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Broken Chiral and Conformal Symmetry in an Effective-Lagrangian Formalism

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The simultaneous breaking of conformal and chiral symmetry is investigated within the framework of nonlinear realizations and effective Lagrangians. The explicit introduction of a massless dilaton field, χ enables conformal invariance to be preserved in Lagrangians for massive matter fields. It is shown that the equation of Callan, Coleman, and Jackiw, $\partial_\mu D_\mu = \theta_{\mu\mu}$, remains valid notwithstanding the introduction of this particle, and also that it is possible to construct Lagrangians which are simultaneously invariant under the chiral and conformal groups. If we introduce a term which explicitly violates both symmetries, then the dilaton acquires a definite (bare) mass which can be expressed in terms of the masses of the chiral bosons —the pion in the case of chiral $SU(2)\times SU(2)$, the pion and kaon in the case of chiral $SU(3)\times SU(3)$. The precise form of this mass relation depends upon the type of symmetry-breaking term adopted.

I. INTRODUCTION

THE well-known techniques of nonlinear-realization theory have been used extensively in the treatment of chiral symmetries. These techniques, which are based upon the concept of group action on homogeneous spaces, have a wider relevance in that they provide a natural vehicle for the description of any symmetry which is spontaneously broken. In particular, they can be usefully employed in discussions of conformal symmetry. The group of conformal transformations on space-time certainly cannot be expected to manifest itself as a symmetry of physical states in that its unitary representations do not include discrete nonvanishing masses. However, it is conceivable that this symmetry may be present at least in the equations of motion although not in their solutions, i.e., that it is a spontaneously broken symmetry. This point of view has been advocated in recent works.[1] In the present paper we consider the problem of setting up effective Lagrangians which are simultaneously invariant under the conformal group and under a chiral group [$SU(2)\times SU(2)$ or $SU(3)\times SU(3)$]. The resulting theory possesses the following features: (a) physical states which are classified according to unitary representations of the direct product of the Poincaré group and the internal-symmetry group [$SU(2)$ or $SU(3)$]; (b) Goldstone particles corresponding to the spontaneous breakdown of the higher symmetries, viz., a massless even-parity spin-zero and chiral-invariant "dilaton" together with an $SU(2)$ triplet [or $SU(3)$ octet] of odd-parity spin-zero massless "chiral bosons"; and (c) some remnants of the higher symmetry which are expected to survive in the tree-graph approximation.

The degenerate vacua which are characteristic of the Goldstone solution can be avoided by introducing explicit symmetry-breaking terms into the Lagrangian. Such terms serve to generate masses for the Goldstone particles. Moreover, by choosing a symmetry-breaking term which belongs to an irreducible representation of the combined conformal and chiral groups, we can relate the (bare) mass of the dilaton to the (bare) masses of the chiral bosons. This mass relation depends strongly on the type of symmetry breaking chosen. Thus, in the case of chiral $SU(2)\times SU(2)$, if we assign the symmetry breaker to the $SU(2)$ scalar part of a chiral four-vector, we find

$$m_\chi^2 = 3m_\pi^2,$$

where m_χ and m_π denote the masses of the dilaton and the pion, respectively. In the case of $SU(3)\times SU(3)$, if we adopt a linear combination of the even-parity $SU(2)$ singlets in $(3,\bar{3})\oplus(\bar{3},3)$, then we find

$$8m_\chi^2 = 3m_\pi^2 + 6m_K^2.$$

On the other hand, if we take the symmetry-breaking terms from the representation (8,8), we obtain

$$3m_\chi^2 = 2m_\pi^2 + 4m_K^2.$$

In Sec. II we review the general method for making any Lorentz-invariant Lagrangian into a conformal-invariant one through the introduction of the dilaton field χ. The equation of motion satisfied by this field can be put into the universal form[2]

$$\tfrac{1}{2}\Box\chi^2 + \theta_{\mu\mu} = \partial_\mu D_\mu,$$

provided there are no derivative-containing symmetry-breaking terms. In this equation, D_μ denotes the current of the generator of dilatations and is of course conserved in a dilatation-invariant theory. The tensor $\theta_{\mu\nu}$ denotes the usual[3] symmetrized form of the canonical energy-momentum tensor. It may be noted that the equation of motion for χ can be put into a form advocated recently

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[1] A. Salam and J. Strathdee, Phys. Rev. **184**, 1750 (1969); **184**, 1760 (1969); C. J. Isham, A. Salam, and J. Strathdee, Phys. Letters **31B**, 300 (1970); Ann. Phys. (N. Y.) (to be published).

[2] A similar result is contained in D. J. Gross and J. Wess, CERN Report No. Th.1076 (unpublished). Our scalar field χ, however, is directly identifiable with the dilaton.

[3] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), p. 22.

by Callan, Coleman, and Jackiw,[4]

$$\bar{\theta}_{\mu\mu}=\partial_\mu \bar{D}_\mu\,,$$

where $\theta_{\mu\nu}$ and $\bar{D}_\mu$ are defined by

$$\bar{\theta}_{\mu\nu}=\theta_{\mu\nu}-\tfrac{1}{6}(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)\chi^2\,,$$
$$\bar{D}_\mu=D_\mu+\partial_\nu[-\tfrac{1}{6}(x_\nu\partial_\mu-x_\mu\partial_\nu)\chi^2+F_{\nu\mu}]\,.$$

An explicit formula for $F_{\nu\mu}=-F_{\mu\nu}$ is contained in Sec. II. Such redefinitions are permissible in that the added terms contribute neither to the space integrals of $\bar{\theta}_{0\nu}$ and $D_{0\nu}$ nor to the four-divergences. We have not examined the renormalizability of $\bar{\theta}_{\mu\nu}$.

In Sec. III we discuss the problem of combining conformal invariance with chiral invariance. The solution is given in the form of a set of simple rules for generating a chiral- and conformal-invariant Lagrangian from one which is only chiral invariant. The remainder of Sec. III is devoted to the construction of symmetry-breaking terms and extracting the associated mass formulas mentioned above.

II. LAGRANGIAN FORMALISM

The conformal transformations of space-time constitute a 15-parameter Lie group which is characterized by the fact that the Jacobian matrix $\partial x_\mu'/\partial x_\nu$ is proportional to a Lorentz transformation. More precisely,

$$\frac{\partial x_\mu'}{\partial x_\alpha}\frac{\partial x_\nu'}{\partial x_\beta}\eta^{\mu\nu}=\left|\det\frac{\partial x'}{\partial x}\right|^{1/2}\eta^{\alpha\beta}\,, \tag{2.1}$$

where $\eta^{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1)$ denotes the Minkowskian metric tensor. The Poincaré group is evidently included as a ten-parameter subgroup. The property (2.1) allows one immediately to extend any representation of the Poincaré group to the full conformal group. Thus, suppose that the set of fields $\psi_\alpha(x)$ transforms under the Poincaré group according to

$$\psi(x)\to\psi'(x')=D(\Lambda)\psi(x)\,, \tag{2.2}$$

where $\Lambda_{\mu\nu}=\partial x_\mu'/\partial x_\nu$ denotes a homogeneous Lorentz transformation. Since, according to (2.1), for any conformal transformation we can write

$$\frac{\partial x_\mu'}{\partial x_\nu}=\left|\det\frac{\partial x'}{\partial x}\right|^{1/4}\times(\text{a Lorentz matrix}),$$

it follows that the transformation law

$$\psi(x)\to\psi'(x')$$
$$=\left|\det\frac{\partial x'}{\partial x}\right|^{l/4}D\left(\left|\det\frac{\partial x}{\partial x'}\right|^{1/4}\frac{\partial x_\mu'}{\partial x_\nu}\right)\psi(x) \tag{2.3}$$

is well defined, provided l is a Lorentz scalar. The behavior of ψ under conformal transformations is completely specified by the action of Lorentz transformations (2.2) and pure dilatations $x_\mu'=\lambda x_\mu$ ($\lambda>0$) for which (2.3) simplifies to the form

$$\psi(x)\to\psi'(x')=\lambda^l\psi(x)\,. \tag{2.4}$$

Since the pure dilatations must commute with the homogeneous Lorentz transformations, we must require that the matrix l commutes with $D(\Lambda)$. [In particular, if $D(\Lambda)$ is irreducible, l will be a pure number.]

The Lagrangian of a conformal-invariant theory must be a Lorentz scalar with $l=-4$, i.e.,

$$L(\psi'(x'))=\left|\det\frac{\partial x}{\partial x'}\right|L(\psi(x))\,, \tag{2.5}$$

in order that the action be invariant. In the absence of spontaneous symmetry breakdown the condition (2.5) can be met only in theories with dimensionless coupling constants and vanishing masses. However, if we suppose that there is spontaneous symmetry breaking in the theory, then we have at our disposal a fundamental scalar field $\chi(x)$ with $l=-1$ and whose vacuum expectation value is nonvanishing.[5] Our basic hypothesis, therefore, is that the conformal-invariant Lagrangian[6]

$$L_\chi=\tfrac{1}{2}(\partial_\mu\chi)^2+\kappa\chi^4 \tag{2.6}$$

is capable of yielding a degenerate or Goldstone solution with $\langle\chi\rangle\neq0$. In principle, the value of $\langle\chi\rangle$ could be determined self-consistently in the manner of Goldstone. Since $l=-1$ for this field, the nonvanishing of $\langle\chi\rangle$ can only mean that the vacuum state is not a conformal invariant and, correspondingly, that the χ particle is massless.

The dilaton field χ can be used to generate effective masses and coupling constants for the other fields ψ in the system. That is, any given Lorentz-invariant Lagrangian density $L(\psi)$ can be turned into a conformal density $L(\psi,\chi)$ by introducing χ and its derivative in the appropriate places. Two fundamental operations are involved. First, the weight of each term in $L(\psi,\chi)$ is brought to the value $l=-4$ through multiplying by the appropriate power—positive or negative—of χ. Second, the ordinary derivative $\partial_\mu\psi$ is replaced by the covariant form

$$D_\mu\psi=\partial_\mu\psi+(lg_{\mu\nu}-iS_{\mu\nu})\psi\frac{\partial_\nu\chi}{\chi}\,, \tag{2.7}$$

where the matrices l and $S_{\mu\nu}$ characterize the behavior of ψ under infinitesimal dilatations and Lorentz trans-

[4] C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) (to be published).

[5] The dilaton field χ was represented in the form $\exp(-g\sigma)$ in Ref. 1. In order to avoid any confusion with the σ of chiral $SU(2)\times SU(2)$, which is the isoscalar component of a chiral four-vector, we are using χ to denote the chiral-invariant dilaton.

[6] By "conformal-invariant" Lagrangian we mean, of course, a Lagrangian which yields a conformal-invariant action. Such Lagrangians are in fact scalar densities as defined by (2.5).

formations, respectively. Evidently this formulation requires $\langle\chi\rangle\neq 0$, since the field χ occurs in the denominator of (2.7).

We turn now to consider the general form of the equation of motion for χ. The discussion is facilitated by first putting the Lagrangian $L(\psi)$ into the canonical form[7]

$$L(\psi)=i\psi^T\Gamma_\mu\partial_\mu\psi-H(\psi)\,, \qquad (2.8)$$

where ψ^T denotes the transpose of ψ, and Γ_μ denotes a set of numerical matrices. The adoption of (2.8) represents no loss of generality since any Lagrangian can be brought into this form by introducing sufficient auxiliary fields.[7] Moreover, we can assume that the components ψ are real. This implies that the spin matrices $S_{\mu\nu}$, defined by

$$\psi'(x')=(1-\tfrac{1}{2}i\epsilon_{\mu\nu}S_{\mu\nu})\psi(x)$$

for infinitesimal Lorentz transformations, are purely imaginary. Lorentz invariance requires that $H(\psi)$ in (2.8) be a scalar invariant, while Γ_μ must satisfy

$$i(S_{\mu\nu}{}^T\Gamma_\lambda+\Gamma_\lambda S_{\mu\nu})=g_{\nu\lambda}\Gamma_\mu-g_{\mu\lambda}\Gamma_\nu\,.$$

Since divergence terms in a Lagrangian are variationally insignificant, no generality is lost in assuming that the matrices Γ_μ are symmetric between fermion fields and antisymmetric between boson fields. Thus, for example, the four-vector $\psi^T\Gamma_\mu\psi$ vanishes identically, while the electric current is represented by the quadratic form $i\psi^T\Gamma_\mu q\psi$, with q an antisymmetric Lorentz scalar matrix.

For the conformal-invariant Lagrangian corresponding to (2.8) we adopt the real form[8]

$$L(\psi,\chi)=i(\chi^{\frac{3}{2}+l}\psi)^T\Gamma_\mu D_\mu(\chi^{\frac{3}{2}+l}\psi)-\chi^4H(\chi^l\psi)+L_\chi\,, \qquad (2.9)$$

where L_χ is given by (2.6) and the covariant operator D_μ by (2.7). Explicitly,

$$L(\psi,\chi)=i\psi^T\chi^{\frac{3}{2}+l^T}\Gamma_\mu\chi^{\frac{3}{2}+l}\left[\partial_\mu\psi+(lg_{\mu\nu}-iS_{\mu\nu})\psi\frac{\partial_\nu\chi}{\chi}\right]$$
$$+\chi^4[\kappa-H(\chi^l\psi)]+\tfrac{1}{2}(\partial_\mu\chi)^2\,. \qquad (2.10)$$

The equation of motion which is obtained from this Lagrangian by varying χ is given by

$$\tfrac{1}{2}\Box\chi^2+\theta_{\mu\mu}=0\,, \qquad (2.11)$$

where $\theta_{\mu\nu}$ denotes the usual symmetrized form[3] of the canonical energy-momentum tensor for the whole system (including χ). The inclusion of a symmetry-breaking term $L_1(\psi,\chi)$ in (2.10) will modify this equation of motion. If we suppose that L_1 contains no derivatives and transforms like a Lorentz scalar with $l_1\neq -4$, then (2.11) is replaced by

$$\tfrac{1}{2}\Box\chi^2+\theta_{\mu\mu}=\partial_\mu D_\mu=-(l_1+4)L_1\,, \qquad (2.12)$$

where D_μ denotes the canonical dilatation current

$$D_\mu=\frac{\partial L}{\partial\psi_{,\mu}}(x_\nu\partial_\nu-l)\psi+\frac{\partial L}{\partial\chi_{,\mu}}(x_\nu\partial_\nu+1)\chi$$
$$-x_\mu(L+L_1)\,. \qquad (2.13)$$

It may be of interest to see that this current can be put into a form similar to that given in Ref. 4. To begin with, we have

$$D_\mu=T_{\mu\nu}x_\nu+\psi^T\chi^{\frac{3}{2}+l^T}\Gamma_\alpha\chi^{\frac{3}{2}+l}S_{\alpha\mu}\psi+\chi\partial_\mu\chi\,,$$

where $T_{\mu\nu}$ denotes the canonical energy-momentum tensor. A little algebra gives

$$D_\mu=\theta_{\mu\nu}x_\nu+\chi\partial_\mu\chi+\partial_\nu F_{\nu\mu}\,, \qquad (2.14)$$

where $F_{\nu\mu}=-F_{\mu\nu}$ is defined by

$$F_{\mu\nu}=\tfrac{1}{2}x_\alpha(H_{\alpha\mu\nu}+H_{\nu\mu\alpha}-H_{\mu\nu\alpha})\,,$$
$$H_{\alpha\mu\nu}=\psi^T\chi^{\frac{3}{2}+l^T}\Gamma_\alpha\chi^{\frac{3}{2}+l}S_{\mu\nu}\psi\,.$$

Define the new, noncanonical currents

$$\bar{D}_\mu=D_\mu-\partial_\nu[F_{\nu\mu}+\tfrac{1}{6}(x_\nu\partial_\mu-x_\mu\partial_\nu)\chi^2]\,,$$
$$\bar{\theta}_{\mu\nu}=\theta_{\mu\nu}-\tfrac{1}{6}(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)\chi^2\,, \qquad (2.15)$$

in terms of which the relation (2.14) takes the particularly simple form

$$\bar{D}_\mu=\bar{\theta}_{\mu\nu}x_\nu\,. \qquad (2.16)$$

It should be emphasized that while this relation has the same appearance as that of Ref. 4, it is not identical with it. Our definitions of $\bar{D}_\mu$ and $\bar{\theta}_{\mu\nu}$ differ from the canonical ones only through the presence of the dilation field χ (in addition to the higher-spin contributions in $F_{\nu\mu}$), whereas Callan, Coleman, and Jackiw employ all of the zero-spin fields in their redefinition. In particular, we make no claims about the renormalizability of $\bar{\theta}_{\mu\nu}$.

We conclude this section on the representation of conformal symmetry in a Lagrangian framework by remarking that the conformal-invariant Lagrangian (2.10) must yield S-matrix elements which, on the mass shell, are independent of l. This follows from the fact that the modification $l\to l'$ can be effected by the field redefinition

$$\psi\to\psi'=\chi^{l-l'}\psi\,,$$

which, according to the well-known equivalence theorems of field theory, leaves the S matrix unchanged. In the presence of the l-dependent symmetry-breaking term

$$L_1(\chi,\psi)=\chi^{-l_1}f(\chi^l\psi)\quad (l_1\neq -4)$$

this statement remains true. It fails in more general broken-symmetry models.

[7] C. Lanczos, Rev. Mod. Phys. **29**, 337 (1957).

[8] If V_μ is a conformal four-vector of weight $l=-3$, then $D_\mu V_\mu=\partial_\mu V_\mu$. Moreover, if $V_\mu=\psi^T\Gamma_\mu\psi$, then $\partial_\mu V_\mu=(D_\mu\psi)^T\Gamma_\mu\psi+\psi^T\Gamma_\mu D_\mu\psi$. Hence the term $-(D_\mu\psi)^T\Gamma_\mu\psi$ is variationally equivalent to $\psi^T\Gamma_\mu D_\mu\psi$, so that (2.9) is real up to a four-divergence.

III. BROKEN CHIRAL AND CONFORMAL SYMMETRIES

There are at least two equivalent methods of generating conformal-invariant Lagrangians from chiral-invariant ones. However, caution must be observed in applying the rules of Sec. II in order that the chiral invariance should not be lost. It will be found that the dilaton field χ is intimately involved in the nonlinear chiral transformations.

The simplest approach—and one which is particularly suited to the chiral $SU(2)\times SU(2)$ case—is to consider chiral-invariant Lagrangians expressed in terms of fields which transform linearly under the chiral group. Since the conformal and chiral transformations are commutative, there will be no difficulty in applying the prescriptions of Sec. II. Conformal invariance is obtained without disturbing the chiral invariance provided only that the dilaton is taken to be a chiral scalar. This we shall assume. Nonlinearity is now achieved by imposing the appropriate covariant constraints. For example, in chiral $SU(2)\times SU(2)$ the pion field is usually assigned to a chiral four-vector $(\sigma,\boldsymbol{\pi})$ which is then constrained according to

$$\sigma^2+\boldsymbol{\pi}^2=f^2\,, \tag{3.1}$$

where f is a constant which, in the tree approximation, can be identified with the pion decay constant F_π. The constraint (3.1) is conformally invariant only if both σ and $\boldsymbol{\pi}$ are assigned the conformal weight $l=0$. This is a perfectly consistent arrangement. However, it is usually more convenient to assign the value $l=-1$ to boson fields (such as $\boldsymbol{\pi}$), since this corresponds to the assignment $l=-4$ for the kinetic-energy term $(\partial_\mu)^2$, which then appears without the encumbering weight factor χ^{2l+2}. The conformal-covariant analog of the constraint (3.1) now takes the form

$$\sigma^2+\boldsymbol{\pi}^2=\chi^2 \quad (l_\sigma=l_\pi=l_\chi=-1)\,, \tag{3.2}$$

which is also chiral invariant since χ is a chiral scalar.[5] Clearly we should now expect $\langle\chi\rangle=F_\pi$. Using the generalized constraints like (3.2), one can readily construct Lagrangians which are invariant under both chiral and conformal transformations. [It may be remarked that the constraint (3.2) is more a redefinition of scalar fields than a genuine constraint. However, this is a peculiarity of the $SU(2)\times SU(2)$ case which does not carry over to $SU(3)\times SU(3)$. In both cases we introduce one chiral scalar—the dilation—into the theory; it just happens that in the nonlinear $SU(2)\times SU(2)$ case only one field was removed by the constraint—as opposed to ten in $SU(3)\times SU(3)$.]

The nonlinear chiral transformations of the pion field which can be deduced by using (3.2) to eliminate σ from the relations $\delta\pi^a=\epsilon^a\sigma$ take the form

$$\delta\pi^a=\epsilon^a\chi[1-(\pi/\chi)^2]^{1/2}.$$

In general, we should obtain a relation of the form

$$\delta\pi^a=\chi f^{ab}(\pi/\chi)\epsilon^b\,, \tag{3.3}$$

where the function f_{ab} depends upon the choice of chiral coordinates.

It is not always convenient to make explicit use of constraint equations in setting up chiral-invariant Lagrangians. The general method is to start with a form which is invariant under the linear subgroup [$SU(2)$ or $SU(3)$] and which does not contain the chiral boson fields (π or π, K, η). Invariance under the full chiral group is then obtained by everywhere replacing the ordinary derivative $\partial_\mu\psi$ by a chiral-covariant form

$$\nabla_\mu\psi=\partial_\mu\psi+\Gamma_{\mu a}(\pi)(\partial_\mu\pi^a)\psi\,.$$

The pion kinetic energy is then expressed in terms of the covariant derivative

$$\nabla_\mu\pi^a=\lambda_b{}^a(\pi)\partial_\mu\pi^b$$

and added to the Lagrangian. In order to achieve conformal invariance as well as chiral invariance, this prescription must be modified. Instead of $\nabla_\mu\psi$ and $\nabla_\mu\pi^a$, we must insert the chiral- *and* conformal-covariant derivatives

$$D_\mu\psi=\partial_\mu\psi+(lg_{\mu\nu}-iS_{\mu\nu})\psi\frac{\partial_\nu\chi}{\chi}+\Gamma_{\mu a}\left(\frac{\pi}{\chi}\right)\partial_\mu\left(\frac{\pi^a}{\chi}\right)\psi\,, \tag{3.4}$$

$$D_\mu\pi^a=\chi\lambda_b{}^a(\pi/\chi)\partial_\mu(\pi^b/\chi)\,.$$

Each term in the resulting Lagrangian is then brought to the correct weight by multiplying with a power of the chiral-invariant dilaton field χ, and, finally, the dilaton Lagrangian (2.6) is added. [Equations (3.4) are based on the assignment $l_\pi=-1$. If some other value of l_π is chosen, then we must replace χ by χ^{-l_π} in them.]

Consider now the problem of symmetry breaking. To the fully invariant Lagrangian, made up according to the prescriptions outlined above, we shall add a non-invariant term L_1. In general, L_1 must be a Lorentz scalar with even parity. We shall require in addition that it contain no derivatives. For the case of chiral $SU(2)\times SU(2)$, we shall assign L_1 to be the $SU(2)$ singlet member of a chiral four-vector and in the case of $SU(3)\times SU(3)$ we shall consider two possible assignments: a mixture of $SU(3)$ singlet and octet components, first in the chiral multiplet $(3,\bar{3})\oplus(\bar{3},3)$, and secondly in $(8,8)$. In all cases we shall take $l_1=-1$.

The point about assigning the symmetry breaker L_1 to a definite irreducible representation of the direct product of the chiral and conformal groups is that one obtains in this way a relation between the divergences of the dilatation and axial-vector currents. Thus,

$$\partial_\mu D_\mu=(l_1+4)L_1\,, \tag{3.5}$$

$$\partial_\mu A_\mu{}^a=(1/i)[Q_5{}^a,L_1]\,, \tag{3.6}$$

where $Q_5{}^a$ denotes the axial-vector charge operator. Combining these formulas with the fundamental relation of Sec. II,

$$\partial_\mu D_\mu=\theta_{\mu\mu}+\tfrac{1}{2}\Box\chi^2=\bar{\theta}_{\mu\mu}\,, \tag{3.7}$$

one could embark upon a current-algebraic investigation of the consequences of partially broken conformal symmetry. For example, in the case of $SU(2)\times SU(2)$, where we take[9]

$$L_1=A\sigma \quad (l_1=-1),$$

with A a numerical constant, Eqs. (3.5) and (3.6) take the respective forms

$$\partial_\mu D_\mu=3A\sigma, \tag{3.8}$$

$$\partial_\mu A_\mu{}^a=A\pi^a, \tag{3.9}$$

and (3.7) then gives the basic formula

$$\bar\theta_{\mu\mu}=i\sum_{a=1}^{3}[Q_5{}^a,\pi^a]. \tag{3.10}$$

The right-hand side of (3.9), sandwiched between appropriate states, is directly related to the so-called "σ terms" of low-energy scattering,[10] while, for zero momentum transfer, the left-hand side yields the mass of the states.

One of the principal functions of the symmetry-breaking term is to generate masses for the various Goldstone particles in the theory. In the tree approximation these masses will be related. A simple procedure for extracting this mass relation is to express the symmetry-breaking term in powers of the Goldstone fields and examine the coefficients of the second-order terms. In this approximation it is necessary to allow for the nonvanishing of $\langle\chi\rangle$ by expressing the dilaton field in the form

$$\chi=\langle\chi\rangle+\tilde\chi$$

and adjusting the parameter κ in L_χ so as to make the coefficient of the linear term in $\tilde\chi$ equal to zero. The coefficient of $-\frac{1}{2}\tilde\chi^2$ will then be interpreted as the dilaton bare mass (squared). We consider the cases in turn.

A. Chiral $SU(2)\times SU(2)$, $L_1\in(2,2)$

The symmetry-breaking term is given by

$$\begin{aligned}L_1=A\sigma&=A[(\langle\chi\rangle+\tilde\chi)^2-\pi^2]^{1/2}\\&=A\langle\chi\rangle+A\tilde\chi-(A/2\langle\chi\rangle)\pi^2+\cdots.\end{aligned}$$

The pure dilaton part of the Lagrangian is given by

$$\begin{aligned}L_\chi&=\tfrac{1}{2}(\partial_\mu\chi)^2+\kappa\chi^4\\&=\tfrac{1}{2}(\partial_\mu\tilde\chi)^2+\kappa\langle\chi\rangle^4+4\kappa\langle\chi\rangle^3\tilde\chi+6\kappa\langle\chi\rangle^2\tilde\chi^2+\cdots.\end{aligned} \tag{3.11}$$

The vanishing of the coefficient of $\tilde\chi$ gives the relation

$$4\kappa\langle\chi\rangle^2=-A/\langle\chi\rangle,$$

which can be used to eliminate κ from the expressions for the bare masses. These are given by

$$m_\chi{}^2=3A/\langle\chi\rangle,\quad m_\pi{}^2=A/\langle\chi\rangle. \tag{3.12}$$

It therefore appears that breaking the chiral $SU(2)\times SU(2)$ symmetry with the isoscalar component of a chiral four-vector of conformal weight $l_1=-1$ yields the mass relation

$$m_\chi{}^2=3m_\pi{}^2. \tag{3.13}$$

From Eq. (3.9) we obtain $A=m_\pi{}^2F_\pi$, which, together with (3.12), implies the expected result

$$\langle\chi\rangle=F_\pi. \tag{3.14}$$

As mentioned above, the constraint (3.2) can be looked upon as a redefinition of the scalar fields. This suggests that the mass relation (3.13) should be obtainable by using (3.2) to eliminate the dilaton field χ rather than σ. Thus we should write

$$\begin{aligned}L&=\cdots+\kappa\chi^4+A\sigma\\&=\cdots+\kappa(\sigma^2+\pi^2)^2+A\sigma.\end{aligned}$$

Applying the same method as above, we do indeed arrive at the relation $m_\sigma{}^2=3m_\pi{}^2$. The existence of this alternative approach is, as has already been remarked, a peculiarity of $SU(2)\times SU(2)$. The larger group $SU(3)\times SU(3)$ requires the first approach.

B. Chiral $SU(3)\times SU(3)$, $L_1\in(3,\bar 3)+(\bar 3,3)$

The symmetry-breaking term is given by

$$L_1=AU_0+BU_8, \tag{3.15}$$

where U_0 and U_8 belong to the linear representation $(3,\bar3)+(\bar3,3)$ which contains a scalar nonet U_i and a pseudoscalar nonet V_i of the diagonal subgroup $SU(3)$. These 18 components are constructed out of the eight independent Goldstone fields M_i, $i=1, 2, \cdots, 8$ (or π, K, $\bar K$, η), which constitute a pseudoscalar octet of $SU(3)$. In terms of exponential coordinates, the 18 linearly transforming components can be represented by the expressions[11]

$$\begin{aligned}U_i(M,\chi)&=\chi\,\mathrm{Tr}[\lambda_i(e^{2i\lambda\cdot M/\chi}+e^{-2i\lambda\cdot M/\chi})],\\&\qquad i=0,\cdots,8\\V_i(M,\chi)&=i\chi\,\mathrm{Tr}[\lambda_i(e^{2i\lambda\cdot M/\chi}-e^{-2i\lambda\cdot M/\chi})],\\&\qquad i=0,\cdots,8\end{aligned} \tag{3.16}$$

where all fields have been assigned the same conformal weight

$$l=-1,\quad\text{and}\quad \lambda\cdot M\equiv\sum_{i=1}^{8}\lambda_iM_i.$$

The dilaton field can be expressed in terms of the components U_i and V_i by a formula of the same type as the constraint (3.2), viz.,

$$\chi^2=\sum_{i=0}^{8}(U_iU_i+V_iV_i).$$

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[9] This term was first considered in the context of dilation symmetry by G. Mack, Nucl. Phys. **B5**, 499 (1968); see also G. Mack and A. Salam, Ann. Phys. (N. Y.) **53**, 174 (1969).

[10] S. P. de Alwis and P. J. O'Donnell, University of Toronto report (unpublished).

[11] C. J. Isham, Nuovo Cimento **61A**, 729 (1969); Nucl. Phys. **B15**, 540 (1970).

Another nine constraints exist among these fields but we shall not make any use of them.

Expanding L_1 in powers of M, we obtain

$$L_1=6A(\sqrt{\tfrac{2}{3}})(\langle\chi\rangle+\tilde{\chi})$$
$$-\frac{8A}{\langle\chi\rangle}\left\{\left[(\sqrt{\tfrac{2}{3}})+\frac{B}{A}\sqrt{\tfrac{1}{3}}\right]\pi^2+\left[(\sqrt{\tfrac{2}{3}})-\frac{B}{2A}\sqrt{\tfrac{1}{3}}\right]\bar{K}K\right.$$
$$\left.+\left[(\sqrt{\tfrac{2}{3}})-\frac{B}{A}\sqrt{\tfrac{1}{3}}\right]\eta^2\right\}+\cdots. \quad (3.17)$$

Incorporating this with L_χ as given in (3.11), we find, upon setting equal to zero the linear term in $\tilde{\chi}$, the condition

$$4\kappa\langle\chi\rangle^2=-(6A/\langle\chi\rangle)\sqrt{\tfrac{2}{3}},$$

which can be used to eliminate κ from the expressions for the bare masses. These are given by

$$m_\pi^2=\frac{16A}{\langle\chi\rangle}\left[(\sqrt{\tfrac{2}{3}})+\frac{B}{A}\sqrt{\tfrac{1}{3}}\right],$$
$$m_K^2=\frac{16A}{\langle\chi\rangle}\left[(\sqrt{\tfrac{2}{3}})-\frac{B}{2A}\sqrt{\tfrac{1}{3}}\right], \quad (3.18)$$
$$m_\eta^2=\frac{16A}{\langle\chi\rangle}\left[(\sqrt{\tfrac{2}{3}})-\frac{B}{A}\sqrt{\tfrac{1}{3}}\right],\quad m_\chi^2=\frac{18A}{\langle\chi\rangle}\sqrt{\tfrac{2}{3}}.$$

The four masses are expressed in terms of two parameters and hence satisfy two relations: the Gell-Mann–Okubo formula and

$$8m_\chi^2=3m_\pi^2+6m_K^2, \quad (3.19)$$

which implies for the dilaton mass $m_\chi\approx 440$ MeV. Finally, the parameter B/A which measures the ratio of $SU(3)$ breaking to $SU(3)\times SU(3)$ breaking takes the usual value ≈ -1.25.

C. Chiral $SU(3)\times SU(3)$, $L_1\in(8,8)$

Although the most popular method of breaking $SU(3)\times SU(3)$ symmetry is via the $(3,\bar{3})+(\bar{3},3)$ representation, this is by no means the only interesting one. Indeed, the use of that particular representation was originally motivated by considerations of a quark model for the vector and axial-vector currents and, it may be argued, should not appear in a pure-current model. In the latter case a more natural symmetry-breaking representation would be (8,8). This possibility has been discussed elsewhere.[12]

The 64 linearly transforming components ω_{ab} of (8,8) can be represented in terms of the Goldstone octet $M_1, \ldots, M_8$ by the formula

$$\omega_{ab}(M,\chi)=\chi\left[\exp\left(\frac{2O\cdot M}{\chi}\right)\right]_{ab},\quad a,b=1,\ldots,8 \quad (3.20)$$

where

$$(O\cdot M)_{ab}=\sum_{c=1}^{8} f_{abc}M_c.$$

For the symmetry-breaking contribution to the Lagrangian we take

$$L_1(M,\chi)=(\sqrt{\tfrac{1}{8}})A\sum_{a=1}^{8}\omega_{aa}+(\sqrt{\tfrac{3}{5}})B\sum_{ab=1}^{8}d_{8ab}\omega_{ab}, \quad (3.21)$$

where the numerical factors are inserted for normalizing purposes in order to facilitate comparison with the $(3,\bar{3})+(\bar{3},3)$ case (they do not affect the value of m_χ^2).

The computation now proceeds in the same manner as before. In particular, expanding L_1 gives

$$L_1=2\sqrt{2}A(\langle\chi\rangle+\tilde{\chi})-\frac{6A}{\langle\chi\rangle}\left\{\left(\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{5}}\frac{B}{A}\right)\pi^2\right.$$
$$\left.+\left(\frac{1}{2\sqrt{2}}-\frac{1}{4\sqrt{5}}\frac{B}{A}\right)\bar{K}K+\left(\frac{1}{2\sqrt{2}}-\frac{1}{2\sqrt{5}}\frac{B}{A}\right)\eta^2\right\}+\cdots.$$

In addition to the Gell-Mann–Okubo mass formula, one obtains the mass relation

$$3m_\chi^2=2m_\pi^2+4m_K^2, \quad (3.22)$$

which implies for the dilaton mass $m_\chi\approx 590$ MeV. Finally, the ratio of $SU(3)$ breaking to $SU(3)\times SU(3)$ breaking is given by

$$\frac{B}{A}=-2\sqrt{2}\frac{m_K^2/m_\pi^2-1}{2m_K^2/m_\pi^2-1}\sqrt{(5/4)}\approx-1.25\sqrt{(5/4)}. \quad (3.23)$$

[12] K. J. Barnes and C. J. Isham, Nucl. Phys. B17, 267 (1970).