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Anomalous Ward Identities, Hard-Meson Calculations, and the $\pi^0\gamma\gamma$ Form Factor*

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Assuming the anomalous axial-vector Ward identity for the $\langle VVA\rangle$ three-point functions, we calculate the ρ-ω-π and ρ-φ-π vertex functions and the off-mass-shell $\pi^0\gamma\gamma$ form factor in the framework of Schnitzer and Weinberg's hard-meson calculations. The mass-shell value of the form factor reduces to a result of Adler; and its off-mass-shell values are used to predict certain electron-positron annihilation cross sections. We compare the results of the present model with those of the conventional vector-meson-dominance model.

I. INTRODUCTION

REACTIONS involving low-energy pions have been, for the most part, successfully accounted for in current-algebra calculations with the partially conserved axial-vector current (PCAC) hypothesis. One difficulty, however, arises in processes involving second-order electromagnetic interaction. It has been shown by Sutherland[1] that PCAC leads to a vanishing amplitude for the decay $\pi^0 \to 2\gamma$, in the limit of zero pion mass. Recently, Bell and Jackiw[2] and Adler[3] found that in the σ model and in spinor electrodynamics, the PCAC equation

$$\partial^\mu A_\mu{}^3(x) = \mu^2 f_\pi \phi_{\pi^0}(x)\,, \tag{1}$$

where μ is the mass of the pion, is incompatible with the gauge invariance as applied to the reaction $\pi^0 \to 2\gamma$, and therefore that the naive manipulations of field equations must be misleading. To resolve this dilemma, Adler[3] has given a modification of (1) which is consistent with gauge invariance and which also results in a successful prediction of the $\pi^0 \to 2\gamma$ width. Subsequent field-theoretical calculations have substantiated Adler's modification.[4]

Taking this result as a cue, model calculations have been used to demonstrate that anomalous terms are also present in other axial-vector-current Ward identities (WI's). Explicit expressions for all of the possible anomalies of n-point function WI's for the $SU(3)$ currents evaluated in a free-quark model have been listed by Brown, Shih, and Young.[5]

Except for the prediction on the $\pi^0 \to 2\gamma$ width[3] and applications to ep scattering in a certain high-energy limit,[6] the aforementioned work as well as the related research into anomalous commutator, etc.,[7] are mainly concerned with the structure of model field theory which possesses internal symmetry. Since we are primarily dealing with strong interacting particles, it is not obvious that arguments of perturbation theory and field-theoretical calculations are really relevant. Hence, additional experimental tests of the existence of WI anomalies and related phenomena are desirable. Very recently, Glashow, Jackiw, and Shei[8] and Gounaris[8] have examined the electromagnetic decays of neutral pseudoscalar mesons in the context of broken $SU(3)\times SU(3)$ and modified PCAC. Predictions of the widths of the decays $X^0 \to 2\gamma$ and $\eta \to 2\gamma$ and other measurable quantities have been obtained.

In this article we have investigated, in the framework of hard-meson calculations[9] and the meson-dominance model (MD), other consequences of the WI anomalies. We have calculated the ρ-ω-π and ρ-φ-π (proper) vertex functions and the off-mass-shell $\pi^0\gamma\gamma$ form factor. By measuring the electron-positron annihilation cross sections for certain final states, tests for the predictions of this form factor can be made.

We begin in Sec. II to derive expressions for the ρ-ω-π and ρ-φ-π vertex functions and obtain the $\pi^0 \to 2\gamma$ amplitude in the limit of zero pion mass. In Sec. III, we use the experimental data on $\omega \to 3\pi$ and the radiative decays of the vector mesons to determine the parameters entering these vertex functions. This enables us to determine the $\pi^0\gamma\gamma$ form factor for low-mass virtual photons. Using this predicted form of the form factor, we discuss in Sec. IV the processes $e^+ + e^- \to \pi^0 + \gamma$ and $e^+ + e^- \to \pi^0 + e^+ + e^-$. The experimental cross sections of these two reactions give direct check on the form factor. A comparison of the results of Secs. III and IV with those predicted in the conventional vector-meson-dominance (VMD) model is given in Sec. V. Section VI is devoted to a discussion.

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[6] R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 975 (1969).
[7] The literature on the related problems is by now abundant. We apologize for not listing any here, but only refer to the following review article for a summary and for the references cited therein: R. Jackiw, CERN Report No. TH 1065, 1969 (unpublished).
[8] S. L. Glashow, R. Jackiw, and S.-S. Shei, Phys. Rev. **187**, 1916 (1969); G. Gounaris, Phys. Rev. D **1**, 1426 (1970).
[9] H. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1969).

II. WARD-IDENTITY ANOMALIES AND $\pi^0\gamma\gamma$ FORM FACTOR

In this section we shall apply the technique of hard-meson calculations[9] in order to obtain the ρ-ω-π and ρ-φ-π vertex functions and the off-mass-shell $\pi^0\gamma\gamma$ form factor. Consider the following three-point functions[10]:

$$T_a^{\mu\nu\lambda}(q,p)=\int d^4x d^4y\; e^{iqx}e^{ipy} \times\langle T(V_a^\mu(x)V_3^\nu(y)A_3^\lambda(0))\rangle_0, \quad (2a)$$

$$T_a^{\mu\nu}(q,p)=\int d^4x d^4y\; e^{iqx}e^{ipy} \times\langle T(V_a^\mu(x)V_3^\nu(y)P_3(0))\rangle_0. \quad (2b)$$

Here $V_d^\mu(A_d^\mu)$ is a vector (axial-vector) current with an $SU(3)$ index d; P_d is a pseudoscalar density which, in the context of PCAC, is the term on the right-hand side of (1). We shall restrict ourselves to $a=8$ or 0. To introduce the ω-φ mixing we follow the procedure of Ref. 11, which gives the results of the current-mixing model.[12] We can write

$$\begin{aligned}T_a^{\mu\nu\lambda}(q,p)=&\,id_{a33}g_A^{-1}g_\rho^{-1}\Delta_\rho^{\nu'\nu}(p)\Delta_A^{\lambda'\lambda}\\&\times[g_{a\varphi}^{-1}\Delta_{a\varphi}^{\mu\mu'}(q)\Gamma_{\mu'\nu'\lambda'}^{\varphi}(q,p)\\&+g_{a\omega}^{-1}\Delta_{a\omega}^{\mu\mu'}(q)\Gamma_{\mu'\nu'\lambda'}^{\omega}(q,p)]\\&+id_{a33}[f_\pi k^\lambda/(k^2-\mu^2)]g_\rho^{-1}\Delta_\rho^{\nu\nu'}(p)\\&\times[g_{a\varphi}^{-1}\Delta_{a\varphi}^{\mu\mu'}(q)\Gamma_{\mu'\nu'}^{\varphi}(q,p)\\&+g_{a\omega}^{-1}\Delta_{a\varphi}^{\mu\mu'}(q)\Gamma_{\mu'\nu'}^{\omega}(q,p)],\end{aligned} \quad (3a)$$

$$\begin{aligned}T_a^{\mu\nu}(q,p)=&-d_{a33}[f_\pi\mu^2/(k^2-\mu^2)]g_\rho^{-1}\Delta_\rho^{\nu\nu'}(p)\\&\times[g_{a\varphi}^{-1}\Delta_{a\varphi}^{\mu\mu'}(q)\Gamma_{\mu'\nu'}^{\varphi}(q,p)\\&+g_{a\omega}^{-1}\Delta_{a\omega}^{\mu\mu'}(q)\Gamma_{\mu'\nu'}^{\omega}(q,p)],\end{aligned} \quad (3b)$$

where $k=p+q$ and $\Delta_i^{\mu\nu}(p)$, $i=\rho$, ω, φ, or A, has the standard form

$$\Delta_i^{\mu\nu}(p)=\int d\sigma^2\frac{\rho_i(\sigma^2)}{p^2-\sigma^2}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{\sigma^2}\right)$$
$$\equiv R_i(p)g^{\mu\nu}-S_i(p)p^\mu p^\nu.$$

The quantities $g_{d\varphi}$, etc., enter the spectral functions $\rho_{d\varphi}(\sigma^2)$, etc., in the form $g_{d\varphi}^2\delta(\sigma^2-m_\rho^2)$, etc.

According to Refs. 3 and 5, the vector WI of (2a) and (2b) need not be modified if the axial-vector WI of (2a) contains an anomalous term:

$$k_\lambda T_a^{\mu\nu\lambda}(q,p)=-iT_a^{\mu\nu}(q,p)+d_{a33}\chi^{\mu\nu\alpha\beta}\epsilon_{\alpha\beta\sigma\tau}q^\sigma p^\tau, \quad (4a)$$

$$q_\mu T_a^{\mu\nu\lambda}=0,\quad p_\nu T_a^{\mu\nu\lambda}=0, \quad (4b)$$

$$q_\mu T_a^{\mu\nu}=0,\quad p_\nu T_a^{\mu\nu}=0. \quad (4c)$$

The second term on the right-hand side of (4a) is the anomaly. Here, the form of $\chi^{\mu\nu\alpha\beta}$ depends on whether or not there are direct interactions between the (octet or nonet) vector mesons and the fundamental constituents of currents, e.g., the quarks. In a renormalizable-field-theory model, the (octet or nonet) vector mesons, through which the spin-1 components of the currents are dominated, should be considered as "composite,"[13] and therefore they make no direct contribution to the WI anomalies. Accordingly, the anomalous term appearing in (4a) is

$$\chi^{\mu\nu\alpha\beta}=\chi g^{\mu\alpha}g^{\nu\beta}. \quad (5)$$

The constant χ will be fixed later.

From (4b) and (4c) we obtain

$$[g_\varphi^{-1}\sigma_\omega^{-1}R_{8\varphi}(0)R_{0\omega}(0)-\sigma_\varphi^{-1}g_\omega^{-1}R_{0\varphi}(0)R_{8\omega}(0)] \times q^\mu\Gamma_{\mu\nu\lambda}^{\omega,\varphi}(q,p)=0, \quad (6)$$

where

$$g_i\equiv g_{8i},\quad \sigma_i\equiv g_{0i},\quad i=\omega,\varphi.$$

We can express g_φ, etc., in terms of the couplings of ω and φ to the hypercharge and baryon currents[14]:

$$g_\varphi=\frac{\sqrt3}{2f_Y}m_\varphi^2\cos\theta_Y,\quad g_\omega=-\frac{\sqrt3}{2f_Y}m_\omega^2\sin\theta_Y,$$

$$\sigma_\varphi=\frac{\sqrt3}{\sqrt2 f_B}m_\varphi^2\sin\theta_B,\quad \sigma_\omega=\frac{\sqrt3}{\sqrt2 f_B}m_\omega^2\cos\theta_B,$$

where f_Y and f_B are defined in terms of the couplings of the hypercharge current $J_\mu^{(Y)}$, and the baryon current $J_\mu^{(B)}$, to the fields of ω and φ,

$$J_\mu^{(Y)}=f_Y^{-1}(\cos\theta_Y\, m_\varphi^2\varphi_\mu-\sin\theta_Y\, m_\omega^2\omega_\mu),$$
$$J_\mu^{(B)}=f_B^{-1}(\sin\theta_B\, m_\varphi^2\varphi_\mu+\cos\theta_B\, m_\omega^2\omega_\mu).$$

Using MD and the above-defined couplings g_φ, etc., we have

$$g_\varphi^{-1}\sigma_\omega^{-1}R_{8\varphi}(0)R_{0\omega}(0)-\sigma_\varphi^{-1}g_\omega^{-1}R_{0\varphi}(0)R_{8\omega}(0) =(3/2f_Bf_Y)m_\varphi^2m_\omega^2\cos\theta(\theta_Y-\theta_B)\neq0.$$

We conclude from (6) that

$$q^\mu\Gamma_{\mu\nu\lambda}^{\omega,\varphi}(q,p)=0. \quad (7a)$$

A similar argument gives

$$p^\nu\Gamma_{\mu\nu\lambda}^{\omega,\varphi}(q,p)=0, \quad (7b)$$

$$q^\mu\Gamma_{\mu\nu}^{\omega,\varphi}(q,p)=0,\quad p^\nu\Gamma_{\mu\nu}^{\omega,\varphi}(q,p)=0. \quad (8)$$

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[10] We use the metric $a_\mu b^\mu=a_0b_0-\mathbf{a}\cdot\mathbf{b}$.

[11] C.-S. Lai and B.-L. Young, Phys. Rev. 169, 1241 (1968).

[12] S. Coleman and H. J. Schnitzer, Phys. Rev. 134, B863 (1964); N. M. Kroll, T. D. Lee, and B. Zumino, *ibid.* 157, 1376 (1967).

[13] We mean the vector mesons as "composite" in the following sense: We consider the vector mesons which dominate the spin-1 channel of the vector currents as generated through a set of ladder diagrams (with strong-interaction gluons) in the channel of the currents. It follows, therefore, from S. Adler and W. Bardeen [Phys. Rev. 182, 1517 (1969)] that only the triangular graphs give rise to axial-vector WI anomalies for the vertex function (2a). The calculation of the anomalies follows directly from, e.g., R. W. Brown *et al.*, Ref. 5.

[14] R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters 19, 1266 (1968). The current-mixing model predicts $\theta_Y\simeq47.2°$, $\theta_B\simeq32.7°$; or $\theta_Y\simeq35°$, $\theta_B\simeq22.5°$. The mass-mixing model predicts $\theta_Y=\theta_B\simeq39°$.

From (4a) and (5) we obtain

$$k^\lambda \begin{pmatrix} \Gamma_{\mu\nu\lambda}{}^{\omega}(q,p) \\ \Gamma_{\mu\nu\lambda}{}^{\varphi}(q,p) \end{pmatrix} = -g_A f_\pi R_A{}^{-1}(0) \begin{pmatrix} \Gamma_{\mu\nu}{}^{\omega}(q,p) \\ \Gamma_{\mu\nu}{}^{\varphi}(q,p) \end{pmatrix} + \chi g_A R_A{}^{-1}(0) \begin{bmatrix} \dfrac{r_\varphi - 1}{r_\varphi - r_\omega} g_\omega \Delta_{8\varphi}{}^{-1}(q)_{\mu\mu'} \\ \dfrac{1-r_\omega}{r_\varphi - r_\omega} g_\varphi \Delta_{8\varphi}{}^{-1}(q)_{\mu\mu'} \end{bmatrix} \times g_\rho \Delta_\rho{}^{-1}(q)_{\nu\nu'} \epsilon^{\mu'\nu'\sigma\tau} q_\sigma p_\tau , \quad (9)$$

where $r_\omega = \sigma_\omega / g_\omega$ and $r_\varphi = \sigma_\varphi / g_\varphi$. We can write[15]

$$\Gamma_{\mu\nu\lambda}{}^{i}(q,p) = \epsilon_{\mu\nu\lambda\sigma}(A_1{}^i q^\sigma + A_2{}^i p^\sigma) + (B_1{}^i q_\mu + B_2{}^i p_\mu)\epsilon_{\nu\lambda\sigma\tau} q^\sigma p^\tau + (C_1{}^i q_\nu + C_2{}^i p_\nu)\epsilon_{\mu\lambda\sigma\tau} q^\sigma p^\tau \quad (10)$$

for $i = \omega, \varphi$. Using (7) and (8) to eliminate $A_1{}^i$ and $A_2{}^i$, we derive from (9)

$$\Gamma_{\mu\nu}{}^{\omega}(q,p) = \left[-\frac{g_A}{f_\pi m_A{}^2}(\alpha_\omega k^2 + \beta_\omega q^2 + \gamma_\omega p^2) + \frac{\chi}{f_\pi} \frac{r_\varphi - 1}{r_\varphi - r_\omega} \frac{q^2 - m_\omega{}^2}{g_\omega} \frac{p^2 - m_\rho{}^2}{g_\rho} \right] \times \epsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau , \quad (11a)$$

$$\Gamma_{\mu\nu}{}^{\varphi}(q,p) = \left[-\frac{g_A}{f_\pi m_A{}^2}(\alpha_\varphi k^2 + \beta_\varphi q^2 \times \gamma_\varphi p^2) + \frac{\chi}{f_\pi} \frac{1 - r_\omega}{r_\varphi - r_\omega} \frac{q^2 - m_\varphi{}^2}{g_\varphi} \frac{p^2 - m_\rho{}^2}{g_\rho} \right] \times \epsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau , \quad (11b)$$

where α_i, etc., are linear combinations of $B_{1,2}{}^i$ and $C_{1,2}{}^i$. In (11a) and (11b), the vector-current propagators are approximated by the corresponding vector-meson propagators.

Let us write

$$i\int d^4x\, e^{ipx} \langle 0 | T(J_\mu{}^{\text{em}}(x) J_\nu{}^{\text{em}}(0)) | \pi^0, k \rangle \equiv \epsilon_{\mu\nu\sigma\tau} p^\sigma q^\tau F(p,q)/\mu . \quad (12)$$

Then, using reduction formulas to define the off-mass-shell $\pi^0\gamma\gamma$ form factor $F(p,q) \equiv F(p^2,q^2,k^2)$, we arrive at the following expression:

$$F(p^2,q^2,k^2) = -\frac{2\chi e^2}{3f_\pi}\mu + \frac{e^2}{3}\frac{g_A}{m_A{}^2}\frac{1}{f_\pi \mu} \times \left\{ \frac{g_\rho}{p^2 - m_\rho{}^2} \left[\frac{g_\omega}{q^2 - m_\omega{}^2}(\alpha_\omega k^2 + \beta_\omega q^2 + \gamma_\omega p^2) + \omega \to \varphi \right] + \frac{g_\rho}{q^2 - m_\rho{}^2}[p \leftrightarrow q] \right\} . \quad (13)$$

In the soft-pion limit, the $\pi^0 \to 2\gamma$ amplitude is

$$F(0,0,0) = -\tfrac{2}{3}\chi e^2 (\mu/f_\pi) . \quad (14)$$

If the axial-vector WI (4a) is the naive one, i.e., $\chi = 0$, (14) clearly reduces to the result of Sutherland: $F(0,0,0) = 0$. In the Han-Nambu or the Maki-Hari model,[16,17]

$$\chi = 3/8\pi^2 , \quad (15)$$

which is also a result of Adler[3,18]:

$$|F(0,0,0)| = \left| \frac{\alpha}{\pi} \frac{\mu}{f_\pi} \right| \simeq 0.46\alpha , \quad (16)$$

while the sign of $F(0,0,0)$ is the negative of the sign of f_π.

Equation (16) agrees very well with the experimental data,[19]

$$|F_{\text{expt}}(0,0,\mu^2)| \simeq 0.45\alpha .$$

The sign of $F(0,0,0)$ also agrees with what is derived from the present experimental information.[20] Let us remark that in the (fractionally charged) quark model,

$$\chi = 1/8\pi ;$$

then $|F(0,0,0)|$ is only one-third of the experimental value.

III. ρ-ω-π AND ρ-φ-π VERTEX FUNCTIONS

From (11) we obtain the ρ-ω-π and ρ-φ-π (proper) vertex functions[21]:

$$g_{\rho\omega\pi}(q,p) = \frac{-g_A}{\sqrt{3} f_\pi m_A{}^2 \mu}(\alpha_\omega k^2 + \beta_\omega q^2 + \gamma_\omega p^2) + \frac{\chi}{\sqrt{3} f_\pi}\mu \frac{q^2 - m_\omega{}^2}{g_\omega} \frac{p^2 - m_\rho{}^2}{g_\rho} \frac{r_\varphi - 1}{r_\varphi - r_\omega} \quad (17a)$$

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[15] In addition to the six tensors used in (10), there are two more, $q_\lambda \epsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau$ and $p_\lambda \epsilon_{\mu\nu\sigma\tau} q^\sigma p^\tau$, which can be expressed in terms of the six in (10) due to the following identity: $g_{\lambda\rho}\epsilon_{\mu\nu\tau\sigma} + g_{\lambda\mu}\epsilon_{\nu\tau\sigma\rho} + g_{\lambda\nu}\epsilon_{\tau\sigma\rho\mu} + g_{\lambda\tau}\epsilon_{\sigma\rho\mu\nu} + g_{\lambda\sigma}\epsilon_{\rho\mu\nu\tau} = 0$.

[16] M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965); Y. Hara, *ibid.* **134**, B701 (1964); Z. Maki, Progr. Theoret. Phys. (Kyoto) **31**, 331 (1964).

[17] There is some controversy as to whether or not χ will be modified by the effects of strong and high-order electromagnetic interactions [e in (14) is, of course, the physical charge]. We refer to a detailed calculation on the renormalization effects by S. Adler and W. Bardeen, Ref. 13; and to the controversy raised by R. Jackiw, Ref. 7.

[18] Here, we use $|f_\pi| \simeq 93$ MeV, which corresponds to the experimental charged pion decay constant $|f_{\pi^\pm}| \simeq 132$ MeV.

[19] N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, A. H. Rosenfeld, P. Söding, C. G. Wohl, M. Roos, and G. Conforto, Rev. Mod. Phys. **41**, 109 (1969).

[20] S. Okubo, Phys. Rev. **179**, 1629 (1969); F. J. Gilman, *ibid.* **184**, 1964 (1969).

[21] We use the following definitions for the $\rho\omega\pi$ and $\omega\pi\gamma$ couplings:

$$\langle \rho, p, \epsilon | j_\pi | \omega, q, \epsilon' \rangle = g_{\rho\omega\pi} \epsilon_{\mu\nu\lambda\sigma} \epsilon^\mu \epsilon'^\nu q^\lambda p^\sigma / \mu ,$$
$$\langle \gamma, p, \epsilon | j_\pi | \omega, q, \epsilon' \rangle = f_{\omega\pi\gamma} \epsilon_{\mu\nu\lambda\sigma} \epsilon^\mu \epsilon'^\nu q^\lambda p^\sigma / \mu .$$

The couplings $\rho\varphi\pi$, $\varphi\pi\gamma$, and $\rho\pi\gamma$ are similarly defined.

and

$$g_{\rho\varphi\pi}(q,p)=\frac{-g_A}{\sqrt{3}f_\pi m_A{}^2\mu}(\alpha_\varphi k^2+\beta_\varphi q^2+\gamma_\varphi p^2)+\frac{\chi}{\sqrt{3}f_\pi}\mu\frac{q^2-m_\varphi{}^2}{g_\varphi}\frac{p^2-m_\rho{}^2}{g_\rho}\frac{1-r_\omega}{r_\varphi-r_\omega},\quad(17\text{b})$$

where p, q, and k are the momenta of ρ, $\omega(\varphi)$, and π. The off-mass-shell vector-meson–pion–photon couplings are given by

$$f_{\omega\pi\gamma}(q,p)=[eg_\rho/(p^2-m_\rho{}^2)]g_{\rho\omega\pi}(q,p)\,,$$
$$f_{\varphi\pi\gamma}(q,p)=[eg_\rho/(p^2-m_\rho{}^2)]g_{\rho\varphi\pi}(q,p)\,,\quad(18)$$
$$f_{\rho\pi\gamma}(q,p)=\frac{e}{\sqrt{3}}\left[\frac{g_\omega}{q^2-m_\omega{}^2}g_{\rho\omega\pi}(q,p)+\frac{g_\varphi}{q^2-m_\varphi{}^2}g_{\rho\varphi\pi}(q,p)\right].$$

The quantities α_ω, etc., are, in general, functions of p and q. Since we shall only consider cases of $|p^2|$ and $|q|^2\lesssim\frac{1}{4}m_\rho{}^2$, and $k^2\leq\mu^2$, we can invoke the smoothness assumption[9] and approximate them by real constants.

In the following, we shall concern ourselves mainly with the Han-Nambu or the Maki-Hari model, but shall briefly discuss the results of the quark model at the end of this section. From the discussion of the last section, where we have arrived at the result $|F(0,0,\mu^2)|\simeq|F(0,0,0)|$, barring an unforeseen cancellation, we see immediately that

$$\alpha_\omega\approx\alpha_\varphi\simeq0\,.\quad(19)$$

From now on, we can drop the pion momentum dependence in (13), (17a), and (17b). This is an explicit construction of amplitudes, which satisfy the soft-pion hypothesis. The remaining four constants can be determined from the various vector-meson decay widths. Notice that the anomalous term does not contribute to the decay amplitudes of the vector mesons.

To determine the rest of the parameters, we shall use the results of the spectral-function sum rules[22] as part of the input. g_ρ and g_A are given by

$$g_\rho{}^2=g_A{}^2=2m_\rho{}^2f_\pi{}^2\,.$$

There are two sets of values for g_ω and g_φ: One is given by Das, Mathur, and Okubo (DMO),[23]

$$g_\omega{}^2\simeq\frac{m_\omega{}^2(3m_\varphi{}^2+m_\rho{}^2-4m_{K^*}{}^2)}{3m_\rho{}^2(m_\varphi{}^2-m_\omega{}^2)}g_\rho{}^2\simeq0.43g_\rho{}^2\,,$$
$$g_\varphi{}^2\simeq\frac{m_\varphi{}^2(4m_{K^*}{}^2-m_\rho{}^2-3m_\omega{}^2)}{3m_\rho{}^2(m_\varphi{}^2-m_\omega{}^2)}g_\rho{}^2\simeq1.03g_\rho{}^2;\quad(20\text{a})$$

the other set is given by Oakes and Sakurai (OS),[14]

$$g_\omega{}^2\simeq\frac{3m_\rho{}^2(m_\varphi{}^2-m_{K^*}{}^2)-m_\varphi{}^2(m_{K^*}{}^2-m_\rho{}^2)}{3m_{K^*}{}^2(m_\varphi{}^2-m_\omega{}^2)}\left(\frac{m_\omega}{m_\rho}\right)^4g_\rho{}^2\simeq0.23g_\rho{}^2,\quad(20\text{b})$$
$$g_\varphi{}^2\simeq\frac{3m_\rho{}^2(m_{K^*}{}^2-m_\omega{}^2)+m_\omega{}^2(m_{K^*}{}^2-m_\rho{}^2)}{3m_{K^*}{}^2(m_\varphi{}^2-m_\omega{}^2)}\left(\frac{m_\varphi}{m_\rho}\right)^4g_\rho{}^2\simeq1.39g_\rho{}^2.$$

The relative sign of g_ρ and g_φ is taken to be positive, while that of g_ρ and g_ω is not determined. However, as we shall see later, this undetermined relative sign does not give rise to further ambiguities, since the signs of β_ω, etc., are now known.

The magnitudes of β_ω and β_φ can be calculated from the experimental widths of $\omega\to\pi+\gamma$ and $\varphi\to\pi\gamma$ [19]:

$$|\beta_\omega|\simeq0.98(1\pm0.07),\quad|\beta_\varphi|\lesssim0.04.\quad(19')$$

γ_φ is calculated from[19] $\Gamma_{\varphi\to\rho\pi}$:

$$|\gamma_\varphi|\lesssim0.18,\quad\gamma_\varphi/\beta_\varphi>0$$
$$\lesssim0.06,\quad\gamma_\varphi/\beta_\varphi<0.\quad(19'')$$

γ_ω can be calculated from $\omega\to3\pi$ in the Gell-Mann–Sharp–Wagner model.[24] To be consistent, we take the $\rho\pi\pi$ coupling from Ref. 9,[25]

$$g_{\rho\pi\pi}(p^2)=(m_\rho/\sqrt{2}f_\pi)[1-\tfrac{1}{4}(1+\delta)p^2/m_\rho{}^2]\,,$$

with $\delta\approx-1/\sqrt{2}$.[26] Here, the pions are on mass shell and the ρ, with 4-momentum p, is off mass shell. Using the above-mentioned $\rho\pi\pi$ coupling together with (17a) to fit the width of $\omega\to3\pi$, we obtain

$$\gamma_\omega=(0.08\pm0.14)\beta_\omega\,,\quad(19''')$$

which predicts that

$$\Gamma_{\rho\to2\pi}=0.3\text{ keV},\quad\text{DMO}$$
$$=0.2\text{ keV},\quad\text{OS}.$$

We have rejected another solution of γ_ω, $\gamma_\omega=(-5.2\pm0.26)\beta_\omega$. It gives a $\rho\to\pi+\gamma$ width of 3.3 MeV for DMO and 1.5 MeV for OS; both are much too large in comparison with the experimental upper limit 0.5±0.1 MeV.

Now that we have determined the parameters, we can proceed to make predictions with the $\pi^0\gamma\gamma$ form

[22] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967). We are aware of the controversy of the values of g_ρ, etc. For a review, see E. Lohrmann, in Proceedings of the Lund International Conference of Elementary Particle Physics, 1969 [DESY report (unpublished)].

[23] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 470 (1967).

[24] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[25] The ρ-π-π vertex function derived in Ref. 9 is still applicable, since the three-point function $\langle AAV\rangle$ does not need modification in any of its WI's. See K. Wilson, Ref. 5.

[26] In Ref. 9, $\delta=-\frac{1}{2}$ is used. It turns out that if the value $\delta=-1/\sqrt{2}$ is used, equally acceptable fits for the ρ and A_1 widths can be obtained. The latter value also predicts the ratio of the axial-vector form factor to the vector form factor for $\pi\to l+\nu+\gamma$ to be 0.4, which agrees excellently with the experimental data [P. Depommier, J. Heintze, C. Rubbia, and V. Soergel, Phys. Letters **7**, 285 (1963)]. The corresponding number for $\delta=-\frac{1}{2}$ is 0.2.

factor. We can immediately calculate the π^0 Dalitz decay ($\pi^0 \to \gamma e^+e^-$) form factor, which is defined as

$$F(\mu^2 y,0) \approx F(0,0,0)(1+a_\pi y)\,, \tag{21}$$

for $y \ll 1$. From (13), we have

$$a_{\pi^0} \simeq -\frac{4\pi^2}{3}\left(\frac{f_\pi}{\mu}\right)^2 \frac{g_\omega}{m_\omega^2}\beta_\omega \times\left[1+\frac{\gamma_\omega}{\beta_\omega}+\frac{g_\varphi/m_\varphi^2}{g_\omega/m_\omega^2}\frac{(\beta_\varphi+\gamma_\varphi)}{\beta_\omega}\right], \tag{22}$$

which gives the following results:

$$0.013 \lesssim |a_{\sigma^0}| \lesssim 0.032, \quad \text{DMO}$$
$$0.010 \lesssim |a_{\pi^0}| \lesssim 0.022, \quad \text{OS}$$
$$\operatorname{sgn} a_{\pi^0} = -\operatorname{sgn}(\beta_\omega g_\omega)\,.$$

Experimentally, there are the following pieces of information:

$$a_{\pi^0} = 0.01 \pm 0.11 \text{ (Ref. 27) and } -0.25 \pm 0.15 \text{ (Ref. 28)}.$$

Unlike the usual practice which assumes that $g_{\rho\varphi\pi}$ is always small and, therefore, that φ does not contribute to the decay $\pi^0 \to 2\gamma$, (17b) hints of an appreciable φ contribution to $\pi^0 \to 2\gamma$. Similar conclusions have been reached in Ref. 29 in a different approach. It is also interesting to note that the off-mass-shell values of $g_{\rho\omega\pi}$ and $g_{\rho\varphi\pi}$ depend on the coupling strengths of ω and φ to the hypercharge current, g_ω and g_φ; and on those to the baryon current, σ_ω and σ_φ. As has been mentioned in Sec. II, σ_ω and σ_φ are functions of θ_B and f_B. The quantity f_B is not known unless additional information on, for example, how the $U(3)$ symmetry is broken in the spectral functions of the vector currents. Specially, the usual assumption that $g_{\rho\varphi\pi} \approx 0$ requires that $r_\sigma \approx 1$ in the present model. This corresponds to a particular $U(3)$ symmetry-breaking effect.[30]

In the quark model, (19) is no longer true, and

$$F(0,\,0,\,k^2=\mu^2) = -\frac{\alpha}{\pi}\frac{\mu}{f_\pi} + \frac{2e^2}{3}\frac{g_A}{m_A^2}\frac{\mu}{f_\pi}\left(\alpha_\omega\frac{g_\omega}{m_\omega^2}+\alpha_\varphi\frac{g_\varphi}{m_\varphi^2}\right).$$

Taking the sign determination of Ref. 20, i.e., $F(0,\,0,\,k^2=\mu^2)$ and the anomalous term have the same sign, we get

$$\alpha_\omega\frac{g_\omega/g_\rho}{m_\omega^2/m_\rho^2}+\alpha_\varphi\frac{g_\varphi/g_\varphi}{m_\varphi^2/m_\rho^2} \approx -9.67\,. \tag{23}$$

[27] S. Devons, P. Némethy, C. Nissim-Sabat, E. Di Capua, and A. Lanzara, Phys. Rev. **184**, 1356 (1969).

[28] N. Samios, Phys. Rev. **121**, 275 (1961); H. Kobrak, Nuovo Cimento **20**, 1115 (1961).

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[30] We can calculate f_B by assuming that φ decouples from the nucleons at small momentum transfers, i.e., $\langle N,p'|j_\mu{}^\varphi(0)|N,p\rangle \simeq 0$ for $(p-p')^2 \simeq 0$. This gives $r_\omega \simeq 2/\sqrt{3}$, $r_\varphi \simeq -(2/\sqrt{3})\tan\theta_B \tan\theta_Y$. Then we have $g_{\rho\varphi\pi}(0,0)/g_{\rho\omega\pi}(0,0) \simeq 0.07$ DMO, 0.08 OS. Indeed, $g_{\rho\varphi\pi}$ is small in comparison with $g_{\rho\omega\pi}$ and hence its contribution to $\pi^0 \to 2\gamma$ decay is negligible. In the present model the smallness of $g_{\rho\varphi\pi}$ and $g_{NN\varphi}$ are consistent with each other.

Equations (19′) and (19‴) are replaced by

$$|-\alpha_\omega(\mu/m_\omega)^2+\beta_\omega| \approx 0.96(1\pm 0.07)$$

and

$$\gamma_\omega \simeq (0.08\pm 0.12)[-\alpha_\omega(\mu/m_\omega)^2+\beta_\varphi]\,.$$

Equation (23) indicates that at least one of α_ω and α_φ is of the order of 10. This will give rise to a large variation in $g_{\rho\omega\pi}$, $f_{\pi\rho\gamma}$, etc., when the pion momenum k is extrapolated from $k^2=\mu^2$ to $k^2=0$.

IV. MEASUREMENT OF $\pi^0\gamma\gamma$ FORM FACTOR IN COLLIDING-BEAM EXPERIMENTS

The Dalitz decay of π^0, $\pi^0 \to \gamma + e^+ + e^-$, provides information on the $\pi^0\gamma\gamma$ form factor for small (timelike) photon masses. A measurement of its behavior for large photon masses can be achieved in colliding-beam experiments such as

$$l^+ + l^- \to \pi^0 + \gamma \tag{24}$$

and

$$l^+ + l^- \to \pi^0 + l'^+ + l'^-, \tag{25}$$

where l and l' are leptons. To lowest order in the electromagnetic interaction, their kinematics are illustrated in the diagrams of Fig. 1. In both cases, the virtual photon momentum can attain arbitrarily high (timelike) values. Their scattering amplitudes are, respectively,

$$T(\pi^0\gamma) = e\bar{v}(p_2)\gamma^\mu u(p_1)\epsilon_{\mu\nu\lambda\sigma}\epsilon^\nu k^\lambda q^\sigma (P^2\mu)^{-1}F(P^2,0) \tag{26}$$

and

$$T(\pi^0 l'^+ l'^-) = e^2\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(q_1)\gamma^\nu v(q_2)\epsilon_{\mu\nu\lambda\sigma}P^\lambda Q^\sigma \times F(P^2,Q^2)/\mu P^2 Q^2\,, \tag{27}$$

where

$$P = p_1 + p_2, \quad Q = q_1 + q_2\,.$$

In the case that l and l' are electrons, the total-cross-section formulas are given by

$$\sigma(\pi^0\gamma) = (\alpha/24\mu^2)(1-\mu^2/s)^3 |F(s,0)|^2\,, \tag{28}$$

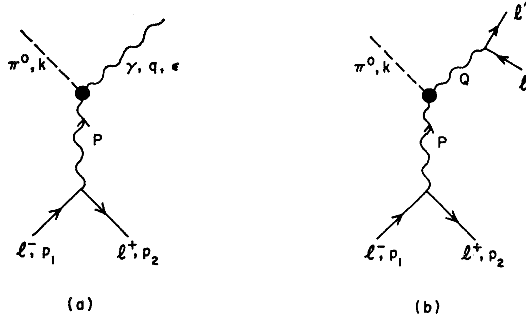

FIG. 1. Kinematics for the reactions (a) $l^+ + l^- \to \pi^0 + \gamma$ and (b) $l^+ + l^- \to \pi^0 + l'^+ + l'^-$.

$$\sigma(\pi^0e^+e^-)=\frac{\alpha^2}{72\pi\mu^2}\int_{x_1}^{x_2}\frac{dx}{x}|F(s,sx)|^2\left(1+\frac{2m^2}{sx}\right)\left(1-\frac{4m^2}{sx}\right)^{1/2}\times\left\{\left[\left(1+\frac{\mu}{s^{1/2}}\right)^2-x\right]\left[\left(1-\frac{\mu}{s^{1/2}}\right)^2-x\right]\right\}^{3/2},\quad(29)$$

where m is the electron mass,

$$s=P^2,\qquad x=Q^2/s,$$
$$x_1=4m^2/s,\quad x_2=(1-\mu/s^{1/2})^2.$$

It is obvious from (28) that a measurement of the energy spectrum of $\sigma(\pi^0\gamma)$ gives $|F(s,0)|^2$ directly.

A quantity similar to a_{π^0} [see (21)] can be defined here:

$$F(s,sx)\approx F(s,0)[1+a_\pi(s)x]$$

for $x\ll1$, where

$$F(s,0)=-\frac{\alpha}{\pi}\frac{\mu}{f_\pi}-\frac{e^2}{3}\frac{g_A}{m_A{}^2}\frac{s}{f_\pi\mu}\times\left[\frac{g_\rho}{s-m_\rho{}^2}\left(\frac{g_\omega}{m_\omega{}^2}\gamma_\omega+\frac{g_\varphi}{m_\varphi}\gamma_\varphi\right)+\frac{g_\rho}{m_\rho{}^2}\left(\frac{g_\omega}{s-m_\omega{}^2}\beta_\omega+\frac{g_\varphi}{s-m_\varphi{}^2}\beta_\varphi\right)\right],\quad(30)$$

$$F(s,0)a_\pi(s)=-\frac{e^2}{3}\frac{g_A}{m_A{}^2}\frac{s}{f_\pi\mu}\left\{\frac{g_\rho}{s-m_\rho{}^2}\left(\frac{sg_\omega}{m_\omega{}^2}\gamma_\omega+\frac{sg_\varphi}{m_\varphi{}^2}\gamma_\varphi+\frac{g_\omega}{m_\omega{}^2}\beta_\omega+\frac{g_\varphi}{m_\varphi{}^2}\beta_\varphi\right)+\frac{g_\rho}{m_\rho{}^2}\left[\frac{s}{m_\rho{}^2}\left(\frac{g_\omega}{s-m_\omega{}^2}\beta_\omega+\frac{g_\varphi}{s-m_\varphi{}^2}\beta_\varphi\right)+\frac{g_\omega\gamma_\omega}{s-m_\omega{}^2}+\frac{g_\varphi\gamma_\varphi}{s-m_\varphi{}^2}\right]\right\}.\quad(31)$$

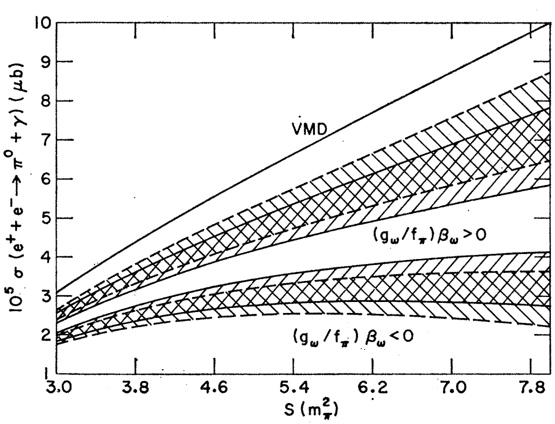

FIG. 2. Predicted cross sections for the reaction $e^++e^-\to\pi^0+\gamma$. The areas bounded by the solid and broken lines are predicted in the present model with, respectively, the OS and DMO values for the vector-meson–photon couplings. The lines themselves are the upper or lower bounds of the predictions. The single line is predicted by VMD with the OS value for the vector-meson–photon couplings.

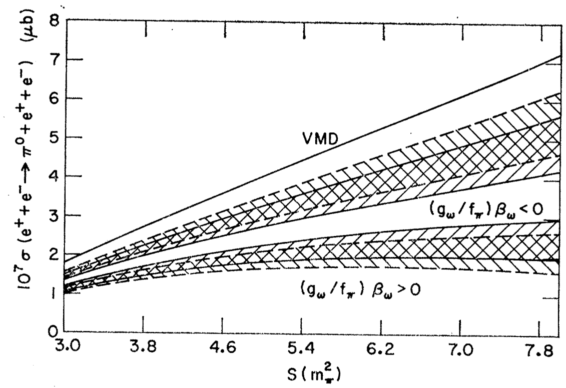

FIG. 3. Predicted cross sections for the reaction $e^++e^-\to\pi^0+e^++e^-$ (see caption of Fig. 2).

Using (13) and (19)–(19'''), we have plotted $\sigma(\pi^0\gamma)$, $\sigma(\pi^0e^+e^-)$, and $|a_\pi(s)|$, respectively, in Figs. 2–4. In the energy range under consideration, $F(s,0)$ is always negative and therefore the sign of $a_\pi(s)$ is that of $g_\omega\beta_\omega/f_\pi$. In these figures, uncertainties in the magnitudes and ambiguities in the signs of β_ω, etc., have been indicated in the plots by marking with hatching.

The reactions which compete with and form back-

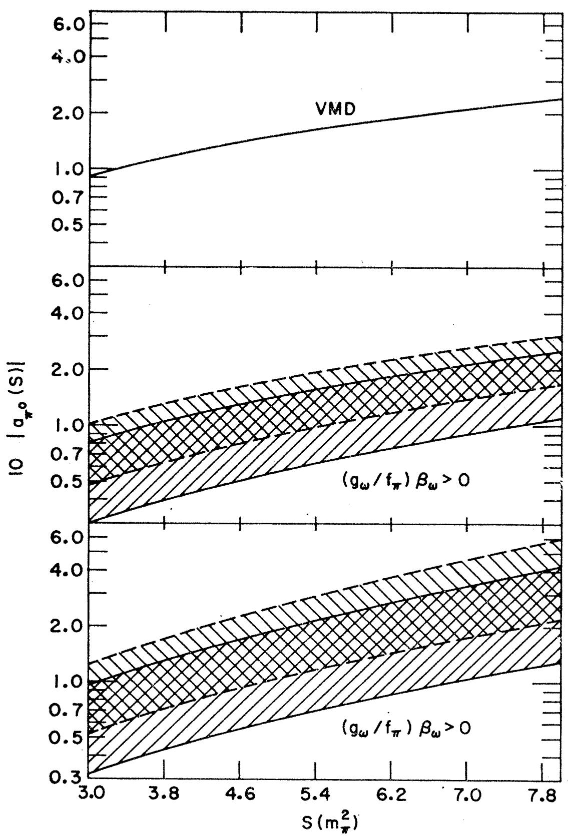

FIG. 4. Predicted values of $|a_{\pi^+}(s)|$ (see caption of Fig. 2).

TABLE I. Summary of the predictions of the anomalous WI and VMD models. If available, the experimental data are also listed.

g_ω, g_φ	Anomalous WI OS	Anomalous WI DMO	VMD OS	VMD DMO	Expt.
$\vert F(0,0)\vert$	0.46α		$0.49\alpha(1\pm0.05)$	$0.68\alpha(1\pm0.05)$	$0.45\alpha(1\pm0.16)$
$\vert a_{\pi^0}\vert$	$\left\{\begin{matrix}\le 0.022\\ \ge 0.01\end{matrix}\right.$	$\left.\begin{matrix}\le 0.032\\ \ge 0.01\end{matrix}\right\}$	0.031		Refs. 28 and 29
$(f_{\rho\pi\gamma}^2/4\pi)\times 10^2$	$\lesssim 0.33\alpha$	$\lesssim 0.4\alpha$	1.4α	2.6α	$<76\alpha(1\pm0.2)$
$\Gamma_{\rho\to\pi\gamma}$ (keV)	$\lesssim 0.22$	$\lesssim 0.26$	0.9	1.7	$<5\pm1$
$\sigma(e^+e^-\to\pi^0\gamma)$	Fig. 2	Fig. 2	Fig. 2	...	...
$\sigma(e^+e^-\to\pi^0e^+e^-)$	Fig. 3	Fig. 3	Fig. 3	...	...
$\vert a_\pi(s)\vert$	Fig. 4	Fig. 4	Fig. 4	...	...

ground of (24) are the electromagnetic processes

$$e^+ + e^- \to 2\gamma \tag{32}$$

and

$$e^+ + e^- \to 3\gamma\,, \tag{33}$$

where (32) is of order α^{-1} larger than (24) and (33) is of the same order of (24). Despite its larger rate, (32) can be readily distinguished from (24) in experiments, even though (33) may simulate reactions like (24). The reactions (24) and (33) can also be distinguished by means of their different distributions in the final (three) photons. Furthermore, if (24) is observed in a selected configuration, the partial rate of (24) is an order of magnitude larger than that of (33), which can be calculated exactly to the lowest order in α. We shall discuss this briefly in the Appendix.

V. COMPARISON WITH VECTOR-MESON-DOMINANCE MODEL

In this section, we compare the results obtained in VMD with those of the previous sections in order to illustrate their differences.

Restricting ourselves to the cases where $|q^2|$ and $|p^2|$ are less than, say, $\frac{1}{4}m_\rho^2$, the $\pi^0\gamma\gamma$ form factor can be written in VMD as

$$F(p^2,q^2)=\frac{e^2}{\sqrt{3}}g_\rho g_\omega g_{\rho\omega\pi}\left[\frac{1}{(p^2-m_\rho^2)(q^2-m_\omega^2)}+\frac{1}{(p^2-m_\omega^2)(q^2-m_\rho^2)}\right], \tag{34}$$

where p and q are the momenta of the (off-mass-shell) photons; eg_ρ and $eg_\omega/\sqrt{3}$ are, respectively, the ρ- and ω-photon coupling constants. The magnitude of $g_{\rho\omega\pi}$, which is calculated from $\omega\to 3\pi$ in the Gell-Mann–Sharp–Wagner model,[31] is

$$g_{\rho\omega\pi}^2/4\pi \simeq 0.51\,.$$

[31] The value $g_{\rho\omega\pi}^2/4\pi\simeq0.51$ is obtained in the original Gell-Mann–Sharp–Wagner model (Ref. 24). The ρ-π-π coupling constant $g_{\rho\pi\pi}$ is given by $g_{\rho\pi\pi}^2/4\pi\simeq2.4$, corresponding to a ρ width of 125 MeV (see Ref. 18).

Then, we obtain

$$\begin{aligned}|F(0,0)|&=(2e^2/\sqrt{3})\,|g_\rho g_\omega g_{\rho\omega\pi}|\,(m_\rho^2m_\omega^2)^{-1}\\&=0.68\alpha(1\pm0.05),\quad \text{DMO}\\&=0.49\alpha(1\pm0.05),\quad \text{OS}\end{aligned}$$

$$a_{\pi^0}=\tfrac{1}{2}m_{\pi^0}^2(m_\rho^{-2}+m_\omega^{-2})\simeq0.031\,,$$

and

$$a_{\pi^0}(s)=\frac{s[m_\omega^4(s-m_\rho^2)+m_\rho^4(s-m_\omega^2)]}{m_\rho^2m_\omega^2[m_\omega^2(s-m_\rho^2)+m_\rho^2(s-m_\omega^2)]}\approx\frac{s}{m_\rho^2}.$$

The prediction of OS for $|F(0,0)|$ agrees with the experimental data. With OS, we also plot the predictions of VMD for $\sigma(\pi^0\gamma)$, $\sigma(\pi^0e^-e^+)$, and $a_{\pi^0}(s)$, respectively, in Figs. 2, 3, and 4. Notice that in VMD a_{π^0} and $a_{\pi^0}(s)$ are positive. In Table I, we summarize the predictions of the present model and those of the VMD.

VI. DISCUSSION

We have used the anomalous WI evaluated in the Han-Nambu or the Make-Hara model to derive an expression for the $\pi^0\gamma\gamma$ form factor of low-mass (virtual) photons. Even though only the upper bounds of the values for some of the parameters in the model are determined, the predictions are already different from those of the conventional VMD model. By confrontation with experiments (i.e., the electron-positron colliding-beam experiments discussed in Sec. IV), we can further check the existence of the axial-vector WI anomaly. To narrow down the values of the parameters, and therefore to subject the present model to more stringent tests, improved measurements of the radiative decay modes of the vector mesons are required.

We have noted that the vertex functions $g_{\rho\omega\pi}(q,p)$ and $g_{\rho\varphi\pi}(q,p)$ are not very smooth. In particular, $g_{\rho\varphi\pi}$ may not always be small.[30] In order to know the off-mass-shell values of $g_{\rho\omega\pi}$ and $g_{\rho\varphi\pi}$, we need to determine the baryon current coupling strengths to the vector mesons or vice versa.

Since with the quark-model prediction one loses the smoothness extrapolation of the pion mass, we have considered only the value of χ obtained in the Han-Nambu or the Maki-Hara model. However, on the ground of the smoothness assumption alone, we cannot rule out the quark model. It will be interesting to carry out the same calculation of Sec. IV, with χ calculated in the quark model. This will enable us to see whether or not the study of reactions (24) and (25) will lead to a distinction of the charges of the quarks, integral or fractional. However, the nonvanishing of α_ω and α_φ makes it impossible to determine all the parameters α_ω, β_ω, etc., with the present experimental information. Should the experimental data, for example, of a_{π^0} be much improved, such a calculation will be desirable.

The program presented here can, in principle, be applied to the case of the η meson. Its Dalitz decay mode, $\eta\to\gamma+e^++e^-$, serves as an ideal test of models for the electromagnetic interactions of hadrons. However, one is beset here by the present experimental uncertainties; the η-X^0 mixing and the validity of the assumption of soft η are further problems in the model. When suitable experimental data, e.g., the widths of $X^0\to 2\gamma$, V(vector meson) $\to\eta\gamma$, etc., are available, we shall be able to investigate this problem in the present model.

ACKNOWLEDGMENTS

The author thanks Dr. R. W. Brown for reading the manuscript and Dr. R. Schult for helpful discussion.

APPENDIX

In this appendix we shall discuss briefly the process

$$e^++e^-\to 3\gamma \tag{33}$$

for the situation which may simulate the reaction (24). The cross-section formula has been given in Jauch and Rohrlich.[32] We shall follow their notation. Let k_1, k_2, and k_3 be the momenta of the photons and p_1 and p_2 those of leptons. Averaging (summing) over the leptons (photon) spins, we obtain

$$\sigma(3\gamma)=\frac{\alpha^3}{2\pi^2m^2[(p_1,p_2)^2-m^4]^{1/2}}\int d^4k_1d^4k_2d^4k_3$$
$$\times\delta_+(k_1^2)\delta_+(k_2^2)\delta_+(k_3^2)\delta^4(k_1+k_2+k_3-p_1-p_2)X. \tag{A1}$$

Denoting

$$m^2K_i=p_ik_i,\quad m^2K_i'=p_2k_i,\quad i=1,2,3$$

we can write

$$\begin{aligned}X=&2(ab-c)[8+ab-c-(a+b)(x+2)]\\&+2x(a^2+b^2)+8c+(4X/AB)\{x^2(z-1)-2z\\&+(aA+bB)[2+z(1/x-1)]-(A+B)(x+1)\}\\&+2\rho[ab+c(1-x)],\end{aligned}$$

where

$$a=\sum_1^3\frac{1}{K_i},\quad b=\sum_1^3\frac{1}{K_i'},\quad c=\sum_1^3\frac{1}{K_iK_i'},$$
$$x=\sum_1^3K_i=\sum_1^3K_i',\quad z=\sum_1^3K_iK_i',$$
$$A=K_1K_2K_3,\quad B=K_i'K_2'K_3',\quad \rho=\sum_1^3\left(\frac{K_i}{K_i'}+\frac{K_i'}{K_i}\right).$$

Let us choose $\mathbf{p}_1$ along the z direction, $\mathbf{k}_1$ in the zx plane. We can perform several of the integrations and obtain

$$d^4\sigma(3\gamma)=\frac{\alpha}{8\pi m^2s}\frac{\omega_1(s-2W\omega_1)}{[W-\omega_1(1-\cos\theta_{12})]^2}\times Xd\omega_1d\cos\theta_1d\Omega_2, \tag{A2}$$

where s is the total energy, $W=s^{1/2}$, ω_1 is the energy of the first photon, and θ_{12} is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. In (A2), ω_2 and ω_3 are evaluated at

$$\omega_2=\frac{s-2W\omega_1}{2[W-\omega_1(1-\cos\theta_{12})]},$$
$$\omega_3=\frac{s-2\omega_1(W-\omega_1)(1-\cos\theta_{12})}{2[W-\omega_1(1-\cos\theta_{12})]}.$$

Since the energy of the photon in (24) is

$$\omega_0=(s-\mu^2)/2W,$$

then (33) will simulate (24) when one of the photons has approximately the energy ω_0. The main contribution to (A1) comes from the configuration that two of the photons lie close to the beam direction and the third photon is soft. Hence, the partial rate of (33) will be minimum when all the photons are in the plane perpendicular to the beam directions. Let us calculate the partial rate of (33) for, say,

$$\tfrac{1}{4}\pi\leq\theta_1,\theta_2\leq\tfrac{3}{4}\pi,\quad 0\leq\varphi_2\leq 2\pi$$
$$\omega_0-\Delta\omega\leq\omega_1\leq\omega_0+\Delta\omega,$$

i.e.,

$$\Delta\sigma(3\gamma)=\int_0^{2\pi}d\varphi_2\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}d\omega_1\int_{-1/\sqrt{2}}^{1/\sqrt{2}}d\cos\theta_1\times\int_{-1/\sqrt{2}}^{1/\sqrt{2}}d\cos\theta_2\frac{d^4\sigma}{d\omega_1d\cos\theta_1d\Omega_2}.$$

Taking $\Delta\omega=20$ MeV, we have

$$\Delta\sigma(3\gamma)\lesssim 0.25\times 10^{-5}\ \mu\text{b}$$

for $3m_\pi^2\leq s\leq 8m_\pi^2$.

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[32] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), p. 270.