$$^2P_{3/2}{}^\circ \to {}^2S_{1/2} = 15\,393(5.667) + 16\,053 - 19\,752/7.667 + 0.4565(6.667)^4 - \tfrac{1}{4}(425.1)$$
$$= 91\,504.5\ \text{cm}^{-1} \to 1092.84\ \text{Å} = \lambda\,,$$
$$^2P_{1/2}{}^\circ \to {}^2S_{1/2} = 91\,079.4\ \text{cm}^{-1} \to 1097.94\ \text{Å} = \lambda\,.$$

The following table gives a comparison of the O VI values with experiment.[9]

		Calculated	Observed (Ref. 9)	Edlén (Ref. 10)
$^2P_{3/2}{}^\circ \to {}^2S_{1/2}$	$\tilde{\nu}$	96 874.8 cm^{-1}	96 907.5 cm^{-1}	96 905.0 cm^{-1}
	λ	1032.05 Å	1031.91 Å	1031.94 Å
$^2P_{1/2}{}^\circ \to {}^2S_{1/2}$	$\tilde{\nu}$	96 358.1 cm^{-1}	96 375.0 cm^{-1}	96 3726 cm^{-1}
	λ	1037.80 Å	1037.61 Å	1037.64 Å

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Source of the Kerr Metric*

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Assuming that the Kerr-Newman metric is the field of a layer of mass and charge distributed over the equatorial disk spanning the ring singularity, the source distribution on the disk is computed explicitly. In the uncharged case, this interpretation automatically excises the noncausal parts of the manifold, so that one obtains the unique source of the causally maximal extension of the vacuum metric. A Newtonian field which gives the same source distribution is exhibited, and shown to be closely analogous to the relativistic case. In particular, the Newtonian particle orbits show the same avoidance of the ring singularity that is such a remarkable feature of geodesics in the Kerr geometry. In the charged case, we examine how the gyromagnetic moment (which is equal to that of the Dirac electron) is reflected in the character of the source distribution.

I. INTRODUCTION

THE gravitational collapse of a star is a highly complex phenomenon whose details depend sensitively on the nature of the asymmetries involved. No matter how diverse the initial conditions, however, it is now widely believed[1–3] that, in the terminal black-hole stage of an irreversible collapse, the external field depends on only two parameters, and in fact is identical with the Kerr vacuum field having the appropriate mass and angular momentum.

In the case of zero angular momentum, this conjecture is already on fairly firm ground. It is known[4] that the Reissner-Nordström spherisymmetric space-times are the only electromagnetic vacuum (EMV) fields which are static, asymptotically flat and possess nonsingular event horizons with the topology of a 2-sphere. Therefore, external asymmetries due to internal sources (such as mass quadrupoles and magnetic dipoles) cannot be statically supported by a black hole. This suggests[4] that in the gravitational collapse of a nonrotating electrically neutral star all such asymmetries should rapidly leak away—partly by radiation to infinity, and partly by falling in through the event horizon after the star itself has collapsed—leaving Schwarzschild's vacuum field as the sole external manifestation of the collapsed object. Direct support for these ideas has come from recent dynamical studies[5] of idealized collapse models with small departures from spherical symmetry.

Justification for extending these arguments to the case of nonvanishing angular momentum hangs at present on two rather slender lines of evidence. First, the charged Kerr-Newman fields[6] are (in several senses) the natural stationary generalizations of the Reissner-Nordström fields; they have the same simple algebraic structure (Petrov Type D) and are the only stationary, asymptotically flat EMV fields having this structure. Secondly, a study of small, axisymmetric stationary EMV perturbations of the Reissner-Nordström fields[2,7–9] shows that the only perturbations which preserve asymptotic flatness and a nonsingular event horizon are members of the Kerr-Newman family.

These considerations, inconclusive through they still are, lend a special interest to recent efforts to

* Work partially supported by the National Research Council of Canada.
[1] B. Carter, Phys. Rev. **174**, 1559 (1968).
[2] K. S. Thorne, in *Proceedings of the International School of Physics "Enrico Fermi"* (Academic, New York, to be published), Course 47.
[3] J. M. Bardeen, Nature **226**, 64 (1970).
[4] V. L. Ginzburg and L. M. Ozernoi, Zh. Eksperim. i Teor. Fiz. **47**, 1030 (1964) [Soviet Phys. JETP **20**, 689 (1965)]; A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Zh. Eksperim i Teor. Fiz. **49**, 170 (1965) [Soviet Phys. JETP **22**, 122 (1966)]; W. Israel, Commun. Math. Phys. **8**, 245 (1968).
[5] V. de la Cruz, J. E. Chase, and W. Israel, Phys. Rev. Letters **24**, 423 (1970).
[6] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. **6**, 918 (1965).
[7] V. de la Cruz and W. Israel, Phys. Rev. **170**, 1187 (1968), Ref. 15.
[8] K. S. Thorne, 1968 (unpublished).
[9] W. Israel, in Proceedings of the Berne Seminar on the Bearings of Topology upon General Relativity, Gravitation et Relatité Generale, Vol. **1**, No. 3 (to be published).

understand the nature of the Kerr geometry and its sources. Of course, there is no reason to expect that such studies will provide direct insight into the actual conditions prevailing in the deepest interior of a black hole; asymmetries which have been sucked inwards in the collapse will presumably begin to make their presence felt at or near the inner of the two Kerr horizons. Nevertheless, an understanding of the unperturbed Kerr space-time is a necessary prelude to more ambitious investigations.

For the moment, we confine attention to the uncharged Kerr vacuum metrics[10] characterized by two parameters m and ma, representing the mass and angular momentum of the source. The Riemann tensor is algebraically degenerate Type D, so that there are two congruences of principal null geodesics, directed inwards and outwards, respectively, which transform into each other under simultaneous reversal of time and sense of rotation.[11] It is useful to describe the manifold in terms of a coordinate r, which can be defined geometrically as an affine parameter along either of these two principal null congruences, and which functions as an asymptotic radial coordinate at spatial infinity.

A key feature of the Kerr geometry is an equatorial disk, centered on the axis of symmetry, which is intrinsically flat and of radius $|a|$. The ringlike boundary of this disk comprises the geometrical singularity of the metric. In addition, the disk itself has a remarkable property: As one approaches it from either above or below, r tends to zero through positive values, but grad r (directed outward from the disk) does not vanish. Since r has an intrinsic meaning, this must be interpreted in one of the following ways.

(i) The complete Kerr manifold is defined so that $r\geq 0$ everywhere, and there is a discontinuity in the normal derivative of the metric across the flat disk.

(ii) Alternatively, the metric remains smooth everywhere away from the ring singularity, but an observer crossing the disk $r=0$ from a region with $r>0$ emerges into a new asymptotically flat space characterized by $r<0$. The two "Riemann sheets" with $r>0$ and $r<0$ are to be considered as joined together on the disk $r=0$, which serves as a branch cut.[12] As Carter[1] first pointed out, the azimuthal vector $\partial/\partial\phi$ becomes timelike for $r<0$, so that the negative-r sheet contains closed timelike curves.[13]

Investigation of the geodesics had to be confined at first to the axis of symmetry[14] and the equatorial plane,[11] until Carter[1] noticed that the Hamilton-Jacobi equation is separable, and was thus able to reduce the solution of the complete geodesic equations to quadratures. A number of curious results emerged from these analyses. For example, it was found that, for a particle falling down the axis, the gravitational force becomes repulsive[14] when $r<|a|$, that particles in the plane of the flat disk have straight orbits,[15] and, most remarkably, that all timelike and null geodesics avoid the ring singularity except for some in the equatorial plane.[1]

Several papers have been devoted to the problem of fitting the Kerr exterior field to rotating material sources of various kinds,[16] including fluid bodies[17] and spherical shells.[7,18] Naturally, this problem does not have a unique solution, since one is at liberty (for example) to choose arbitrarily the boundary between the exterior vacuum and the source. The arbitrariness disappears if one asks about the source of the *maximally extended* vacuum metric, but here we are thwarted by our present inability to interpret singularities of Einstein's field equations. Newman and Janis[19] conjectured that Kerr's solution is the field of a spinning ring of mass, but (as they themselves recognized in a note appended to their paper) this fails to take account of the peculiar geometry of the disk.[20]

In the present paper, we shall follow up this concluding note in the Newman-Janis paper.[19] Adopting interpretation (i) of the Kerr geometry, we consider the lack of smoothness of the metric at $r=0$ to be caused by a layer of mass spread over the disk. Since the theory of surface layers in general relativity is well understood,[21] the physical properties of the layer can be unambiguously derived.[22] Using a "subtraction argument" we can then infer the general characteristics of the ring singularity. This approach has the added advantage that, by cutting out the negative-r sheet, we eliminate (at any rate for the uncharged case[13]) all noncausal features of the space. We obtain in this way

[10] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963). Most of our remarks apply without change to the charged EMV generalization of the Kerr solution discovered by Newman *et al.* (Ref. 6).

[11] R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967).

[12] R. P. Kerr, in *Quasistellar Sources and Gravitational Collapse*, edited by I. Robinson, A. Schild, and E. L. Schucking (Chicago U. P., Chicago, 1965), p. 99.

[13] In the charged Kerr solutions these noncausal curves extend some way into the positive-r sheet.

[14] B. Carter, Phys. Rev. **141**, 1242 (1966); see also R. L. Gautreau, Nuovo Cimento **50A**, 120 (1967).

[15] This follows from the formulas of Ref. 1 after adjusting for the fact that the quantity P in Eqs. (64)–(71) of this reference enters with the wrong sign. (The conclusions of the paper are unaffected.)

[16] W. C. Hernandez, Phys. Rev. **159**, 1070 (1967); **166**, 1263 (1968).

[17] R. H. Boyer, Proc. Cambridge Phil. Soc. **61**, 527 (1965); **62**, 495 (1966); M. Trümper, Z. Naturforsch. **22a**, 1347 (1967).

[18] J. M. Cohen, J. Math. Phys. **8**, 1477 (1967).

[19] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965).

[20] Hernandez, Ref. 16 (1968), has given additional reasons why this simple interpretation will not work.

[21] C. Lanczos, Ann. Physik **74**, 518 (1924); W. Israel, Nuovo Cimento **44B**, 1 (1966); **48B**, 463 (1967); A Papapetrou and A. Hamoui, Ann. Inst. Henri Poincaré **9**, 179 (1968).

[22] W. B. Bonnor and A. Sackfield [Commun. Math. Phys. **8**, 338 (1968)] have applied a similar approach to the interpretation of certain static axisymmetric vacuum fields with disk singularities.

the unique source of the causally maximal extension of Kerr's space-time.

Our main results may be briefly symmarized as follows. The (uncharged) Kerr disk has a negative surface density, "effectively" (i.e., taking kinetic and gravitational potential energy into account) equal to

$$\sigma_{\text{eff}}=-\tfrac{1}{2}(m/\pi a^2)(1-\rho^2/a^2)^{-3/2}, \qquad (1)$$

where ρ is the radial coordinate on the (intrinsically flat) disk. The material of the disk rotates with supra-aluminal speed and angular velocity (as measured by a stationary observer at infinity) $\omega=a/\rho^2$; the annulus from ρ to $\rho+d\rho$ contributes angular momentum

$$-\tfrac{1}{2}m(\rho/a)^3(1-\rho^2/a^2)^{-3/2}d\rho.$$

Since both the mass and angular momentum of the disk diverge to $-\infty$ as $\rho\rightarrow|a|$, we infer that the singular ring must have positively infinite mass and angular momentum in order to yield the finite net observed values m and ma.

In terms of this mass distribution, we can understand at once why the gravitational field becomes repulsive close to the disk. Going a step further, we can derive the Newtonian potential of the surface distribution (1), assuming the disk surrounded by a massive ring which brings the total mass up to m. *The orbits of particles in this Newtonian field closely resemble the local description*[23] *of timelike geodesics in the Kerr geometry.* The Newtonian analog cannot, of course, reproduce inertial dragging effects of the Lense-Thirring type. However, it does exhibit the same avoidance of the ring by test particles, showing that this is not a new nonlinear effect of Einstein's theory, but is explainable more simply in terms of the peculiar dipole-like structure of the source near the ring.

These results are useful in giving us a better intuitive feeling for the geodesic structure of the Kerr geometry [test particles behave *as if* they were in the Newtonian field of the surface distribution (1)]. But it is clear from the unstable and unphysical character of the relativistic distribution that we are not much closer to an understanding of the real conditions within the inner Kerr horizon. In any realistic collapse situation it is very possible that strong, time-dependent, random perturbations will overwhelm the background metric in this region, making a radically new approach to the problem necessary.

II. SURFACE ENERGY TENSOR OF CHARGED KERR DISK

For the sake of completeness, we consider the charged, EMV generalization[6] of the Kerr metric, which can

[23] Globally, there may be far-reaching differences resulting from the existence of relativistic event horizons in the case of low angular momentum ($|a|\leq m$).

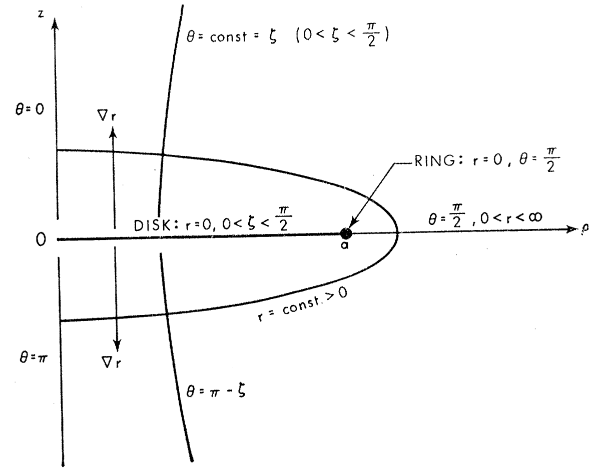

FIG. 1. Map of a 2-space of constant ϕ and t, showing sections of the Kerr disk and the ring singularity. The horizontal and vertical axes represent pseudocylindrical coordinates ρ and z defined by $z\equiv r\cos\theta$, $\rho\equiv(r^2+a^2)^{1/2}\sin\theta$. The same diagram illustrates the character of the oblate spheroidal coordinates r and θ used to discuss the field of the Newtonian disk of Sec. V; ρ and z are now genuine cylindrical coordinates in Euclidean space.

be written

$$ds^2=(r^2+a^2\cos^2\theta)\left(d\theta^2+\frac{dr^2}{r^2-2mr+a^2+e^2}\right)+(r^2+a^2)\sin^2\theta\,d\phi^2-dt^2+\frac{2mr-e^2}{r^2+a^2\cos^2\theta}\times(dt-a\sin^2\theta\,d\phi)^2 \qquad (2)$$

in terms of the Schwarzschild-like coordinates r, θ, ϕ, and t introduced by Boyer and Lindquist[11] for the uncharged metric and extended to the charged case by several authors.[1,24,25] The associated vector potential is

$$\begin{aligned}A_1&=A_2=0,\\ A_3&=(ear\sin^2\theta)/(r^2+a^2\cos^2\theta),\\ A_4&=-er/(r^2+a^2\cos^2\theta).\end{aligned} \qquad (3)$$

If regions of negative r are excluded from the manifold, and if we wish to regard r, θ, and ϕ as asymptotically spherical coordinates, then dr/ds must change sign and θ must change to $\pi-\theta$ when a curve θ, ϕ, $t=$const crosses $r=0$ (Fig. 1). To avoid confusion, we introduce a new angular variable ζ, with range $0\leq\zeta\leq\frac{1}{2}\pi$, such that $\theta=\zeta$ and $\theta=\pi-\zeta$, respectively, in the upper and lower half-spaces in Fig. 1.

[24] F. J. Ernst, Phys. Rev. **168**, 1415 (1968). Note that Ernst's sign conventions for a and A_μ are opposite to ours. Apart from this, there is a persistent sign anomaly in Ernst's work which is most easily corrected by reversing the sign of ω in his basic metric form (first equation of Sec. II). Our signs agree with those of Ref. 1.

[25] J. M. Cohen, J. Math. Phys. **9**, 905 (1968).

In terms of intrinsic coordinates[26] $\xi^a \equiv \zeta$, ϕ, and t, the 3-metric of $r=0$ is

$$g_{ab}d\xi^a d\xi^b = a^2 \cos^2\zeta\, d\zeta^2 + a^2 \sin^2\zeta\, d\phi^2 - dt^2 - (e^2/a^2 \cos^2\zeta)(dt - a\sin^2\zeta\, d\phi)^2. \quad (4)$$

In the special case $e=0$, this hypersurface is flat (radial coordinate $\rho = a \sin\zeta$), the history of a disk of radius $|a|$. For general e, the rim of the "disk" ($r=0$, $\theta=\frac{1}{2}\pi$) marks a singularity of the 4-metric (2) and of the four-dimensional Riemann tensor.

Let n_α, proportional to $\partial_\alpha r$, be the unit normal pointing *upwards* from the disk. Then

$$n_\alpha = k\cos\theta\,\delta_\alpha{}^1 = \epsilon k \cos\zeta\,\delta_\alpha{}^1, \quad (5)$$

where $k=(1+e^2/a^2)^{-1/2}$ and $\epsilon=+1$ and -1 on the upper and lower faces, respectively.

The extrinsic curvature of the disk is defined by

$$K_{ab} = n_{\alpha|\beta}\, e_{(a)}{}^\alpha e_{(b)}{}^\beta.$$

Here, the stroke indicates covariant differentiation with respect to the 4-metric (2), and $e_{(a)}{}^\alpha = \partial x^\alpha/\partial\xi^a$ are the tangential base vectors associated with the intrinsic coordinates ζ, ϕ, and t: $e_{(\zeta)}{}^\alpha = \epsilon\delta_2{}^\alpha$, $e_{(\phi)}{}^\alpha = \delta_3{}^\alpha$, and $e_{(t)}{}^\alpha = \delta_4{}^\alpha$. Straightforward calculation yields for the nonvanishing components

$$K_{\phi\phi} = (\epsilon m \sin^4\zeta)/k\cos^3\zeta, \quad K_{tt} = \epsilon m/ka^2\cos^3\zeta,$$
$$K_{\phi t} = -(\epsilon m \sin^2\zeta)/k\cos^3\zeta,$$

which change sign across the disk.

We consider all the gravitational and electromagnetic sources to be concentrated on the disk in the form of a surface layer. The surface energy tensor S_{ab} is then given by[21]

$$-8\pi S_{ab} = [K_{ab}] - g_{ab}g^{cd}[K_{cd}] \quad (6)$$

in terms of the jump $[K_{ab}]$ in the extrinsic curvature on crossing the layer in the direction of n_α. The final result can be written in the form

$$S_a{}^b = \sigma(u_a u^b + \zeta_a \zeta^b), \quad (7)$$

where

$$\sigma = -km/4\pi a^2 \cos\zeta; \quad (8)$$

$\zeta_a = (a\cos\zeta, 0, 0)$ is a unit radial vector; and

$$u^a = (0, 1/a\sin^2\zeta, 1)\tan\zeta \quad (9)$$

is an orthogonal unit *spacelike* vector. The disk is thus composed of material having negative proper surface density σ, rotating with supraluminal velocity u^a and prevented from flying off radially by a radial tension $|\sigma|$. The angular velocity measured by a stationary observer at infinity is

$$\omega = u^\phi/u^t = 1/a\sin^2\zeta. \quad (10)$$

Consider now the stationary field of any axisymmetric, steadily spinning shell, in which the metric has the asymptotic form

$$g_{44} \approx -(1-2m/r), \quad g_{34} \approx -(2ma\sin^2\theta)/r \quad (r\to\infty).$$

(We are using quasispherical coordinates which are canonical in the sense that all metric coefficients are independent of ϕ and t, and g_{34} is the only off-diagonal component.) In the case where S_{ab} is finite everywhere, it can be shown that[7]

$$m = \int_{\text{shell}} \sigma_{\text{eff}}\, d\Sigma_2, \quad ma = \int_{\text{shell}} \lambda\, d\Sigma_2, \quad (11)$$

where $d\Sigma_2$ is the invariant element of 2-area of the shell, and

$$\sigma_{\text{eff}} = -(-g^{44})^{-1/2}(S_t{}^t - S_\phi{}^\phi - S_\zeta{}^\zeta)$$
and
$$\lambda = (-g^{44})^{-1/2} S_\phi{}^t \quad (12)$$

can be interpreted as "effective" surface densities of mass and angular momentum, including contributions from the gravitational field and energy of rotation. For the Kerr disk, Eqs. (12) give

$$\sigma_{\text{eff}}\, d\Sigma_2 = -(m/2\pi a^2\cos^3\zeta)\, a\cos\zeta\, d\zeta\, a\sin\zeta\, d\phi, \quad (13)$$

$$\lambda\, d\Sigma_2 = -[(m\sin^2\zeta)/4\pi a\cos^3\zeta]\, a\cos\zeta\, d\zeta\, a\sin\zeta\, d\phi. \quad (14)$$

The integrals of (13) and (14) diverge as $\zeta\to\frac{1}{2}\pi$. Hence, if Eqs. (11) are assumed to retain a meaning in the present case, the singular ring $r=0$, $\zeta=\frac{1}{2}\pi$ must contribute infinite positive mass and angular momentum.

III. SURFACE CHARGE AND CURRENT

We begin by deriving the electromagnetic analog of the gravitational jump conditions (6) for an arbitrary charged surface layer.

Starting with arbitrary intrinsic coordinates ξ^2, ξ^2, and ξ^4 in the layer, we extend these to a system of four-dimensional coordinates x^α by a Gaussian construction[27]: $x^a = \xi^a$, $x^1 = \pm$(normal geodesic distance from layer), so that $g_{1\alpha} = \delta_{1\alpha}$. Maxwell's equations, $(4\pi)^{-1} \times \partial_\mu[(-g)^{1/2}F^{\lambda\mu}] = (-g)^{1/2}J^\lambda = (-g)^{1/2}j^\lambda\delta(x^1)$, integrate to $[F^{\lambda 1}] = 4\pi j^\lambda$. Here j^λ is the surface current and $[\cdots]$ denotes a jump across the layer. Our result can be more conveniently written

$$[e_{(a)}{}^\alpha F_{\alpha\beta} n^\beta] = 4\pi j_a, \quad (15)$$

in which $j^\lambda = j^a e_{(a)}{}^\lambda$, and the left-hand side may now be evaluated in *arbitrary* four-dimensional coordinates, since it is a 4-scalar dependent only on the intrinsic coordinates ξ^a.

Consider now a stationary field due to any (nonsingular) axisymmetric surface distribution of charge and current such that (in canonical quasispherical

[26] Greek indices run from 1 to 4. Lower case italic indices refer to the subspace $r=0$ and run from 2 to 4. Capitalized italic indices (in Sec. VI) run from 1 to 3.

[27] J. L. Synge, *Relativity, The General Theory* (North-Holland, Amsterdam, 1960), p. 35.

coordinates)

$$A_4 \approx -e/r, \quad A_3 = O(r^{-1}) \quad (r \to \infty).$$

By integrating Maxwell's equations over all space, it is readily shown that

$$e = \int_{\text{shell}} (-g^{44})^{-1/2} j^t d\Sigma_2,$$
$$0 = \int_{\text{shell}} (-g^{44})^{-1/2} j^\phi d\Sigma_2. \tag{16}$$

To obtain the charge and current distribution on the Kerr disk, we compute $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ at $r=0$ from (3). The nonvanishing components are

$$F_{13} = (e/a)\tan^2\zeta, \quad F_{14} = -(e/a^2)\sec^2\zeta. \tag{17}$$

We then find, from (5) and (15),

$$j^a = qv^a, \tag{18}$$

where

$$v^a = (0, a^{-1}, 1)k \sec\zeta \tag{19}$$

is a unit timelike vector orthogonal to u^a, and the surface charge density is

$$q = -(e/2\pi a^2)\sec^2\zeta. \tag{20}$$

The integrals in (16) diverge to $-\infty$ as $\zeta \to \frac{1}{2}\pi$, indicating that the ring carries infinite positive charge.

IV. A CLASSICAL MODEL FOR THE ELECTRON?

From a comparison of (3) with the asymptotic formula

$$A_3 \approx \mu(\sin^2\theta)/r \quad (r \to \infty) \tag{21}$$

for an axisymmetric stationary magnetic field of dipole moment μ, a curious result emerges[28]: The Kerr-Newman field is characterized by a gyromagnetic ratio $\mu/ma = e/m$. This ratio is twice that of a classical distribution having a constant ratio of charge and mass densities, and coincides with that of the Dirac electron. It is of interest to examine whether the "unrenormalized" distribution over the Kerr disk exhibits the same gyromagnetic "anomaly".

In *Euclidean* space the magnetic moment defined by (21) can be re-expressed as an integral over the source distribution. We postulate a disk source, represented by $r=0$ in oblate spheroidal coordinates [see Eq. (25)], in terms of which the flat space-time metric is

$$ds^2 = (r^2 + a^2\cos^2\theta)[dr^2/(r^2+a^2)] + (r^2+a^2)\sin^2\theta\, d\phi^2 - dt^2. \tag{22}$$

Then the integral for μ is

$$\mu = \frac{1}{2}\int_0^{\pi/2}\int_0^{2\pi} j_\phi a \cos\zeta\, d\zeta\, a \sin\zeta\, d\phi, \tag{23}$$

with ζ defined as in Sec. II.

[28] G. C. Debney, R. P. Kerr, and A. Schild, J. Math. Phys. **10**, 1842 (1969). I am much indebted to Dr. E. T. Newman for calling this reference to my attention.

In general, there is no analog of (23) in a curved space. However, it is possible to associate a flat space-time with the Kerr-Newman manifold by identifying points having the same coordinate labels in (2) and (22). [Because the Kerr-Newman metric (2) can be invariantly decomposed[1] as $(2) = (22) + (k_\alpha dx^\alpha)^2$, where k_α is either of the two Debever principal null vectors, this apparently arbitrary association actually has an invariant significance.] Equation (23) is then an identity provided we insert a new "effective" source distribution j_a computed from the field (3) by using the auxiliary flat metric (22) in place of (2). The only change in our previous formulas (18)–(20) is that k is replaced by unity, and we obtain

$$j_\phi = -(e/2\pi a)(\sin^2\zeta)/(\cos^3\zeta).$$

Comparing with (14), we obtain for every annulus of the disk the gyromagnetic ratio e/m.

It thus appears possible to account for some of the spin properties of the electron on a purely classical basis by visualizing it as a Kerr-Newman disk having angular momentum $ma = \frac{1}{2}\hbar$ and diameter $2a = \hbar/m$ equal to the Compton wavelength. Since the disk material has negative mass, the electrostatic repulsive forces normal to the plane of the disk would tend to draw the material closer together, and the thin disk structure would actually be stable for $|e| > m$. Whether such a model is to be taken seriously is a question we shall not pursue further here.

V. NEWTONIAN ANALOG

What Newtonian field yields the mass distribution that we found in Eq. (13) for the Kerr disk? For simplicity, we shall confine ourselves here to the uncharged case.

Consider the scalar field in Euclidean 3-space:

$$V = -\tfrac{1}{2}m(R^2 - 2iaR\cos\Theta - a^2)^{-1/2} - \tfrac{1}{2}m(R^2 + 2iaR\cos\Theta - a^2)^{-1/2} \tag{24}$$

(R, Θ, and ϕ are spherical polar coordinates). This function is obviously harmonic, being formally the potential due to two masses of $\frac{1}{2}m$ placed at imaginary points ($R = ia$, $\Theta = 0$) and ($R = ia$, $\Theta = \pi$) on the axis of symmetry. To make V single-valued we postulate that in (24) the positive square roots be taken on the equator $\Theta = \frac{1}{2}\pi$, $R > a$, and that V be extended to other parts of the complex $Re^{i\Theta}$ plane by analytic continuation. It is then necessary to consider the disk $\Theta = \frac{1}{2}\pi$, $R \leq a$ as a branch cut.

Introduce oblate spheroidal coordinates r, θ defined by

$$R\cos\Theta = r\cos\theta, \quad R\sin\Theta = (r^2+a^2)^{1/2}\sin\theta \tag{25}$$

(see Fig. 1). Then (24) simplifies to

$$V = -mr/(r^2 + a^2\cos^2\theta). \tag{26}$$

The potential becomes singular at the ring $r=0$, $\theta=\frac{1}{2}\pi$, and its normal derivative is discontinuous on the disk $r=0$. For the surface density of the disk, we obtain

$$\sigma=\frac{1}{4\pi}\left(\frac{\partial V}{\partial n}\right)_{r=0}$$

$$=-m/2\pi a^2\cos^3\zeta\,,$$

in concordance with (13) (ζ is defined as in Sec. II).

Expansion of (24) in spherical harmonics gives

$$V=\frac{m}{R}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{R}\right)^{2n}P_{2n}(\cos\Theta)$$

$$=-m/R+(ma^2/R^3)P_2(\cos\Theta)-\cdots\,.$$

The quadrupole moment ma^2 agrees with that of the Kerr metric.[2,16]

VI. ORBITS IN THE NEWTONIAN FIELD

For a particle on the axis of symmetry in the Newtonian field (26), conservation of energy is expressed by

$$\tfrac{1}{2}\dot{r}^2-mr/(r^2+a^2)=E\,,$$

the dot denoting a time derivative. For $r<|a|$, the gravitational force becomes repulsive. Only a particle with sufficient energy to escape to infinity can penetrate the repulsive potential barrier of the disk. Since $V=0$ when $r=0$, particles moving in the disk experience no gravitational force. These results are in full agreement with the relativistic analysis.[14,15]

The general orbital equations can be integrated by separating variables in the Hamilton-Jacobi equation. The kinetic energy in oblate spheroidal coordinates is

$$\tfrac{1}{2}g_{AB}\dot{x}^A\dot{x}^B=\tfrac{1}{2}(r^2+a^2\cos^2\theta)\left(\frac{\dot{r}^2}{r^2+a^2}+\dot{\theta}^2\right)+\tfrac{1}{2}(r^2+a^2)\sin^2\theta\,\dot{\phi}^2.$$

Two constants of the motion are the angular momentum

$$p_\phi=(r^2+a^2)\sin^2\theta\,\dot{\phi}\equiv\Phi \tag{27}$$

and the total energy

$$E=\tfrac{1}{2}g_{AB}\dot{x}^A\dot{x}^B+V\,. \tag{28}$$

The Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\tfrac{1}{2}g^{AB}\frac{\partial S}{\partial x^A}\frac{\partial S}{\partial x^B}+V=0$$

has the separable solution

$$S=-Et+\Phi\phi+S_1(r)+S_2(\theta)\,,$$

where S_1 and S_2 satisfy

$$-(r^2+a^2)(dS_1/dr)^2+2Er^2+2mr+a^2\Phi^2/(r^2+a^2)$$
$$=K=(dS_2/d\theta)^2-2Ea^2\cos^2\theta+\Phi^2/(\sin^2\theta)$$

and K is a constant. The equations $p_\theta=dS_2/d\theta$ and $p_r=dS_1/dr$ now yield

$$(r^2+a^2\cos^2\theta)\dot{\theta}=\pm[K+2Ea^2\cos^2\theta-\Phi^2/\sin^2\theta]^{1/2}, \tag{29}$$

$$\frac{(r^2+a^2\cos^2\theta)}{r^2+a^2}\dot{r}=\pm\left(-K+2Er^2+2mr+\frac{a^2\Phi^2}{r^2+a^2}\right)^{1/2}. \tag{30}$$

Equations (27), (29), and (30) determine ϕ, θ, and r as functions of time. Essentially the only difference between the Newtonian and relativistic equations[1] is that the latter contain extra terms in $a\Phi$ corresponding to rotational dragging effects on local inertial frames, which naturally cannot be reproduced in a purely scalar theory.

We conclude by showing that, just as in the relativistic case,[1] no freely moving particle can strike the ring singularity $r=0$, $\theta=\frac{1}{2}\pi$ unless its orbit lies in the equatorial plane.

If $r=0$, $\theta=\frac{1}{2}\pi$ is a point on the orbit,

$$K=\Phi^2$$

follows from (29) or (30). By combining these two equations we then obtain

$$\int\frac{d\theta}{\cos\theta\,(2a^2E-\Phi^2/\sin^2\theta)^{1/2}}=\pm\int\frac{dr}{[2mr+2Er^2-r^2\Phi^2/(r^2+a^2)]^{1/2}(r^2+a^2)^{1/2}},$$

assuming the orbit does not lie entirely in the plane $\theta=\frac{1}{2}\pi$. The last equation is incompatible with our assumption that the orbit includes $r=0$, $\theta=\frac{1}{2}\pi$, since (for $m>0$) the right-hand integral tends to a finite limit as $r\rightarrow 0$, whereas the left-hand integral becomes infinite as $\theta\rightarrow\frac{1}{2}\pi$. This contradiction establishes our statement.

Note added in proof. Dr. W. J. Sarill kindly points out that H. Keres[29] has devised a "correspondence principle" for associating Newtonian with general relativistic fields, and has applied his principle to the Kerr metric, thus arriving by a completely independent route at the Newtonian analog of Sec. V.

[29] H. Keres, Zh. Eksperim. i Teor. Fiz. **52**, 768 (1967) [Soviet Phys. JETP **25**, 504 (1967)].