the amplitudes for any process. For other renormalizable, and, except for masses, scale-invariant theories, the same can be done. However, the relevance of such theories to the strong interactions observed in nature is unclear. In fact, we shall argue that these scale-invariant theories do not have asymptotic behavior corresponding to Regge poles but rather to fixed cuts in angular momentum. Regge behavior is inherently a non-scale-invariant phenomenon.

There is one other application we have in mind, namely, high-energy inelastic lepton-hadron scattering at large momentum transfers. In this case, the manifestly non-scale-invariant contributions of discrete states are rather small, owing to the rapidly falling form factors of these states and the large momentum transfers. Therefore, there is hope that one might see a scale-invariant background,[3] and in fact recent data indicate strongly that this is so.[4] We therefore would hope to impose conformal invariance on these amplitudes and to determine their asymptotic forms.

The outline of the paper is as follows. In Sec. II we discuss the formal properties of scale and conformal transformations. We prove that conformal invariance is a consequence of scale invariance in many Lagrangian theories. Section III is devoted to a derivation of the low-energy theorem for the emission of gravitons (or the energy-momentum tensor). In Sec. IV we use this low-energy theorem to derive the equations which the S matrix should satisfy at high energies. These equa-

[3] J. D. Bjorken, Phys. Rev. **179**, 1547 (1969).

[4] E. D. Bloom *et al.*, presented at the Fourteenth International Conference on High-Energy Physics, Vienna, 1968 (unpublished); W. K. Panofsky, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968).

tions are derived for all processes involving scalar, spin-$\frac{1}{2}$, and spin-1 particles. In the Appendix we show the equivalence of these equations to those derived from the transformation properties of the fields. Section V is devoted to applications of our equations. The first case considered is scalar-scalar scattering. We show that conformal invariance is automatic once the scattering amplitude is scale invariant, and we discuss the consequences of the latter. A comparison is made with the studies of the asymptotic behavior of scalar-scalar scattering in the literature. Secondly, we consider spinor-scalar and photon-scalar scattering. Here we show that conformal invariance requires the helicity-flip amplitudes to vanish asymptotically, and that no restrictions are imposed on the helicity-nonflip amplitudes.

II. SCALE AND CONFORMAL INVARIANCE IN LAGRANGIAN FIELD THEORIES

On purely group-theoretical grounds there is no reason why scale invariance should imply conformal invariance. The generators of the Poincaré group and of scale transformations form a closed algebra, which is a subalgebra of the full conformal algebra.[5] We shall show, however, that for a wide class of field theories, the breaking of conformal invariance is related to the breaking of scale invariance.

Scale and conformal transformations are generated by the time components of the currents $S_\mu(x)$ and $C_{\mu\nu}(x)$, respectively. We shall prove that the following relation exists between the divergences of these currents for a large class of theories[6]:

$$\partial^\mu C_{\mu\nu}(x) = -2x_\nu \partial^\mu S_\mu(x). \tag{2.1}$$

The class of theories for which this relation holds will be specified below; it includes all Lagrangian theories of scalar, vector, and spin-$\frac{1}{2}$ fields without derivative couplings.

The form of the scale and conformal currents for special theories has been derived in detail in Ref. 1. Here we outline the derivation of these currents for an arbitrary theory. If we perform an infinitesimal scale transformation,

$$x_\mu \to x_\mu' = x_\mu + \epsilon x_\mu, \tag{2.2}$$

a field $\phi_i(x)$ will transform according to[1]

$$\phi_i(x) \to \phi_i'(x') = (1 - d_i\epsilon)\phi_i(x). \tag{2.3}$$

We have defined d_i to be the dimension of the field ϕ_i, i.e., $\phi_i \approx (\text{mass})^{d_i}$:

$$\begin{aligned} d_i &= 1, \quad \text{scalar field} \\ &= \tfrac{3}{2}, \quad \text{spinor field} \\ &= 1, \quad \text{vector field.} \end{aligned} \tag{2.4}$$

Similarly, when an infinitesimal conformal transformation is performed,

$$x_\mu \to x_\mu' = x_\mu + \alpha_\mu x^2 - 2x_\mu \alpha\cdot x, \tag{2.5}$$

the field $\phi_i(x)$ will transform according to[1]

$$\phi_i(x) \to \phi_i'(x') = \{[1 + 2d_i\alpha x]\phi_i(x) + 2\alpha^\mu x^\nu \textstyle\sum_{\mu\nu}{}^{ij}\phi_j(x)\}, \tag{2.6}$$

where $\Sigma_{\mu\nu}{}^{ij}$ is defined by the transformation properties of the field $\phi_i(x)$ under Lorentz transformations. If $M_{\mu\nu}$ are the generators of Lorentz transformations, then

$$[M_{\mu\nu}, \phi_i(0)] = i\,\Sigma_{\mu\nu}{}^{ij}\varphi_i(0). \tag{2.7}$$

For scalars, $\Sigma_{\mu\nu} = 0$; for spinors, $\Sigma_{\mu\nu} = -\frac{1}{2}i\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$; and for vector fields,

$$\Sigma_{\mu\nu}{}^{\alpha\beta} = (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}).$$

We now use Noether's theorem to derive the currents that generate these transformations, i.e.,

$$J_\mu = \sum_i \left\{ \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi_i)}\Delta\phi_i + \left[g_{\mu\nu}\mathcal{L} - \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi_i)}\partial_\nu\phi_i \right]\Delta x^\nu \right\}, \tag{2.8}$$

where $\Delta\phi_i(x) = \phi_i'(x') - \phi_i(x)$ and $\Delta x_\mu = x_\mu' - x_\mu$. The result is that

$$S_\mu = \Theta_{\mu\nu}x^\nu - \tfrac{1}{2}\sum_i \partial_\nu\phi_i{}^2 - F_\mu, \tag{2.9}$$

$$C_{\mu\nu} = \Theta_{\mu\lambda}(g_\nu{}^\lambda x^2 - 2x^\lambda x_\nu) + \sum_i (x_\nu\partial_\mu\phi_i{}^2 - g_{\mu\nu}\phi_i{}^2) + 2x_\nu F_\mu, \phi_i, \tag{2.10}$$

where $\Theta_{\mu\nu}$ is the symmetric energy-momentum tensor,[7] ϕ_i are the scalar fields in the Lagrangian, and F_μ is defined to be[8]

$$F_\mu = \sum_i \frac{\delta\mathcal{L}_{\text{int}}}{\delta(\partial_\nu\phi_i)}[d_i g_{\mu\nu}\phi_i + \Sigma_{\mu\nu}{}^{ij}\phi_j]. \tag{2.11}$$

We see that the charges, $S = \int S_0(x)d^3x$ and $C_\nu = \int C_{0\nu} \times (x)d^3x$ are essentially given by moments of the

[5] For the algebraic structure of the conformal group, see Ref. 1.

[6] That scale invariance accompanies conformal invariance for specific models is a fact known to many authors. After completion of this work our attention was drawn to a recent paper by G. Mack and A. Salam [Ann. Phys. (N. Y.) **53**, 174 (1969)], in which the same theorem is derived. These authors, however, claim that F_μ must vanish for the theory to be conformally invariant, whereas we argue that it is sufficient for F_μ to be curlless.

[7] The energy-momentum tensor is not determined uniquely by the canonical formalism. One can always add additional terms which are symmetric and conserved. This can be stated in another fashion. For bosons the symmetric energy-momentum tensor is defined by $\Theta_{\mu\nu} = \delta\mathcal{L}/\delta g^{\mu\nu} - g_{\mu\nu}\mathcal{L}$. However, if we add to the Lagrangian a total divergence, it will appear in $\Theta_{\mu\nu}$. Thus, equivalent Lagrangians generate different $\Theta_{\mu\nu}$'s. All the various $\Theta_{\mu\nu}$'s have the same physical content since they give the same four-momentum. Our definition of $\Theta_{\mu\nu}$ is the standard one [see, for example, J. M. Jauch and R. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), pp. 20–22].

[8] We wish to thank C. Callan, S. Coleman, and R. Jackiw for pointing out that F_μ can be expressed in this particularly elegant form.

energy-momentum tensor, except for the terms involving the scalar fields and F_μ.

If we take the divergences of these currents, we obtain

$$\partial^\mu S_\mu = \Theta_\mu{}^\mu - \tfrac{1}{2}\sum_i \Box\phi_i{}^2 - \partial_\mu F^\mu, \tag{2.12}$$

$$\partial^\mu C_{\mu\nu} = -2x_\nu \partial_\mu S^\mu + 2F_\nu. \tag{2.13}$$

Therefore (2.1) will be true whenever $F_\nu = 0$. This will certainly be the case if there are no derivative couplings present in the interaction Lagrangian $\mathfrak{L}_{\text{int}}$. However, certain combinations of derivatives can appear in the interaction without contributing to F_μ. These include couplings involving a current of the form $\sum_{i,j} \phi_i \partial_\mu \phi_j C_{ij}$, $C_{ij} = -C_{ji}$, for scalar fields; and couplings involving the tensor $\partial_\mu A_\nu - \partial_\nu A_\mu$ for a vector field A_μ.

There is a larger class of theories where F_μ does not vanish but is curl free, i.e.,

$$\partial_\mu F_\nu - \partial_\nu F_\mu = 0$$

and so can be written as a total divergence, $F_\mu = \partial_\mu \Lambda$. For these theories we can define a slightly different conformal current

$$\tilde{C}_{\mu\nu} = C_{\mu\nu} - 2g_{\mu\nu}\Lambda \tag{2.14}$$

for which Eq. (2.1) holds. The interaction

$$\mathfrak{L}_{\text{int}} = \partial_\mu \phi \partial^\mu \phi F(\phi)$$

is an example of such a theory. In this class of theories, which is the largest for which there exists a relation of the form (2.1), the divergence of the scale and conformal currents is proportional to the trace of the energy-momentum tensor, up to a term which is of the form $\Box\eta$, where η is a local operator. This will be of importance for the derivation of the low-energy theorems in Sec. IV.

Ultimately, we are interested in theories where scale invariance is broken only by the presence of finite masses, i.e., theories with no dimensional coupling constants. Such theories we will call "essentially scale invariant." If we ignore interaction Lagrangians which cannot be expanded in power series in the fields then for all essentially scale-invariant theories, Eq. (2.1) is true. Of course, the number of possible interactions of this type is quite limited; in fact, the most general essentially scale-invariant interaction Lagrangian is of the form

$$\begin{aligned}\mathfrak{L}_{\text{int}} = {}& \phi\bar{\psi}(b + a\gamma_5)\psi + c\phi^4 \\ & + A^\mu[\bar{\psi}(d\gamma_\mu + e\gamma_5\gamma_\mu)\psi + f\phi\overleftrightarrow{\partial}_\mu\phi] \\ & + h(A^\mu \mathbf{x} A^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ & \qquad + g(A^\mu \mathbf{x} A^\nu)(A_\mu \mathbf{x} A_\nu).\end{aligned} \tag{2.15}$$

This includes all renormalizable interactions except for the ϕ^3 coupling. For all of the above interactions, $F_\mu = 0$.

Couplings of the form $A_\mu{}^i \phi^l \partial_\mu \phi^k$ or $A_\mu{}^i A_\nu{}^j \partial^\mu A^{k,\nu}$ would seem to provide a counterexample to the claim just made, since they are scale invariant and yet do not yield a vanishing F_μ. However, these couplings are not really essentially scale invariant. In fact, if we were to set the vector-meson mass equal to zero they would lead to inconsistent field equations, since A_μ is not coupled to a conserved current. In other words, there is no consistent zero-mass limit to this interaction. In fact, if we make the usual decomposition of A_μ, i.e., $A_\mu = V_\mu - (1/m)\partial_\mu \phi$, where $\partial_\mu V^\mu = 0$, we see that these interactions effectively involve dimensional coupling constants. When A_μ is coupled to a conserved current, as in (2.13), the resulting interaction is indeed conformally invariant and F_μ vanishes.

One can derive the transformation properties of Green's functions and scattering amplitudes directly from the transformation properties of the fields. This we do in the Appendix. However, we prefer to proceed in a somewhat indirect fashion and derive the equations that express the conformal and scale invariance of a scattering amplitude from the low-energy theorem for the emission of the trace of the energy-momentum tensor. One advantage of this method is that we can work solely in terms of the energy-momentum tensor, without introducing fields or Lagrangians. In that case, we simply define an essentially scale-invariant theory to be one in which the matrix elements of $\Theta_\mu{}^\mu + \Box\eta$, where η is some local operator, vanish when all masses approach zero.

III. LOW-ENERGY THEOREM FOR GRAVITONS

The amplitude for the emission of a photon from an arbitrary process has been determined up to terms linear in the photon momenta by Low.[9] The derivation rests upon the gauge invariance of the electromagnetic current. It is therefore not surprising that a similar theorem can be derived for the emission of a graviton from an arbitrary process, since the graviton couples to a conserved tensor, i.e., the energy-momentum tensor $\Theta^{\mu\nu}$.

Consider the matrix element

$$M^{\mu\nu} = \langle \alpha | \Theta^{\mu\nu}(0) | \beta \rangle. \tag{3.1}$$

The amplitude for graviton emission in the process $\alpha \to \beta +$graviton is given by $\epsilon_{\mu\nu}(k,\lambda) M^{\mu\nu}$, where $\epsilon_{\mu\nu}(k,\lambda)$ is the polarization tensor of the graviton whose momentum is $k = p_\alpha - p_\beta$.

We shall now prove that $M^{\mu\nu}$ can be determined up to terms *quadratic* in k. The reason that the linear terms in k are also determined is that $\Theta^{\mu\nu}$ is symmetric and conserved in both indices. In this sense, our result is similar to the case of graviton elastic scattering off matter, for which one of us (D.G.) and Jackiw have derived a low-energy theorem valid up to fourth-order terms in the graviton energy,[10] as compared to the

[9] F. E. Low, Phys. Rev. 96, 1428 (1954).
[10] D. J. Gross and R. Jackiw, Phys. Rev. 166, 1287 (1968).

low-energy theorem for Compton scattering which is only valid to second order in the photon momentum.[11]

We first write $M_{\mu\nu}$ as a sum of two terms: $M_{\mu\nu}{}^I$, the sum of the pole terms where the graviton couples to one of the external particles, and $R_{\mu\nu}$, which consists of the remainder of the amplitude where the graviton couples to internal particles. $M_{\mu\nu}{}^I$ contains the infrared singularities as $k_\mu \to 0$, and is completely determined by the form factors of the external particles and the amplitude for the process $\alpha \to \beta$. $R_{\mu\nu}$ is nonsingular in this limit:

$$M_{\mu\nu}=M_{\mu\nu}{}^I+R_{\mu\nu}. \tag{3.2}$$

The low-energy theorem is derived by finding a function $\Delta_{\mu\nu}(k)$, symmetric in μ and ν, which is nonsingular in the limit $k_\mu \to 0$, and which has the property that

$$k^\mu(M_{\mu\nu}{}^I+\Delta_{\mu\nu})=O(k^3). \tag{3.3}$$

One then notes that $R_{\mu\nu}-\Delta_{\mu\nu}=M_{\mu\nu}-M_{\mu\nu}{}^I-\Delta_{\mu\nu}$ is nonsingular in the limit $k_\mu\to 0$ and satisfies (since $k^\mu M_{\mu\nu}=0$)

$$k^\mu(R_{\mu\nu}-\Delta_{\mu\nu})=O(k^3). \tag{3.4}$$

Therefore, $R_{\mu\nu}-\Delta_{\mu\nu}$ is of order k^2 and

$$M_{\mu\nu}(k)=M_{\mu\nu}{}^I(k)+\Delta_{\mu\nu}(k)+O(k^2). \tag{3.5}$$

We now proceed to determine $\Delta_{\mu\nu}$ explicitly in terms of $M_{\mu\nu}{}^I$. We claim that $\Delta_{\mu\nu}$ can be chosen to be equal to

$$\Delta_{\mu\nu}(k)=\frac{1}{2}\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu}[k^\alpha k^\beta M_{\alpha\beta}{}^I(k)] -\frac{\partial}{\partial k^\mu}[k^\alpha M_{\alpha\nu}{}^I(k)]-\frac{\partial}{\partial k^\nu}[k^\beta M_{\mu\beta}{}^I(k)]. \tag{3.6}$$

To prove this we establish the following.

(1) $\Delta_{\mu\nu}$ is symmetric in μ and ν. This follows trivially from the fact that $M_{\mu\nu}{}^I$ is a symmetric tensor.

(2) $\Delta_{\mu\nu}$ is nonsingular in the limit $k_\mu\to 0$. This is true because $k^\mu M_{\mu\nu}{}^I=-k^\mu R_{\mu\nu}$. Since $R_{\mu\nu}$ is nonsingular in the limit $k_\mu \to 0$, therefore $k^\mu M_{\mu\nu}{}^I$ must be nonsingular in this limit. Since $\Delta_{\mu\nu}$ is expressed in terms of $k^\mu M_{\mu\nu}{}^I$, it is also nonsingular in this limit.

(3) $k^\mu(M_{\mu\nu}{}^I+\Delta_{\mu\nu})=O(k^3)$. To prove this, we once again use the fact that $k^\mu M_{\mu\nu}{}^I=-k^\mu R_{\mu\nu}(k)$. Since $R_{\mu\nu}(k)$ is nonsingular as $k_\mu\to 0$ it has the following expansion:

$$R_{\mu\nu}(k)=R_{\mu\nu}{}^0+k^\alpha R_{\mu\nu,\alpha}{}^1+O(k^2), \tag{3.7}$$

where $R_{\mu\nu}{}^0$ and $R_{\mu\nu,\alpha}{}^1$ are independent of k and symmetric in μ and ν. Therefore,

$$k^\mu M_{\mu\nu}{}^I=-k_\mu R_{\mu\nu}{}^0-k^\mu k^\alpha R_{\mu\nu,\alpha}{}^1+O(k^3). \tag{3.8}$$

We then evaluate $k^\mu\Delta_{\mu\nu}$ from Eq. (3.6):

$$k^\mu\Delta_{\mu\nu}=-\tfrac{1}{2}(k^\mu R_{\nu\mu}{}^0+k^\mu R_{\mu\nu}{}^0)+k^\mu R_{\mu\nu}{}^0+k^\mu R_{\nu\mu}{}^0 +k^\mu k^\alpha R_{\mu\nu,\alpha}{}^1+O(k^3)=-k^\mu M_{\mu\nu}{}^I+O(k^3). \tag{3.9}$$

We therefore have found the appropriate $\Delta_{\mu\nu}$. If we carry out the differentiations in Eq. (3.6), we can cast the low-energy theorem into an extremely compact form, namely,[12]

$$M_{\mu\nu}=\tfrac{1}{2}k^\alpha k^\beta\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k^\nu}M_{\alpha\beta}{}^I+O(k^2). \tag{3.10}$$

To derive the low-energy theorem for graviton emission from an arbitrary process, one merely has to calculate $M_{\mu\nu}{}^I$, which is determined by the gravitational form factors of the external particles, and insert in (3.10).

The gravitational form factors are almost completely determined by the mass and the spin of the appropriate particle. Consider first scalar particles, with the states chosen with the normalization

$$\langle p\,|\,p'\rangle=2p_0(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'). \tag{3.11}$$

The most general form for the graviton scalar vertex is

$$\langle p_1\,|\,\Theta_{\mu\nu}(0)\,|\,p_2\rangle=\tfrac{1}{2}P_\mu P_\nu F_1(k^2) +(k^2g_{\mu\nu}-k_\mu k_\nu)F_2(k^2), \tag{3.12}$$

$$P=p_1+p_2,\quad k=p_1-p_2,\quad p_1{}^2=p_2{}^2=M^2.$$

Since

$$\langle p_1|\,H\,|\,p_2\rangle=2p_0{}^2(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')=\langle p_1|\int d^3x\,\theta_{00}(x)\,|\,p_2\rangle$$
$$=2p_0{}^2(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')F_1(0),$$

we must have

$$F_1(0)=1. \tag{3.13}$$

The second form factor $F_2(k^2)$ is unrestricted.[13]

If one considers spin-$\frac{1}{2}$ particles, then there exist three gravitational form factors

$$\langle p_1\sigma_1|\,\Theta_{\mu\nu}(0)\,|\,p_2\sigma_2\rangle=\bar{u}(p_1\sigma_1)\{\tfrac{1}{2}(\gamma_\mu P_\nu+\gamma_\nu P_\mu)G_1(k^2) +\tfrac{1}{2}P_\mu P_\nu G_2(k^2)+(g_{\mu\nu}k^2-k_\mu k_\nu)G_3(k^2)\}u(p_2\sigma_2), \tag{3.14}$$

where our choice of spinors is such that

$$\bar{u}(p\sigma)\gamma^\mu u(p\sigma)=p^\mu. \tag{3.15}$$

The requirement that $\Theta_{0\mu}(x)$ be the momentum density, and that $x_\mu\Theta_{0\nu}(x)-x_\nu\Theta_{0\mu}(x)$ be the angular momentum

[11] G. Mack, Nucl. Phys. **B5**, 499 (1968), has given a procedure for calculating the matrix elements of $\partial^\mu S_\mu$ up to first order in the momentum associated with the current. As (2.12) shows, this is equivalent to a low-energy theorem for $\Theta_\mu{}^\mu$ up to terms linear in the graviton momenta.

[12] The analogous low-energy theorem for photon emission, Ref. 1, can be also put in this simple form:
$$M_\mu=-k^\alpha(\partial/\partial k^\mu)M_\alpha{}^I+O(k).$$

[13] In fact, different choices of $\Theta_{\mu\nu}$, as described in Ref. 5, lead to different F_2 form factors. However, the value of F_2 is totally irrelevant insofar as symmetry transformations are concerned (this is obvious for Poincaré transformations since $g_{\mu\nu}k^2-k_\mu k_\nu$ vanishes as k^2 when $k_\mu\to 0$), including, as we shall show, scale and conformal transformations. Moreover, the coupling of gravitons does not involve F_2. The only way F_2 would acquire physical content would be if there existed spin-zero particles coupled to the trace of $\Theta_{\mu\nu}$.

density, restricts the values of $G_1(0)$ and $G_2(0)$ to

$$G_1(0)=1,\quad G_2(0)=0, \tag{3.16}$$

whereas $G_3(k^2)$, as in the scalar case, is unrestricted.

One of the useful properties of the low-energy theorem, as it appears in Eq. (3.10), is that we can treat each external particle separately. This is because $M_{\mu\nu}{}^I$ is a sum of terms, each of which corresponds to the graviton coupling to a distinct external particle.

Consider the contribution to $M_{\mu\nu}{}^I$ of a scalar particle, of mass M, with momentum p:

$$M_{\mu\nu}{}^I=\langle p|\Theta_{\mu\nu}|p\pm k\rangle[(p\pm k)^2-M^2]^{-1}\times A(p\pm k,\ldots), \tag{3.17}$$

where $A(p\pm k,\ldots)$ is the amplitude, off shell, for the process $\alpha\to\beta$, and the sign in (3.17) is $+$ $(-)$ if the scalar particle is outgoing (incoming). Therefore,

$$M_{\mu\nu}{}^I=\frac{(\pm)}{k(2p\pm k)}[\tfrac{1}{2}(2p\pm k)_\mu(2p\pm k)_\nu F_1(k^2)+(g_{\mu\nu}k^2-k_\mu k_\nu)F_2(k^2)]\Big[A(p,\ldots)\pm k\frac{\partial}{\partial p}A(p,\ldots)+\tfrac{1}{2}k^\alpha k^\beta\frac{\partial}{\partial p^\alpha}\frac{\partial}{\partial p^\beta}A(p,\ldots)+O(k^3)\Big]. \tag{3.18}$$

Since $k^\mu M_{\mu\nu}{}^I$ does not depend on $F_2(k^2)$, it follows from (3.4) that $\Delta_{\mu\nu}$ gets no contribution from F_2, which therefore contributes a term

$$\pm[k(2p\pm k)]^{-1}(g_{\mu\nu}k^2-k_\mu k_\nu)F_2(0)\times A(p,\ldots)+O(k^2) \tag{3.19}$$

to the amplitude. The contribution of the term involving F_1 is calculated by inserting (3.18) into (3.10); it is

$$\pm\tfrac{1}{2}F_1(k^2)\bigg\{\bigg[\frac{(2p\pm k)_\mu(2p\pm k)_\nu}{k(2p\pm k)}\mp g_{\mu\nu}\bigg]\times\bigg(1\pm k\cdot\frac{\partial}{\partial p}+\tfrac{1}{2}k\cdot\frac{\partial}{\partial p}k\cdot\frac{\partial}{\partial p}\bigg)\mp\bigg(p_\nu\frac{\partial}{\partial p^\mu}+p_\mu\frac{\partial}{\partial p^\nu}\bigg)\bigg(1\pm k\cdot\frac{\partial}{\partial p}\bigg)+k\cdot p\frac{\partial}{\partial p^\mu}\frac{\partial}{\partial p^\nu}\bigg\}A(p,\ldots)+O(k^2). \tag{3.20}$$

Contrary to what one might expect, there is no off-mass-shell dependence in (3.20). Indeed, if $A(p,\ldots)$ has terms of the form $(p^2-M^2)\tilde{A}(p,\ldots)$, their contribution to (3.20), as can be easily checked, vanishes when $p^2=M^2$.

The contribution to $M_{\mu\nu}{}^I$ of an outgoing spin-$\frac{1}{2}$ particle of momenta p and mass M is

$$M_{\mu\nu}{}^I=\bar{u}(p,\sigma)\{\tfrac{1}{2}[\gamma_\mu(2p+k)_\nu+\gamma_\nu(2p+k)_\mu G_1(k^2)]+[(2p+k)_\nu(2p+k)_\mu](1/2M)G_2(k^2)+[g_{\mu\nu}k^2-k_\mu k_\nu]G_3(k^2)\}\frac{(\not p+\not k+M)}{k(2p+k)}\tilde{A}(p+k,\ldots), \tag{3.21}$$

where $\bar{u}(p,\sigma)\tilde{A}(p,\ldots)$ is the amplitude for the process $\alpha\to\beta$. As in the scalar case $\Delta_{\mu\nu}$ gets no contribution from $G_3(k^2)$, which contributes to $M_{\mu\nu}$ a term

$$[k(2p+k)]^{-1}(g_{\mu\nu}k^2-k_\mu k_\nu)G_3(0)M\bar{u}(p,\sigma)\tilde{A}(p,\ldots)+O(k^2). \tag{3.22}$$

The contribution of the other term is easily calculated by inserting (3.21) into (3.10). We get the following contribution:

$$\bar{u}(p,\sigma)G_1(k^2)\bigg\{\frac{1}{2}\bigg[\frac{(2p+k)_\mu(2p+k)_\nu}{k\cdot(2p+k)}-g_{\mu\nu}\bigg]\bigg(1+k\cdot\frac{\partial}{\partial p}+\tfrac{1}{2}k\cdot\frac{\partial}{\partial p}k\cdot\frac{\partial}{\partial p}\bigg)-\bigg(p_\nu\frac{\partial}{\partial p^\mu}+p_\mu\frac{\partial}{\partial p^\nu}\bigg)\bigg(1+k\cdot\frac{\partial}{\partial p}\bigg)+k\cdot p\frac{\partial}{\partial p^\mu}\frac{\partial}{\partial p^\nu}+\frac{(2p+k)_\mu[\not k,\gamma_\nu]+(2p+k)_\nu[\not k,\gamma_\mu]}{8k\cdot(2p+k)}\bigg(1+k\cdot\frac{\partial}{\partial p}\bigg)+\tfrac{1}{8}[\not k,\gamma_\nu]\frac{\partial}{\partial p^\mu}+\tfrac{1}{8}[\not k,\gamma_\mu]\frac{\partial}{\partial p^\nu}\bigg\}\tilde{A}(p,\ldots)+\bar{u}(p,\sigma)G_2(k^2)\frac{4p_\mu p_\nu}{k\cdot(2p+k)}\tilde{A}(p,\ldots)+O(k^2). \tag{3.23}$$

Once again one can check that the off-mass-shell dependence of $\tilde{A}(p,\ldots)$ [i.e., $\tilde{A}(p)\sim(\not p-M)\tilde{A}'(p,\ldots)$] does not appear in (3.23) when $p^2=M^2$. A similar contribution, with a change of sign for p_μ, arises from incoming fermions.

The final form of the low-energy theorem for the emission of a soft graviton from an arbitrary process involving scalar and fermion particles is arrived at by collecting the terms in (3.21) and (3.20) for each fermion.

One can explicitly check that the sum of all these terms is gauge invariant up to order k^3. In fact, the term of order (k^0) vanishes due to momentum conservation, the term of order $(k)^1$ vanishes due to angular momentum conservation, and the term of order $(k)^2$ vanishes identically.

One can derive similar low-energy theorems for processes involving higher-spin particles, the only complication being due to the increased number of gravitational form factors.

Finally, we note that if one is actually interested in the amplitude for graviton emission, i.e., $\epsilon^{\mu\nu}(k,\lambda)M_{\mu\nu}$, then the unknown form factors $F_2(k^2)$ and $G_3(k^2)$ do not contribute, since $\epsilon^{\mu\nu}(k,\lambda)k_\mu=\epsilon^{\mu\nu}(k,\lambda)k_\nu=0$.

IV. SCALE AND CONFORMAL EQUATIONS

With the aid of the low-energy theorem, derived in Sec. III, we can now deduce the constraints placed on the high-energy form of on-mass-shell amplitudes. We use the fact that the trace of the energy-momentum tensor is intimately related to the divergence of the currents that generate scale and conformal transformations for essentially scale-invariant theories, as proved in Sec. II.

First we consider the low-energy theorem for the trace of $\Theta_{\mu\nu}$. From (3.19) and (3.20) it follows that a scalar particle of momentum p and mass M contributes a term

$$\left\{\pm\frac{2M^2F_1(k^2)}{k(2p\pm k)}[1\pm k\cdot\partial+\tfrac{1}{2}(k\cdot\partial)^2] \pm\frac{k^2[6F_2(0)-F_1(0)]}{2k(2p\pm k)}\right\}A(p,\ldots) +F_1(k^2)\{(1+p\cdot\partial)\pm(k\cdot\partial p\cdot\partial-\tfrac{1}{2}k\cdot p\partial\cdot\partial)\}\times A(p,\ldots)+O(k^2). \tag{4.1}$$

The contribution of an outgoing fermion of momentum p and mass M to the low-energy theorem is equal to the trace of (3.22) and (3.23):

$$\left\{\pm\frac{2M^2G_1(k^2)+4M^2G_2(k^2)}{k(2p\pm k)}[1+k\cdot\partial+\tfrac{1}{2}(k\cdot\partial)^2] +\frac{3k^2G_3(0)M}{k(2p\pm k)}\right\}A(p,\ldots)-G_1(k^2)[(\tfrac{3}{2}+p\cdot\partial) \pm(k\cdot\partial p\cdot\partial-\tfrac{1}{2}k\cdot p\partial\cdot\partial+\gamma_\mu\partial^\mu \mathbf{k})]\times A(p,\ldots)+O(k^2). \tag{4.2}$$

In both equations we have abbreviated $\partial/\partial p^\mu=\partial_\mu$. In Eq. (4.2) the derivatives do not operate on the external spinor wave functions.

We have shown in Sec. II that for a wide class of theories, the divergence of the scale and conformal currents is essentially proportional to $\Theta_\mu{}^\mu$, i.e.,

$$\partial_\mu C^{\mu\nu}=-2x^\nu\partial_\mu S^\mu=-2x^\nu(\Theta_\mu{}^\mu+\Box\eta),$$

where η is a local operator. For these theories, we can derive the equations expressing conformal invariance independently of the details of the interaction. We consider the low-energy theorem for emission of the operator $\Theta_\mu{}^\mu+\Box\eta$ which must vanish for scale-invariant theories. Since we have already determined the low-energy theorem for $\Theta_\mu{}^\mu$, it remains only to add the contribution of $\Box\eta$. The matrix elements of $\Box\eta$ vanish like k^2 as $k_\mu\to 0$, i.e.,

$$\langle\alpha|\Box\eta|\beta\rangle=(p_\alpha-p_\beta)^2\langle\alpha|\eta|\beta\rangle=k^2\langle\alpha|\eta|\beta\rangle, \tag{4.3}$$

and thus the only contribution of $\Box\eta$ to the low-energy theorem comes from the infrared terms where $\Box\eta$ couples to the external particles and

$$\langle\alpha|\eta|\beta\rangle\underset{k\to 0}{\sim}1/k.$$

These can be determined by the knowledge of the one-particle matrix element of $\Box\eta$, which is known since $\Box\eta=-\Theta_\mu{}^\mu$. For a massless scalar particle, we have

$$\langle p_1|\Box\eta|p_2\rangle=-\langle p_1|\Theta_\mu{}^\mu|p_2\rangle =\tfrac{1}{2}k^2F_1(k^2)-3k^2F_2(k^2), \tag{4.4}$$

whereas for a massless fermion we have that

$$\langle p_1\sigma_1|\Box\eta|p_2\sigma_2\rangle=-\langle p_1\sigma_1|\Theta_\mu{}^\mu|p_2\sigma_2\rangle=O(k^3). \tag{4.5}$$

[Note that with our normalization, $\bar{u}(p\sigma)u(p\sigma)=0$ for massless fermions.]

The $\Box\eta$ contributes to the low-energy theorem a term

$$\pm\frac{k^2[F_1(0)-6F_2(0)]}{2k(2p\pm k)}A(p,\ldots)+O(k^3) \tag{4.6}$$

for each scalar particle of momentum p, and contributes nothing for fermions.

When (4.6) is combined with (4.1) and (4.2), we finally obtain for scale-invariant theories

$$\sum_i [(d_i+p_i\cdot\partial_i)+\zeta(d_ik\cdot\partial_i+p_i\partial_ik\cdot\partial_i-\tfrac{1}{2}k\cdot p_i\partial_i\partial_i)]A -\sum_{\substack{\text{outgoing}\\\text{fermions}}} i\sigma_{\mu\nu}\partial_i{}^\mu k^\nu A-\sum_{\substack{\text{incoming}\\\text{fermions}}} A_i\overleftarrow{\partial}_i{}^\mu k^\nu\sigma_{\mu\nu}, \tag{4.7}$$

where the first sum runs over all particles, and

$$\begin{aligned} d_i&=1, &&\text{scalars}\\ &=\tfrac{3}{2}, &&\text{fermions}\\ \zeta_i&=+, &&\text{outgoing}\\ &=-, &&\text{incoming}. \end{aligned} \tag{4.8}$$

The S matrix for the process $\alpha\to\beta$ is equal to the matrix A taken between the appropriate spinors for the external fermions.

We can now equate to zero the coefficients in (4.7) of $(k^\mu)^0$ and of k^μ. The coefficient of $(k^\mu)^0$ is just the equation expressing scale invariance. We define the operator

$$\mathfrak{D}_i = d_i + p^i \cdot \partial_i \tag{4.9}$$

as the scale operator for scalars and for fermions (as in Sec. II, $d_i = 1$ for scalars and $d_i = \frac{3}{2}$ for fermions). The scale-invariant amplitude satisfies

$$\mathfrak{D}A = \sum_i \mathfrak{D}_i A = 0 \tag{4.10}$$

In a similar fashion, we define the conformal operator for a particle of momentum p_i, dimension d_i, to be

$$\mathfrak{C}_{i,\mu} = \zeta_i [d_i \partial_\mu{}^i + p^{i\nu} \partial_{\nu i} \partial_\mu{}^i - \tfrac{1}{2} p_\mu \partial_i{}^\nu \partial_\nu{}^i] - \Sigma_{\mu\nu} \partial_i{}^\nu . \tag{4.11}$$

The spin operator $\Sigma_{\nu\mu i}$ is defined as in (2.10), so that for a scalar, $\Sigma_{\mu\nu}{}^{(0)} = 0$; for a fermion, $\Sigma_{\nu\mu}{}^{(1/2)} = -i\sigma_{\nu\mu}$; and

$$\begin{aligned} \zeta_i \overleftrightarrow{\partial}_i &= \overrightarrow{\partial}_i , \quad \text{outgoing fermion} \\ &= \overleftarrow{\partial}_i , \quad \text{incoming fermion.} \end{aligned}$$

The conformal equations are then

$$\mathfrak{C}_\mu A = \sum_i \mathfrak{C}_{i\mu} A = 0 , \tag{4.12}$$

where for outgoing (incoming) fermions the conformal operator acts to the left (right) of A.

Vector particles can be treated in an identical fashion. The result is obvious, namely, that Eqs. (4.9) and (4.11) define the scale and conformal operators for vector particles, if $d_i = 1$ and $\Sigma_{\mu\nu}{}^i \equiv \Sigma_{\mu\nu}{}^{\alpha\beta} = (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})$. Thus, the contribution of an outgoing vector particle with momentum p_i to the conformal equation is

$$\begin{aligned} \epsilon^\alpha(p_i) (\mathfrak{C}_{i\mu})_{\alpha\beta} A_\beta(p_i, \ldots) \\ = \epsilon^\alpha(p_i) [(\partial_\mu{}^i + p_i{}^\nu \partial_\nu{}^i \partial_\mu{}^i - \tfrac{1}{2} p_\mu{}^i \partial_i{}^\nu \partial_\nu{}^i] g_{\alpha\beta} \\ + (g_{\alpha\mu} \partial_\beta{}^i - g_{\beta\mu} \partial_\alpha{}^i)] A^\beta(p_i, \ldots) , \end{aligned} \tag{4.13}$$

where the S matrix is $\epsilon^\alpha(p_i) A_\alpha(p_i, \ldots)$.

The amplitude A contains a momentum conservation δ function

$$A(p_i \cdots) = \delta^4(\sum_i \zeta_i p_i) \mathfrak{M}(p_i \cdots) . \tag{4.14}$$

Using momentum and angular momentum conservation, and scale invariance of A, one can commute the scale and conformal operators with this δ function to obtain

$$\begin{aligned} \mathfrak{D} \delta^{(4)}(\sum_i \zeta_i p_i) \mathfrak{M}(p_i) \\ = \delta^{(4)}(\sum_i \zeta_i p_i) [-4 + \mathfrak{D}] \mathfrak{M}(p_i) = 0 , \end{aligned} \tag{4.15}$$

$$\begin{aligned} \mathfrak{C}_\mu \delta^{(4)}(\sum_i \zeta_i p_i) \mathfrak{M}(p_i) \\ = \delta^{(4)}(\sum_i \zeta_i p_i) \mathfrak{C}_\mu \mathfrak{M}(p_i) = 0 . \end{aligned} \tag{4.16}$$

That these equations do not involve derivatives with respect to the masses results from the following equations, which can easily be derived:

$$\mathfrak{D}_i{}^0 p_i{}^2 = \bar{u}(p_i) \mathfrak{D}_i{}^{1/2} \not{p}_i = \epsilon^\mu(p_i) \mathfrak{D}_{i\mu\nu}{}^1 p_i{}^2 = 0 , \tag{4.17}$$

$$\mathfrak{C}_i{}^0 p_i{}^2 = \bar{u}(p_i) \mathfrak{C}_i{}^{1/2} \not{p}_i = \epsilon^\mu(p_i) \mathfrak{C}_{\mu\nu}{}^1 p_i{}^2 = 0 , \tag{4.18}$$

when $p_i{}^2 = 0$. Furthermore, the scale and conformal transformations for massless vector mesons preserve gauge invariance; i.e., if $p_i \cdot A_i(p_i, \ldots) = 0$, then

$$p_i{}^\mu \mathfrak{C}_{i\mu\nu}{}^1 A^\nu(p_i, \ldots) = 0 . \tag{4.19}$$

The above equations hold for theories which are essentially scale invariant and in which all masses have been set equal to zero. In what sense are they valid for finite masses? From (4.1) and (4.2) it is seen that finite-mass corrections to the low-energy theorems are all of the order $(M^2/p^2)A$, where p is some momentum variable. This must be compared to the terms which generate the scale and conformal equations, which are of order A. Therefore, *these corrections can be neglected if all energy variables are large compared to the external masses.* In addition, when the masses are finite, the amplitude for the emission of $\Theta_\mu{}^\mu$ does not vanish. We therefore must assume that this amplitude (which vanishes when all masses are zero) is small compared to A at large energies. For essentially scale-invariant Lagrangian theories, $\Theta_\mu{}^\mu$ is explicitly proportional to the masses of the particles that appear in the Lagrangian. We would therefore expect that its matrix elements, to any finite order of perturbation theory, are of order $(M/p)A$ for large energies. This estimate is, however, not strictly correct, owing to the infrared divergences that would appear if all masses were set equal to zero. These divergences invalidate the dimensional argument given above. For finite masses and asymptotic energies this problem manifests itself in the appearance of logarithmic terms of the form $\ln(p/M)$, which violate the scale and conformal invariance of the amplitude. To any finite order in perturbation theory there will only appear finite powers of such logarithms. Furthermore, the ratio of the matrix elements of $\Theta_\mu{}^\mu$ to A will approach zero logarithmically at large energies. We therefore make the further assumption that when one works to all orders in perturbation theory, these logarithms in the amplitude do not build up to give power behavior. More precisely, we assume that $\Theta_\mu{}^\mu/A$ (or $\mathfrak{D}A/A$ and $\mathfrak{C}_\mu A/A$) tends to zero logarithmically as all energies become large compared to the masses. In that case the solution of the scale and conformal equations given above will represent the asymptotic behavior of the amplitudes when all energies are large, up to logarthmic terms.

As stated in the Introduction we hope to apply these equations to the study of the asymptotic behavior of model field theories where our assumptions stand a good chance of being true. Quantum electrodynamics, of course, is an essentially scale-invariant theory and thus falls into this class. The strong interactions, however, do not seem to be scale invariant. This can be explained simply if the basic strong interactions involve dimensional couplings. However, there seems to be another way of understanding the non-scale-invariant (Regge) asymptotic behavior of strong amplitudes which is more appealing to us. The mass spectrum of hadrons includes states with very large masses, and in fact there is much speculation that there exist narrow resonances, on Regge trajectories, that increase linearly to infinity. If this is the case then, of course, energies are never large compared to all masses, i.e., there exists no scale-invariant region. It is interesting to note in this connection the strong relation, implied by the idea of duality,[14] and exemplified by the Veneziano model,[15] between infinitely increasing masses (and spins) and Regge behavior. In any case, we do not believe that one can fruitfully apply scale invariance to the study of strong-interaction amplitudes.[16] For this very reason we are skeptical whether the usual renormalizable field theories can serve as a useful guide for high-energy hadronic amplitudes. The only renormalizable field theory which is not essentially scale invariant is the ϕ^3 interaction, which also happens to be the only interaction for which it has been shown that Regge poles dominate the asymptotic behavior of the scattering amplitude.

As noted in the Introduction, high-energy inelastic electroproduction at large momentum transfers seems to yield scale-invariant amplitudes.[4] In this case, the above reasoning against scale invariance for strong amplitudes is inapplicable, since the contributions of the massive discrete states are strongly damped by the rapidly falling electromagnetic form factors. We hope, in a forthcoming publication, to return to this problem and apply conformal invariance to these amplitudes.

V. APPLICATIONS

In this section we will consider the constraints placed upon two-body scattering amplitudes by scale and conformal invariance. The solution to the equations of scale invariance is simply, namely that the amplitude $\mathfrak{M}(p_1,p_2\cdots)$ has the property

$$\mathfrak{M}(p_1,p_2\cdots)=\lambda^{-d}\mathfrak{M}(\lambda p_1,\lambda p_2\cdots)\,, \qquad (5.1)$$

where d is the dimension of $\mathfrak{M}$, $\mathfrak{M}\approx(\text{mass})^d$. The conformal equations, divided in Sec. IV, seem, at first glance, to be quite complicated. In particular, they appear to be second-order differential equations. However, we shall show that in the case of four-point functions, the conformal equations can always be reduced to first-order differential equations.

Consider the scattering amplitude for an arbitrary set of four particles, with momenta p_i, $i=1,\ldots,4$ $(p_1+p_2+p_3+p_4=0)$. It can be decomposed as follows:

$$\mathfrak{M}(p_1,p_2,p_3,p_4)=\sum_j T_j(p_i)\mathfrak{M}_j[p_1\cdot p_2,p_1\cdot p_3]\,, \qquad (5.2)$$

where $T_i(p_i)$ are a complete set of tensor amplitudes and $\mathfrak{M}_j$ are the invariant amplitudes. The amplitudes $\mathfrak{M}_j$ are scalar functions of, say, the energy variables $s=p_1\cdot p_2$ and $t=p_1\cdot p_3$ and have a dimension D_j so that

$$D_j\mathfrak{M}_j=\sum_{i=1}^{4}\left(p^i\cdot\frac{\partial}{\partial p_i}\right)\mathfrak{M}_j=2\left(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t}\right)\mathfrak{M}_j\,. \qquad (5.3)$$

We shall now prove that when the conformal operator is applied to $\mathfrak{M}$, the resulting equation will only involve first-order derivatives of the invariant amplitudes $\mathfrak{M}_i$. The only place that second-order derivatives appear is in the spin-independent part of the conformal operator

$$\zeta_i[d_i\partial_\mu{}^i+p_i{}^\nu\partial_\nu{}^i\partial_\mu{}^i-\tfrac{1}{2}p_\mu{}^i\partial_i{}^\nu\partial_\nu{}^i]\,,$$

and only when these operators act on $\mathfrak{M}_j$. We therefore consider the expression

$$\sum_{i=1}^{4}[d_i\partial_\mu{}^i+p_i{}^\nu\partial_\nu{}^i\partial_\mu{}^i-\tfrac{1}{2}p_\mu{}^i\partial_i{}^\nu\partial_\nu{}^i]\mathfrak{M}_j(s,t)\,.$$

Expressing the derivatives in terms of s and t, we obtain

$$\{[p_\mu{}^1(\partial_s+\partial_t)+p_\mu{}^2\partial_s+p_\mu{}^3\partial_t](s\partial_s+t\partial_t)+p_\mu{}^1[(d_2-1)\partial_s+(d_3-1)\partial_t]+p_\mu{}^2[(d_1-1)\partial_s]+p_\mu{}^3(d_1-1)\partial_t\}\mathfrak{M}_j(s,t)\,,$$

and using Eq. (5.3), we have

$$\{p_\mu{}^1[(d_2+\tfrac{1}{2}D_j-1)\partial_s+(d_3+\tfrac{1}{2}D_j-1)\partial_t]+(p_\mu{}^2\partial_s+p_\mu{}^3\partial_t)(d_1+\tfrac{1}{2}D_j-1)\}\mathfrak{M}_j(s,t)\,. \qquad (5.4)$$

In other words, scale invariance eliminates all second-order derivatives, and the resulting differential equations for the invariant amplitudes will be linear.

A. Scalar-Scalar Scattering

Consider the scattering amplitude for four particles of spin zero, $\mathfrak{M}$ $(s=p_1\cdot p_2,\ t=p_1\cdot p_3)$. Since $\mathfrak{M}$ is dimensionless with our normalization, scale invariance implies that

$$(s\partial_s+t\partial_t)\mathfrak{M}(s,t)=0\,. \qquad (5.5)$$

Conformal invariance places no additional restrictions on the amplitude. This follows from Eq. (5.4) if we remember that $d_i=1$ and that the dimension of $\mathfrak{M}$ is zero. Thus we have the rather surprising result that once scale invariance is satisfied, the scattering amplitude for four scalar particles is automatically conformal invariant. This is not generally the case. We shall show below that conformal invariance places nontrivial restrictions on the scattering of spinning particles.

---

[14] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968).

[15] G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[16] One could, of course, consider $\Theta_\mu{}^\mu$ as an interpolating field for scalar particles (say the σ meson) and use the low-energy theorem derived in Sec. III to relate the amplitudes for σ emission, at small four-momenta, to the breaking of scale and conformal invariance. However, if scalar mesons exist they are probably too massive to justify the extrapolation which is necessary to make contact with experiment.

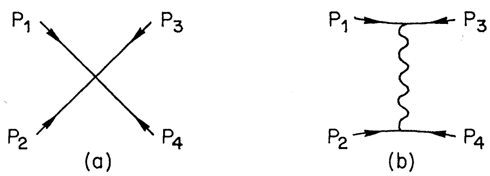

FIG. 1. (a) Born term for ϕ^4 coupling; (b) Born term for $\phi\overleftrightarrow{\partial}_\mu\phi A^\mu$ coupling.

Insofar as conformal invariance is concerned spin is an essential complication.

Equation (5.5), which implies that $\mathfrak{M}$ is a function of the dimensionless variable s/t, is to be interpreted, for essentially scale-invariant theories, as the statement that

$$\mathfrak{M}(s,t) \underset{s,t,u\gg M^2}{\approx} \mathfrak{M}\left(\frac{s}{t}\right)F\left[\ln\left(\frac{s}{M^2}\right),\ln\left(\frac{t}{M^2}\right),\ln\left(\frac{u}{M^2}\right)\right] +O(M^2/s,M^2/t,M^2/u)\,, \quad (5.6)$$

where

$$u=\sum_{i=1}^{4} M_i^2-s-t\,,\quad M^2=\max(m_i^2)\,.$$

We have allowed for logarithmic terms that spoil exact asymptotic scale invariance in the function F, which, however, is restricted not to build up power behavior, i.e.,

$$\ln F/\ln s \underset{s\to\infty}{\longrightarrow} 0\,. \quad (5.7)$$

We now inquire whether perturbation theory, for essentially scale-invariant scalar theories, does in fact yield scale-invariant amplitudes.

We consider the couplings $\alpha\phi^4$ and $\beta\phi_\mu\phi\overleftrightarrow{\partial}A^\mu$, where $\phi(A^\mu)$ is a scalar (vector) field. The Born terms (see Fig. 1) are clearly scale invariant. For ϕ^4 coupling, $\mathfrak{M}=$ const, and for the vector coupling

$$\mathfrak{M}=\text{const}\,\frac{(p_1-p_3)(p_2-p_4)}{(p_1-p_3)^2-M^2}\underset{t\gg M^2}{\longrightarrow}\frac{s-u}{t}\,.$$

When we calculate radiative corrections, we will encounter logarithmic (infrared) breaking of scale invariance. For example, consider the chain of iterated bubbles in the s channel, Fig. 2. The nth-order term yields, for $s\gg M^2$, a contribution $\beta^n \ln^{n-1}(s/M^2)$. When these are summed, we obtain

$$\mathfrak{M}(s,t)\approx\beta[1-\beta\ln(s/M^2)]^{-1}\approx[\ln(s/M^2)]^{-1}.$$

Since the chain has no t dependence, and $\mathfrak{M}$ is dimensionless, exact scale invariance would imply that $\mathfrak{M}(s,t)$ $=\mathfrak{M}(s)=$ const. The infrared problem spoils this; however, we note that *the logarithmic terms do not build up to give power behavior and the predictions of asymptotic scale invariance are true up to logarithmic terms*.[17]

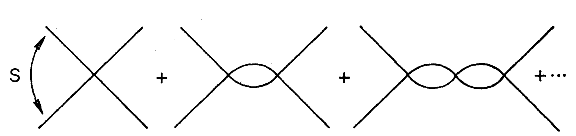

FIG. 2. Iterated bubbles in the s channel.

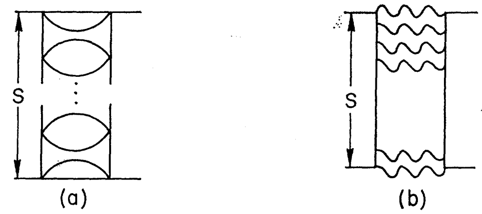

FIG. 3. (a) Ladder diagram for scalars with ϕ^4 coupling. (b) Ladder diagram for scalars with $\phi\overleftrightarrow{\partial}_\mu\phi A^\mu$ coupling.

More interesting are the sums of ladder diagrams, Fig. 3, where, in analogy with the ϕ^3 theory, we might expect to find Regge asymptotic behavior. On the other hand, scale invariance leads us to believe that asymptotic behavior will not be controlled by Regge poles. In the case of zero masses, with an infrared cutoff, this is clearly implied by (5.6); for if we write $\mathfrak{M}$ as a function of $z_s=1+2s/t$, then

$$\mathfrak{M}(z_s,t)=\mathfrak{M}(z_s)F(\ln(s/\lambda),\,\ln(t/\lambda),\,\ln(u/\lambda))\,,$$

where λ is the infrared cutoff. If our assumption that the logarithmic terms do not build up to powers is correct, i.e., Eq. (5.7), then the massless theory cannot exhibit Regge behavior. Instead the asymptotic behavior will be that corresponding to fixed singularities in the angular momentum plane. For finite masses, one cannot strictly rule out Regge poles, since Eq. (5.6) will be satisfied as long as the Regge trajectory $\alpha(t)$ approaches a constant value for large t in the following way:

$$\alpha(t)\underset{-t\gg M^2}{\approx}\text{const}+M^2/t\,.$$

The asymptotic behavior of the sum of the ladder diagrams for both ϕ^3 and $\phi\overleftrightarrow{\partial}_\mu\phi A^\mu$ coupling has been discussed by many authors.[18,19] Sawyer[18] found, by summing the most singular term in each ladder diagram, that the amplitude behaved for large s as follows:

$$\mathfrak{M}(z_s,t)\approx(z_s)^\alpha[\ln(z_s)]^{-3/2}\,,$$

where

$$\alpha=g^2/16\pi^2 \quad\text{for}\quad \phi^3\,,$$
$$=g/2\pi^2\sqrt{8}\quad\text{for}\quad \phi\overleftrightarrow{\partial}_\mu\phi A^\mu\,.$$

This agrees with our expectation that the asymptotic behavior is governed by fixed cuts. Other authors,[19,20]

[17] This example, however, may be misleading since we are summing the perturbation series in a region where it clearly diverges $[\beta\ln(s/M^2)\gg1]$. This point has been discussed by N. N. Bogoliubov and D. V. Shirkov, *Theory of Quantized Fields* (Interscience, New York, 1959), pp. 528, 529.

[18] R. Sawyer, Phys. Rev. **131**, 1384 (1963).

[19] M. Baker and I. J. Muzinich, Phys. Rev. **132**, 2291 (1963); M. K. Banerjee, M. Kugler, C. Levinson, and I. J. Muzinich, *ibid.* **137**, B1280 (1965).

[20] J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963).

using different methods, have confirmed that fixed cuts are indeed the leading singularities for these scale-invariant theories[21] (of particular interest is Ref. 19 in which the connection between scale invariance and fixed singularities in angular momentum is emphasized).

B. Scalar-Spinor Scattering

Consider the amplitude for the scattering of a massless scalar off a massless fermion

$$0(q_1)+\tfrac{1}{2}(p_1)\to 0(q_2)+\tfrac{1}{2}(p_2)$$

(the labels refer to the spins of the particles and $p_1+q_1+p_2+q_2=0$). The amplitude can be written as

$$\bar{u}(p_2)[A\,(s=p_1q_1,\,u=p_1q_2)+(\mathbf{q}_1+\mathbf{q}_2)\times B(s=p_1\cdot q_1,\,u=p_1\cdot q_2)]u(p_1)\,. \quad (5.8)$$

The dimension of A (B) is -1 (-2), so that scale invariance implies that

$$(-\tfrac{1}{2}+s\partial_s+u\partial_u)A\,(s,u)=(-1+s\partial_s+u\partial_u)B(s,u)=0\,. \quad (5.9)$$

Therefore,

$$A\,(s,u)=(su)^{-1/4}f(s/u)\,,\quad B(s,u)=(su)^{-1/2}f'(s/u)\,. \quad (5.10)$$

We now apply the conformal operators. Using the results established previously, we can eliminate all second-order derivatives. The result is that B completely decouples and is unrestricted by conformal invariance. However, A must satisfy the equation

$$\bar{u}(p_2)[p_1^{\mu}(\partial_s+\partial_u)+q_1^{\mu}\partial_s+q_2^{\mu}\partial_u-(\mathbf{q}_1\partial_s-\mathbf{q}_2\partial_u)\gamma^{\mu}]Au(p_1)=0\,, \quad (5.11)$$

which clearly implies that $\partial_sA=\partial_uA=0$, or that A must vanish.

In the massless case, A (B) is proportional to the s-channel helicity-flip (-nonflip) amplitude. Thus, our result is that for spin-zero–spin-$\frac{1}{2}$ scattering in an essentially scale-invariant theory, the spin-flip amplitude vanishes and the spin-nonflip amplitude is an arbitrary scale-invariant function. In the massive case this is translated to imply that

$$A(s,u)\underset{s,t,u\gg M^2}{=}O\left(\frac{M}{\sqrt{s}},\frac{M}{\sqrt{t}},\frac{M}{\sqrt{u}}\right)\times F\left(\ln\frac{s}{M^2},\ln\frac{t}{M^2},\ln\frac{u}{M^2}\right) \quad (5.12)$$

and

$$B(s,u)=\frac{1}{(su)^{1/2}}\left[B\left(\frac{s}{u}\right)F\left(\ln\frac{s}{M^2},\ln\frac{t}{M^2},\ln\frac{u}{M^2}\right)+O\left(\frac{M}{\sqrt{s}},\frac{M}{\sqrt{t}},\frac{M}{\sqrt{u}}\right)\right]. \quad (5.13)$$

The corrections here are of order $M/\sqrt{s}$ owing to the presence of fermions.

The Born terms for all the essentially scale-invariant couplings in (2.15) agree with Eqs. (5.12) and (5.13). We also have verified that these equations are satisfied in second-order perturbation theory for these couplings. As far as summations of an infinite number of perturbation-theory diagrams is concerned, we refer once more to the calculations in the literature where it has been argued[22] that the asymptotic behavior of pion-nucleon scattering, with scalar or vector exchange, is governed by fixed cuts in angular momentum.

C. Scalar-Photon Scattering

We have worked out the restrictions placed on the Compton scattering of massless scalar particles by scale and conformal invariance. The procedure is straightforward and the result is that the helicity-nonflip amplitude is automatically conformal invariant once scale invariance is satisfied, and that the helicity-flip amplitude is forced, by conformal invariance, to vanish.

The result of the three cases discussed above can easily be summarized. When one considers the elastic scattering of scalar and spinning particles, the only additional constraints placed by conformal invariance are that helicity-flip amplitudes vanish. It would probably be much simpler to derive this result if we had formulated the conformal equations directly in terms of helicity amplitudes. Preliminary investigation indicates that the constraints imposed by conformal invariance on amplitudes not involving scalar particles are more severe. We intend, in a forthcoming publication, to develop the helicity formulation of conformal invariance, and apply it to spinor-spinor and spinor-vector scattering.

APPENDIX

The restrictions placed by scale and conformal invariance on Green's functions can be derived from the transformation law of the fields if we assume the vacuum to be invariant. Invariance then leads to the following equation:

$$\langle 0|T\{\phi_1'(x_1)\cdots\phi_n'(x_n)\}|0\rangle=\langle 0|T\{\phi_1(x_1)\cdots\phi_n(x_n)\}|0\rangle=G(x_1\cdots x_n)\,. \quad (A1)$$

[21] An alternative heuristic argument for the presence of fixed cuts in angular momentum in scale-invariant theories is the potential theory analog. The only scale-invariant potential is λ/r^2 [which scales as the kinetic energy $(1/2M)\Delta$], and, as is well known, leads to a fixed cut at $l=-\frac{1}{2}\pm(\lambda-\frac{1}{4})^{1/2}$. However, there is one counterexample to this phenomena of fixed cuts governing the asymptotic behavior of essentially scale-invariant theories. This is the Reggeization of the electron in massive quantum electrodynamics, at least to sixth order, as shown by M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964); H. Cheng and T. T. Wu, *ibid.* **140**, B465 (1965).

[22] G. Cosenza, L. Sertorio, and M. Toller, Nuovo Cimento **31**, 1086 (1964).

For scale invariance,

$$\phi_i'(x)=\phi_i(x)-\epsilon\left[d_i\phi_i(x)+x^\mu\frac{\partial}{\partial x^\mu}\phi_i(x)\right], \quad \text{(A2)}$$

or

$$\sum_{i=1}^{n}\left(d_i+x_i\frac{\partial}{\partial x_i}\right)G(x_1,\ldots,x_n)=0. \quad \text{(A3)}$$

For conformal invariance,

$$\phi_i'(x)=\phi_i(x)+(2d_i\alpha\cdot x\delta_{ij}+2\alpha^\nu x^\mu\Sigma_{\mu\nu}{}^{ij})\phi_j(x) - (\alpha^\mu x^2-2x^\mu\alpha\cdot x)\frac{\partial}{\partial x}\phi_i(x), \quad \text{(A4)}$$

or

$$\sum_{i=1}^{n}\left\{2d^i\alpha\cdot x_i+2\alpha^\nu x_i{}^\mu\Sigma_{\mu\nu}{}^i - (\alpha^\mu x_i{}^2-2x_i{}^\mu\alpha\cdot x_i)\frac{\partial}{\partial x_i{}^\mu}\right\}G(x_1,\ldots,x_n)=0. \quad \text{(A5)}$$

Note that these differential operators commute with the time ordering.

For the Fourier transform of the Green's function,

$$G(x_1,\ldots,x_n)=\int dp_1\cdots dp_n e^{-i\Sigma_r p_r x_r}\tilde{G}(p_1,\ldots,p_n),$$

Eqs. (A3) and (A5) become

$$\sum_{i=1}^{n}\left[(4-d_i)+p^i\cdot\frac{\partial}{\partial p^i}\right]\tilde{G}(p_1,\ldots,p_n)=0, \quad \text{(A6)}$$

$$\sum_{i=1}^{n}\left[\alpha\cdot p_i\frac{\partial^2}{\partial p_i{}^2}-2p_i\frac{\partial}{\partial p_i}\alpha\cdot\frac{\partial}{\partial p_i}+2(d_i-4)\alpha\cdot\frac{\partial}{\partial p_i} +2\alpha^\mu\frac{\partial}{\partial p_i{}^\nu}\Sigma^{\mu\nu}\right]\tilde{G}(p_1,\ldots)=0. \quad \text{(A7)}$$

To see what restrictions these equations imply for the S matrix, we have to remove the propagators. We consider, for example, the case of scalar and spinor particles, and define, with an obvious identification of particles

$$\tilde{G}=\frac{1}{p_1}\cdots\frac{1}{p_r}\frac{1}{q_1{}^2}\cdots\frac{1}{q_s{}^2}A\frac{1}{p_1'}\cdots\frac{1}{p_t'}.$$

Commuting the operators in Eqs. (A3) and (A5) through the propagators (no double poles appear when this is done), we finally obtain for A the Eq. (4.7) derived in Sec. IV from the low-energy theorem.[23] For strictly conformal-invariant theories these equations would impose very severe restrictions, leaving only trivial solutions. As a consequence of scale invariance along the propagator of a fermion field would have to be of the form $C/\gamma\cdot p$, where C is a constant. For C finite, we would be left with a free-field propagator. Thus a strictly scale-invariant theory must yield a trivial S matrix. Clearly this is not what we expect to happen (when masses are set equal to zero in an essentially scale-invariant theory). The resolution of this apparent contradiction is that our assumption of an invariant vacuum is by no means justified. Owing to the fact that scale invariance forces us to deal with massless particles,[1] we will encounter infrared divergences which exclude solutions with an invariant vacuum.

Alternatively, we will have to introduce an infrared cutoff which can serve as a scale. In perturbation theory this cutoff will enter logarithmically as is the case in massless quantum electrodynamics.[24] Of course, there is always the possibility that such logarithmic terms give rise to a power behavior when the perturbation series is summed. (This is the case in the Thirring model.) Our assumption is that this does not take place for on-mass-shell amplitudes.

The purpose of this paper, however, is not to discuss strictly conformal-invariant theories. We are rather interested in what we have called essentially scale-invariant theories, i.e., theories for which scale and conformal invariance might be an approximate symmetry when all the energies are big compared to the masses in the theory. Then we expect that the equations, derived from scale and conformal invariance will be satisfied up to logarithmic terms. This will at least be true for any finite order of perturbation theory, allowing us to make predictions for the leading terms at high energies. Hopefully, it may be valid to all orders in perturbation theory, as indeed the comparison of our predictions with perturbation theory (see Sec. V) indicates.

[23] In the derivation of the low-energy theorem for graviton emission, the d_i were the dimensions of the asymptotic fields, and were fixed by normalization conditions. Here the d_i are the dimensions of the interpolating fields and we have implicitly assumed that they are identical to the dimensions of the asymptotic fields. This assumption might very well be incorrect; in fact, it fails for the Thirring model. In this case the dimension of the interacting spinor field is different from the free-field dimension and depends on the coupling constant. See K. Johnson, Nuovo Cimento **20**, 773 (1961).

[24] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).