Summing Soft Pions*

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Matrix elements are calculated for high-energy reactions in which an unlimited number of soft pions are emitted or exchanged.

I. INTRODUCTION

THE use of current algebra[1] allows us to calculate matrix elements for the emission and absorption of any definite number of soft pions in an arbitrary hadron reaction. The same results can also be obtained by the use of chiral-invariant phenomenological Lagrangians in the tree approximation.[2] It has long been hoped that, by the use of current algebra or of chiral Lagrangians, we might transcend these simple soft-pion theorems, and learn to deal with problems involving *unlimited* numbers of soft pions.

Electrodynamics provides us with examples of the sort of calculation we might attempt. For example, we know how to express general soft-photon matrix elements in closed form, and we can sum up the emission rates to obtain cross sections for the inner bremsstrahlung of arbitrary numbers of real soft photons.[3] Also, by using the eikonal approximation, recently it has been possible to sum up an infinite series of diagrams involving the exchange of unlimited numbers of virtual soft photons.[4] We are thus presented with a challenge: Can we sum up emission rates for real soft pions,[5] and can we, taking the chiral Lagrangians seriously, also sum up the effects of virtual soft pions?

The obstacle to meeting this challenge has been a complex of ferocious technical difficulties not present in electrodynamics. Pion couplings are determined by noncommutative matrices[6] **X** and **T**, and pions couple nonlinearly[2] to themselves and other hadrons. The purpose of this article is to show how these difficulties may be overcome.

It turns out that the complications encountered in summing soft pions cancel each other, but *only* if the Lagrangian used is chiral invariant, if the eikonal approximation is employed, and if enough hadron resonances are included in the problem so that the pion-coupling matrices **X** and **T** form a representation of the chiral algebra.[6] These assumptions are explained in detail in Sec. II.

Sections III and IV are devoted to the solution of technical subproblems: the summation of soft-pion insertions on the external lines of a "hard" hadron process, and the summation of pion trees. By tying these pion trees onto the external hadron lines of a general process, two "physical" problems are then solved in Secs. V and VI: the effects of soft-virtual-pion exchange, and the emission of soft real pions in a coherent state.

A definite pattern seems to emerge from these calculations. Soft pions can be produced profusely in a high-energy reaction only if the S matrix violates algebraic chiral symmetry.[7] In compensation, the exchange of soft virtual pions suppresses all the terms in the S matrix which violate algebraic chirality; any term belonging to an $(\frac{1}{2}N, \frac{1}{2}N)$ representation[8] of $SU(2)\times SU(2)$ gets suppressed by a factor

$$\exp\left(-\frac{N(N+2)\Lambda^2}{4\pi^2 F_\pi{}^2}\right), \tag{1.1}$$

where Λ is the maximum momentum allowed for the virtual soft pions, and $F_\pi \simeq 190$ MeV is the usual pion

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[1] For a general review, see S. L. Adler and R. F. Dashen, *Current Algebra* (Benjamin, New York, 1969); S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 258; S. Fubini and G. Furlan (unpublished).

[2] S. Weinberg, Phys. Rev. Letters **18**, 188 (1966).

[3] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937). For a translation of this classic into modern notation, see S. Weinberg, *ibid*. **140**, B516 (1965); S. N. Gupta, *ibid*. **98**, 1502 (1955).

[4] H. Cheng and T. T. Wu, Phys. Rev. D **1**, 459 (1970); *ibid*. **1**, 467 (1970); and talk at the Third Topical Conference on High-Energy Collisions of Hadrons, Stony Brook, New York, 1969 (unpublished); H. D. I. Abarbanel and C. Itzykson, Phys. Rev. Letters **23**, 53 (1969); M. Lévy and J. Sucher (unpublished); J. Yellin (unpublished).

[5] Preliminary attempts in this direction have been made by R. Perrin [Phys. Rev. **162**, 1343 (1967)] and by L. N. Chang, C. Callen, S. Adler, and R. F. Dashen (unpublished).

[6] S. Weinberg, Phys. Rev. **177**, 2604 (1969).

[7] I distinguish here between dynamic chiral symmetry, which is an invariance of the Lagrangian, and algebraic chiral symmetry, which if valid would mean that the S matrix commutes with the pion-coupling matrix **X**. This distinction is discussed in detail in Ref. 6; also see S. Weinberg, *Contemporary Physics* (International Atomic Energy Agency, Vienna, 1969), p. 261. Presumably, dynamic chirality is broken only by terms in the Lagrangian of order $m_\pi{}^2$, while in contrast there is no *a priori* reason to expect **X** to commute with the S matrix.

[8] Terms in the S matrix are classified in various representations of $SU(2)\otimes SU(2)$ according to their commutation properties with the matrices **X** and **T** of Ref. 6.

decay amplitude. This suggests that *the Pomeranchuk poles or cuts which dominate elastic scattering at high energy may be algebraically chiral invariant, or nearly so.*[9]

Any such conclusion, based on the calculations in this paper, must be regarded as only tentative, since there are unsolved problems in justifying the application of these calculations to the real world. (Some of these problems are discussed at the end of Sec. II.) For the present, I would be content to have this work regarded as the exploration of a mathematical model of soft-pion dynamics. A sufficient reason for doing these calculations is to demonstrate the surprising fact that they can be done.

II. GENERAL ASSUMPTIONS

This article will deal with the emission and absorption or exchange of *soft* virtual or real pions in a reaction $\alpha \to \beta$ involving *hard* hadrons. (Some of these hadrons may be hard pions.) The treatment of this problem will rest on three key assumptions.

A. Chiral Dynamics

The demands of chiral symmetry will be met here by letting all soft pions be absorbed and emitted from the external hard-particle lines of the process $\alpha \to \beta$, using for this purpose the chiral-invariant interaction Lagrangian[2]

$$\mathcal{L}' = -F_\pi^{-1}[\mathbf{A}^\mu \cdot \mathbf{D}_\mu(\boldsymbol{\pi}) + \mathbf{V}^\mu \cdot \mathbf{E}_\mu(\boldsymbol{\pi})]. \tag{2.1}$$

Here $F_\pi \simeq 190$ MeV is the usual pion decay amplitude, $\mathbf{V}^\mu$ and $\mathbf{A}^\mu$ are phenomenological vector and axial-vector currents, given by sums of terms like $i\bar{N}\gamma^\mu \mathbf{t} N$ and $i(g_A/g_V)\bar{N}\gamma^\mu\gamma_5\mathbf{t}N$, and $\mathbf{D}_\mu(\boldsymbol{\pi})$ and $\mathbf{E}_\mu(\boldsymbol{\pi})$ are nonlinear functions[10] of the pion field:

$$\mathbf{D}_\mu(\boldsymbol{\pi}) = \partial_\mu \boldsymbol{\pi} + \frac{1}{\boldsymbol{\pi}^2}\left[1 - \frac{\sin[2F_\pi^{-1}(\boldsymbol{\pi}^2)^{1/2}]}{2F_\pi^{-1}(\boldsymbol{\pi}^2)^{1/2}}\right] \times [\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})], \tag{2.2}$$

$$\mathbf{E}_\mu(\boldsymbol{\pi}) = (F_\pi/\boldsymbol{\pi}^2)\sin^2[F_\pi^{-1}(\boldsymbol{\pi}^2)^{1/2}][\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}]. \tag{2.3}$$

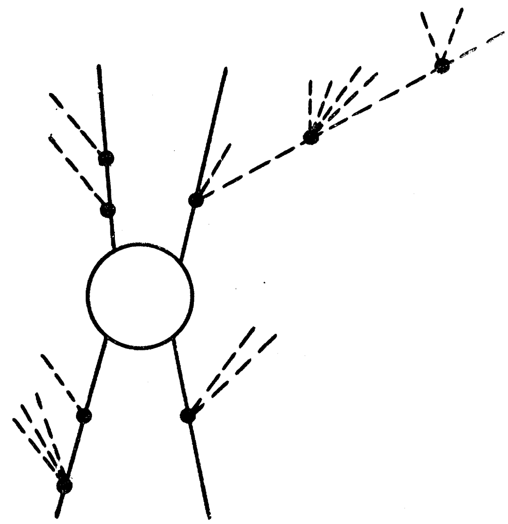

FIG. 1. Typical diagram for soft-pion emission. (Solid lines are hard hadrons; dashed lines are soft pions.)

Pion-pion interactions will be taken into account by using a chiral-invariant pion Lagrangian

$$\mathcal{L}_\pi = -\tfrac{1}{2}\mathbf{D}_\mu(\boldsymbol{\pi})\mathbf{D}^\mu(\boldsymbol{\pi}), \tag{2.4}$$

with the pion part of each diagram limited to certain classes of trees, to be specified later. (The pion mass is neglected throughout.) Typical diagrams for the emission or the exchange of soft pions in a hard-hadron scattering process are shown in Figs. 1 and 2. Evidently a miracle is needed to make the summation of these graphs possible; the particular assumptions and

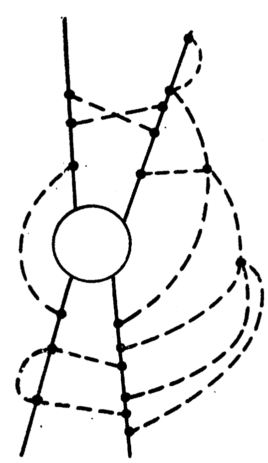

FIG. 2. Typical diagram for soft-pion exchange.

[9] Note the contrast between the asymptotic algebraic chiral symmetry suggested here, and the asymptotic chirality proposed by T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **18**, 701 (1967); *ibid* **19**, 470, (1967). The former is an approximate property of the S matrix which derives from the dynamical effects of soft virtual pions; the latter is a supposedly exact property of current propagators which can either be assumed directly or derived from assumptions about Schwinger terms or field algebra. Both ideas tend to confirm our underlying suspicion that chiral symmetry ought somehow to emerge as a good symmetry of the strong interactions at high energy, but they do so in different, and apparently unrelated, ways.

[10] In general, the "covariant derivatives" of the pion field $\boldsymbol{\pi}$ and any other field ψ are given, respectively, by $\mathbf{D}_\mu(\boldsymbol{\pi})$ and $[\partial_\mu + 2iF_\pi^{-1}\mathbf{E}_\mu\cdot\mathbf{t}]\psi$, where $\mathbf{t}$ is the isospin matrix of ψ. The specific form of the functions $\mathbf{D}_\mu$ and $\mathbf{E}_\mu$ depends in part on how we decide to define the pion field $\boldsymbol{\pi}$, although of course the answer to any physical question cannot depend on how this field is defined. In the present work it proves extraordinarily convenient to adopt the definition of the pion field given by S. Coleman, J. W. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1968); C. G. Callen, S. Coleman, J. W. Wess, and B. Zumino, *ibid.* **177**, 2247 (1968) (referred to below as the CCWZ pion field), rather than the definition used in Ref. 2. Formulas (2.2) and (2.3) give $\mathbf{D}_\mu$ and $\mathbf{E}_\mu$ as functions of the CCWZ field, and can be obtained by setting $f = v^{-1} = (\boldsymbol{\pi}^2)^{1/2}\cot[2F_\pi^{-1}(\boldsymbol{\pi}^2)^{1/2}]$ in Eqs. (4.5) and (4.19) of S. Weinberg, *ibid.* **166**, 1568 (1968). If any pion field other than that of CCWZ were used here, the calculations performed in Secs. III and IV would be much more complicated, but the complications would cancel in the final results obtained in Secs. V and VI.

approximations made here are chosen to ensure that this miracle will occur.

B. Soft Pions and Hard Hadrons

It is assumed here that in some Lorentz frame[11] all hard-hadron momenta p^μ are so large, and all soft-pion momenta q^μ are so small, that they satisfy the inequalities

$$|\mathbf{p}|\gg m\gg|\mathbf{q}|, \tag{2.5}$$

where m is a typical hadron mass. Then m and q^μ may be neglected in the pion-hadron vertices, and also in the numerators of the hadron propagators, which become just projection operators onto the positive-energy hadron spin states.[12] In consequence, a string of pion-hadron vertices sandwiched between propagator numerators may be calculated by just taking the product of mass-shell covariant matrix elements[13] of $\mathcal{L}'$ for each pion-hadron vertex. To calculate these matrix elements, we note that for $|\mathbf{p}|\to\infty$ the covariant matrix elements of the currents are[14]

$$(2\pi)^3(4p'^0p^0)^{1/2}\langle n'\lambda'\mathbf{p}|\mathbf{A}^\mu|n\lambda\mathbf{p}\rangle \to 4p^\mu\delta_{\lambda'\lambda}[\mathbf{X}(\lambda)]_{n'n}, \tag{2.6}$$

$$(2\pi)^3(4p'^0p^0)^{1/2}\langle n'\lambda'\mathbf{p}|\mathbf{V}^\mu|n\lambda\mathbf{p}\rangle \to 4p^\mu\delta_{\lambda'\lambda}[\mathbf{T}]_{n'n}, \tag{2.7}$$

where λ is the helicity, n is a discrete index running over particle types, $\mathbf{T}$ is the isospin matrix, and $\mathbf{X}(\lambda)$ is the coupling matrix defined in earlier work.[6] The matrix element we seek will thus involve a string of $\mathbf{X}$ and $\mathbf{T}$ matrices multiplied into the internal and final particle labels of a "core" matrix element, shown as a circle in Figs. 1 and 2. It is explicitly assumed that all the p^μ are so large and all the q^μ are so small that this core matrix element can be evaluated as an ordinary on-mass-shell S-matrix element for a reaction among hard hadrons with the same 3-momenta as those in the initial and final states of the original process.

There remain the denominators of the hard-hadron propagators. Consider a virtual hard hadron of mass m' and momentum $p^\mu+q^\mu$, where p^μ is the momentum of the hadron when it leaves the initial state or arrives in the final state, with $p^2=-m^2$. The denominator of its propagator is

$$D\equiv(p+q)^2+m'^2-i\epsilon=2p\cdot q+m'^2-m^2+q^2-i\epsilon.$$

It is assumed that the only virtual hadrons which contribute appreciably here are those whose mass m' satisfies the inequality

$$|m'^2-m^2|\ll 2|p\cdot q|. \tag{2.8}$$

Since $2|p\cdot q|$ is also much greater than $|q^2|$, the denominator takes the eikonal form[4]

$$D\simeq 2p\cdot q-i\epsilon.$$

The numerator of the propagator has already been incorporated into the matrix elements (2.6) and (2.7), so the rule will be to insert a factor

$$-i\Delta_{\text{eik}}=-iD^{-1}=-i(2p\cdot q-i\epsilon)^{-1} \tag{2.9}$$

for each virtual hard-hadron line.

C. Complete Chiral Multiplets

It is assumed that the matrices $\mathbf{X}(\lambda)$ and $\mathbf{T}$ satisfy the commutation relations of $SU(2)\times SU(2)$:

$$[T_a,T_b]=i\epsilon_{abc}T_c, \tag{2.10}$$

$$[T_a,X_b(\lambda)]=i\epsilon_{abc}X_c(\lambda), \tag{2.11}$$

$$[X_a(\lambda),X_b(\lambda)]=i\epsilon_{abc}T_c. \tag{2.12}$$

Equations (2.10) and (2.11) are consequences of isospin invariance alone and need no defense, but Eq. (2.12) does require some explanation. This relation is nothing but an algebraic formulation of the general Adler-Weisberger sum rule saturated by single-particle states,[6] and is therefore valid as long as we sum over enough states in calculating the matrix product. However, the single-particle states which can be included among the hard-hadron branches of graphs like Figs. 1 and 2 are restricted by Eq. (2.8) to a limited band Δm^2 of squared masses. There are two different ways that this condition can be met.

Semisoft pions. We can consider typical pion momenta $\mathbf{q}$, which, although small in the center-of-mass frame of the hard-hadron reaction $\alpha\to\beta$, are large (say of order 1 GeV or more) in the *rest frames* of all of the hard hadrons. Then $|p\cdot q|$ will be large, and Eq. (2.8) will allow us to include among the hard-hadron lines of our graphs enough resonant states to satisfy Eq. (2.12).

Very soft pions. If we consider typical pion momenta $\mathbf{q}$, which are small (say of order 200 MeV or less) in the center-of-mass frame of the reaction *and* in the rest frames of all of the hard hadrons, then $|p\cdot q|$ will be small, and (2.8) will compel us to restrict the hard-hadron branches in our graphs to states that are degenerate with the initial or final hadrons. In a few cases, the matrix $\mathbf{X}(\lambda)$ will still satisfy Eq. (2.12) approximately even when restricted to such a degenerate set of single-particle states, and then the calculation can go through. Such is the case for nucleons in the approximation that $|g_A/g_V|$ is unity instead of 1.2. [Here $\mathbf{X}(\pm\frac{1}{2})\simeq\pm\mathbf{T}$.] Such is definitely not the case

[11] In the usual case this Lorentz frame is the center-of-mass system, and the process of emitting soft real or virtual photons is known as pionization. See J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. D (to be published).

[12] Detailed discussions of this point for the case of spin $\frac{1}{2}$ are given in the papers of Ref. 4.

[13] In the sense used here, a covariant matrix element is just the usual matrix element, but with factors $(2\pi)^{-3/2}(2E)^{-1/2}$ omitted.

[14] See Ref. 6. It follows from Lorentz invariance alone that these matrix elements have the form given in Eqs. (2.6) and (2.7), with $\mathbf{X}(\lambda)$ and $\mathbf{T}(\lambda)$ unknown p-independent matrices. The identification of $\mathbf{X}$ and $\mathbf{T}$ can be achieved by using (2.6) and (2.7) to calculate the matrix elements of $\mathbf{A}^0+\hat{p}\cdot\mathbf{A}$ and $\mathbf{V}^0$.

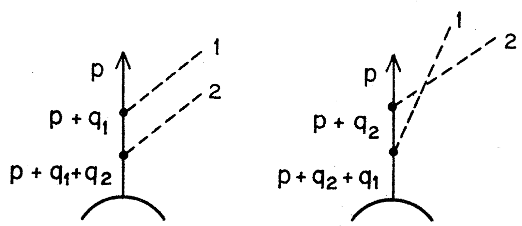

FIG. 3. Two diagrams that contribute to the matrix element for emission of two soft photons or pions.

for pions, kaons, ρ mesons, etc. Thus the results obtained in this article can be applied to the emission or exchange of very soft pions, but only if we restrict ourselves to reactions like nucleon-nucleon or nucleon-antinucleon scattering, and then only for $|g_A/g_V|=1$. The cases of semisoft and very soft pions will be treated together throughout this article.

In order to appreciate the crucial role played here by the commutation relations (2.10)–(2.12), compare the matrix elements for emission of two soft photons or soft pions from a particular outgoing hard-hadron line. For photons, the two diagrams of Fig. 3 yield a matrix element proportional to a factor

$$f_{2\gamma}=(p\cdot e_1)(p\cdot q_1)^{-1}(p\cdot e_2)(p\cdot[q_1+q_2])^{-1} \\ +(p\cdot e_2)(p\cdot q_2)^{-1}(p\cdot e_1)(p\cdot[q_2+q_1])^{-1}, \quad (2.13)$$

where $q_1{}^\mu$ and $q_2{}^\mu$ are the photon momenta and $e_1{}^\mu$ and $e_2{}^\mu$ are the corresponding polarization vectors. [Compare Eq. (2.9).] By combining denominators, we find that this sum factors into a product of contributions for each photon:

$$f_{2\gamma}=\left(\frac{p\cdot e_1}{p\cdot q_1}\right)\left(\frac{p\cdot e_2}{p\cdot q_2}\right). \quad (2.14)$$

This wonderful factorization occurs for any number of soft photons. Indeed, it is precisely this circumstance that allows us to sum up infinite series of graphs in electrodynamics, both in the Block-Nordsieck calculation[3] and in the eikonal approximation.[4]

In contrast, the matrix element for emission of two soft pions receives from the two diagrams of Fig. 3 a contribution proportional to the factor

$$f_{2\pi}=(p\cdot q_1)X_a(p\cdot q_1)^{-1}(p\cdot q_2)X_b(p\cdot[q_1+q_2])^{-1} \\ +(p\cdot q_2)X_b(p\cdot q_2)^{-1}(p\cdot q_2)X_a(p\cdot[q_2+q_1])^{-1},$$

where a and b are the isovector indices of pions 1 and 2. [The vertex factors $p\cdot q\mathbf{X}$ arise from the first term of Eq. (2.1).] Since X_a and X_b do not commute for $a\neq b$, it is impossible to combine these two terms into a factorized product like Eq. (2.14).

Fortunately, there is another diagram. The second term in Eq. (2.1) generates the diagram shown in Fig. 4, and contributes to the matrix element a term proportional to a factor

$$f_{2\pi}'=\tfrac{1}{2}i(p\cdot[q_1-q_2])\epsilon_{abc}T_c(p\cdot[q_1+q_2])^{-1}.$$

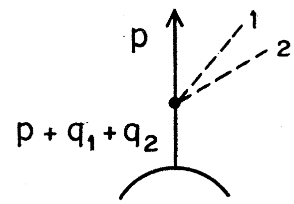

FIG. 4. Another diagram for emission of a pair of soft pions.

Using Eq. (2.12), we see that $f_{2\pi}'$ cancels the part of $f_{2\pi}$ antisymmetric in a and b, leaving a symmetrized product of individual pion factors

$$f_{2\pi}+f_{2\pi}'=\tfrac{1}{2}\{X_a,\,X_b\}. \quad (2.15)$$

It will be seen in Sec. III that *this factorization occurs for any number of soft pions*. Without Eq. (2.12), however, we would find an extra term in Eq. (2.15) which would be proportional to $p\cdot[q_1-q_2]/p\cdot[q+q_2]$, and which would blight any hope of summing soft pions.

Honesty compels me to point out that the dynamical framework laid out in the above assumptions is not very well grounded in current algebra. Chiral-invariant Lagrangians like that given here by Eqs. (2.1)–(2.4), when used in the tree approximation, are guaranteed[2] to reproduce the results of current algebra for the emission and absorption of *very soft real pions*. How do we know that such Lagrangians can be used in the tree approximation for the semisoft pions discussed above, or for virtual pions? The answer, if there is one, is reserved for future papers. In the meanwhile, the calculations performed here may be regarded as merely the summation of infinite sets of tree graphs in a Lagrangian model which happens to be consistent with current algebra.

III. HARD-HADRON PROCESSES IN EXTERNAL SOFT-PION FIELD

It is necessary, as a prelude to the calculations in Secs. V and VI, first to calculate the effect of an external soft-pion field $\boldsymbol{\pi}(x)$ on the matrix element for a general hard-hadron reaction:

$$\mathbf{p}_1\lambda_1n_1,\;\; \mathbf{p}_2\lambda_2'n_2,\;\ldots\rightarrow \mathbf{p}_1'\lambda_1'n_1',\;\; \mathbf{p}_2'\Lambda_2'n_2',\;\ldots. \quad (3.1)$$

(The λ's are helicities, and the n's label particle types.) Pion-pion interactions will be ignored in this section; they are discussed in Sec. IV and brought together with

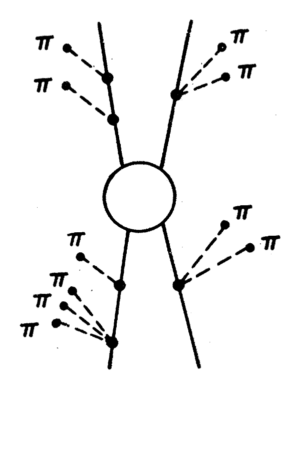

FIG. 5. Typical diagram of the class summed in Sec. III.

the results of this section in Secs. V and VI. The task here is to add up all diagrams of the form illustrated in Fig. 5, with pion-hadron vertices given by Eqs. (2.1)–(2.3), the field $\boldsymbol{\pi}(x)$ being now understood as a prescribed c-number function.

The answer is amazingly simple. The effect of the external pion field is just that the factor $(2\pi)^4\delta^4(q)$, which ensures momentum conservation in the absence of external fields, is replaced with a matrix

$$\Omega(q)=\int d^4x\, e^{-iq\cdot x}\exp[2iF_\pi^{-1}\Delta\mathbf{X}\cdot\boldsymbol{\pi}(x)]. \quad (3.2)$$

Here q^μ is the net momentum loss

$$q^\mu\equiv(p_1^\mu+p_2^\mu+\cdots)-(p_1'^\mu+p_2'^\mu+\cdots) \quad (3.3)$$

and $\Delta\mathbf{X}$ is a "net chiral boost"

$$\Delta\mathbf{X}=\mathbf{X}_1'(\lambda_1')\oplus\mathbf{X}_2'(\lambda_2')\oplus\cdots\oplus-\mathbf{X}_1(\lambda_1)\oplus-\mathbf{X}_2(\lambda_2)\oplus\cdots, \quad (3.4)$$

it being understood that all primed X matrices act *from the left* on the n-labels on the S matrix of the respective hadrons in the final state, while all unprimed X matrices act *from the right* on the n-labels on the S matrix of the respective hadrons in the initial state. That is,

$$\begin{aligned}(\Delta\mathbf{X}\,S)_{n_1'\lambda_1',n_2'\lambda_2',\cdots;n_1\lambda_1,n_2\lambda_2,\cdots}\equiv\sum_n[(\mathbf{X}(\lambda_1'))_{n_1'n}S_{n\lambda_1',n_2'\lambda_2',\cdots;n_1\lambda_1,n_2\lambda_2,\cdots}\\+(\mathbf{X}(\lambda_2'))_{n_2'n}S_{n_1'\lambda_1',n\lambda_2',\cdots;n_1\lambda_1,n_2\lambda_2,\cdots}\\+\cdots-S_{n_1'\lambda_1',n_2'\lambda_2',\cdots;n\lambda_1,n_2\lambda_2,\cdots}(\mathbf{X}(\lambda_1))_{nn_1}\\-S_{n_1'\lambda_1',n_2'\lambda_2',\cdots;n_1\lambda_1,n\lambda_2,\cdots}(\mathbf{X}(\lambda_2))_{nn_2}-\cdots]. \quad (3.5)\end{aligned}$$

(Here S can be any matrix, not just the S matrix.)

To derive this result, it will be convenient to work in coordinate space rather than momentum space. The propagator for a hard-hadron line running upwards from x_1^μ to x_2^μ (i.e., with x_1^μ closer to the initial state or x_2^μ closer to the final state) is given by the Fourier transform of Eq. (2.9):

$$\begin{aligned}-i\Delta_{\rm eik}(x_2-x_1;p)\equiv-i(2\pi)^{-4}\int d^4q\,(2p\cdot q-i\epsilon)^{-1}\\\times\exp[i(p+q)\cdot(x_2-x_1)]\\=\int_0^\infty d\tau\,\delta^4(x_2-x_1-2p\tau)\exp(2ip^2\tau),\end{aligned}$$

where p^μ is the momentum of the hadron line when it starts in the initial state or ends in the final state. For $|\mathbf{p}|\to\infty$ with p^2 fixed, this gives

$$-i\Delta_{\rm eik}(x_2-x_1;p)=(2|\mathbf{p}|)^{-1}\delta^3(\mathbf{u}_2-\mathbf{u}_1)\theta(v_2-v_1), \quad (3.6)$$

where $\mathbf{u}$ and v are a convenient set of auxiliary space-time coordinates,

$$\mathbf{u}\equiv\mathbf{x}-\hat{\mathbf{p}}x^0,\quad v\equiv\tfrac{1}{2}(\hat{\mathbf{p}}\cdot\mathbf{x}+x^0). \quad (3.7)$$

The vertex insertions are given by the covariant matrix elements of the interaction Lagrangian (2.1). Using (2.6) and (2.7), this gives for $|\mathbf{p}|\to\infty$ the vertex factors

$$i(2\pi)^3(4p'^0p^0)^{1/2}\langle n'\lambda'\mathbf{p}|\mathcal{L}'(x)|n\lambda\mathbf{p}\rangle\to2i|\mathbf{p}|\delta_{\lambda'\lambda}\Gamma_{n'n}(x;\lambda), \quad (3.8)$$

where Γ is the matrix

$$\Gamma(x;\lambda)=-2F_\pi^{-1}[\mathbf{X}(\lambda)\cdot\mathbf{D}_v(x)+\mathbf{T}\cdot\mathbf{E}_v(x)], \quad (3.9)$$

with

$$\mathbf{D}_v\equiv\frac{\partial\boldsymbol{\pi}}{\partial v}+\frac{1}{\boldsymbol{\pi}^2}\left[1-\frac{\sin[2F_\pi^{-1}(\boldsymbol{\pi}^2)^{1/2}]}{2F_\pi^{-1}(\boldsymbol{\pi}^2)^{1/2}}\right]\times\left[\boldsymbol{\pi}\times\left(\boldsymbol{\pi}\times\frac{\partial\boldsymbol{\pi}}{\partial v}\right)\right], \quad (3.10)$$

$$\mathbf{E}_v=\left(\frac{F_\pi}{\boldsymbol{\pi}^2}\right)\sin^2[F_\pi^{-1}(\boldsymbol{\pi}^{-2})^{1/2}]\left(\boldsymbol{\pi}\times\frac{\partial\boldsymbol{\pi}}{\partial v}\right). \quad (3.11)$$

The effect of the pion field on an outgoing hard-hadron line with final momentum p^μ and helicity λ is that the final "wave function" $e^{ip\cdot x}$ in the S matrix is replaced with

$$e^{ip\cdot x}F(x;\mathbf{p},\lambda), \quad (3.12)$$

where F is a matrix, given by (3.8) as

$$\begin{aligned}F(x;\mathbf{p},\lambda)=1+\sum_{n=1}^\infty\int d^4x_1\cdots d^4x_n[2i|\mathbf{p}|\Gamma(x_1;\lambda)]\\\times[-i\Delta_{\rm eik}(x_1-x_2;p)][2i|\mathbf{p}|\Gamma(x_2;\lambda)]\cdots\\\times[2i|\mathbf{p}|\Gamma(x_1;\lambda)][-i\Delta_{\rm eik}(x_n-x;p)].\end{aligned}$$

Using (3.6) for $\Delta_{\rm eik}$, we find that all factors $2|\mathbf{p}|$ cancel, and in terms of the auxiliary coordinates (3.7), we now have

$$\begin{aligned}F(\mathbf{u},v;\mathbf{p},\lambda)=1+\sum_{n=1}^\infty i^n\int_v^\infty dv_n\int_{v_n}^\infty dv_{n-1}\cdots\\\times\int_{v_2}^\infty dv_1\Gamma(\mathbf{u},v_1;\lambda)\Gamma(\mathbf{u},v_2;\lambda)\cdots\Gamma(\mathbf{u},v_n;\lambda). \quad (3.13)\end{aligned}$$

In the same way, the effect of the external pion field on an incoming hard-hadron line with initial momentum p^μ and helicity λ is that the initial "wave function" $e^{-ip\cdot x}$ in the S matrix is replaced with

$$e^{-ip\cdot x}I(x;p,\lambda), \quad (3.14)$$

where I is the matrix

$$I(\mathbf{u},v;\,p,\lambda)=1+\sum_{n=1}^{\infty} i^n \int_{-\infty}^{v} dv_1 \int_{-\infty}^{v_1} dv_2\cdots \times\int_{-\infty}^{v_{n-1}} dv_n \Gamma(\mathbf{u},v_1;\,\lambda)\Gamma(\mathbf{u},v_2;\,\lambda)\cdots\Gamma(\mathbf{u},v_n;\,\lambda)\,. \quad (3.15)$$

To calculate the matrices F and I, we note that they obey the differential equations

$$\frac{\partial}{\partial v}F(\mathbf{u},v;\,\mathbf{p},\lambda)=-iF(\mathbf{u},v;\,\mathbf{p},\lambda)\Gamma(\mathbf{u},v;\,\lambda)\,, \quad (3.16)$$

$$\frac{\partial}{\partial v}I(\mathbf{u},v;\,\mathbf{p},\lambda)=i\Gamma(\mathbf{u},v;\,\lambda)I(\mathbf{u},v;\,\mathbf{p},\lambda)\,, \quad (3.17)$$

with the initial conditions

$$F(\mathbf{u},\infty\,;\,\mathbf{p},\lambda)=I(\mathbf{u},\,-\infty\,;\,\mathbf{p},\lambda)=1\,. \quad (3.18)$$

Also recall that the functions E_v and $\mathbf{D}_v$ appearing in Γ are defined by CCWZ[10] so that

$$\Gamma(\mathbf{u},v;\,\lambda)=i\exp[-2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(\mathbf{u},v)] \times\frac{\partial}{\partial v}\exp[2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(\mathbf{u},v)]\,, \quad (3.19)$$

provided that the matrices $\mathbf{X}(\lambda)$ *and* $\mathbf{T}$ *obey the chiral commutation relations* (2.10)–(2.12). [It is at just this point that these commutation relations, together with the dynamic chiral symmetry of the Lagrangian (2.1), play their crucial roles.] The differential equations (3.16) and (3.17) may thus be written

$$\frac{\partial}{\partial v}\{F(\mathbf{u},v;\,\mathbf{p},\lambda)\exp[-2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(\mathbf{u},v)]\}=0\,, \quad (3.20)$$

$$\frac{\partial}{\partial v}\{\exp[2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(\mathbf{u},v)]I(\mathbf{u},v;\,\mathbf{p},\lambda)\}=0\,. \quad (3.21)$$

We are tacitly assuming that $\boldsymbol{\pi}$ vanishes for $v\to\pm\infty$ with $\mathbf{u}$ fixed [otherwise the integrals in (3.13) and (3.15) make no sense], so the solutions which satisfy the initial conditions (3.18) are

$$F(\mathbf{u},v;\,\mathbf{p},\lambda)=\exp[2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(\mathbf{u},v)]\,,$$
$$I(\mathbf{u},v;\,\mathbf{p},\lambda)=\exp[-2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(\mathbf{u},v)]\,.$$

Note that F and I no longer depend on p^μ, either explicitly or through the definitions of the auxiliary coordinates $\mathbf{u}$ and v, so we may drop the label p and write simply

$$F(x;\,\lambda)=\exp[2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(x)]\,, \quad (3.22)$$

$$I(x;\,\lambda)=\exp[-2iF_\pi{}^{-1}\mathbf{X}(\lambda)\cdot\boldsymbol{\pi}(x)]\,. \quad (3.23)$$

The S matrix for a general hard-hadron reaction will now be

$$S(p_1\lambda_1,p_2\lambda_2,\ldots\to p_1'\lambda_1',p_2'\lambda_2',\ldots) =\int d^4x_1'd^4x_2'\cdots d^4x_1d^4x_2\cdots \times[e^{ip_1'\cdot x_1'}F_1'(x_1';\,\lambda_1')\otimes e^{ip_2'\cdot x_2'}F_2'(x_2';\,\lambda_2')\otimes\cdots] \times M_0(x_1\lambda_1,x_2\lambda_2,\ldots\to x_1'\lambda_1',x_2'\lambda_2',\ldots) \times[e^{-ip_1\cdot x_1}I_1(x_1,\lambda_1)\otimes e^{-ip_2\cdot x_2}I_2(x_2,\lambda_2)\otimes\cdots]\,, \quad (3.24)$$

where M_0 is the coordinate-space matrix element in the absence of the pion field. (Labels are included on the F and I matrices to indicate on which particle labels the X matrices act.) Since M_0 is translation invariant, it is convenient to introduce coordinates x^μ, $\xi_n{}^\mu$, and $\xi_n{}'^\mu$:

$$x_n{}^\mu=x^\mu+\xi_n{}^\mu\,,\quad x_n{}'=x^\mu+\xi_n{}'^\mu\,,$$

with the $\xi_n{}^\mu$ and $\xi_n{}'^\mu$ constrained so that

$$\xi_1{}^\mu+\xi_2{}^\mu+\cdots+\xi_1{}'^\mu+\xi_2{}'^\mu+\cdots=0.$$

Then M_0 depends only on the $\xi_n{}^\mu$ and $\xi_n{}'^\mu$, not on x^μ, and (3.24) therefore reads

$$S(p_1\lambda_1,p_2\lambda_2,\ldots\to p_1'\lambda_1',p_2'\lambda_2',\ldots) =\int d^4x d^4\xi_1'd^4\xi_2'\cdots d^4\xi_1d^4\xi_2\cdots \times\delta^4(\langle\xi\rangle)e^{-iq\cdot x}[e^{ip_1'\cdot\xi_1'}F_1'(x+\xi_1';\,\lambda_1') \otimes e^{ip_2'\cdot\xi_2'}F_2'(x+\xi_2';\,\lambda_2')\otimes\cdots] \times M_0(\xi_1\lambda_1,\xi_2\lambda_2,\ldots\to\xi_1'\lambda_1',\xi_2'\lambda_2',\ldots) \times[e^{-ip_1\cdot\xi_1}I_1(x+\xi_1;\,\lambda_1) \otimes e^{-ip_2\cdot\xi_2}I_2(x+\xi_2;\,\lambda_2)\otimes\cdots]\,, \quad (3.25)$$

where $\langle\xi^\mu\rangle$ is the average of all $\xi_n{}^\mu$ and $\xi_n{}'^\mu$, and q^μ is the momentum transferred to the pion field

$$q^\mu\equiv(p_1{}^\mu+p_2{}^\mu+\cdots)-(p_1{}'^\mu+p_2{}'^\mu+\cdots)\,.$$

The pion field is supposed to be "soft," which means that it varies very little over the support of M_0. Hence the arguments $x^\mu+\xi_n{}'^\mu$ and $x^\mu+\xi_n{}^\mu$ in the F and I matrices may all be replaced with x^μ, and (3.25) becomes

$$S(p_1\lambda_1,p_2\lambda_2,\ldots\to p_1'\lambda_1',p_2'\lambda_2',\ldots) =\int d^4x\,e^{-iq\cdot x}[F_1'(x;\,\lambda_1')\otimes F_2'(x;\,\lambda_2')\otimes\cdots] \times T_0(p_1\lambda_1,p_2\lambda_2,\ldots\to p_1'\lambda_1',p_2'\lambda_2',\ldots) \times[I_1(x;\,\lambda_1)\otimes I_2(x;\,\lambda_2)\otimes\cdots]\,, \quad (3.26)$$

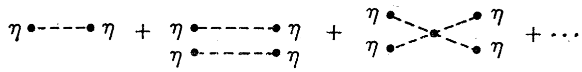

FIG. 6. Series of tree diagrams for the functional $\Pi[\boldsymbol{\eta}]$.

where T_0 is the momentum-space matrix element for zero pion field:

$$T_0(p_1\lambda_1,p_2\lambda_2,\ldots \to p_1'\lambda_1',p_2'\lambda_2',\ldots)$$
$$\equiv \int d^4\xi_1' d^4\xi_2' \cdots d^4\xi_1 d^4\xi_2 \cdots \delta^4(\langle\xi\rangle)$$
$$\times e^{ip_1'\cdot\xi_1'} e^{ip_2'\cdot\xi_2'} \cdots e^{-ip_1\cdot\xi_1} e^{-ip_2\cdot\xi_2}\cdots$$
$$\times M_0(\xi_1\lambda_1,\xi_2\lambda_2,\ldots \to \xi_1'\lambda_1',\xi_2'\lambda_2',\ldots). \quad (3.27)$$

Note that for zero pion field, F and I are unity, and (3.26) gives an S matrix

$$S_0=(2\pi)^4\delta^4(q)T_0, \quad (3.28)$$

as of course it must.

At this point we can condense our notation by introducing the "net chiral-boost generator" $\Delta\mathbf{X}$ defined by Eq. (3.4) or (3.5). Since X matrices for different hadrons commute, the exponentials (3.22) and (3.23) may be combined in (3.26) into a single exponential. Equation (3.26) then reads

$$S=\int d^4x\, e^{-iq\cdot x} \exp[2iF_\pi^{-1}\Delta\mathbf{X}\cdot\boldsymbol{\pi}(x)]T_0, \quad (3.29)$$

as was to be proven.

If the original hard-hadron process conserves chirality in the ordinary algebraic sense,[7] then $\Delta\mathbf{X}T_0$ vanishes [see (3.5)] and the S matrix (3.29) becomes equal to the S matrix (3.28) for zero pion field. Thus *hard hadrons undergo no scattering in an external soft-pion field unless they participate in a reaction in which algebraic chiral invariance is violated.* In particular, a single hard hadron which does not participate in any reaction at all will not in the eikonal approximation be able to transfer momentum to or from an external soft-pion field. None of these conclusions would be altered if we took the pion-pion interactions into account by attaching trees of soft pions at all pion-hadron vertices, as in Figs. 1 and 2; so we can further conclude that *soft pions can neither be absorbed, emitted, scattered, nor exchanged by hard hadrons unless these hadrons participate in some reaction which violates algebraic chiral symmetry.*[7]

IV. PION TREES

Now that we know the effect of soft-pion insertions on the external lines of a hard-hadron process, the next step is to calculate the sums of pion trees which connect these pion vertices.

The sum of all pion trees (connected or not), with n external pion lines, labeled with isovector indices $a_1\cdots a_n$ and space-time coordinates $x_1\cdots x_n$, will be denoted $\Pi_{a_1\cdots a_n}{}^{(n)}(x_1\cdots x_n)$. For example,

$$\Pi_{ab}{}^{(2)}(x,y)=-i\Delta_\pi(x-y)\delta_{ab}, \quad (4.1)$$

where Δ_π is the free-pion propagator

$$\Delta_\pi(x-y)\equiv(2\pi)^{-4}\int d^4q\,(q^2-i\epsilon)^{-1}e^{iq\cdot(x-y)}. \quad (4.2)$$

Instead of dealing directly with the individual $\Pi^{(n)}$, it is both easier and more useful to work with the generating functional:

$$\Pi[\boldsymbol{\eta}]\equiv 1+\sum_{n=1}^{\infty}\frac{i^n}{n!}\int d^4x_1\cdots d^4x_n$$
$$\times\Pi_{a_1\cdots a_n}{}^{(n)}(x_1\cdots x_n)\eta_{a_1}(x_1)\cdots\eta_{a_n}(x_n). \quad (4.3)$$

This quantity is just the sum of all vacuum diagrams (in the tree approximation) for a Lagrangian with an external pion current $\boldsymbol{\eta}(x)$:

$$\mathcal{L}(x)=\mathcal{L}_\pi(x)+\boldsymbol{\eta}(x)\cdot\boldsymbol{\pi}(x), \quad (4.4)$$

where $\mathcal{L}_\pi$ is given by Eqs. (2.4) and (2.2) (see Fig. 6).

A prescription for calculating the functional $\Pi[\boldsymbol{\eta}]$ is provided by the work of Nambu.[15] First, it is necessary to calculate a c-number function $\boldsymbol{\phi}(x)$ by solving the nonlinear field equation

$$\Box^2\boldsymbol{\phi}(x)=-\boldsymbol{\eta}(x)-\mathbf{J}(\boldsymbol{\phi}(x),\partial_\mu\boldsymbol{\phi}(x),\partial_\mu\partial_\nu\boldsymbol{\phi}(x)), \quad (4.5)$$

which follows from the Lagrangian (4.4), using causal boundary conditions. (That is, $\boldsymbol{\phi}$ should contain only positive frequencies for $t\to+\infty$ and only negative frequencies for $t\to-\infty$.) Here $\mathbf{J}$ is the pion current arising from pion-pion interactions,

$$J_a(\boldsymbol{\pi},\partial_\mu\boldsymbol{\pi},\partial_\mu\partial_\nu\boldsymbol{\pi})\equiv\frac{\partial\mathcal{L}_\pi'(\boldsymbol{\pi},\partial_\mu\boldsymbol{\pi})}{\partial\pi_a}-\partial_\mu\frac{\partial\mathcal{L}_\pi'(\boldsymbol{\pi},\partial_\mu\boldsymbol{\pi})}{\partial(\partial_\mu\pi_a)}, \quad (4.6)$$

with $\mathcal{L}_\pi'$ the interaction part of $\mathcal{L}_\pi$,

$$\mathcal{L}_\pi\equiv-\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi})+\mathcal{L}_\pi'. \quad (4.7)$$

Equations (2.2) and (2.4) give

$$\mathcal{L}_\pi'=d(\boldsymbol{\pi}^2)[\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}]\cdot[\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi}]$$
$$-\tfrac{1}{2}d^2(\boldsymbol{\pi}^2)[\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})]\cdot[\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})], \quad (4.8)$$

$$\mathbf{J}=\partial_\mu[d(\boldsymbol{\pi}^2)\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})]+d(\boldsymbol{\pi}^2)\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\Box^2\boldsymbol{\pi})$$
$$+d(\boldsymbol{\pi}^2)\boldsymbol{\pi}\times\{\boldsymbol{\pi}\times\partial_\mu[d(\boldsymbol{\pi}^2)\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})]\}$$
$$-2F_\pi^{-1}e(\boldsymbol{\pi}^2)(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})\times\{\partial_\mu\boldsymbol{\pi}+d(\boldsymbol{\pi}^2)$$
$$\times[\boldsymbol{\pi}\times(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})]\}, \quad (4.9)$$

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[15] Y. Nambu, Phys. Letters **26B**, 626 (1968). This article is primarily concerned with the derivation of the tree approximation as a semiclassical limit, so a certain amount of work must be done to adapt its results to the present context. Also see D. G. Boulware and L. S. Brown, Phys. Rev. **172**, 1628 (1968); and L. V. Prokhorov, *ibid.* **183**, 1515 (1969).

where

$$d(z)\equiv\frac{1}{z}\left[1-\frac{\sin(2F_\pi^{-1}z^{1/2})}{2F_\pi^{-1}z^{1/2}}\right], \qquad (4.10)$$

$$e(z)\equiv(F_\pi/z)\sin^2(F_\pi^{-1}z^{1/2}). \qquad (4.11)$$

Once $\boldsymbol{\phi}$ is known, the functional $\Pi(\boldsymbol{\eta})$ can be calculated from the formula

$$\Pi[\boldsymbol{\eta}]=\exp\Bigg[\tfrac{1}{2}i\int\boldsymbol{\eta}(x)\cdot\boldsymbol{\eta}(y)\Delta_\pi(x-y)\,d^4x d^4y$$

$$+i\int\mathfrak{L}'(\boldsymbol{\phi}(x),\partial_\mu\boldsymbol{\phi}(x))\,d^4x$$

$$-\tfrac{1}{2}\int d^4xd^4y\,\mathbf{J}(\boldsymbol{\phi}(x),\partial_\mu\boldsymbol{\phi}(x),\partial_\mu\partial_\nu\boldsymbol{\phi}(x))$$

$$\cdot\mathbf{J}(\boldsymbol{\phi}(y),\partial_\mu\boldsymbol{\phi}(y),\partial_\mu\partial_\nu\boldsymbol{\phi}(y))\Delta_\pi(x-y)\,d^4xd^4y\Bigg]. \qquad (4.12)$$

It will turn out in Sec. V that the calculation of $\Pi[\boldsymbol{\eta}]$ is nowhere near so formidable as it looks.

In order to prove Eq. (4.12), it must first be noted that $\phi_a(x)$ is nothing but the sum of all *connected* tree graphs with one external pion line, labeled with isovector index a and coordinates x^μ (see Fig. 7). This interpretation becomes obvious if we write Eq. (4.5), with its boundary conditions, as an integral equation:

$$\boldsymbol{\phi}(x)=\int d^4y\,\Delta_\pi(x-y)[\boldsymbol{\eta}(y)+\mathbf{J}(y)], \qquad (4.13)$$

where, for brevity, we write

$$\mathbf{J}(y)\equiv\mathbf{J}(\boldsymbol{\phi}(y),\partial_\mu\boldsymbol{\phi}(y),\partial_\mu\partial_\nu\boldsymbol{\phi}(y)). \qquad (4.14)$$

The first term in the integrand generates the first term in Fig. 7, while the second term generates all terms of higher order in $\boldsymbol{\eta}$.

Next, note that since $\Pi[\boldsymbol{\eta}]$ is the sum of all vacuum tree graphs, its variational derivative with respect to $i\eta_a(x)$ is the sum of all tree graphs, connected or not, with one external pion line. The disconnected contributions to this latter sum are just vacuum graphs, so they add up to a factor $\Pi[\boldsymbol{\eta}]$, which just multiplies the connected part $\phi_a(x)$ of this sum. Thus $\Pi[\boldsymbol{\eta}]$ obeys the functional differential equation

$$\frac{\delta\Pi[\eta]}{\delta\eta_a(x)}=i\phi_a(x)\Pi[\boldsymbol{\eta}], \qquad (4.15)$$

which obviously determined $\Pi[\eta]$ uniquely as a power series in $\boldsymbol{\eta}$.

It now only remains to show that Eq. (4.12) does satisfy the functional equation (4.15). Direct calculation gives

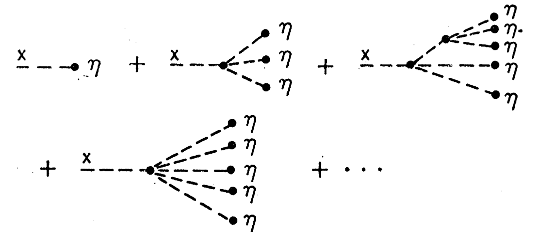

FIG. 7. Series of tree diagrams for $\boldsymbol{\phi}(x;\boldsymbol{\eta})$.

$$\frac{\delta\Pi[\eta]}{\delta\eta_a(x)}=\left[i\int d^4y\,\Delta_\pi(x-y)\eta_a(y)+i\int d^4y\,J_b(y)\frac{\delta\phi_b(y)}{\delta\eta_a(x)}\right.$$

$$\left.-i\int d^4yd^4z\,J_b(y)\frac{\delta J_b(z)}{\delta\eta_a(x)}\Delta_\pi(y-z)\right]\Pi[\eta]. \qquad (4.16)$$

By taking the variational derivative of Eq. (4.13) with respect to $\boldsymbol{\eta}$, we find

$$\frac{\delta\phi_b(y)}{\delta\eta_a(x)}=\Delta_\pi(x-y)\delta_{ab}+\int d^4z\,\Delta_\pi(y-z)\frac{\delta J_b(z)}{\delta\eta_a(x)}. \qquad (4.17)$$

Using (4.17) in (4.16), we find that the terms involving $\delta J_b/\delta\eta_a$ cancel, and we are left with

$$\frac{\delta\Pi[\eta]}{\delta\eta_a(x)}=i\int d^4y\,\Delta_\pi(x-y)[\eta_a(y)+J_a(y)]\Pi[\eta],$$

thus verifying that the formula (4.12) for $\Pi[\boldsymbol{\eta}]$ does obey the functional equation (4.15).

V. SOFT-PION EXCHANGE IN HARD-HADRON PROCESS

We now turn to our first "physical" problem, the summation of diagrams in which soft pions are exchanged among the external lines of a hard-hadron reaction. A question immediately confronts us: What shall we do with the pion-pion interactions? Current algebra is not much help with virtual pions, so an element of guesswork necessarily enters the calculation at this point. We have however one guide which seems highly reasonable: The graphs included should be selected according to a topological criterion, such that the final answer will not depend upon how the pion field is defined. For instance, it would not do to arbitrarily discard all pion-pion interactions, because the result will then depend on whether we take the pion field to be the CCWZ field $\boldsymbol{\pi}(x)$ used in the above sections, or some arbitrary function of $\boldsymbol{\pi}(x)$. The simplest selection criterion which gives a result independent of the definition of the pion field is to *include all graphs in which the purely pionic part is a tree*, connected or not. In particular, this includes graphs like Fig. 2. Note that the over-all graph is not a tree, because the pion tree is attached to hard-hadron lines, but there are no purely pionic loops.

Although not strictly necessary, it will be convenient to go on using the CCWZ definition of the pion field. The pion trees are then generated by the pion Lagrangian given by Eqs. (2.2) and (2.4). As in Sec. IV, define $\Pi_{a_1\cdots a_n}{}^{(n)}(x_1\cdots x_n)$ as the sum of all pion trees with n external pion lines. When the tips of these pion lines are tied to the external lines of a hard-hadron reaction, we obtain a matrix element

$$S=\int d^4x \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2i}{F_\pi}\right)^n (\Delta X)_{a_1}\cdots(\Delta X)_{a_n}T_0 \times\Pi_{a_1\cdots a_n}{}^{(n)}(x,x,\ldots)e^{-iq\cdot x} \quad (5.1)$$

[see Eq. (3.29)]. But $\Pi^{(n)}$ is translation invariant, so

$$\Pi^{(n)}(x,x,\ldots)=\Pi^{(n)}(0,0,\ldots),$$

and therefore (5.1) gives

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2i}{F_\pi}\right)^n \times\Pi_{a_1\cdots a_n}{}^{(n)}(0,0,\ldots,0)(\Delta X)_{a_1}\cdots(\Delta X)_{a_n}S_0, \quad (5.2)$$

where S_0 is the S matrix (3.28) for the original hard-hadron reaction, without soft-pion corrections. Comparing with the definition (4.3) of the functional $\Pi[\boldsymbol{\eta}]$, we see that

$$S=\Pi[(2/F_\pi)\Delta\mathbf{X}\delta^4(x)]S_0. \quad (5.3)$$

To complete the calculation it is only necessary to compute the function

$$\Upsilon(\mathbf{e})\equiv\Pi[\mathbf{e}\delta^4(x)], \quad (5.4)$$

where $\mathbf{e}$ is an arbitrary constant 3-vector. (Isospin invariance ensures that Υ actually depends only upon $\mathbf{e}^2$.) Once Υ is known, the effect of soft-pion exchange on the S matrix can be determined from the formula

$$S=\Upsilon(2F_\pi^{-1}\Delta\mathbf{X})S_0. \quad (5.5)$$

According to the results of Sec. IV, the first step in calculating Υ is to solve the differential equation (4.5) with causal boundary conditions. In our present problem, the external pion current is

$$\boldsymbol{\eta}(x)=\mathbf{e}\delta^4(x), \quad (5.6)$$

and (4.5) reads

$$\Box^2\boldsymbol{\phi}(x)=-\mathbf{e}\delta^4(x)-\mathbf{J}(\boldsymbol{\phi}(x),\partial_\mu\boldsymbol{\phi}(x),\partial_\mu\partial_\nu\boldsymbol{\phi}(x)).$$

An obvious solution is

$$\boldsymbol{\phi}(x)=\mathbf{e}\Delta_\pi(x), \quad (5.7)$$

because Eq. (4.9) gives for the CCWZ field

$$\mathbf{J}(\mathbf{e}\Delta_\pi,\mathbf{e}\partial_\mu\Delta_\pi,\mathbf{e}\partial_\mu\partial_\nu\Delta_\pi)=0. \quad (5.8)$$

Also, Eq. (4.8) gives, in this case,

$$\mathcal{L}_\pi'(\mathbf{e}\Delta_\pi,\mathbf{e}\partial_\mu\Delta_\pi)=0. \quad (5.9)$$

Using Eqs. (5.7)–(5.9) in Eq. (4.12), we see that

$$\Upsilon(\mathbf{e})=\exp[\tfrac{1}{2}i\mathbf{e}\Delta_\pi(0)]. \quad (5.10)$$

That is, *pion-pion interactions make no contribution here.* Using Eq. (5.10) in Eq. (5.5) yields our general answer

$$S=\exp[2iF_\pi^{-2}(\Delta\mathbf{X})^2\Delta_\pi(0)]S_0. \quad (5.11)$$

The coefficient $\Delta_\pi(0)$ is, of course, infinite. Here, as in the Bloch–Nordsieck calculation,[3] it is necessary to set some upper limit Λ on the momenta of pions which qualify as "soft." The propagator is then

$$\Delta_\pi(x)=\frac{i}{(2\pi)^3}\int\frac{d^3q}{2|\mathbf{q}|}\exp[i\mathbf{q}\cdot\mathbf{x}-i|\mathbf{q}||x^0|]\theta(\Lambda-|\mathbf{q}|),$$

and so

$$\Delta_\pi(0)=i\Lambda^2/8\pi^2. \quad (5.12)$$

Putting this into (5.11) gives the effect of virtual soft pion-exchange as[16]

$$S=\exp\left[-\frac{\Lambda^2}{4\pi^2F_\pi^2}(\Delta\mathbf{X})^2\right]S_0. \quad (5.13)$$

(Note that S_0 depends on Λ, because it includes the effects of all virtual pions with momenta greater than Λ.)

If the uncorrected S matrix S_0 is algebraically chiral invariant,[7] then, as already noted in Sec. III, $\Delta\mathbf{X}S_0$ vanishes, and the virtual soft-pion exchange has no effect on the S matrix. More generally, *the exponential factor in* (5.13) *tends to suppress any parts of the S matrix which do not commute with* $\mathbf{X}$. To make this suppression explicit, suppose we expand S_0 in terms which belong to irreducible representations[8] of the chiral algebra. Since S_0 does commute with isospin, this sum can receive contributions only from terms $S_0{}^N$ which transform like the isoscalar part of the representation $(\frac{1}{2}N,\frac{1}{2}N)$ of $SU(2)\otimes SU(2)$, or, equivalently, like the $00\cdots0$ component of a symmetric traceless $SO(4)$ tensor of Nth rank. It is an elementary exercise in $SO(4)$-tensor algebra to show that[17]

$$(\Delta\mathbf{X})^2S_0{}^N=N(N+2)S_0{}^N. \quad (5.14)$$

Thus, if we write the uncorrected S matrix as an expression

$$S_0=\sum_{N=0}^{\infty}S_0{}^N, \quad (5.15)$$

[16] A similar exponential factor is found by R. Perrin, Ref. 5. Perrin deals only with neutral pions having a linear derivative coupling, so a direct comparison with his results is difficult.

[17] The algebra is done in Sec. VI of S. Weinberg, Phys. Rev. 166, 1568 (1968). Note that this earlier paper dealt with terms in the Lagrangian which break dynamic chiral symmetry, rather than terms in the S matrix which break algebraic chiral symmetry. However, the algebra is the same.

then the effect of virtual soft-pion exchange is to change this expression to read

$$S=\sum_{N=0}^{\infty}\exp\left(-\frac{N(N+2)\Lambda^2}{4\pi^2F_\pi{}^2}\right)S_0{}^N. \qquad (5.16)$$

The critical energy appearing in Eq. (5.16) has the numerical value

$$2\pi F_\pi\simeq 1.2 \text{ BeV},$$

so if we arbitrarily set $\Lambda=m_\rho$, then $\Lambda^2/4\pi^2F_\pi{}^2\simeq 0.4$. With this cutoff, the $N=1$ term is suppressed by a factor of order 0.3, the $N=2$ term is suppressed by a factor of order 0.04, and all higher terms become negligible.

The reason for this suppression of chiral-nonvariant terms in the S matrix is the same as the reason for the suppression of the whole S matrix by virtual infrared photons for reactions involving charged particles.[3] That is, the rates for reactions in which real soft pions or photons are not emitted are suppressed to avoid the corresponding rates for pion or photon emission from being large enough to violate unitarity.[16] One important difference between the suppression factors produced by virtual photon or pion exchange is that for photons these factors involve both an infrared and an ultraviolet logarithmic divergence, while for pions they involve a quadratic ultraviolet divergence only. Also, the suppression is far more dramatic for pions than for photons.

These remarks may also be applied to hadronic weak and electromagnetic interactions. For instance, if all strong interactions except for soft-pion exchanges are neglected in high-energy electron-positron or electron-antineutrino annihilation, then the uncorrected matrix elements S_0 are linear combinations, respectively, of T_3 and Y (hypercharge) or T_1+iT_2 and X_1+iX_2. But, Eqs. (2.10)–(2.12) give

$$\Delta\mathbf{X}^2\ \mathbf{T}\equiv[X_a,[X_a,\mathbf{T}]]=2\mathbf{T},$$
$$\Delta\mathbf{X}^2\ \mathbf{X}\equiv[X_a,[X_a,\mathbf{X}]]=2\mathbf{X},$$

so the isovector part of the electron-positron annihilation matrix element and the whole electron-antineutrino annihilation matrix element are suppressed by factors

$$\exp(-\Lambda^2/2\pi^2F_\pi{}^2),$$

while the isoscalar part of the electron-positron annihilation matrix element is not suppressed by soft-pion-exchange effects.

It is worth emphasizing that the pion-pion interactions really play a very important role in this calculation, despite the fact that they do not contribute in the final results. The point is that if we had used any definition of the pion field other than that of CCWZ,[10] then (5.8) and (5.9) would not apply, and pion-pion interactions *would* contribute to the S matrix. However, in this case the results of Sec. III would also be much more complicated, and the effect of the pion-pion interactions would just be to cancel these complications.

VI. SOFT-PION EMISSION IN HARD-HADRON PROCESS

Now we shall consider the calculation of matrix elements for the emission of soft pions in a hard-hadron reaction. In this case, current algebra makes a definite statement about how to calculate matrix elements[2]: They are given by the sum of all tree graphs like Fig. 1, with all external soft-pion lines attached to pion trees which are rooted, each at a single point, to the incoming and outgoing hard-hadron lines.

Instead of trying to use this prescription to derive expressions in closed form for matrix elements for the production of arbitrary numbers of pions, it proves far easier to sum the diagrams for emission of pions in a *coherent state*.[18] The general coherent state is of the form

$$|z\rangle=\prod_{\mathbf{q},b}\exp[z(\mathbf{q},b)a^\dagger(\mathbf{q},b)]|0\rangle, \qquad (6.1)$$

where $a^\dagger(\mathbf{q},b)$ is the creation operator for a pion of momentum $\mathbf{q}$ and isovector index b, $|0\rangle$ is the pion vacuum, and the coefficients $z(\mathbf{q},b)$ are an infinite set of complex numbers which characterize the state. (Box normalization is used to define the product.) Coherent states are so named because they are eigenstates of the annihilation operators

$$a(\mathbf{q},b)|z\rangle=z(\mathbf{q},b)|z\rangle, \qquad (6.2)$$

and hence also of the positive-frequency part of the free pion field

$$\boldsymbol{\pi}^{(+)}(x)|z\rangle=\mathbf{e}^{(z)}(x)|z\rangle, \qquad (6.3)$$

where

$$\pi_b{}^{(+)}(x)\equiv\sum_{\mathbf{q}}(2V|\mathbf{q}|)^{-1/2}e^{iq\cdot x}a(\mathbf{q},b), \qquad (6.4)$$

$$e_b{}^{(z)}(x)=\sum_{\mathbf{q}}(2V|\mathbf{q}|)^{-1/2}e^{iq\cdot x}z(\mathbf{q},b), \qquad (6.5)$$

and V is the normalization volume.

The use of Eq. (6.3), together with the above current-algebra prescription for computing pion-emission matrix elements, leads immediately to a statement of how to calculate matrix elements for emission of pions in a coherent state: We must use the results of Sec. III for hard-hadron reactions in an external pion field, but with pion-pion interactions taken into account by replacing the external field $\boldsymbol{\pi}(x)$ with a function $\boldsymbol{\phi}^{(z)}(x)$, defined as $\mathbf{e}^{(z)}(x)$ plus the sum of all pion trees with factors $\mathbf{e}^{(z)}(y)$, etc., at the branch tips. (See Fig. 7). Following the reasoning of Sec. IV, this means that

[18] R. J. Glauber, Phys. Rev. **131**, 2766 (1963). Coherent states are used to treat soft-photon emission and infrared divergences by V. Chung, *ibid.* **140**, B1110 (1965); and T. W. B. Kibble, *ibid.* **173**, 1527 (1968).

$\boldsymbol{\phi}^{(z)}(x)$ is the solution of the nonlinear integral equation

$$\boldsymbol{\phi}^{(z)}(x)=\mathbf{e}^{(z)}(x)+\int d^4y\, \Delta_\pi(x-y) \times\mathbf{J}(\boldsymbol{\phi}^{(z)}(y),\partial_\mu\boldsymbol{\phi}^{(z)}(y),\partial_\mu\partial_\nu\boldsymbol{\phi}^{(z)}(y)), \quad (6.6)$$

with Δ_π and $\mathbf{J}$ given by Eqs. (4.2) and (4.9). With $\boldsymbol{\phi}^{(z)}(x)$ put in place of $\boldsymbol{\pi}(x)$ in Eq. (3.29), the S matrix for production of a coherent soft-pion state z in a hard-hadron reaction $\alpha\to\beta$ is

$$S(\alpha\to\beta+z)=\int d^4x\, e^{-i(p_\alpha-p_\beta)\cdot x} \times\{\exp[2iF_\pi^{-1}\Delta\mathbf{X}\cdot\boldsymbol{\phi}^{(z)}(x)]T_0\}_{\beta\alpha}. \quad (6.7)$$

The exchange of trees of soft virtual pions could be taken into account here by adding a term $-\Lambda^2(\Delta\mathbf{X})^2/4\pi^2F_\pi{}^2$ in the exponential.

The matrix elements for emission of any definite number of soft pions can in principle be determined by taking functional derivatives of $S(\alpha\to\beta+z)$ with respect to z:

$$S(\alpha\to\beta+\mathbf{q}_1b_1+\cdots+\mathbf{q}_nb_n) =\left[\frac{\delta^n}{\delta z(\mathbf{q}_1b_1)\cdots\delta z(\mathbf{q}_nb_n)}S(\alpha\to\beta+z)\right]_{z=0}. \quad (6.8)$$

However, it is not necessary to square, integrate, and sum these n-pion matrix elements in order to calculate the total soft-pion emission rate—the calculation can be done directly in terms of the coherent states. These states form a complete set, with[18]

$$1=\int dz|z\rangle\langle z|, \quad (6.9)$$

where dz is a weighted-volume element in the infinite-dimensional space of the coefficients $z(\mathbf{q},b)$:

$$dz\equiv\prod_{\mathbf{q},b}\frac{1}{\pi}\exp[-|z(\mathbf{q},b)|^2]d\,\mathrm{Re}z(\mathbf{q},b)d\,\mathrm{Im}z(\mathbf{q},b). \quad (6.10)$$

The total rate for emission of soft pions in the reaction $\alpha\to\beta$ is thus

$$\Gamma(\alpha\to\beta)=\int dz\int d^4x\, e^{-i(p_\alpha-p_\beta)\cdot x} \times\{\exp[2iF_\pi^{-1}\Delta\mathbf{X}\cdot\boldsymbol{\phi}^{(z)}(x)]T_0\}_{\beta\alpha} \times\{\exp[2iF_\pi^{-1}\Delta\mathbf{X}\cdot\boldsymbol{\phi}^{(z)}(0)]T_0\}_{\beta\alpha}{}^*. \quad (6.11)$$

Note that there is just one x integration here, because the rate Γ should contain only one momentum-conservation factor $(2\pi)^4\delta^4(\;)$, not two.

The functional integral in (6.11) could perhaps be done by a Monte Carlo method. Instead of using the plane-wave decomposition (6.5), it would be better to choose a set of, say, 10 suitable spherical wave functions $u_n(x)$, with covariant orthonormalization. A particular set of 30 complex quantities $z(n,b)$ would be chosen at random, using the Gaussian probability distribution prescribed in Eq. (6.10). Then one would have to calculate the function $\boldsymbol{\phi}^{(z)}(x)$ by solving the nonlinear integral equation (6.6), with

$$e_b{}^{(z)}(x)\equiv\sum_n u_n(x)z(n,b),$$

and then compute the x integral in Eq. (6.11). This calculation would be repeated for a number of different random choices of the 30 z's, and the result would then be averaged over all these choices. This task is left as an exercise for the reader.

Note added in proof. Lowell Brown has pointed out to me that Eq. (5.13) needs to be completely symmetrized in $\Delta\mathbf{X}$ to each order in the power-series expansion of the exponential, so that

$$S=\sum_{n=0}^{\infty}\frac{(-)^n\lambda^{2n}}{n!}\delta(a_1\ldots a_{2n})\Delta X_{a1}\ldots\Delta X_{a2n}S_0,$$

where $\lambda=\Lambda/2\pi F_\pi$, and δ is a product of n Kronecker-δ symbols, averaged over all permutations of $a_1\ldots a_{2n}$. As a result, the suppression factor is no longer given by Eq. (5.16). The correct suppression factors for the first few chiral tensors are now

$N=0$: 1,
$N=1$: $(1-2\lambda^2)e^{-\lambda^2}$,
$N=2$: $\frac{1}{3}+\frac{2}{3}(1-8\lambda^2)e^{-4\lambda^2}$,
$N=3$: $\frac{1}{2}(1-18\lambda^2)e^{-9\lambda^2}+\frac{1}{2}(1-2\lambda^2)e^{-\lambda^2}$

instead of $e^{-N(N+2)\lambda^2}$. For large λ^2, all chiral tensors of odd rank are still strongly suppressed, but the suppression factors in the even tensors remain finite even for $\lambda\to\infty$.