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Convergence of Perturbation Theory for Static Models

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The unrenormalized perturbation series for the charged-scalar static model with finite cutoff is shown to have a finite radius of convergence. Although only the two-point function is explicitly considered, the result generalizes to three- and four-point functions.

INTRODUCTION

THE question of convergence of perturbation series for field-theoretical amplitudes has recently been discussed for the ϕ^4 theory.[1] The purpose of this paper is to establish the convergence of bare perturbation series for the extended-source static models.

Many of the innovations in field theory have been prompted by pilot studies with static models.[2] Recently, an off-shell approach to the charged-scalar model has been developed by Freeman, Rubin, and North.[3] The computational scheme consists of deriving integral equations based on bare perturbation theory. The convergence of the bare perturbation theory establishes the validity of such derivations provided analytic continuation of solutions may be done in the g_0 plane.

We confine ourselves to the charged-scalar-meson field coupled to a fixed source, but it should be clear that the proof generalizes to any static model. We also confine the detailed proof to the study of the proper self-energy series for the nucleon evaluated at the physical nucleon mass. However, the proof easily generalizes to the case of proper vertex functions and proper four-point functions. Unfortunately, we can say nothing about the important case of the point-source or local-field-theory limit.

BARE PERTURBATION THEORY

The rules of noncovariant perturbation theory are well known and are illustrated here only for reference. For charged-scalar theory, the interaction is of the Yukawa type:

$$H_I = g_0 \sum_{\mathbf{k}} \rho(\omega)[\tau_- a_{\mathbf{k}} + \tau_+ a_{\mathbf{k}}^\dagger + \tau_- b_{\mathbf{k}}^\dagger + \tau_+ b_{\mathbf{k}}], \quad (1)$$

where $a_{\mathbf{k}}$ destroys a π^+ meson, $b_{\mathbf{k}}$ destroys a π^- meson,

$$\tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have also

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}, \quad [b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'},$$

$$\sum_{\mathbf{k}} \rho^2(\omega) f(\omega) \to \int \frac{d^3k}{8\pi^3} \rho^2(\omega) f(\omega) \to \int_1^\infty \frac{\omega k}{2\pi^2} \rho^2(\omega) f(\omega) d\omega$$

$$\equiv \int_1^\infty \sigma(\omega) f(\omega) d\omega, \quad \omega \equiv (k^2+1)^{1/2}.$$

$\rho(\omega_k)$ is usually taken to be such that

$$\rho^2(\omega) = 1/2\omega(1+\alpha^2 k^2), \quad (2)$$

where α may be regarded as the "radius" of the source. The limit $\alpha \to 0$ is the point-source or local-field-theory limit.

As an illustration of the rules for construction of the integrals, we consider a self-energy or propagator modification graph, e.g., Fig. 1. The nucleon line is always drawn as a horizontal line; meson lines are curved. In this example, there are five propagation denominators, which may be written

$$D = \frac{1}{E-m_0-z} \frac{1}{E-m_0-x-z} \frac{1}{E-m_0-x-z-y} \times \frac{1}{E-m_0-x-z} \frac{1}{E-m_0-z}, \quad (3)$$

where E is the external energy, m_0 is the bare nucleon

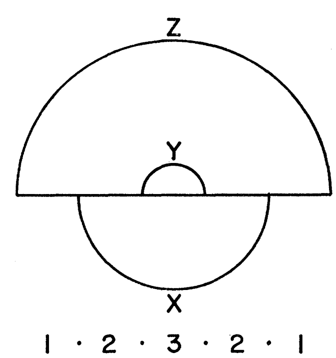

FIG. 1. Feynman graph for a particular contribution to the third-order (in g_0^2) contribution to the nucleon propagator. For reference, the number of mesons in each intermediate state is noted.

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mass, and x, y, and z are the meson energies. Finally, one must multiply by a factor of $g_0^2\sigma(\omega)$ for each closed meson loop and perform an integration with respect to each meson energy. The result for the graph of Fig. 1 is

$$S_A^{(3)}(E,m_0)=g_0^6\int_1^\infty\int_1^\infty\int_1^\infty \sigma(x)\sigma(y)\sigma(z) \times D(x,y,z)dxdydz. \quad (4)$$

Any graph can be constructed from the rules illustrated by the previous example.

Now consider the modified neutron or proton propagator $S(E)$. A standard result from field theory is that

$$S^{-1}(E)=E-m_0-\Sigma(E,m_0). \quad (5)$$

$\Sigma(E,m_0)$ consists of only the sum of proper graphs; a "proper" graph cannot be divided into two parts connected by a single line. The bare mass m_0 is chosen so that the modified propagator has a pole at the physical neutron mass, i.e., $E=0$. That is, m_0 is the root of

$$m_0=-\Sigma(0,m_0). \quad (6)$$

PROOF OF CONVERGENCE

The theorem we wish to prove is that the series representing $\Sigma(0,m_0)$ converges when $g_0^2\equiv\lambda$ is taken sufficiently small. The result holds only for extended-source models since the radius of convergence may tend to zero as the source volume decreases.

The first observation is that all terms in $\Sigma(0,m_0)$ contain an odd number of denominators. It follows that every graph is negative, and so henceforth only the absolute magnitude of the series will be considered. For any mth-order (in λ) diagram, there are $2m-1$ factors in the denominator of the corresponding integral. Of these factors, m differ from the preceding factor by the emission of an intermediate meson, and $m-1$ correspond to an absorption.

Consider the following modification of the first group of (emission) factors: All meson energies except that of the newly emitted meson are set equal to zero. The resulting integral will be strictly greater than the integral exactly representing the diagram. Next the $m-1$ absorption factors are modified as follows: m_0 is set equal to zero and *all* meson energies are replaced by their lower limits. These terms then factor out of the integral, giving a term which is the inverse of a product p of $m-1$ integers which are equal exactly to the *number of mesons* present after each absorption. This modification of the integral again produces an upper bound so that our arbitrary mth-order graph is less than or equal to

$$\frac{\lambda^m}{p}\left[\int_1^\infty \frac{\sigma(x)dx}{m_0+x}\right]^m. \quad (7)$$

Each mth-order graph is characterized by a sequence of $2m-1$ integers representing the number of mesons present in each propagator. For example, the sequence for the graph in Fig. 1 is $1\cdot2\cdot3\cdot2\cdot1$. We may either absorb or emit a meson at each vertex subject to the following conditions: We start with one meson in the first slot and return to one in the last slot; there may never be less than one meson present so as to avoid improper graphs; and the total number never exceeds m. Therefore, the total number of such sequences

$$p_1\cdot p_2\cdots p_m\cdot p_{m+1}\cdots p_{2m-1}, \quad (8)$$

with

$$p_k-p_{k\pm1}=\pm1, \quad (9)$$

is strictly less than 2^{2m}.

There may be several graphs corresponding to a given sequence. These "branches" arise only during absorption; if a positive meson is absorbed and there are seven positives in the preceding slot, there will be a total of seven branches at that slot. For example, the graphs in Figs. 1 and 2 have the same sequence due to the fact that there are two positive mesons which may be absorbed between the third and fourth slots. For the proton propagator, for example, the absorption of a negative meson to reach an intermediate state of p_k mesons introduces a multiplicity of $\frac{1}{2}p_k+\frac{1}{2}$; p_k must be greater than or equal to 1. The absorption of a positive meson introduces a multiplicity of $\frac{1}{2}p_k+1$; $p_k\geq 2$ in this case, since the last positive meson is never absorbed. The total number of branches is thus the product of these factors for the $m-1$ integers p_k which denote a slot following an absorption.

The total contribution of any sequence of graphs is bounded by the upper bound for any graph in the sequence (7) times the number of branches for that sequence. The ratio of the total number of branches to p is a product of $m-1$ factors, each of which is less than or equal to 1. The total contribution of any sequence is therefore bounded by (7) with the factor p removed. Therefore, the total contribution in the mth order is less than or equal to the upper bound on the number of sequences times this bound,

$$|\Sigma^m(0,m_0)|\leq 2^{2m}\lambda^m\left[\int_1^\infty \frac{\sigma(x)dx}{m_0+x}\right]^m, \quad (10)$$

which is a term of a geometric series. The perturbation

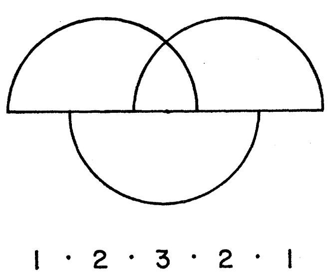

FIG. 2. The graph illustrated is similar to that in Fig. 1 in that the sequence of numbers of mesons in each intermediate state is the same. These are the only two third-order graphs having this denominator sequence.

series therefore converges for all

$$\lambda < 1 \Bigg/ \left[4 \int_1^\infty \frac{\sigma(x)dx}{m_0 + x} \right]. \tag{11}$$

CONCLUSIONS

It is apparent that the argument for the convergence of the series making up the proper vertex part goes through exactly as with the proper self-energy parts provided $E \leq m_0$. For $E > m_0$ the argument above requires rethinking because of the tendency for cancellation in the denominators. We have not bothered with this point because of its irrelevance to the basic questions of existence of a solution and convergence at the points necessary for renormalization.

It is not surprising that we have found convergence in the case considered in view of the fact that, for example, in neutral scalar theory the fields correspond to a harmonic oscillator with a linear perturbation in contrast to the anharmonic systems which are qualitatively very different for large $\langle\phi\rangle$.

Finally, we would like to point out that the models covered by this proof are nontrivial as compared to such truncated models as the Lee model in which the series for $\Sigma(0,m_0)$ consists of one term.

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Renormalization and Unitarity in the Dual-Resonance Model*

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All the one-loop graphs of the dual-resonance model are explicitly calculated. These graphs fall into three categories: planar, nonorientable, and orientable nonplanar. Using the properties of various elliptic functions, we are able to generalize the renormalization procedure, obtained previously for the planar diagrams, to the other two categories. The orientable nonplanar diagrams turn out to be particularly interesting. First, their integration regions have to be reduced from the ones naively obtained in order to avoid multiple counting. Secondly, they give rise to new singularities (branch points) in channels that are naturally identified as having vacuum quantum numbers. These singularities are probably related to the Pomeranchukon. The question of unitarity is explored at the one-loop level, i.e., to the first nontrivial order in the perturbation series. Although the counting of diagrams is somewhat subtle, a rather simple result emerges: All inequivalent diagrams (with respect to duality transformations) should be counted with equal weight. Finally, it is indicated that three of the four primitive renormalized loop operators of the theory can be obtained from the formulas of this paper.

I. INTRODUCTION

An attractive attitude towards the generalized Veneziano model[1,2] is that it provides one with the Born approximation to a theory of hadrons. The The implementation of this idea is seemingly straightforward. One proceeds to factorize the Born term,[3] i.e., the N-point tree graphs, thereby deducing the level structure implicit in the model and the Feynman rules of the theory.[4] These rules are then to be utilized in the construction of a unitary perturbation series.

The one-loop planar diagrams were in fact easily derived[5] once it had been demonstrated that the tree graphs were completely factorizable.[3] Further progress has, however, been impeded by a series of technical difficulties and by the fundamental problem of renormalization. The technical difficulties relate to the inclusion of twisted vertices and to the removal of the spurious states which are found by naive factorization.

The twisted vertices are necessary for the construction of twisted loop diagrams, some of which correspond to nonplanar Feynman graphs and some of which (the

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