Laplace Transforms and the Diagonalization of Bethe-Salpeter Equations for Absorptive Parts*

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A Laplace transform is developed for the crossed-channel partial-wave analysis of Bethe-Salpeter–like equations for the absorptive part of scattering matrix elements. The transform requires no assumption about rotating contours to a Euclidian region and allows from the outset power growth in energy of the transformed absorptive part. This eliminates the need for any analytic continuation after the transformation. The diagonalization of the forward and nonforward equations with arbitrary irreducible kernels is explicitly carried out.

I. INTRODUCTION

THE partial-wave analysis of S-matrix elements as a concise expression of the underlying symmetry of the physical problem has reached a remarkable level of sophistication since the fundamental work of Jacob and Wick a decade ago.[1,2] Of particular interest of late has been the technique of crossed-channel partial-wave expansions which directly reflect the properties associated with the little group of the four-momentum transfer vector. Since the little groups associated with lightlike, spacelike, and null momenta are noncompact, the conventional partial-wave expansions are restricted to scattering functions which are square integrable over the group manifold. This is indeed an unfortunate circumstance, since one of the physical goals behind making the expansions is that one will end up with a formalism in which the high-energy behavior of the collision phenomenon of interest will be expressible in simple terms (usually meromorphy) in the partial-wave plane. Restriction to square-integrable functions leads to an elegant expression of the so-called background integral, but misses the leading Regge-pole or -cut behavior. It has become standard form to have a footnote recognizing this fact which concludes with the hope that some kind of analytic continuation in the l-plane can subsequently be made.

In a study of the Bethe-Salpeter equation for the absorptive part of the scattering amplitude,[3] we have found an integral transform which makes the usual partial diagonalization of the relevant integral equation and yet allows power growth of the absorptive part. It is in essence a Laplace transform on the little group of the momentum transfer and is in a natural way the alteration of the usual harmonic or Fourier analysis described in Ref. 2 for the treatment of non-square-integrable functions over the group. After stating the basic idea in a precise manner, we will demonstrate the usefulness of the transform we introduce by explicitly diagonalizing the absorptive-part equation. We carry out our analysis for spinless particles only, but this is primarily a pedagogical device since the Laplace transform has obvious extensions to spinning particles and is likely to find its most fruitful applications in multiperipheral or multi-Regge analyses. We must confess that we have not been able to state our procedure in a "proper" group-theoretic context, but we content ourselves with the straightforward analysis given below.

II. LAPLACE TRANSFORM

We discuss in this section the idea of the Laplace transform, which we use below to diagonalize the forward (null momentum transfer) and nonforward (spacelike momentum transfer) absorptive-part equations. Recall that one arrives at the usual Laplace transform by taking the representation functions for the unitary representations of the translation group, namely $\sin kx$, and splitting them up into "functions of the second kind" by $\sin kx = (1/2i)(e^{ikx} - e^{-ikx})$. Then one of these functions of the second kind, say e^{ikx}, is continued in k away from the real line to reach nonunitary representations. This gives a nonunitary harmonic which allows, because of the nice decrease of the function of the second kind in a certain half-plane, the Laplace analysis of fuctions which are not square integrable on the line.

All this is completely elementary, but is precisely the idea one needs to perform the same analysis on the little groups of the Lorentz group. In particular, let us begin with the little group for null momentum transfer, $SO(1,3)$. The harmonics of this group have been thoroughly discussed in any number of the papers listed in Ref. 2. We use here the material from the review papers by Bander and Itzykson.[4] The lowest

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[1] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[2] A complete exposition of the techniques used and a full bibliography of past work is given in lectures delivered by P. Winternitz at the Dublin Summer School, 1969 (unpublished); Rutherford Laboratory Report No. RPP/T/3 (unpublished).
[3] The set of review lectures "Multiperipheral Dynamics" given by M. L. Goldberger at the 1969 Erice Summer School, Princeton University report (unpublished), gives a lucid development of the basic equation and an extensive discussion of its properties.
[4] M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966); **38**, 346 (1966).

harmonic on $SO(1,3)$ corresponding to the principal series of unitary representations is well known to be

$$Z_N(\theta)=\sin N\theta/N\sinh\theta\,, \tag{1}$$

with N real and positive and θ the hyperbolic angle on the timelike 1+3 hyperboloid. We divide this into two functions $S_N(\theta)$ of the second kind:

$$Z_N(\theta)=[S_N(\theta)+S_{-N}(\theta)]/2i\,, \tag{2}$$

so

$$S_N(\theta)=e^{iN\theta}/N\sinh\theta\,, \tag{3}$$

and carry out the (trivial) continuation in N away from the positive real N axis. This gives us the Laplace harmonic (dropping an irrelevant N)

$$L_N(\theta)=e^{-N\theta}/\sinh\theta \tag{4}$$

on $SO(1,3)$.

Now a scattering amplitude $A(s)$ is conveniently parametrized by $\cosh\theta\propto s$ and the transform with (4) of such an amplitude will be

$$A_N=\int_0^\infty d\theta(\sinh\theta)^2L_N(\theta)A(\theta)\,, \tag{5}$$

where the measure appropriate to the hyperboloid has been introduced.[5] The inversion formula is almost too well known to be recorded, but for completeness

$$\sinh\theta A(\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dN\,e^{N\theta}A_N\,. \tag{6}$$

Now we wish to contemplate the growth of $A(\theta)$ as s^α or $(\cosh\theta)^\alpha$, so the contour in (6) should be taken to the right of $\alpha+1$ in the N plane.

For spacelike momentum transfers, the lowest harmonic functions on the little group $SO(1,2)$ are Legendre functions $P_l(y)$ with $l=-\frac{1}{2}+iN$, N real, and $y=\cosh\theta$, where θ is the hyperbolic angle on the 1+2 timelike hyperboloid. The functions of the second kind are the old favorites $Q_l(y)$, related to $P_l(y)$ by the familiar

$$\pi P_l(y)/\tan\pi l=Q_l(y)-Q_{-l-1}(y)\,. \tag{7}$$

A scattering amplitude $A(s,t)$ will again have s proportional to $\cosh\theta$, and the Laplace transform

$$A_l(t)=\int_0^\infty d\theta\,\sinh\theta Q_l(\cosh\theta)A(\theta,t)$$
$$=\int_1^\infty dy\,Q_l(y)A(y,t) \tag{8}$$

is suggested.[6] For an amplitude with power growth no greater than y^α, $A_l(t)$ is analytic in the half l plane to the right of $\mathrm{Re}\,l=\alpha$. The inversion formula to recover $A(y,t)$ from (8) is found with the aid of the integral[7]

$$\int_1^\infty dy\,P_l(y)Q_{l'}(y)=\frac{1}{(l'-l)(l'+l+1)}\,, \tag{9}$$

valid for $\mathrm{Re}\,l'-\mathrm{Re}\,l>0$ and $\mathrm{Re}(l'+l)>-1$, and is

$$A(y,t)=\int_{c-i\infty}^{c+i\infty}\frac{dl}{2\pi i}(2l+1)P_l(y)_l(t) \tag{10}$$

with c to the right of α.

We will not treat the lightlike case below, although there is no reason in principle not to do so, but only mention that the argument above leads one immediately to the transforms of Meijer[8] on Bessel functions of imaginary argument. The relevance of this to the little group of lightlike vectors, $E(2)$, is clear from Ref. 2.

Before turning to the diagonalization of the absorptive-part equations, we note that since in the large-s limit, $s\simeq e^\theta$ in both examples discussed, the Laplace transform becomes essentially the Mellin transform which is conventionally employed in the discussion of these equations.[9]

III. DIAGONALIZATION OF ABSORPTIVE-PART EQUATION

A. Forward Scattering

This case has been discussed by Nussinov and Rosner,[5] using the transform in (5). We repeat it here since it sets our notation and kinematics and, further, it forms a significant part of the work involved in the nonforward problem. The equation we have in mind is given graphically in Fig. 1 when Q is set equal to zero. That is,

$$A(P,K)=I(P,K)+2\int\frac{d^4P'}{(2\pi)^4}\frac{I(P,P')A(P',K)}{(m^2-P'^2)^2}\,, \tag{11}$$

where A is the absorptive part of the scattering amplitude in the s channel with t fixed. The "potential" I is the sum of all two-line irreducible contributions to A, and the propagators of the horizontal lines have been assigned a mass m.

The first step in the analysis is to change to invariant variables which we choose at this point to be $s=(P+K)^2$, $u=P^2$, $v=K^2$, $s'=(P'+K)^2$, $u'=P'^2$, and

[5] It should be noted at this point that S. Nussinov and J. Rosner [J. Math. Phys. **7**, 1670 (1966)] have also introduced this Laplace transform to diagonalize the forward absorptive-part equation.

[6] M. Toller, in his paper on the $SO(1,2)$ harmonic analysis of a scattering amplitude [Nuovo Cimento **37**, 631 (1965)] discusses a "Laplace Transform" on the group. The relation between that notion and the one discussed here is obscure to the present authors. In the end he requires his amplitudes to fall as $s^{-1/2}$ anyway.

[7] *Higher Transcendental Functions*, edited by A. Erdélyi *et al.* (McGraw-Hill, New York, 1953), Vol. I.

[8] Reference 7, Vol. II.

[9] One can trace the literature on this from Ref. 3 where the technique is extensively employed.

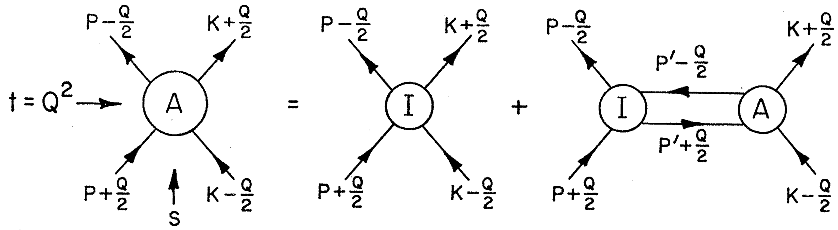

FIG. 1. The kinematics of the Bethe-Salpeter equation for the s-channel absorptive part of a scattering amplitude. All particles are spinless. In the text the forward ($Q=0$) and nonforward ($Q^2<0$) cases are considered.

$s_0=(P'-P)^2$. The Jacobian of this transformation is discussed in many places[3,5] and we record the result:

$$A(s,u,v)=I(s,u,v)+\frac{\theta(s-4L^2)}{16\pi^3\Delta^{1/2}(s,u,v)}\times\int_{L^2}^{(s^{1/2}-L)^2}ds_0\int_{L^2}^{(s^{1/2}-s_0^{1/2})^2}ds'\int_{u-'}^{u+'}du'\times\frac{I(s_0,u,u')A(s',u',v)}{(m^2-u')^2}. \quad (12)$$

The limits on the integration come from physical requirements on the absorptive part and from the imposition of a lower limit, called L^2 here, on contributions in the invariant energies s, s_0, and s'. The limits on the internal mass u' are

$$u_\pm'=s_0+u-\frac{(s+u-v)(s+s_0-s')}{2s}\pm\frac{\Delta^{1/2}(s,u,v)\Delta^{1/2}(s,s_0,s')}{2s}, \quad (13)$$

and the usual triangle function

$$\Delta(a,b,c)=a^2+b^2+c^2-2(ab+ac+bc) \quad (14)$$

has been introduced. When u and v are taken negative, $u_\pm<0$, and the equation is defined in the region $-\infty<(\text{masses})^2\leq 0$.

We wish to define now an integral transform of Eq. (12) and, as a trial, use

$$\tilde{A}_l(u,v)=\int_{L^2}^{\infty}ds\,f_l(s,u,v)A(s,u,v). \quad (15)$$

One may perform the indicated changes of integration limits to cast the transformed equation into the form

$$\tilde{A}_l(u,v)=\tilde{I}_l(u,v)+\frac{1}{16\pi^3}\int_{-\infty}^{0}du'\int_{L^2}^{\infty}ds_0\int_{L^2}^{\infty}ds'\int_{\sigma}^{\infty}ds\times\frac{f_l(s,u,v)I(s_0,u,u')A(s',u',v)}{\Delta^{1/2}(s,u,v)(m^2-u')^2}, \quad (16)$$

with

$$\sigma-u-v=\frac{(-1)}{2u'}[(s'-u'-v)(s_0-u'-u)+\Delta^{1/2}(s',u',v)\Delta^{1/2}(s_0,u',u)]. \quad (17)$$

At this point we take a clue from the group-theoretical treatment of forward scattering and define the variables

$$\cosh\theta=(s-u-v)/2(uv)^{1/2}, \quad (18)$$

$$\cosh\theta_0=(s_0-u-u')/2(uu')^{1/2}, \quad (19)$$

and

$$\cosh\theta'=(s'-u'-v)/2(u'v)^{1/2}, \quad (20)$$

which casts the lower limit of the s integration into the neat form $\theta_{\min}=\theta_0+\theta'$. Since $\Delta^{1/2}(s,u,v)=2(uv)^{1/2}\sinh\theta$, choosing $f_l(s,u,v)=e^{-(l+1)\theta}$ allows us to carry out the θ integration in an elementary manner and results in

$$A_l(u,v)=I_l(u,v)+\frac{1}{(l+1)(2\pi)^3}\int_{-\infty}^{0}du'\times\frac{(-u')I_l(u,u')A_l(u',v)}{(m^2-u')^2}, \quad (21)$$

where the transform suggested in the previous section has been introduced, and

$$A_l(u,v)=\int_0^\infty d\theta(\sinh\theta)^2\left(\frac{e^{-(l+1)\theta}}{\sinh\theta}\right)A(u,v,\cosh\theta). \quad (22)$$

The lower limit in (22) is really set by θ functions involving the minimum mass we have called L^2, i.e., $A(s,u,v)=0$ for $s<L^2$.

Equation (21) is the central result for $t=0$. One may quickly check that when one restricts himself to the ladder model in which

$$I(s_0,u,v)=\pi g^2\delta(s_0-m_0^2), \quad (23)$$

the results of Amati *et al.*[10] as recorded in Ref. 3 are reproduced.

B. Nonforward Scattering

Now we treat the kinematically more challenging case of the Bethe-Salpeter absorptive-part equation for $t=Q^2<0$. Again we refer to Fig. 1 for definitions and write

$$A(P,K,Q)=I(P,K,Q)+2\int\frac{d^4P'\,I(P,P',Q)A(P',K,Q)}{(2\pi)^4[m^2-(P'+\frac{1}{2}Q)^2][m^2-(P'-\frac{1}{2}Q)^2]}. \tag{24}$$

Since the choice of variables is crucial in making the diagonalization, we proceed in the following manner. Choose now as independent variables $s=(P+K)^2$, $u=P^2$, $a=P\cdot Q$, $s_0=(P-P')^2$, $u'=P'^2$, $a'=P'\cdot Q$, $s'=(P'+K)^2$, $v=K^2$, $\alpha=K\cdot Q$, and, of course, $t=Q^2$. This change of variables turns the integral equation into

$$A(s,u,a,v,\alpha,t)=I(s,u,a,v,\alpha,t)+\theta(s-4L^2)/32\pi^4\times\int_{L^2}^{(s^{1/2}-L)^2}ds_0\int_{L^2}^{(s^{1/2}-s_0^{1/2})^2}ds'\int_{u-'}^{u+'}du'\int da'\times\frac{I(s_0,u,a,u',a',t)A(s',u',a',v,\alpha,t)\theta(-\tilde{D})}{[(m^2-u'-\frac{1}{4}t)^2-a'^2](-\tilde{D})^{1/2}}, \tag{25}$$

with the limits on the s_0 and s' integrations set as before; $u_\pm'$ have the same structure as (13) with the recognition that u, u', and v are no longer the invariant masses of the legs. The limits on the a' integration are set by the θ function which requires the Jacobian

$$\tilde{D}=\begin{vmatrix} t & a' & a & \alpha \\ a' & u' & -\frac{1}{2}(s_0-u-u') & \frac{1}{2}(s'-u'-v) \\ a & -\frac{1}{2}(s_0-u-u') & u & \frac{1}{2}(s-u-v) \\ \alpha & \frac{1}{2}(s'-u'-v) & \frac{1}{2}(s-u-v) & v \end{vmatrix} \tag{26}$$

to be negative.

Since the variables t, v, and α play a purely passive role, we will suppress them in the argument to follow. Also note at this point that the correspondence between the $t=0$ equation, (12), and the first three integrations in (25) suggests that we restrict ourselves to u and v negative. Then the range of u' integration will be over negative u' only and the equation will be defined over a restricted domain. The question of continuing the answer A to regions of positive u and v is certainly not a severe one.[11]

As in the forward scattering case, we now make the transform of the equation with a function $f_l(s,u,a)$ by defining

$$\tilde{A}_l(u,a)=\int_{L^2}^{\infty}ds\,f_l(s,u,a)A(s,u,a). \tag{27}$$

The various integration limits may now be interchanged simply by reference to the $t=0$ equation, and we find

$$\tilde{A}_l(u,a)=\tilde{I}_l(u,a)+\frac{1}{32\pi^4}\int_{-\infty}^{0}du'\int_{L^2}^{\infty}ds_0\int_{L^2}^{\infty}ds'\int_{\sigma}^{\infty}ds\int da'\times\frac{I(s_0,u,a,u',a')A(s',u',a')\theta(-\tilde{D})f_l(s,u,a)}{[(m^2-u'-\frac{1}{4}t)^2-a'^2](-\tilde{D})^{1/2}}. \tag{28}$$

The structure of the determinant is now quite clearly exhibited if we scale out the momentum transfer t and the masslike variables u, u', and v. This leads us to introduce θ, θ_0, and θ' just as we did in (18)–(20) and, in addition,[12]

$$z=a/(tu)^{1/2}=P\cdot Q/(P^2Q^2)^{1/2}, \tag{29}$$

$$z'=a'/(tu')^{1/2}=P'\cdot Q/(P'^2Q^2)^{1/2}, \tag{30}$$

and

$$\zeta=\alpha/(tv)^{1/2}=K\cdot Q/(K^2Q^2)^{1/2}. \tag{31}$$

This enables us to remove a factor $tuu'v$ from the determinant and write $\tilde{D}=(tuu'v)D$, where

$$D=\begin{vmatrix} -1 & z' & z & \zeta \\ z' & -1 & -\cosh\theta_0 & \cosh\theta' \\ z & -\cosh\theta_0 & -1 & \cosh\theta \\ \zeta & \cosh\theta' & \cosh\theta & -1 \end{vmatrix}. \tag{32}$$

The limits on the z' integration are set by the vanishing of D. When we interchange the θ and the z' integrations (which replaced s and a'), it is useful to note that the absolute upper and lower limits on z' are ±1, coming from the vanishing of the 2×2 minor which is the coefficient of $(\cosh\theta)^2$ in D. The z' integration may thus be put to the end, allowing us to write the integral equation as

$$\tilde{A}_l(u,z)=\tilde{I}_l(u,z)+\frac{2(uv)^{1/2}}{8\pi^4}\times\int_{-\infty}^{0}du'(-u')\int_{-1}^{1}dz'\int^{\infty}d(\cosh\theta_0)\times\int^{\infty}d(\cosh\theta')\int^{\infty}d(\cosh\theta)\times\frac{\theta(-D)I(\theta_0,u,z,u',z')A(\theta',u',z')f_l(\theta,u,z)}{(-D)^{1/2}[(m^2-u'-\frac{1}{4}t)^2-tu'z'^2]}, \tag{33}$$

[10] D. Amati *et al.*, Nuovo Cimento **26**, 896 (1962).

[11] We make this statement aware of its lack of proof, but the work reported in Ref. 3 makes us confident of its validity.

[12] S. Pinsky and W. I. Weisberger (unpublished) have also introduced these variables in their study of the nonforward multiperipheral equation.

with the lower limit on the $\cosh\theta$ integration determined by the vanishing of D.

To determine the appropriate choice of variables for the carrying out of the θ integration (our argument from the previous section tells us that f_l will be a Q_l of the variable), we recall the salient feature of the group-theoretical discussion. In that approach, one sits in the Lorentz frame where $Q=(0,0,0,(-t)^{1/2})$ and notes that the little group of this vector is $SO(1,2)$. This suggests that the other vectors of the problem, P, P', and K, be decomposed into a part along Q and a part orthogonal to Q. The variables of integration then adopted are the projection on Q, the length of the three-vector orthogonal to Q, and the two angles specifying the orientation of the three-vector. For the vector P, say, this means decompose P_λ into $(P\cdot Q)Q_\lambda/Q^2$ and the three-vector (written in four-component form)

$$\tilde{P}_\lambda=P_\lambda-Q_\lambda P\cdot Q/Q^2\,, \tag{34}$$

with length

$$\tilde{P}^2=P^2-(P\cdot Q)^2/Q^2=u(1-z^2)\leqslant 0\,, \tag{35}$$

which means it is spacelike in our problem. The energy variables are now chosen to be the hyperbolic angles between the respective three-vectors $\tilde{P}$, $\tilde{K}$, and $\tilde{P}'$,

$$y=\frac{\tilde{P}\cdot\tilde{K}}{(\tilde{P}^2\tilde{K}^2)^{1/2}}=\frac{\cosh\theta+z\zeta}{[(1-z^2)(1-\zeta^2)]^{1/2}}\,, \tag{36}$$

$$y_0=\frac{-\tilde{P}\cdot\tilde{P}'}{(\tilde{P}^2\cdot\tilde{P}'^2)^{1/2}}=\frac{\cosh\theta_0-zz'}{[(1-z^2)(1-z'^2)]^{1/2}}\,, \tag{37}$$

and

$$y'=\frac{\tilde{P}'\cdot\tilde{K}}{(\tilde{P}'^2\tilde{K}^2)^{1/2}}=\frac{\cosh\theta'+z'\zeta}{[(1-z'^2)(1-\zeta^2)]^{1/2}}\,. \tag{38}$$

At last the reward for our kinematic diligence comes when we express the Jacobian determinant in the y, y_0, and y' variables and rewrite the integral equation with them:

$$\begin{aligned}\tilde{A}_l(u,z)=\tilde{I}_l(u,z)&+(2/8\pi^4)(uv)^{1/2}(1-z^2)^{1/2}(1-\zeta^2)^{1/2}\\ &\times\int_{-\infty}^{0}(-u')du'\int_{-1}^{1}dz'(1-z'^2)^{1/2}\\ &\times\int_1^\infty dy_0\int_1^\infty dy'\int_1^\infty dy\frac{I(y_0,u,z,u',z')A(y',u',z')}{[(m^2-u'-\frac{1}{4}t)^2-u'tz'^2]}\\ &\times\frac{Q_l(y)\theta(y^2+y_0^2+y'^2-1-2yy_0y')}{(y^2+y_0^2+y'^2-1-2yy_0y')^{1/2}}\,,\end{aligned} \tag{39}$$

where we have also chosen our transforming function f_l to be $Q_l(y)$. The lower limits here on the various y integrations are in fact greater than 1 and are set by L^2. It is possible to carry out the y integration in (39) with the following result, as shown in the Appendix:

$$\int_1^\infty dy\,Q_l(y)\frac{\theta(y^2+y_0^2+y'^2-1-2yy_0y')}{(y^2+y_0^2+y'^2-1-2yy_0y')^{1/2}}=Q_l(y_0)Q_l(y')\,. \tag{40}$$

It behooves us to define the integral transform over the $Q_l(y)$ in the obvious fashion,

$$A_l(u,z)=\int_1^\infty dy\,Q_l(y)A(y,u,z)\,, \tag{41}$$

so (39) becomes simply (restoring the passive variables)

$$\begin{aligned}A_l(u,z,v,\zeta,t)=I_l(u,z,v,\zeta,t)&+\frac{1}{8\pi^4}\int_{-\infty}^{0}du'(-u')\int_{-1}^{1}dz'(1-z'^2)^{1/2}\\ &\times\frac{I_l(u,z,u',z',t)A_l(u',z',v,\zeta,t)}{[(m^2-u'-\frac{1}{4}t)^2-tu'z'^2]}\,,\end{aligned} \tag{42}$$

where the range of u and v is over the negative real line and z and ζ are taken between ±1. From the solution to this two-dimensional integral equation over one finite and one infinite range, one recovers the full absorptive part by the inverse transformation (10). It is of clear interest to study the l-plane behavior of this absorptive part since the singularities furthest to the right determine in the usual manner the leading asymptotic behavior of $A(y)$.

IV. OBSERVATIONS

The Laplace transformation of the absorptive-part equation has been carried out here for forward and nonforward scattering of spinless particles. By working directly on the equation, with judicious hints from the group-theoretical treatment of the problem, we have carried out the diagonalization with an arbitrary "potential" or irreducible kernel without requiring a continuation of the equation first to a Euclidian region. Further, by transforming with functions of the second kind which have good decrease properties in a half-plane, we are able to allow from the outset for power growth of the absorptive parts and need not perform diagonalization.

Amusing as (21) and (42) are, they remain only kinematic skeletons until some dynamical muscle is provided in the form of some specific irreducible kernel. However, it is not unreasonable to imagine that many general properties of these equations may be extracted for wide classes of kernels because of their simple structure. For example, the recent electroproduction

[13] E. Bloom *et al.*, Phys. Rev. Letters **23**, 930 (1969); M. Breidenbach *et al.*, *ibid.* **23**, 935 (1969).

experiments at SLAC[13] pose the question of the dependence of the Regge residues on the external masses. The dynamics entailed in this program, as well as the generalization of the Laplace transform to spinning particles and Reggeons, will be the subject of further work.

Note added in manuscript. The works of the following authors on the diagonalization of absorptive-part equations using the more standard harmonic analysis have been brought to our attention: (1) M. Ciafaloni, C. Detar, and M. N. Misheloff, Phys. Rev. **188**, 2252 (1969); (2) A. H. Mueller and I. J. Muzinich, Ann. Phys. (N. Y.) **57**, 500 (1970). Also, using the medium of Mehler transforms, Regge and his collaborators have tabulated a number of interesting integrals involving Legendre functions, one of which is our Eq. (40). See V. de Alfaro, T. Regge, and C. Rossetti, Nuovo Cimento **26**, 1029 (1962).

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APPENDIX

In this appendix, we drive the integral (40) which is of the nature of an addition theorem on functions of the second kind. The starting point is the similar integral on P_l functions which is derived from the unitarity relation by Goldberger and Watson,[14] namely,

$$\frac{1}{\pi}\int_{-1}^{1} d\mu\, P_l(\mu)\frac{\theta(1-\mu^2-\mu_1{}^2-\mu_2{}^2+2\mu\mu_1\mu_2)}{(1-\mu^2-\mu_1{}^2-\mu_2{}^2+2\mu\mu_1\mu_2)^{1/2}} = P_l(\mu_1)P_l(\mu_2)$$

valid for μ_1, μ_2 in the interval -1 to $+1$. By using the integral definition of $Q_l(z)$,

$$Q_l(z)=\frac{1}{2}\int_{-1}^{1}\frac{dz'}{z-z'}P_l(z'),$$

the basic integral becomes

$$Q_l(z_1)Q_l(z_2)=\frac{1}{4\pi}\int_{-1}^{1} d\mu\, P_l(\mu)\int_{-1}^{1}\frac{d\mu_1}{z_1-\mu_1}\int_{-1}^{1}\frac{d\mu_2}{z_2-\mu_2} \times\frac{\theta(1-\mu^2-\mu_1{}^2-\mu_2{}^2+2\mu\mu_1\mu_2)}{(1-\mu^2-\mu_1{}^2-\mu_2{}^2+2\mu\mu_1\mu_2)^{1/2}}.$$

The integral

$$I=\int d\Omega_q\frac{1}{(\hat{q}\cdot\hat{k}-z_1)(\hat{q}\cdot\hat{k}'-z_2)}$$

can be cast into the two forms

$$I=\int_{-1}^{1}\frac{d\mu_1}{\mu_1-z_1}\int_{-1}^{1}\frac{d\mu_2}{\mu_2-z_2}\int d\Omega_q\delta(\mu_1-\hat{q}\cdot\hat{k})\delta(\mu_2-\hat{q}\cdot\hat{k}')$$

$$=2\int_{-1}^{1} d\mu_1\frac{1}{\mu_1-z_1}\int_{-1}^{1} d\mu_2 \times\frac{\theta(1-\mu^2-\mu_1{}^2-\mu_2{}^2+2\mu\mu_1\mu_2)}{(\mu_2-z_2)(1-\mu^2-\mu_1{}^2-\mu_2{}^2+2\mu\mu_1\mu_2)^{1/2}},$$

setting $\mu=\hat{k}\cdot\hat{k}'$, and[15]

$$I=4\pi\int_{1}^{\infty}\frac{dz\,\theta(z^2+z_1{}^2+z_2{}^2-1-2zz_1z_2)}{(z-\mu)(z^2+z_1{}^2+z_2{}^2-1-2zz_1z_2)^{1/2}},$$

which observation leads directly to

$$Q_l(z_1)Q_l(z_2)=\int_{1}^{\infty} dz\, Q_l(z)\frac{\theta(z^2+z_1{}^2+z_2{}^2-1-2zz_1z_2)}{(z^2+z_1{}^2+z_2{}^2-1-2zz_1z_2)^{1/2}},$$

the announced result.

[14] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 595.

[15] Reference 14, p. 605. Note the misprint in the quantity called κ.