[where $D(x)$ is the divergence of the Cabibbo current], as one readily sees from the Bjorken limit. The commutators (4) do not contribute to the divergence, as in Sec. IV. If one takes PCAC literally, so that $D(x)$ is related to canonical spin-zero fields, then (44) is a harmless c number, and first-order weak processes would be finite.

It is intriguing that, in the quark model of Sec. III, the commutators are not expressible in terms of traceless tensors unless the quark mass vanishes. This may have something to do with the idea that part of the Lagrangian of the model is scale invariant.[25] One would hope that whatever breaks scale invariance does not affect the commutators (4), (5), and (7), in much the same way that mass terms in Yang-Mills theories do not affect the usual current-algebra commutators.

Note added in manuscript. After this manuscript was finished, it was brought to my attention that a paper by R. A. Brandt, Phys. Rev. D (to be published), deals with some of the topics of the present paper in somewhat different form.

[25] See, e.g., G. Mack and A. Salam, Ann. Phys. (N. Y.) **53**, 174 (1969); D. J. Gross and J. E. Wess, Phys. Rev. D (to be published); P. Carruthers and M. Gell-Mann (unpublished).

ACKNOWLEDGMENTS

It is a pleasure to acknowledge many stimulating conversations with Professor David Gross, which were essential to the ideas in this paper. I would also like to thank Professor J. Prentki of the Theory Division at CERN for his hospitality during the period when this work was initiated.

PHYSICAL REVIEW D VOLUME 2, NUMBER 3 1 AUGUST 1970

Absorption Model Applied to π-π Scattering

L. CHAN* AND V. HAGOPIAN†
The University of Pennsylvania, Philadelphia, Pennsylvania 19104 and The Florida State University, Tallahassee, Florida 32306

AND

P. K. WILLIAMS
The Florida State University, Tallahassee, Florida 32306

(Received 1 April 1970)

A study of the possibility of using an absorption model to obtain a unique s-wave $I=0$ $\pi\pi$ phase shift from the reaction $\pi^-p\to\pi^-\pi^+n$ was made. Applying the model to combined data from several latoratories, we obtained a set of resaonable (but twofold ambiguous) values for the isospin-zero s-wave phase shift $\delta_0{}^0$, using the *isotropic* moment of the dipion decay distribution. Values for $\delta_0{}^0$ deduced by using the s-p wave *interference* term in the dipion decay cross section agreed well with previous (twofold ambiguous) results obtained by extrapolation procedures. Comparing the two sets of twofold ambiguous solutions for $\delta_0{}^0$, the lower or "down" branch was preferred marginally in the dipion mass region below the ρ meson. No unique conclusion could be drawn in the higher-mass region. We emphasize the model dependence of these results, but we believe that similar studies with improved statistics for $-t>\mu^2$ would yield similar results. Good data for $-t<\mu^2$ are required to do definitive model-independent analyses.

I. INTRODUCTION

SEVERAL methods have been used recently to deduce $\pi\pi$ phase shifts $\delta_l{}^I$ in the dipion energy range 500–900 MeV from the reactions[1,2]

$$\pi^-p\to\pi^-\pi^0p \qquad (1)$$

and

$$\pi^-p\to\pi^-\pi^+n. \qquad (2)$$

These methods generally use "evasive" Chew-Low[3] extrapolations of polynomials representing $(\mu^2-t)^2d\sigma/dt$, and various moments thereof, which are constrained to vanish at $t=0$, where μ is the charged pion mass and t is the square of the four-momentum transfer to the nucleon. In particular, using the $I=2$ s-wave phase shifts from Baton and Laurens,[2] Marateck *et al.*[1] obtained sets of $I=0$ s-wave $\pi\pi$ phase shifts $\delta_0{}^0$ from the extrapolated moments A_2 and A_1 in the decay distribution of the dipion in reaction (2) in the $\pi\pi$ c.m. frame,

$$W(\theta)=A_0+A_1\cos\theta+A_2\cos^2\theta\,, \qquad (3)$$

* Present address: Physics Department, Georgia Institute of Technology.
† Present address: Physics Department, The Florida State University.
[1] S. Marateck *et al.*, Phys. Rev. Letters **21**, 1613 (1968); for many details of the collaboration data see also V. Hagopian, in Proceedings of the Conference on $\pi\pi$ and $K\pi$ Scattering, Argonne National Laboratory, 1969, p. 149 (unpublished). The data of the Florida State University group at 2.26 GeV/c were reported by B. Reynolds *et al.*, Phys. Rev. **184**, 1424 (1969). The nonevasive production mechanism is indicated in a recent study by J. Scharenguivel *et al.*, Phys. Rev. Letters **24**, 332 (1970).
[2] J. Banton and G. Laurens, Phys. Rev. **176**, 1574 (1968).
[3] G. Chew and F. Low, Phys. Rev. **113**, 1640 (1959).

where θ is the polar dipion scattering angle. We review briefly some of the problems connected with δ_0^0 deduced in Ref. 1: (a) Charged-pion photoproduction and, by vector-dominance arguments, also reaction (2) reveal some structure at $-t \leq +\mu^2$, in apparent contradiction with simple vanishing of $d\sigma/dt$ at $t=0$, as assumed in Ref. 1. This low-$|t|$ behavior can be partially understood in the context of absorption models or possibly other "nonevasive" peripheral models. (b) The simple-polynomial extrapolated moment for A_0 yields a result for the s-wave cross section which exceeds the limits imposed by unitarity[1] by a factor of ~ 3. Again, this is qualitatively suggestive of the effects of absorption—in this case, effects of depolarization "leakage" of the resonant $\pi\pi$ p wave from A_2 into the isotropic moment A_0. (c) "Evasive" extrapolation of A_0 with presently available data therefore yields no directly useful information about δ_0^0. The extrapolated moments A_1 and A_2 can be combined to give a twofold ambiguous solution for δ_0^0. As the two branches merge together in the 700-MeV region, there are manifested four possible combinations of solutions for δ_0^0, labeled "up-down," "down-up," "down-down," and "up-up," using now standard nomenclature.[1] It should be noted that the "up" solutions of different analyses are not always identical, and likewise of course for the "down" solutions.

From the above discussion, it is obvious that one way to attempt to resolve the present ambiguity in δ_0^0 mentioned above is to examine the relative consistency of these solutions with a set of meaningful values for δ_0^0 obtained from the A_0 moment by the use of a theoretical model which incorporates correctly the effects of absorption (specifically, we mean depolarization of the p wave and nonevasion). This can be done by introducing parameters (in an extrapolation) which describe the qualitative features of absorptive effects, or by using a more quantitative absorption model. The method of Kane and Ross[4] is an eight-free-parameter description inspired by qualitative behavior of the absorption mechanism at high energies and low $|t|$. With the best available experimental data, the fit to this model is not considered practical because of insufficient statistics and the assumption of nucleon spin-flip dominance. The method of Williams,[5] however, includes some quantitative features of absorption at moderate energies with only four parameters (including the three most important $\pi\pi$ phase shifts). In this analysis, we apply this method[5] of introducing absorptive effects to data limited to the extreme forward angles, $1.6\mu^2 \leq |t| \leq 4\mu^2$, in reaction (2) in order to deduce $\pi\pi$ phase shifts. The region of $|t|$ was so chosen as to partially minimize kinematic cutoff effects of data combined from several laboratories, to accommodate specific assumptions in the model, and also to minimize effects of non-pion-exchange background. We use, for this purpose, the collaboration data of Ref. 1 combined with the data at 2.26 GeV/c from the Florida State University group. This is by far the largest supply of data available (4817 events were included). We shall examine, with a suitably chosen set of conditions, the possibility of obtaining a set of reasonably consistent solutions for the phase shifts from the model. We will show that the incorporation of absorption in the moment A_1 gives essentially the same (ambiguous) solutions for δ_0^0 as those of Ref. 1, but with somewhat smaller statistical errors (we emphasize, however, the model dependence of our results). It will also be shown that the additional information extracted from A_0 allows us to only marginally prefer the "down" solution in the lower $m_{\pi\pi}$ regions (i.e., $m_{\pi\pi} \leq 700$ MeV). However, both solutions for $m_{\pi\pi} > 700$ MeV are consistent, so we make no claims about resolving the δ_0^0 puzzle.

In Sec. II, a brief description of the method proposed in Ref. 5 is given. In Sec. III, the procedure followed in fitting the model to experimental data is described. Section IV contains the results obtained from using the above procedure on data, and Sec. V contains discussions of the results and possibilities for improving upon the present method as well as on the simple polynomial extrapolations, should a larger supply of data be available in the future.

II. DESCRIPTION OF MODEL

A full treatment of the method used in this analysis (hereafter referred to as I) can be found in the literature.[5] We shall discuss the main aspects of the model in this section.

The OPE (one-pion exchange) amplitude for the physical process

$$\pi(A)+N(B) \to \pi(1)+\pi(2)+N(3)$$

was written in I in the form

$$M(AB \to 123) = \sum_{l,\lambda} F_l B[AB \to (12)_l{}^\lambda + 3] Y_l{}^\lambda(\theta', \phi'), \quad (4)$$

where

$$F_l = [4\pi(2l+1)]^{1/2} P_A{}^{-l} A_{\pi\pi}{}^{\text{off}}(l,\sigma,t), \quad (5)$$

$$B[AB \to (12)_l{}^\lambda + 3] = [P_A{}^l d_{\lambda,0}{}^l(\psi)] \frac{\bar{u}(P_3)\gamma_5 u(P_1)}{\mu^2 - t} \frac{G_{\pi NN}}{(4\pi)^{1/2}}. \quad (6)$$

In Eqs. (4)–(6), all explicit angles and three-momenta are evaluated in the dipion rest frame; l and λ are the spin and helicity of the $\pi\pi$ system, $t=(P_B-P_3)^2$ and $\sigma=(P_1+P_2)^2=m_{\pi\pi}^2$, θ' is the angle between P_1 and $-P_3$, ϕ' is the azimuthal angle of P_1 with respect to

[4] G. Kane and M. Ross, Phys. Rev. **177**, 2353 (1969); see also G. Kane, in Proceedings of the Conference on $\pi\pi$ and $K\pi$ Scattering, Ref. 1, p. 533.

[5] P. K. Williams, Phys. Rev. D **1**, 1312 (1970); see also G. Kane, in Ref. 4.

the production plane, and the argument ψ of the rotation function $d_{\lambda,0}{}^l$ is the angle between P_A and $-P_3$. One can envision F_l as the effective strength of the quasi-two-body reaction $AB\to(12)_l{}^\lambda+3$ described by the amplitude B in Eq. (6).

A simplifying assumption, the validity of Selleri's off-shell factors,[6] is then suggested for moderately small σ and t. Under this assumption F_l can be rewritten in terms of the *on-shell* $\pi\pi$ scattering amplitude as

$$F_l=[4\pi(2l+1)]^{1/2}P_1{}^{-l}A_{\pi\pi}{}^{\mathrm{on}}(l,\sigma,\mu^2)\,, \qquad (7)$$

which is independent of t. Thus the t dependence of the amplitude appears only in B, which is proportional to the helicity projection of the *usual invariant* production amplitude, to which the usual absorption prescriptions can be applied.

The normalization of $M(AB\to123)$ is such that

$$A_{\pi\pi}{}^{\mathrm{on}}(l,\sigma,\mu^2)=(\sqrt{\sigma}/P_1)e^{i\delta(l,\sigma)}\sin\delta(l,\sigma) \qquad (8)$$

with the cross section, denoted by N, given by

$$N=(8\pi sP_*{}^2)^{-1}\int\cdots\int dt\,d\sigma\,d\cos\theta'\,d\phi'\frac{P_1}{\sqrt{\sigma}}\,\bar{\Sigma}\,|M|^2, \qquad (9)$$

where $\delta(l,\sigma)$ is the on-shell l-wave $\pi\pi$ phase shift, $s=(P_A+P_B)^2$, P_* is the initial c.m. momentum, and $\bar{\Sigma}$ denotes the usual spin sum average.

A simple prescription for introducing absorption to B was given in I. It involves including first the major helicity-dependent effects of absorption in B and representing the remaining helicity-independent effect of absorption by multiplication of the amplitude with a collimation factor $F_c(t)$, taken to be

$$F_c(t)=\exp[A(t-\mu^2)]. \qquad (10)$$

The major helicity-dependent effects of absorption, on the other hand, are assumed to be given by removing from B the large "exceptional" terms in the lower partial waves.[7] This subtraction can be accomplished with a simple procedure which eliminates the need for partial-wave decomposition of B. To do this, one first writes each Born helicity amplitude B as the product of the pion propagator term, some minimal angular factors required for angular momentum conservation, and a polynomial in t. Replacing t by μ^2 in this polynomial then exactly accomplishes the removal of the exceptional terms.

Guided by the behavior of the collimation factor at high energies, Williams[5] partially justified the assumption that A is essentially independent of l. We found it necessary to make the further assumption that A is only weakly dependent on σ in the region of interest. We shall in fact choose a *constant* value for A in our study. This will be accomplished by fitting the t dependence of the theoretical p-wave cross section (as a function of t) to data (from which the most probable background has been removed) restricted to the ranges $740\le m_{\pi\pi}\le 800$ MeV and $2\mu^2\le|t|\le 12\mu^2$.

Finally, denoting the helicity projections of B in Eq. (6) by $\langle\lambda\lambda_3|B_l(t)|\lambda_B\rangle$, and applying the above absorptive prescription to them, gives

$$\langle\lambda\lambda_3|M_l{}^{\mathrm{abs}}(s,t,\sigma,\theta',\phi')|\lambda_B\rangle=F_lY_l{}^\lambda(\theta',\phi')\langle\lambda\lambda_3|B_l{}^{\mathrm{abs}}|\lambda_B\rangle$$
$$=\langle\lambda\lambda_3|m_l|\lambda_B\rangle Y_l{}^\lambda(\theta',\phi')\,, \qquad (11)$$

where

$$\langle|B_l{}^{\mathrm{abs}}|\rangle=\frac{e^{A(t-\mu^2)}}{\mu^2-t}\left(\frac{1-x}{1-z}\right)^{|\alpha-\beta|/2}\left(\frac{1+x}{1+z}\right)^{|\alpha+\beta|/2}\times[(\mu^2-t)\langle|B_l|\rangle]_{t=\mu^2},$$

where $x\equiv\cos\theta_*$ (θ_* is the c.m. scattering angle) and $z=[x]_{t=\mu^2}$; $\alpha=\lambda$ and $\beta=\lambda_3-\lambda_A$.

From the absorptive OPE amplitudes given in Eq. (11), it is routine to arrive at expressions for the density-matrix elements describing the decay of the dipion in the helicity frame of reference (suppressing kinematic variables s and t):

$$\rho_{\lambda,\lambda'}{}^h\equiv\rho_{\lambda\lambda'}{}^{11},\quad \rho_{s\lambda}{}^h\equiv\rho_{0\lambda}{}^{01},\quad \rho_s{}^h\equiv\rho_{00}{}^{00},$$

where the (weighted) density-matrix elements are given by

$$\rho_{\lambda'\lambda}{}^{l'l}=\sum_s f_s\sum_{\lambda_3,\lambda_b}\langle\lambda'\lambda_3|m_{l'}|\lambda_B\rangle^*\langle\lambda\lambda_3|m_l|\lambda_B\rangle\Big/\sum_s f_s\sum_{l,\lambda}\sum_{\lambda_3\lambda_B}|\langle\lambda\lambda_3|m_l|\lambda_B\rangle|^2. \qquad (12)$$

Here f_s is the number of events per microbarn for the experiments at c.m. energy $\sqrt{s}$. In terms of the weighted density-matrix elements, the decay distribution of events of the dipion in the helicity frame is

$$W^h(\theta')=\sum_s f_s\int_{t_0}^{t_0+\Delta t}dt\,\Delta(\sqrt{\sigma})\frac{d^2N}{dt\,d\sqrt{\sigma}}\bar{W}(\theta')$$
$$=A_0{}^h+A_1{}^h\cos\theta'+A_2{}^h\cos^2\theta'\,, \qquad (13)$$

where $2(A_0{}^h+\frac{1}{3}A_2{}^h)=4817$ events, and

$$\bar{W}(\theta')=\tfrac{1}{2}[(\rho_s{}^h+3\rho_{11}{}^h)+2\sqrt{3}\rho_{s0}{}^h\cos\theta'+3(\rho_{00}{}^h-\rho_{11}{}^h)\cos^2\theta'].$$

The corresponding expression for the decay distribution in the Gottfried-Jackson (GJ) frame[8] is given by replacing the polar angle θ' by the corresponding polar angle in the GJ frame, and transforming the helicity frame density matrix to the GJ frame. From Eq. (13) one sees that the leakage term in the isotropic moment A_0 is (for $\Delta t\to 0$)

$$A_{\mathrm{leak}}=\sum_s f_s\frac{d^2N}{dtd\sqrt{\sigma}}\Delta t\Delta(\sqrt{\sigma})\times\tfrac{3}{2}\rho_{11}=A_2\frac{\rho_{11}}{\rho_{00}-\rho_{11}}\,. \qquad (14)$$

[6] F. Selleri, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colo., 1964).
[7] P. K. Williams, Phys. Rev. **181**, 1963 (1969).
[8] K. Gottfried and J. D. Jackson, Nuovo Cimento **34**, 757 (1964).

It can be seen from Eqs. (9) and (11)–(13) that, using the suggested method, one can obtain information about $\delta(l,\sigma)$ by directly fitting Eq. (13) to the decay distribution of the dipion. Since only four parameters are involved (in contrast to seven in Ref. 1), the statistical uncertainties of the model-deduced phase shifts are expected to be smaller than those in Ref. 1. Also, it will be possible to restrict oneself to extreme forward momentum transfer regions in which the non-pion-exchange background is expected to be minimal. On the other hand, the information about δ_0^0 deduced here is inherently model dependent. It depends specifically on the following: (a) the accuracy to which the model describes the "real" decay matrix element of the p wave. Since absorptive models probably do not in general describe all decay density-matrix elements equally well in different reference frames, frame dependence is also expected of the results of such a fit. We shall partially test the frame dependence by fitting data both in the helicity frame and in the GJ frame.[8] (b) The $m_{\pi\pi}$ dependence of the model-deduced phase shifts is sensitive to the relative accuracy to which the model is able to predict the p-wave density-matrix elements at different values of $m_{\pi\pi}$. Provided the $m_{\pi\pi}$-dependent error is not significant, the main effects of this error will be manifested in A_{leak} (which depends on ρ_{00}/ρ_{11}). In the mass ranges away from the ρ mass the accuracy of A_{leak} predicted by the model is more uncertain. However, this error is not expected to affect significantly the qualitative features of the information extracted, since the leakage term is small in comparison to A_0 in this region. In the ρ region, the accuracy of theoretical density-matrix elements can be studied. We see in Fig. 1 that the helicity frame ρ_{00} (in the region $1.6\mu^2\leq|t|\leq4\mu^2$) predictions of the model agrees very well with the experimental ρ_{00}. The agreement is not quite as good in the GJ frame. (c) We are sensitive to other detailed assumptions used with the model, specifically, the use of Selleri's off-shell factors, the use of the exponential form $\exp[-A(\mu^2-t)]$ for collimation, and the assumption that A is spin independent and $m_{\pi\pi}$ independent.

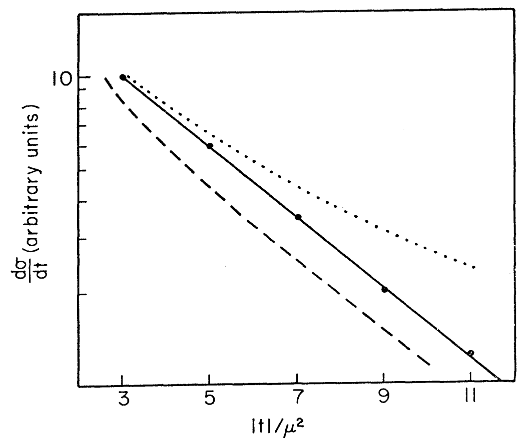

FIG. 2. Differential cross section $d\sigma/dt$ for reaction (2) in the region $740\leq m_{\pi\pi}\leq800$ MeV. The dotted curve represents the experimental data, the solid curve is experimental data minus probable background crudely estimated from a study of the corresponding dipion effective-mass distributions, and the dashed curve is the theoretical prediction with $A=0.10\mu^{-2}$ (arbitrary normalization).

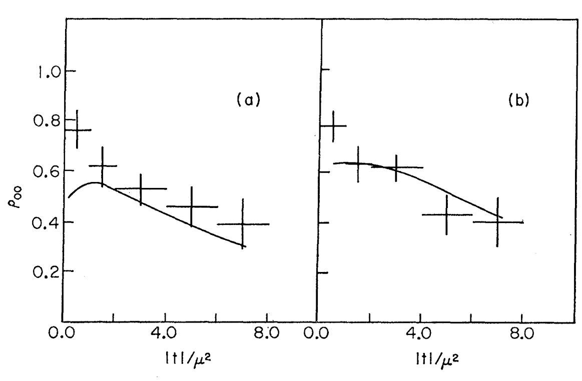

FIG. 1. Comparison of the theoretical (solid curves) and experimental density-matrix element for events in the ρ^0 band (700–800 MeV) in (a) the Gottfried-Jackson frame and (b) the helicity frame. Theoretical curves are calculated from the model of Ref. 7 with the additional assumption that the s-wave cross section is 15% of the p-wave cross section.

It must be mentioned that the method proposed in I allows $F_c(t)$ to be an essentially arbitrary function of t. If we could further allow $F_c(t)$ to be l dependent and $m_{\pi\pi}$ dependent, we would in effect have a nonevasive, but otherwise essentially model-independent method (however, the on-shell $\pi\pi$ phase shifts would still be directly used as parameters). Our specific assumptions about the functional form of $F_c(t)$ correspond roughly to the specification of the form of parametrization (linear, quadratic, etc.) in extrapolation procedures. The assumptions made here corresponding to a four-parameter description are motivated by statistical necessity.

III. ANALYSIS OF DATA

The constant A in the collimating factor $F_c(t)$ essentially determines the shape as a function of t of the differential cross section. The value of A was found on the basis of fits in the ρ region (see Fig. 2) to be

$$A=(0.1\pm0.015)\mu^{-2}.$$

This was done through a study of $dN/d\sigma$ for $2\mu^2\leq|t|\leq12\mu^2$ to determine the background (bg) under the ρ peak as a function of t. Then, fitting the shape of $d(N-\text{bg})/dt$ for $740\leq\sqrt{\sigma}\leq800$ MeV with the model using p wave only, the value of A given above was determined.

For the rest of the analysis, we chose to consider only the restricted t region $1.6\mu^2\leq|t|\leq4\mu^2$ to minimize background and kinematic effects, as mentioned above.

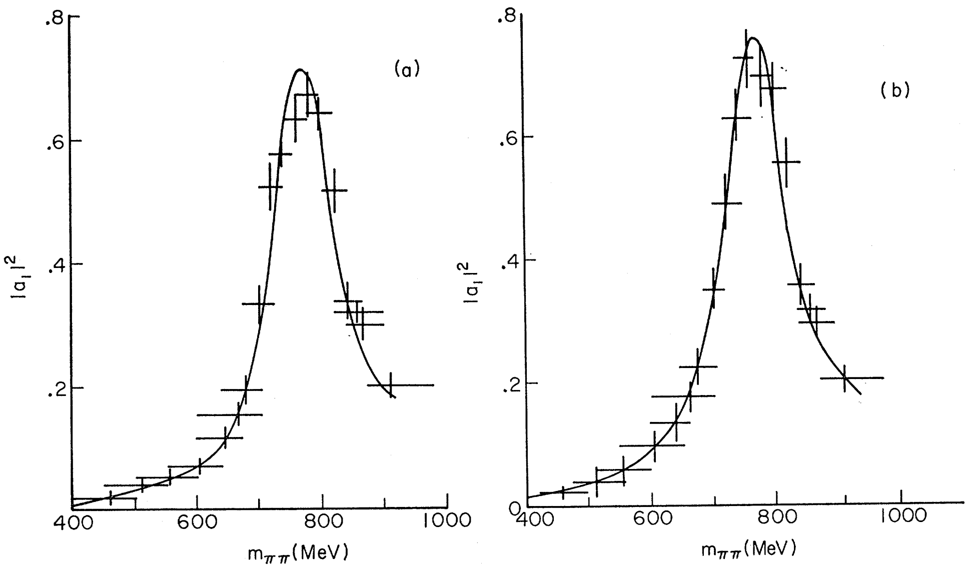

FIG. 3. Plot of $|a_1|^2$ versus $m_{\pi\pi}$ [see Eq. (15)] in the region $400 \leq m_{\pi\pi} \leq 980$ MeV in (a) the helicity frame and (b) the Gottfried-Jackson frame.

For future reference, we use the notation

$$a_0 = \tfrac{2}{3} e^{i\delta_0^0} \sin\delta_0^0 + \tfrac{1}{3} e^{i\delta_0^2} \sin\delta_0^2, \qquad (15)$$
$$a_1 = e^{i\delta_1^1} \sin\delta_1^1,$$

where δ_l^I is the real phase shift for isospin I and spin l. Fitting Eq. (13) to the experimental distribution of events for the decay of the dipion system gives directly the quantities $|a_1|^2$, $|a_0|^2$, and the interference term $\mathrm{Re}(a_0 a_1^*) \equiv |a_0||a_1| \cos\beta$.

The results of fitting for the quantities $|a_0|^2$, $|a_1|^2$, and $|a_0||a_1|\cos\beta$ are not entirely satisfactory, as will be discussed in the next section. The quantity $|a_0|^2$ is poorly determined, with large relative statistical errors. The quantity $|a_1|^2$ does not reach its unitarity limit in the ρ region. Consequently, it is not reasonable to fit the phase shifts δ_0^0, δ_0^2, and δ_1^1 to these quantities through Eq. (15). Instead, we adopt the following procedure (see Ref. 1 for details). We renormalize $|a_1|^2$ so that it reaches unity at its highest point. We take values of δ_0^2 from Ref. 2. Then the renormalized values of $|a_1|^2$ and $|a_0||a_1|\cos\beta$ can be combined to give the usual twofold ambiguous solutions for δ_0^0. Using the values of $|a_0|^2$ instead of $|a_0||a_1|\cos\beta$ in this procedure gives another set of twofold ambiguous solutions for δ_0^0. We then look for consistency between the sets of solutions. Assuming that partial waves higher than p wave are insignificant,[9] and that the non-pion-exchange background is small in the t region under consideration, we performed the least-squares fitting indicated by Eq. (13) to deduce the values of the parameters involved. The integral over the t range $1.6\mu^2 \leq |t| \leq 4.0\mu^2$ was done numerically. For each $\pi\pi$ mass bin, the integrand was evaluated at the median mass.

[9] P. Johnson *et al.* Phys. Rev. **163**, 1497 (1967).

IV. RESULTS OF FIT

We examine here the results of the method of fitting discussed in the previous section to the decay data in both the helicity frame and the GJ frame. We first take up that in the helicity frame.

A plot of $|a_1|^2$ versus $m_{\pi\pi}$ is shown in Fig. 3(a). The interesting feature here is that the peak of the spectrum only reaches $(72\pm5)\%$ of its unitarity limit. It is entirely possible that this indicates an overestimate of the theoretical p-wave production amplitude due, perhaps, to our choice of $F_c(t)$. A similar disagreement, obtained by extrapolation, was observed for this quantity in Ref. 1. Following Ref. 1, we solve for δ_1^1 by *renormalizing* the above spectrum to its unitarity limit and fitting with a smooth Breit-Wigner shape. A plot of $|a_0|^2$ versus $m_{\pi\pi}$ given in Fig. 4(a) shows that the unitarity limit ($\approx 4/9$) is reached within errors. The solutions for δ_0^0 from $|a_0|^2$ and from $|a_0||a_1|\cos\beta$ are given in Figs. 5(a) and 5(b). [In obtaining δ_0^0, we have used the renormalized (Breit-Wigner) values of δ_1^1.] The figure indicates eight separate combinations of solutions. The symbols L and H indicate lower and higher dipion mass regions, respectively; the symbols U and D indicate up and down solutions, respectively; and the symbols I and S indicate solutions taken from the "interference term" $|a_0||a_1|\cos\beta$ and from the the "s-wave term" $|a_0|^2$, respectively. We can see a certain degree of consistency between the branches LDS and LDI for the lower-mass regions. We cannot, however, see any inconsistency between the four branches in the higher-mass regions. If we take this as an indication that the "down" solution in the lower-mass regions is preferred, we will still have left the possibilities "down-down" and "down-up."

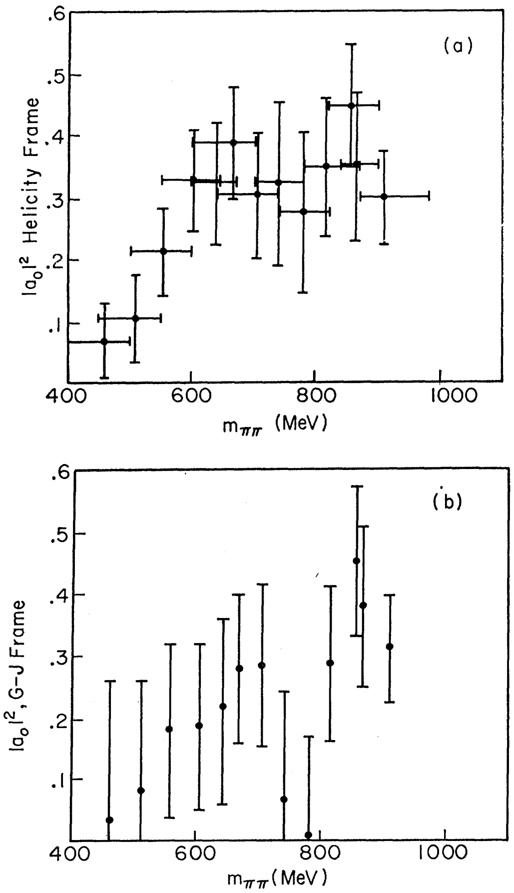

FIG. 4. Plot of $|a_0|^2$ versus $m_{\pi\pi}$ [see Eq. (15)] in the region $400 \leq m_{\pi\pi} \leq 980$ MeV in (a) the helicity frame and (b) the Gottfried-Jackson frame.

A similar set of plots is given in Figs. 3(b), 4(b), 5(c), and 5(d) for quantities fitted in the GJ frame. The features of this set of results are the following. (a) In Fig. 3(b), the sudden drop in the spectrum of $|a_0|^2$ at the ρ peak could indicate an overestimate of the leakage contribution in the isotropic decay moment A_0. This agrees with the fact that the ρ_{11} given by the model in this frame is somewhat too large in comparison with the experimental value. (b) A similar consistency between *LDS* and *LDI* is also indicated in the GJ frame. In fact, it can be seen that corresponding results obtained from analysis in *both* frames agree with each other within errors.

V. DISCUSSION AND CONCLUSIONS

The results of the fitting discussed above indicated the following points. (a) The phase shifts $\delta_0{}^0$ deduced from $|a_0|^2$ in the mass range 400–700 MeV seem to be more qualitatively consistent with the down solution *LDI* of $\delta_0{}^0$ deduced from the interference term. Quantitatively, there is still a small but bothersome difference between the branches *LDI* and *LDS*, especially for the 600–700 MeV range. It must be stressed here that the solutions from the s-wave term $|a_0|^2$ should be taken only as a guide in discriminating between the twofold ambiguous solutions from the interference term because the latter is less model dependent and more statistically reliable. (b) The major problem in determining the preferred solution for $\delta_0{}^0$ by using both A_0 and A_1 is in the higher-$m_{\pi\pi}$ regions where effects due to a possible rising d wave would be more significant. Much better statistics would be necessary to study the d-wave effects. Further lack of reliable experimental knowledge about the d-wave decay density matrix (e.g., in the f^0 region) unfortunately rendered our effort to include the d wave hopeless. (c) We found a high sensitivity of the fitted values of $|a_0|^2$ to the theoretical prediction for the quantity $\rho_{00}/(\rho_{00}+2\rho_{11})$. (d) The deduced p-wave cross section falls below its unitary limit. If this is not a physical effect, causes for this effect could be easily traced to the model description of the t dependence of the differential cross section; on the other hand, an easily calculable effect due to the presence of *any* d wave tends to decrease the moment A_2. (e) The present simple choice of functional form for $F_c(t)$ is necessary because of the limitations in statistics. In the event that higher statistics were to be available

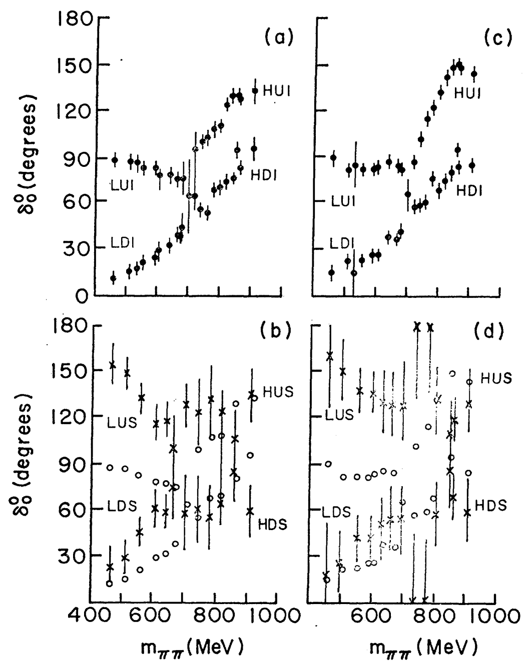

FIG. 5. Plot of $\delta_0{}^0$ versus $m_{\pi\pi}$ in the region $400 \leq m_{\pi\pi} \leq 980$ MeV as deduced from the p-wave decay moment A_2, the phase shift $\delta_0{}^2$ taken from Ref. 2, and (a) the interference moment A_1 in the helicity frame, (b) the isotropic moment A_0 in the helicity frame, (c) the interference moment A_1 in the G-J frame, and (d) the isotropic moment in the Gottfried-Jackson frame. Symbols H and L indicate, respectively, the higher and lower $m_{\pi\pi}$ region, D and U indicate, respectively, down and up branches, and I and S indicate, respectively, phase shifts from interference and isotropic decay moments. Open circles in (b) and (d) are points plotted from (a) and (c), respectively, for comparison.

in the future, it would be possible to deduce the phase shift δ_0^0 by fitting the t dependence of various decay moments to the model with a more flexible form involving more parameters for $F_c(t)$, thereby removing the assumptions of spin independence and perhaps $m_{\pi\pi}$ independence. However, much higher statistics (especially for $-t<\mu^2$) would also permit a "nonevasive" polynomial *extrapolation* (with sufficient number of parameters) which is, in principle, model independent. However, we believe that the use of a reasonably "good" absorption model with somewhat better (say, doubled) data would still give a value for δ_0^0 deduced from the "s-wave" term more reliable than that obtained by nonevasive polynomial extrapolation.

In conclusion, we have seen that while an absorption model can be used on present data to give a set of reasonable δ_0^0 from A_0, these results are perhaps too sensitive to the amount of depolarization predicted by *any* absorption model to be by themselves definitive. However, these results can be used to discriminate between the twofold ambiguous solutions for δ_0^0 obtained from A_1 in the region $500\leq m_{\pi\pi}\leq 700$ MeV. With present data, it is not possible to make a similar statement on the twofold ambiguity in the higher-$m_{\pi\pi}$ region. We believe that a simple doubling of the present data for $-t>\mu^2$ would be helpful in settling, perhaps once and for all, the ambiguity at least in the low-$m_{\pi\pi}$ region. *Even more* statistics would be necessary to look into the same problem at the higher-$m_{\pi\pi}$ regions because of the possibility of important d-wave effects and other processes contributing to the same three-body final state; also, it is desirable for discrimination to have good data for $-t<\mu^2$.

ACKNOWLEDGMENTS

The authors gratefully thank Professor W. Selove and Professor J. E. Lannutti for many discussions and continuing interest in this work. This work could not have been completed without the availability of the Pennsylvania-Florida State-Purdue-Colorado-UCRL (Berkeley)-Wisconsin-Saclay-Toronto collaboration data of Ref. 1. Finally the programming was facilitated at all stages with the help of Dr. J. P. Chandler and A. Snyderman.

PHYSICAL REVIEW D VOLUME 2, NUMBER 3 1 AUGUST 1970

Unitarity Upper Bounds and Elastic Scattering*

D. G. RAVENHALL AND W. J. PARDEE†
Physics Department, University of Illinois, Urbana, Illinois 61801
(Received 30 March 1970)

A derivation is given in terms of the eikonal formalism of a recently derived upper bound on the absorptive part of a spin-zero elastic scattering amplitude. The multivaluedness inherent in the problem is explored, and considerably smaller upper bounds are obtained by imposing conditions of reasonability on the profile function. Experimental differential cross sections for $\pi^{\pm}p$ and pp elastic scattering over a wide energy range are found to fall on a universal curve, which lies just below our most stringent upper bound over a range of three decades in $d\sigma/dt$.

It was shown by MacDowell and Martin[1] that unitarity puts a bound on $[dA(t)/dt]_0/A(0)$, the slope of the absorptive part $A(t)$ of the elastic scattering amplitude in the forward direction. In a recent Letter, Singh and Roy[2] extended those considerations of unitarity to obtain an upper bound on the non-spin-flip absorptive amplitude $A(t)/A(0)$ apparently for general (negative) values of t. The purpose of this comment is threefold: (1) to give a simpler derivation of the latter result, in terms of the eikonal formalism; (2) in this framework to show that the functional character of the solution is multivalued; and (3) to show that the assumption of reasonable additional restrictions on the profile function gives a considerably reduced upper bound, which lies close to experimental values of $d\sigma/dt$ over three decades.

As is made clear in Refs. 1 and 2, these bounds are useful mainly in the energy regions where there is strong absorption and many partial waves contribute. This is the region where the eikonal approximation is appropriate. We therefore use this approximation. In terms of the impact parameter $b=l/k$, the momentum transfer q, assumed to be transverse, where $t=-q^2$, and the imaginary part of the partial-wave amplitude $\Gamma(b)$ (the profile function), we must examine the quantity

$$F(q)=A(q^2)/A(0)$$

$$=\int d^2b\,\Gamma(b)\,e^{iq\cdot b}\Big/\int d^2b\,\Gamma(b) \tag{1a}$$

$$=\int_0^\infty bdb\,\Gamma(b)J_0(qb)\Big/\int_0^\infty bdb\,\Gamma(b)\,. \tag{1b}$$

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* Supported in part by the National Science Foundation, under Grant Nos. NSF GP 13671 and NSF GP 19433.
† General Electric Company Predoctoral Fellow.
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