and ϕ_{+-} coincides with ϕ_W within the experimental error of a few degrees.[22] The $\pi^+\pi^-$ contribution to (12) is therefore 1.3×10^{-3}. There is far greater uncertainty about η_{00}, but it is consistent with all reported measurements to say that it has a magnitude comparable to $|\eta_{+-}|$. The only reported[7] measurement of ϕ_{00} yielded a value in the first quadrant, which assures a positive contribution to (12). We can therefore predict a finite nonvanishing T asymmetry [Eq. (10)] of several parts in a thousand, independent of any symmetry assumptions. The expected asymmetry could vanish only in the unlikely circumstance that $|\eta_{00}|$ were significantly larger than $|\eta_{+-}|$ and had a phase ϕ_{00} differing from ϕ_W by considerably more than 90°.

I wish to thank Professor K. E. Eriksson and NORDITA for hospitality at the Chalmers Institute of Technology, where the basic argument was first presented at a Seminar.

[22] D. A. Jensen *et al.*, Phys. Rev. Letters **23**, 615 (1969).

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Current Algebra, $\bar{K}_{l3}{}^0$ Form Factors, and Radiative $\bar{K}_{l3}{}^0$ Decay*

HAROLD W. FEARING†
The Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

AND

EPHRAIM FISCHBACH
The Niels Bohr Institute, University of Copenhagen, Copenhagen, Denmark‡
and
The Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

AND

JACK SMITH
The Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

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The complete gauge-invariant matrix element for the decays $\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma$ ($l=e$ or μ) is derived using the soft-photon theorems of Low and of Adler and Dothan. These theorems, along with several corollaries of them, are reviewed in detail and their application demonstrated by reference to the radiative $\bar{K}_{l3}{}^0$ decay mode. The square of the matrix element is calculated using the theorem of Burnett and Kroll, and is compared with the result of a direct computer evaluation of the appropriate traces. Structure-dependent terms are discussed, and the dominant terms among those linear in the photon energy are estimated in the soft-pion and kaon limits. Results for the radiative photon spectra are given, together with the decay rates for a specific value of the minimum photon energy E_0.

I. INTRODUCTION

THE present paper, which is the sequel to a previous paper[1] on radiative $K_{l3}{}^+$ decays, has two purposes. One is to discuss in detail the matrix element for the radiative decay $\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma$ ($l=e$ or μ), and then calculate results for decay branching ratios and photon spectra. The other is to use this calculation as a vehicle for reviewing a number of soft-photon theorems and corollaries which are useful for discussing radiative processes in general. Of particular interest are the theorems of Low[2] and Adler and Dothan[3] for the radiative matrix element, and Burnett and Kroll[4] and Bell and Van Royen[5] for the square of the radiative matrix element. Additional references to radiative decays are given in I.

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† Air Force Office of Scientific Research Postdoctoral Fellow, 1968–1969. Address after Sept. 1, 1970: Los Alamos Scientific Laboratory, Los Alamos, N. M.

‡ Present address.

[1] E. Fischbach and J. Smith, Phys. Rev. **184**, 1645 (1969); hereafter called I. A compilation of rates and spectra for both charged and neutral K decays (with different values of E_0) has also been published. See H. W. Fearing, E. Fischbach, and J. Smith, Phys. Rev. Letters **24**, 189 (1970). With regard to the comment made in this Letter on the first number in the branching ratio $\Gamma(K\to\pi\mu\nu)/\Gamma(K\to\pi e\nu)$ published by N. Cabibbo in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), p. 34, we would like to thank Dr. Cabibbo for confirming the fact that this number was misprinted and should read 0.6457 (for charged K decays) and not 0.6487. Details of Dr. Cabibbo's calculation have been given by C. T. Murphy in University of Michigan Bubble Chamber Group Research Note No. 58/66 (unpublished). The correct branching ratios for charged and neutral K decays can be obtained from Eqs. (2)–(5) of the Letter referred to above, and are given in Appendix B below.

[2] F. E. Low, Phys. Rev. **110**, 974 (1958).

[3] S. L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966).

[4] T. H. Burnett and N. M. Kroll, Phys. Rev. Letters **20**, 86 (1968).

[5] J. S. Bell and R. Van Royen, Nuovo Cimento **60A**, 62 (1969).

Although soft-photon theorems have been applied to many radiative processes,[6] radiative K_{l3} decays are of particular interest since the corresponding nonradiative modes have been studied extensively both experimentally and theoretically. Thus these modes provide a unique opportunity for checking the predictions of soft-photon theorems (which relate the radiative modes to the corresponding nonradiative modes) and, in particular, the presence of derivative terms, which are absent in two-body radiative decays and usually difficult to observe in scattering processes owing to the lack of a simple theory of the elastic scattering matrix element. Although radiative $\bar{K}_{l3}{}^0$ decays are formally similar to radiative $K_{l3}{}^+$ decays considered before, there exist some additional technical problems due to the fact that in this case both of the charged particles are in the final state. This necessitates the calculation of many additional phase-space integrals. There is, in the $\bar{K}_{l3\gamma}{}^0$ decay mode, the additional question of a possible CP violation which we would expect on the basis of the known CP violation in the corresponding nonradiative modes.[7] Throughout this paper, we will, however, assume that CP is conserved and reserve for a future paper a discussion of CP violation. We will assume, in addition, the usual $V-A$ theory, μ-e universality, and the $|\Delta \mathbf{I}|=\frac{1}{2}$ rule.

On the experimental side, a recent experiment by Evans *et al.*[8] found a few events consistent with radiative leptonic $K_L{}^0$ decay and quoted a preliminary branching ratio

$$\Gamma(K_L{}^0 \to \pi^\pm e^\mp \nu\gamma)/\Gamma(K_L{}^0 \to \pi^\pm e^\mp \nu) = (0.75\pm 0.4)\times 10^{-2}. \quad (1.1)$$

However, their results have not yet been fully analyzed. In particular, the branching ratio depends logarithmically on the energy E_0, below which the apparatus does not detect photons, and E_0 is not given by the authors. In our paper, we calculate the branching ratios

$$R_1 = \Gamma(\bar{K}^0 \to \pi^+ e^- \bar{\nu}\gamma;\, E_\gamma > E_0)/\Gamma(\bar{K}^0 \to \pi^+ e^- \bar{\nu})$$

and

$$R_2 = \Gamma(\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}\gamma;\, E_\gamma > E_0)/\Gamma(\bar{K}^0 \to \pi^+ \mu^- \bar{\nu})$$

for a typical value of $E_0 = 30$ MeV.[1] We also plot the photon spectrum for representative values of the $\bar{K}_{l3}{}^0$ form factors. This spectrum diverges as k^{-1} for small photon momentum k.

The outline of the present paper is as follows. In Sec. II we discuss soft-photon theorems in general and then apply our results to the specific mode $\bar{K}^0 \to \pi^+ e^- \bar{\nu}\gamma$. In Sec. III the Burnett-Kroll method is used to derive the k^{-2} and k^{-1} terms in the square of the matrix element, which is then compared to the result of the explicit trace calculation. In Sec. IV we estimate the values of those structure-dependent form factors which we expect to be most important, and then finally in Sec. V we give our results. In Appendix A we list the complete square of the matrix element actually used in our computation, and in Appendix B we give the technical details of the phase-space integrations and the corresponding decay rates for the nonradiative $\bar{K}_{l3}{}^0$ modes.

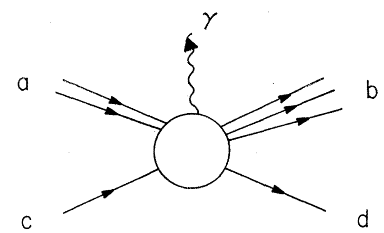

FIG. 1. Feynman diagram for the process $a+c \to b+d+\gamma$; a and b refer to bosons, while c and d are fermion lines.

II. DERIVATION OF $\bar{K}_{l3\gamma}{}^0$ MATRIX ELEMENT

Our main task in this section is the derivation of the T matrix for the radiative $\bar{K}_{l3}{}^0$ decay. We know that the terms in T of order k^{-1} and k^0 can be completely determined in terms of the nonradiative matrix element[9] using the Low-Adler-Dothan procedure as was done explicitly in I. Rather than simply repeat that calculation here, we propose to outline the derivation of these terms for a more general process. The resulting formula will serve as a starting point for our discussion of the application of the Burnett-Kroll theorem. Hopefully it will also be of value to those wishing to apply soft-photon techniques to processes other than $K_{l3\gamma}$ decays. In the process we want to collect in one place a number of corollaries and comments regarding soft-photon theorems which are at present widely scattered through the literature. Once the general formulas have been obtained, it is a relatively simple exercise to obtain the specific results we require for $\bar{K}_{l3\gamma}{}^0$ decay.

Thus we consider a general radiative process $a+c \to b+d+\gamma$ shown in Fig. 1. Here a and b are incoming and outgoing states containing an arbitrary number of spin-

[6] Some recent examples are: $p+p \to p+p+\gamma$, E. Nyman, Phys. Rev. **170**, 1628 (1968); $\eta \to \pi^+\pi^-\pi^0\gamma$, R. Ferrari, Nuovo Cimento **48A**, 898 (1968); $K \to 3\pi\gamma$, R. Ferrari and M. Rosa-Clot, *ibid.* **56A**, 582 (1968).

[7] J. Steinberger, in Proceedings of the Topical Conference on Weak Interactions, CERN Report No. 69-7, 1969, p. 291 (unpublished); D. Dorfan, J. Enstrom, D. Raymond, M. Schwartz, S. Wojcicki, D. H. Miller, and M. Paciotti, Phys. Rev. Letters **19**, 987 (1967); S. Bennett, D. Nygren, H. Saal, J. Steinberger, and J. Sutherland, *ibid.* **19**, 993 (1967); **19**, 997 (1967); S. Bennett, D. Nygren, H. Saal, J. Sutherland, J. Steinberger, and K. Kleinknecht, Phys. Letters **27B**, 244 (1968).

[8] G. R. Evans, M. Golden, J. Muir, K. J. Peach, I. A. Budakov, H. W. K. Hopkins, W. Krenz, F. A. Nezrick, and R. G. Worthington, Phys. Rev. Letters **23**, 427 (1969).

[9] This statement must be qualified a little. Even when a process involves only soft photons, there are problems with the Low theorem when the matrix elements have resonances, particularly if they are narrow resonances. This situation has been commented on by F. E. Low, Ref. 2; by S. Barshay and T. Yao, Phys. Rev. **171**, 1708 (1968); and by H. Feshbach and D. R. Yennie, Nucl. Phys. **37**, 150 (1962).

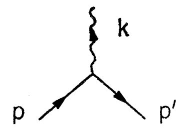

FIG. 2. Electromagnetic vertex with p an incoming boson or fermion and p' an outgoing boson or fermion.

zero bosons, while c and d are states containing a single incoming and outgoing spin-one-half fermion, respectively. (We limit ourselves to one fermion line to simplify the notation. Additional fermions present no difficulties, in principle.) For such a process the contributions to T of order k^{-1} and k^0, which we call T_L, can be obtained using a simple recipe due to Low[2] and Adler and Dothan,[3] which we now state:

(1) Write down T_{ex}, the sum of the contributions in which the photon is radiated from an external charged line.

(2) Expand T_{ex} with respect to the explicit k dependence about $k=0$ and drop all terms in the result which are explicitly independent of k or which are of order k or higher. Denote the result of this step by T_{ex}'.

(3) Add to T_{ex}' a contribution ΔT, *independent* of k so as to make $T_{\text{ex}}'+\Delta T$ gauge invariant. Then

$$T_L=T_{\text{ex}}'+\Delta T. \qquad (2.1)$$

We want now to study the application of this recipe to the general process of Fig. 1. Let T represent the T matrix for the N-particle nonradiative process $a+c\to b+d$ and define $T_0(\ldots)$ by

$$T=\bar{u}(p_d)T_0(\ldots)u(p_c), \qquad (2.2)$$

where p_c and p_d are the momenta of the two fermions. $T_0(\ldots)$ is the off-mass-shell amplitude (a matrix) with all spinors factored out. $T_0(\ldots)$ may involve such quantities as $\gamma\cdot p_i$, $\gamma\cdot p_i\gamma\cdot p_j$, etc., and will contain scalar functions, or form factors, which depend on the N invariants p_i^2 $(i=1,\ldots,N)$ and the $\frac{1}{2}N(N-1)-N$ other possible scalar invariants $p_i\cdot p_j$, e.g.,

$$T_0(\ldots)=T_0(p_i^2,p_j^2,\ldots,p_i\cdot p_j\ldots,\gamma\cdot p_i\ldots). \qquad (2.3)$$

Note that we consider all momenta p_i, $i=1,\ldots,N$, as *independent*, because we will eventually want to consider them as radiative variables satisfying $p_a+p_c=p_b+p_d+k$ rather than $p_a+p_c=p_b+p_d$. In general, $T_0(\ldots)$ will involve a part which has the same form as the on-mass-shell nonradiative amplitude plus off-mass-shell parts which can be taken to be proportional to $p_i^2+m_i^2$ in the spin-zero case and $i\gamma\cdot p_i+m_i$ in the spin-one-half case. Let the electromagnetic vertex be given by $\Gamma_\mu(p,p')$ with $p=p'+k$ as in Fig. 2. For spin-zero particles,

$$\Gamma_\mu(p,p')=iQ(p+p')_\mu F(p^2,p'^2,k^2), \qquad (2.4)$$

while for spin-one-half particles,

$$\Gamma_\mu(p,p')=-[Q\gamma_\mu F_1(p^2,p'^2,k^2)+\sigma_{\mu\nu}k_\nu(\kappa/2m)F_2(p^2,p'^2,k^2)], \qquad (2.5)$$

where $\sigma_{\mu\nu}=-\frac{1}{2}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$, κ is the anomalous magnetic moment of the spin-one-half particle, and Q is the charge of the particle (in units of $e>0$). The form factors F, F_1, and F_2 are normalized so that in each case $F(-m^2,-m^2,0)=1$, where m is the mass of the corresponding particle. With these preliminaries and using the Feynman rules in the Pauli metric where $p_\mu=(\mathbf{p},p_4=ip_0)$, we can execute steps (1)–(3) of the recipe.

Step 1. Write the contribution of radiation from the external lines of Fig. 1:

$$\begin{aligned}T_{\text{ex}}=\epsilon_\mu\sum_{i\in a}\Gamma_\mu(p_i,p_i-k)\frac{i}{2k\cdot p_i}\bar{u}(p_d)T_0(p_i-k)u(p_c)+\epsilon_\mu\sum_{j\in b}\Gamma_\mu(p_j+k,p_j)\frac{-i}{2k\cdot p_j}\bar{u}(p_d)T_0(p_j+k)u(p_c)\\ +\epsilon_\mu\bar{u}(p_d)T_0(p_c-k)i\frac{-i\gamma\cdot(p_c-k)+m_c}{2k\cdot p_c}\Gamma_\mu(p_c,p_c-k)u(p_c)\\ +\epsilon_\mu\bar{u}(p_d)\Gamma_\mu(p_d+k,p_d)(-i)\frac{-i\gamma\cdot(p_d+k)+m_d}{2k\cdot p_d}T_0(p_d+k)u(p_c). \qquad (2.6)\end{aligned}$$

In this equation the notation $T_0(p_i\pm k)$ means replace the variable p_i in $T_0(p_i^2,p_j^2,\ldots,p_i\cdot p_j,\ldots\gamma\cdot p_i\ldots)$ by $p_i\pm k$, leaving the other momenta untouched.

Step 2. Expand T_{ex} in powers of k, dropping those terms which are independent of k. Before proceeding with the expansion of Eq. (2.6) in powers of k, it is useful to make several comments. (a) Note that the expansion is made with respect to the *explicit* k dependence only, i.e., the p_i and k are considered as independent variables and the conservation of four-momentum equation relating k to the other momenta p_i is *not* used at this stage. (b) Note that the terms arising from the expansion of p_i^2 terms in $\Gamma_\mu(p^2,p'^2,k^2)$ and $T_0(p_i^2,p_j^2\ldots p_i\cdot p_j\ldots\gamma\cdot p_i\ldots)$ make no contribution to T_L. Thus we can effectively let $p_i^2=-m_i^2$ in Γ_μ and $T_0(\ldots)$ before writing Eq. (2.6). To see this consider a general function of p^2, say $f(p^2)$. After replacing $p\to p\pm k$ for the off-mass-shell line and expanding about $k=0$, we get $f((p\pm k)^2)=f(-m^2)\pm 2p\cdot k(\partial f/\partial p^2)|_{p^2=-m^2}+O(k^2)$. However, the $2p\cdot k$ in the second term will be canceled by the propagator factor $(2p\cdot k)^{-1}$ associated with the off-mass-shell line.

Thus the $\partial f/\partial p^2$ term is independent of k [or of $O(k)$ or higher] and according to our recipe is to be dropped. As immediate consequences of this argument, we note that the electromagnetic form factors $F(p^2,p'^2,k^2)$ do not contribute to T_L and the off-mass-shell part of $T_0(p_i^2,\ p_j^2\ldots p_i\cdot p_j\ldots\gamma\cdot p_i\ldots)$ for the spin-zero case also does not contribute, since it is proportional to $p_i^2+m_i^2$. (c) Note that in the spin-one-half case the off-mass-shell part also does not contribute to T_L for similar reasons, i.e., the $i\gamma\cdot(p_i\pm k)+m_i$ factor of the off-mass-shell part, when multiplied by the associated propagator $[-i\gamma\cdot(p_i\pm k)+m_i]/2p_i\cdot k$, leads to terms independent of k or $O(k)$ or higher which are to be dropped according to our recipe.

We are now able to expand Eq. (2.6) in powers of k. In accordance with the observations above we replace all form factors $F(p^2,p'^2,k^2)$ in Γ_μ by unity and expand $T_0(p_i\pm k)$ as

$$T_0(p_i\pm k)=T_0\pm k\cdot\frac{\partial T_0}{\partial p_i},$$

where T_0 with no arguments is

$$T_0(-m_i^2,\ -m_j^2\ldots p_i\cdot p_j\ldots\gamma\cdot p_i\ldots).$$

Thus T_0 is the on-mass-shell nonradiative amplitude, considered as a function of the independent *radiative* variables p_i, $i=1,\ \ldots,\ N$. Note that $\partial T_0/\partial p_i$ may contain derivatives of the scalar functions in T_0 with respect to the remaining independent scalars $p_i\cdot p_j$ as well as terms coming from derivatives of possible explicit p_i dependence, e.g., derivatives of $\gamma\cdot p_i$ terms which may in general appear. With these substitutions and some Dirac algebra, Eq. (2.6) becomes

$$T_{\rm ex}'=\sum_{i=\text{all particles}}\bar{u}(p_d)\left[Q_i\frac{\epsilon\cdot p_i}{k\cdot p_i}\left(\eta_iT_0+k\cdot\frac{\partial T_0}{\partial p_i}\right)\right]u(p_c)+\bar{u}(p_d)T_0\left[Q_c+(-i\gamma\cdot p_c+m_c)\frac{\kappa_c}{2m_c}\right]\frac{\gamma\cdot k\gamma\cdot\epsilon}{2k\cdot p_c}u(p_c)$$
$$+\bar{u}(p_d)\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p_d}\left[Q_d+(-i\gamma\cdot p_d+m_d)\frac{\kappa_d}{2m_d}\right]T_0u(p_c).\quad(2.7)$$

In this equation $\eta_i=+1\ (-1)$ for outgoing (incoming) particles and Q_i is the charge of the ith particle.

Step 3. Add a ΔT *independent of* k, which makes $T_L=T_{\rm ex}'+\Delta T$ gauge invariant. Replacing ϵ_μ by k_μ everywhere in the equation for $T_{\rm ex}'$, we see that the last two terms automatically vanish since $k^2=0$, and the term proportional to η_i vanishes by charge conservation since

$$\sum_i\eta_iQ_i=\sum Q_{\rm out}-\sum Q_{\rm in}=0.$$

Thus the ΔT we need to make $T_{\rm ex}'+\Delta T$ gauge invariant is

$$\Delta T=-\sum_i Q_i\bar{u}(p_d)\epsilon\cdot\frac{\partial T_0}{\partial p_i}u(p_c),\quad(2.8)$$

where the summation extends over all particles. Thus we obtain for the general radiative amplitude

$$T_L=T_{\rm ex}'+\Delta T=\sum_i\eta_iQ_i\frac{\epsilon\cdot p_i}{k\cdot p_i}\bar{u}(p_d)T_0u(p_c)+\sum_iQ_iD_\lambda(p_i)\bar{u}(p_d)\frac{\partial T_0}{\partial p_{i\lambda}}u(p_c)$$
$$+\bar{u}(p_d)T_0\left[Q_c+(-i\gamma\cdot p_c+m_c)\frac{\kappa_c}{2m_c}\right]\frac{\gamma\cdot k\gamma\cdot\epsilon}{2k\cdot p_c}u(p_c)+\bar{u}(p_d)\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p_d}\left[Q_d+(-i\gamma\cdot p_d+m_d)\frac{\kappa_d}{2m_d}\right]T_0u(p_c),\quad(2.9)$$

where

$$D_\lambda(p_i)=\frac{\epsilon\cdot p_i}{k\cdot p_i}k_\lambda-\epsilon_\lambda.\quad(2.10)$$

Equation (2.9) gives the expression for the radiative matrix element for our general process $a+c\rightarrow b+d+\gamma$ up to but not including terms of order k. The next terms in the expansion of the radiative matrix element in powers of k are necessarily of order k or higher and represent the so-called "structure-dependent" effects.

A number of comments regarding the general formula are now perhaps in order.

(a) Recall that T_0 is essentially the nonradiative matrix element on the mass shell, but considered as a function of the radiative variables, i.e., those satisfying $\sum p_{\rm in}=\sum p_{\rm out}+k$. Thus to obtain T_0 for a particular process, one just writes down an explicit form for the nonradiative matrix element in terms of the $\frac{1}{2}N(N-1)-N$ possible scalar invariants $p_i\cdot p_j\ (i\neq j)$ and the appropriate quantities $\gamma\cdot p_i$, etc., and uses that form in Eq. (2.9).

(b) In many cases several expressions for T_0 exist (related to each other by Dirac algebra) which are identical for the nonradiative process but which may

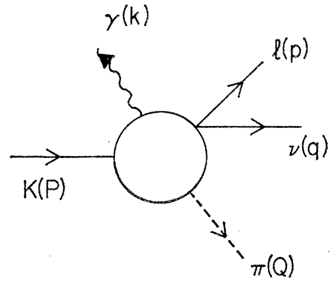

FIG. 3. Diagram for the decay $K \to \pi l \nu \gamma$.

differ by terms $O(k)$ when the momenta are considered as radiative variables. It has been shown, however, by Ferrari and Rosa-Clot[6] and Bell and Van Royen[5] that different choices of T_0 lead to radiative matrix elements which differ only by terms of order k or higher. Thus various choices of T_0 are essentially equivalent.

(c) Because two different choices of T_0 are equivalent, one is free to choose that form of T_0 which makes T_L as simple as possible. It is clear from the final result, Eq. (2.9), that the appropriate choice for T_0 will normally be the one in which as much of the explicit momentum dependence ($\gamma \cdot p_i$, etc.) as possible is given in terms of momenta of *neutral* particles, since this minimizes the number of contributions from the $Q_i D_\lambda(p_i) \partial T_0 / \partial p_{i\lambda}$ term.

(d) Observe that when $N=3$, e.g., $\Lambda \to p\pi$, there are only three independent scalar invariants, p_1^2, p_2^2, and p_3^2, which in accordance with the previous arguments must be replaced by $-m_1^2$, $-m_2^2$, and $-m_3^2$. Hence any invariant functions, i.e., form factors, appearing in T_0 are constants, and thus in this case no terms involving derivatives of these form factors appear in T_L. This was originally noted by Chew[10] and independently by Pestieau.[11]

The matrix element in Eq. (2.9) was derived on the assumption that radiation was emitted by particles and not antiparticles. When there are antiparticles in the initial (final) state with momentum, charge, and anomalous magnetic moment p, Q, and κ we treat them for the purposes of Eq. (2.9) as particles in the final (initial) state with momentum, charge, and anomalous magnetic moment $-p$, $-Q$, and $-\kappa$. T_0 is then calculated by the usual Feynman rules for antiparticles, with the appropriate replacements $u(p) \to v(p)$, etc.

Let us now apply our general formula to radiative $\bar{K}_{l3}{}^0$ decay. We first discuss nonradiative $\bar{K}_{l3}{}^0$ decay, so as to establish notation and normalization conventions, and then proceed to the derivation of the radiative matrix element. Consider the decay $\bar{K}^0(P) \to \pi^+(Q) + l^-(p) + \bar{\nu}(q)$, where P, Q, p, and q are the four-momenta of the respective particles. The matrix element for this process is given by

[10] H. Chew, Phys. Rev. **123**, 377 (1961).

[11] J. Pestieau, Phys. Rev. **160**, 1555 (1967).

$$\mathfrak{M} = {}_{\rm out}\langle \pi^+ l^- \bar{\nu} | \bar{K}^0 \rangle_{\rm in} = -i(2\pi)^4 \delta^4(P-Q-p-q) \left(\frac{m m_\nu}{4 P_0 Q_0 p_0 q_0 V^4} \right)^{1/2} \times \frac{G \sin\theta}{\sqrt{2}} T(\bar{K}_{l3}{}^0), \quad (2.11)$$

where $T(\bar{K}_{l3}{}^0)$ is defined by

$$T(\bar{K}_{l3}{}^0) = (4 P_0 Q_0 V^2)^{1/2} \langle \pi^+(Q) | V_\nu{}^{4+i5}(0) | \bar{K}^0(P) \rangle l_\nu = [f_+(t)(P+Q)_\nu + f_-(t)(P-Q)_\nu] l_\nu, \quad (2.12)$$

with $l_\nu = i\bar{u}(p)\gamma_\nu(1+\gamma_5)v(q)$ and $t = -(P-Q)^2$. In Eq. (2.11) the Fermi constant $G = 1.435 \times 10^{-49}$ erg cm^3, and θ is the Cabibbo angle, where $\sin\theta \simeq 0.21$.[12] We use M, μ, m, and m_ν to denote the mass of the kaon, pion, lepton (e or μ), and neutrino, respectively. (As usual, the limit $m_\nu \to 0$ at the end of the calculation is well defined.) The $\Delta S = 1$ polar-vector current $V_\nu{}^{4+i5}(x)$ and the $\Delta S = 1$ axial-vector current $A_\nu{}^{4+i5}(x)$ (which contributes to the radiative matrix element) are given by

$$V_\nu{}^{4+i5}(x) = \mathfrak{F}_\nu{}^4(x) + i\mathfrak{F}_\nu{}^5(x),$$
$$A_\nu{}^{4+i5}(x) = \mathfrak{F}_{5\nu}{}^4(x) + i\mathfrak{F}_{5\nu}{}^5(x),$$

where the $\mathfrak{F}$'s are assumed to obey the usual $SU(3) \times SU(3)$ commutation relations[13] given in I.[14] Note that $T(\bar{K}_{l3}{}^0)$ can be written in the equivalent forms

$$T(\bar{K}_{l3}{}^0) = \bar{u}(p)[f_1(t) i\gamma \cdot P + f_2(t) i\gamma \cdot Q] \times (1+\gamma_5)v(q) \quad (2.13a)$$

or

$$T(\bar{K}_{l3}{}^0) = \bar{u}(p)[2 f_+(t) i\gamma \cdot P + m f_2(t)] \times (1+\gamma_5)v(q) \quad (2.13b)$$

by use of the Dirac equation. Equation (2.13b), with $f_2(t)$ defined by the relations $2f_+(t) = f_1(t) + f_2(t)$ and $2f_-(t) = f_1(t) - f_2(t)$, is more convenient to use than either Eq. (2.12) or Eq. (2.13a) for the derivation of the radiative matrix element, since it involves explicitly only the neutral momentum P. In the $SU(3)$ limit, $f_+(0) = 1$ and $f_-(0) = 0$. We also define $\xi = f_-(0)/f_+(0)$ and $\zeta = f_2(0)/f_+(0)$, which are related by $1 - \xi = \zeta$. The general momentum dependence of the form factors is expressed by

$$f(t) = f(0)(1 + \Lambda t / M^2), \quad (2.14)$$

where the connection with the usual notation is $\Lambda = \lambda M^2/\mu^2$ and $f(t) = f_+(t)$ or $f_2(t)$. The relation between the first-order quantities is

$$\zeta \Lambda_2 = \Lambda_+ - \xi \Lambda_-. \quad (2.15)$$

[12] N. Brene, M. Roos, and A. Sirlin, Nucl. Phys. **B6**, 255 (1968).

[13] M. Gell-Mann, Physics **1**, 63 (1964).

[14] We alert the reader to a change of conventions from those employed in I. In the present paper the leptons e^- and μ^- are considered as particles, while e^+ and μ^+ are considered as antiparticles. The opposite convention was used in I. As the rate for $K^+ \to \pi^0 l^+ \nu\gamma$ and $K^- \to \pi^0 l^- \nu\gamma$ are equal, as are the rates for $K^0 \to \pi^- l^+ \nu\gamma$ and $\bar{K}^0 \to \pi^+ l^- \nu\gamma$, it does not really matter which convention is used.

Expressing the momentum dependence in terms of t/M^2 rather than t/μ^2 means that we are expanding in a quantity which is numerically less than unity. This normalization is more convenient for numerical computation. At the end of Appendix B, we give the $\bar{K}_{e3}{}^0$ and $\bar{K}_{\mu 3}{}^0$ decay rates as functions of the parameters $f_+(0)$, ζ, Λ_+, and Λ_2.

With these preliminaries, we begin our derivation of the matrix element for the process $\bar{K}^0(P)\to\pi^+(Q)+l^-(p)+\bar{\nu}(q)+\gamma(k)$ shown in Fig. 3. We have

$$\mathfrak{M}(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)={}_{\text{out}}\langle\pi^+l^-\bar{\nu}\gamma|\bar{K}^0\rangle_{\text{in}}$$
$$=-i(2\pi)^4\delta^4(P-Q-p-q-k)\left(\frac{mm_\nu}{8P_0Q_0p_0q_0k_0V^5}\right)^{1/2}\frac{eG\sin\theta}{\sqrt{2}}T(\bar{K}_{l3\gamma}{}^0),\tag{2.16}$$

where e is the electric charge ($e>0$, $e^2/4\pi=\alpha=1/137$). Using the form of $T(\bar{K}_{l3}{}^0)$ given in Eq. (2.13b), we immediately find from Eq. (2.9)

$$T_L(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)=\left(\frac{\epsilon\cdot Q}{Q\cdot k}-\frac{\epsilon\cdot p}{p\cdot k}\right)\bar{u}(p)[2f_+(t)i\gamma\cdot P+mf_2(t)](1+\gamma_5)v(q)+\bar{u}(p)\left[D_\lambda(Q)\frac{\partial}{\partial Q_\lambda}-D_\lambda(p)\frac{\partial}{\partial p_\lambda}\right]$$
$$\times[2f_+(t)i\gamma\cdot P+mf_2(t)](1+\gamma_5)v(q)-\bar{u}(p)\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}[2f_+(t)i\gamma\cdot P+mf_2(t)](1+\gamma_5)v(q)$$
$$=\bar{u}(p)\left(\frac{\epsilon\cdot Q}{Q\cdot k}-\frac{\epsilon\cdot p}{p\cdot k}-\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[2f_+(t)i\gamma\cdot P+mf_2(t)](1+\gamma_5)v(q)$$
$$+2P\cdot D(Q)\bar{u}(p)\left[2\frac{\partial}{\partial t}f_+(t)i\gamma\cdot P+m\frac{\partial}{\partial t}f_2(t)\right](1+\gamma_5)v(q).\tag{2.17}$$

It is of interest at this point to confirm the comment made above that equivalent forms of the nonradiative matrix element give the same result for $T_L(\bar{K}_{l3\gamma}{}^0)$. If we had used Eq. (2.13a) rather than Eq. (2.13b), then $T_L(\bar{K}_{l3\gamma}{}^0)$ would be given from Eq. (2.9) by

$$T_L(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)=\bar{u}(p)\left(\frac{\epsilon\cdot Q}{Q\cdot k}-\frac{\epsilon\cdot p}{p\cdot k}-\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[f_1(t)i\gamma\cdot P+f_2(t)i\gamma\cdot Q](1+\gamma_5)v(q)$$
$$+\bar{u}(p)f_2(t)i\gamma\cdot D(Q)(1+\gamma_5)v(q)+2P\cdot D(Q)\bar{u}(p)\left[\frac{\partial}{\partial t}f_1(t)i\gamma\cdot P+\frac{\partial}{\partial t}f_2(t)i\gamma\cdot Q\right](1+\gamma_5)v(q).\tag{2.18}$$

Thus with this choice of T_0, we get an extra term in the radiative matrix element. It is easy to show, however, using the relation $f_1(t)=2f_+(t)-f_2(t)$ and some Dirac algebra, that Eqs. (2.17) and (2.18) differ only by a term of order k, i.e.,

$$-2P\cdot D(Q)\bar{u}(p)\frac{\partial}{\partial t}f_2(t)i\gamma\cdot k(1+\gamma_5)v(q),$$

which we are instructed to neglect. This establishes the equivalence of Eqs. (2.13a) and (2.13b) for the purposes of deriving $T_L(\bar{K}_{l3\gamma}{}^0)$.

What we have done so far is equivalent to the evaluation of leading terms in k of the Feynman graphs in Fig. 4, which correspond to radiation emitted by the external charged lines plus a seagull graph necessary to maintain gauge invariance. It remains now to discuss the structure-dependent terms of order k and higher. We refer the reader to I, where a lengthy discussion was given of the structure-dependent terms and their relation to T_L. Below we briefly summarize the principal results. We define these structure-dependent terms through the relation

$$T_S=\epsilon_\mu(M_{\mu\nu}{}^V+M_{\mu\nu}{}^A)l_\nu.\tag{2.19}$$

The vector and axial-vector matrix elements $M_{\mu\nu}{}^V$ and $M_{\mu\nu}{}^A$ may be covariantly decomposed as follows[15]:

$$M_{\mu\nu}{}^V=A\delta_{\mu\nu}+Bk_\mu k_\nu+CQ_\mu Q_\nu+DP_\mu P_\nu+Ek_\mu P_\nu+FP_\mu k_\nu+GP_\mu Q_\nu+HQ_\mu P_\nu+IQ_\mu k_\nu+Jk_\mu Q_\nu,\tag{2.20}$$

$$M_{\mu\nu}{}^A=\epsilon_{\mu\nu\alpha\beta}(aP_\alpha Q_\beta+bP_\alpha k_\beta+cQ_\alpha k_\beta)+\epsilon_{\mu\alpha\beta\gamma}P_\alpha k_\beta Q_\gamma(dP_\nu+ek_\nu+fQ_\nu)+\epsilon_{\nu\alpha\beta\gamma}P_\alpha k_\beta Q_\gamma(gP_\mu+hk_\mu+jQ_\mu).\tag{2.21}$$

In Eqs. (2.20) and (2.21) the coefficients are in general

[15] In Paper I we limited ourselves to structures bilinear in the particle momenta and retained only the first three terms in the axial-vector matrix element. A more general form for these amplitudes has been discussed independently by G. W. Intemann, Phys. Rev. **181**, 1866 (1969). His expressions, however, involve amplitudes which are not all independent.

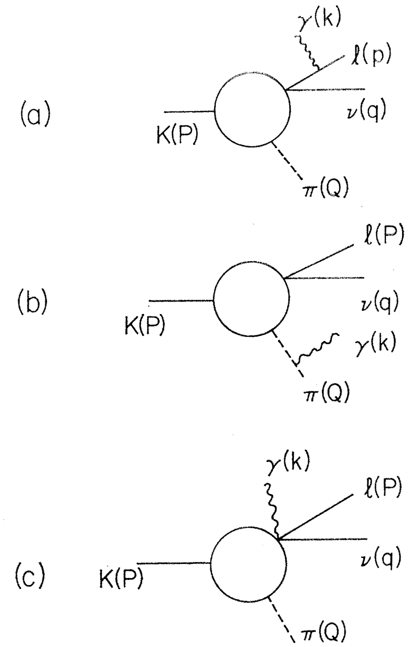

FIG. 4. (a) Inner bremsstrahlung from the lepton line. (b) Inner bremsstrahlung from the pion line. (c) Seagull term necessary to maintain gauge invariance.

functions of the variables $t=-(P-Q)^2$, $P\cdot k$, and $Q\cdot k$ and are assumed to be free of kinematic singularities. Observe first that the terms B, E, J, and h do not contribute, since $\epsilon\cdot k=0$. Also, when $M_{\mu\nu}{}^A$ is contracted with l_ν, the terms d, e, and f are no longer independent of the others by virtue of the identity, valid for arbitrary four-vectors A_α, B_β, and C_γ,

$$\epsilon_{\mu\alpha\beta\gamma}A_\alpha B_\beta C_\gamma(\gamma\cdot C)=\epsilon_{\mu\sigma\alpha\beta}\gamma_\sigma[(A\cdot C)B_\alpha C_\beta-(B\cdot C)A_\alpha C_\beta +C^2A_\alpha B_\beta]+C_\mu\epsilon_{\sigma\alpha\beta\gamma}\gamma_\sigma A_\alpha B_\beta C_\gamma\,, \quad (2.22)$$

which allows us to express the d, e, and f terms as linear combinations of the a, b, c, g, h, and j terms. Thus we are left with twelve independent terms, seven in $M_{\mu\nu}{}^V l_\nu$ and five in $M_{\mu\nu}{}^A l_\nu$. These terms must now be made gauge invariant (remember T_L is separately gauge invariant). To do this in such a way as not to introduce spurious kinematic singularities, we use the procedure developed by Bardeen and Tung.[16] Thus we first multiply $(M_{\mu\nu}{}^V+M_{\mu\nu}{}^A)l_\nu$ by the projection operator $I_{\rho\mu}=\delta_{\rho\mu}-P_\rho k_\mu/P\cdot k$. Then we eliminate the $(P\cdot k)^{-1}$ singularity from $I_{\rho\mu}$ by taking linear combinations of various terms or, if this is not possible, by multiplying by $P\cdot k$. Thus we obtain the general gauge-invariant result

$$\begin{aligned}\epsilon_\mu(M_{\mu\nu}{}^V+M_{\mu\nu}{}^A)l_\nu=\epsilon_\mu\{&C_1[\delta_{\mu\nu}P\cdot k-P_\mu k_\nu]\\ &+C_2[\delta_{\mu\nu}Q\cdot k-Q_\mu k_\nu]+[Q_\mu P\cdot k-P_\mu Q\cdot k]\\ &\times[C_3Q_\nu+C_4P_\nu]+\epsilon_{\mu\nu\alpha\beta}[C_5P_\alpha k_\beta+C_6Q_\alpha k_\beta]\\ &+C_7\epsilon_{\nu\alpha\beta\gamma}P_\alpha Q_\beta[\delta_{\mu\gamma}Q\cdot k-Q_\mu k_\gamma]\\ &+C_8\epsilon_{\nu\alpha\beta\gamma}P_\alpha Q_\beta[\delta_{\mu\gamma}P\cdot k-P_\mu k_\gamma]\}l_\nu\,,\end{aligned} \quad (2.23)$$

where the C_i are free of kinematic singularities or zeros.[17] Note that there are a total of eight independent quantities, just as there are eight independent helicity amplitudes.

As we have no reason to believe that any of the coefficients C_i are abnormally large, and as our experience indicates that the presence of each additional momentum in a structure-dependent term suppresses its contribution to the decay rate, we will for the purposes of this calculation neglect the C_3, C_4, C_7, and C_8 terms which involve four powers of the momenta. If at some future date one finds reason to believe that their influence could be important, they can be included. Thus in this approximation the final form for the complete matrix element is

$$\begin{aligned}T(\bar{K}_{l3\gamma}{}^0)&=T_L(\bar{K}_{l3\gamma}{}^0)+T_S(\bar{K}_{l3\gamma}{}^0)\\ &=\bar{u}(p)\left(\frac{\epsilon\cdot Q}{Q\cdot k}-\frac{\epsilon\cdot p}{p\cdot k}-\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)[2f_+(t)i\gamma\cdot P+mf_2(t)](1+\gamma_5)v(q)\\ &\quad-2\bar{u}(p)\left(\epsilon\cdot P-\frac{\epsilon\cdot Q}{Q\cdot k}P\cdot k\right)\left(2\frac{\partial f_+(t)}{\partial t}i\gamma\cdot P+m\frac{\partial f_2(t)}{\partial t}\right)(1+\gamma_5)v(q)+\frac{A}{M^2}(\epsilon\cdot lP\cdot k-\epsilon\cdot Pl\cdot k)\\ &\quad+\frac{B}{M^2}\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu P_\alpha k_\beta+\frac{C}{M^2}(\epsilon\cdot lQ\cdot k-\epsilon\cdot Ql\cdot k)+\frac{D}{M^2}\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu Q_\alpha k_\beta\,,\end{aligned} \quad (2.24)$$

[16] W. A. Bardeen and W. K. Tung, Phys. Rev. **173**, 1423 (1968).

[17] In I, Eq. (2.35) we made a different choice of the scalar invariants which are essentially different combinations of the ones chosen here. Note that we can add and subtract the term $(I'/M^2)(\epsilon\cdot lP\cdot k-\epsilon\cdot Pl\cdot k)Q\cdot k$ to Eq. (2.35) and find the result

$$[A'/M^2+(I'/M^2)Q\cdot k](\epsilon\cdot lP\cdot k-\epsilon\cdot Pl\cdot k)+(I'/M^2)P\cdot k(\epsilon\cdot Ql\cdot k-\epsilon\cdot lQ\cdot k),$$

which has the same invariant structure as Eq. (2.23) above. The choice made here ensures that all of the scalar invariants are free of kinematic singularities, which was not the case in I. This does not affect the results of Paper I because these terms vanish in the soft-pion limit.

where we have redefined $A=C_1M^2$, $B=C_5M^2$, $C=C_2M^2$, and $D=C_6M^2$ to make our final notation conform more closely to that of I. We will refer to these terms as the main term, the derivative term, and the four structure-dependent terms, respectively.

III. SQUARE OF MATRIX ELEMENT

To obtain the photon spectrum and the decay rate from the matrix element of Eq. (2.24), we must square the matrix element and sum over the lepton spins and photon polarizations. Once the square of the matrix element is written down, the sum over photon polarizations is trivially performed using

$$\sum_{\text{polarizations } m} \epsilon_\mu{}^{(m)}\epsilon_\nu{}^{(m)}=\delta_{\mu\nu}\,. \tag{3.1}$$

The sum over lepton spins may be carried out by use of the usual trace theorems and is a tedious but straightforward calculation. In I the trace calculation was done by hand and checked against the results of the program SCHOONSCHIP, which gives a symbolic evaluation of the trace in terms of the scalar products of four-vectors. In this section we discuss a theorem due to Burnett and Kroll[4] which allows one to obtain the k^{-2} and k^{-1} terms in the square of the matrix element with much less effort than that required for explicit evaluations of the trace. As in Sec. II, we will first discuss a general process and then specialize to the particular decay of interest here.

Consider the matrix element T_L+T_S corresponding to the general process, and call it for simplicity $\mathfrak{M}$. We can write $\mathfrak{M}$ symbolically in the form $\mathfrak{M}=\mathfrak{M}_{-1}+\mathfrak{M}_0+\mathfrak{M}_1$, where $\mathfrak{M}_{-1}$ and $\mathfrak{M}_0$ are of order k^{-1} and k^0, respectively, and are known exactly from Low's theorem. $\mathfrak{M}_1$ is of order k and represents the additional unknown structure-dependent contributions. Thus $|\mathfrak{M}^2|$ summed over spins will involve terms of order k^{-2} coming from $\sum|\mathfrak{M}_{-1}|^2$ and terms of order k^{-1} arising from $\sum(\mathfrak{M}_{-1}\mathfrak{M}_0{}^*+\mathfrak{M}_0\mathfrak{M}_{-1}{}^*)$, both of which involve only known quantities. The terms in $\sum|\mathfrak{M}|^2$ of order k^0 arise from two sources, as we have noted previously. One contribution which arises from $\sum|\mathfrak{M}_0|^2$ is known from T_L and can be calculated directly. The other contribution arises from the interference terms $\sum(\mathfrak{M}_{-1}\mathfrak{M}_1{}^*+\mathfrak{M}_1\mathfrak{M}_{-1}{}^*)$ and involves the unknown structure-dependent part T_S. Burnett and Kroll derived a relatively simple formula for $\sum[|\mathfrak{M}_{-1}|^2+(\mathfrak{M}_{-1}\mathfrak{M}_0{}^*+\mathfrak{M}_0\mathfrak{M}_{-1}{}^*)]$. For completeness, we summarize below the derivation of this formula.

Define the quantity $\hat{Q}$ by

$$\hat{Q}=\sum_i \eta_i Q_i\frac{\epsilon\cdot p_i}{k\cdot p_i}\,, \tag{3.2}$$

where η_i and Q_i appear in Eq. (2.7). From Eq. (2.9) we have

$$\sum_{\text{spins}}|\mathfrak{M}_{-1}|^2=\hat{Q}^2\sum_{\text{spins}}|\bar{u}(p_d)T_0u(p_c)|^2\,. \tag{3.3}$$

For the terms of order k^{-1}, we write

$$O(k^{-1})=\sum_{\text{spins}}(\mathfrak{M}_{-1}\mathfrak{M}_0{}^*+\mathfrak{M}_0\mathfrak{M}_{-1}{}^*)=\sum_{\text{spins}}\hat{Q}\bar{u}(p_d)T_0u(p_c)\left\{\sum_i Q_iD_\lambda(p_i)\bar{u}(p_d)\frac{\partial T_0}{\partial p_{i\lambda}}u(p_c)\right.$$
$$\left.+\bar{u}(p_d)\left[T_0\left(Q_c+(-i\gamma\cdot p_c+m_c)\frac{\kappa_c}{2m_c}\right)\frac{\gamma\cdot k\gamma\cdot\epsilon}{2k\cdot p_c}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p_d}\left(Q_d+(-i\gamma\cdot p_d+m_d)\frac{\kappa_d}{2m_d}\right)T_0\right]u(p_c)\right\}^*$$
$$+\text{complex conjugate}\,. \tag{3.4}$$

With some simple Dirac algebra given explicitly in Ref. 4, one immediately obtains the important result that all terms involving the magnetic moments exactly cancel, and thus do not contribute to the $O(k^{-2})$ or $O(k^{-1})$ parts of the radiative amplitude. Following Burnett and Kroll, we use the identities

$$(m-i\gamma\cdot p)\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}-\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}(m-i\gamma\cdot p)=-i\gamma\cdot D(p) \tag{3.5}$$

and

$$-i\gamma_\lambda=\frac{\partial}{\partial p_\lambda}(m-i\gamma\cdot p)=2m\frac{\partial}{\partial p_\lambda}\Big[\sum_{\text{spins}}u(p,s)\bar{u}(p,s)\Big] \tag{3.6}$$

to write the charge parts of the last two terms in Eq. (3.4) in terms of derivatives. Thus the entire $O(k^{-1})$ contribution can be written as

$$\hat{Q}\sum_i Q_iD_\lambda(p_i)\frac{\partial}{\partial p_{i\lambda}}\sum_{\text{spins}}|\bar{u}(p_d)T_0u(p_c)|^2\,. \tag{3.7}$$

Combining Eqs. (3.3) and (3.7), we obtain the final result of Burnett and Kroll:

$$\sum_{\text{spins}}|\mathfrak{M}|^2=\left[\hat{Q}^2+\hat{Q}\sum_i Q_iD_\lambda(p_i)\frac{\partial}{\partial p_{i\lambda}}\right]\times\sum_{\text{spins}}|T|^2+O(k^0)\,. \tag{3.8}$$

As discussed above and in Refs. 4 and 5, T in Eq. (3.8) is to be interpreted as the nonradiative amplitude considered as a function of particle momenta satisfying the momentum-conservation equation

$$\sum_{\text{initial}} p_i - \sum_{\text{final}} p_i = k \tag{3.9}$$

rather than

$$\sum_{\text{initial}} p_i - \sum_{\text{final}} p_i = 0. \tag{3.10}$$

Thus although $\sum |T|^2$ has the same form as the unpolarized nonradiative cross section, it is a function of the radiative momenta. The significance of this distinction is evident if we consider a two-body decay such as $\Lambda \to p\pi$, which is conventionally described by an amplitude $T_0 = A + B\gamma_5$, where A and B are constants. Thus

$$\sum_{\text{spins}} |T|^2 = (|A|^2 - |B|^2) - \frac{p_p \cdot p_\Lambda}{m_p m_\Lambda}(|A|^2 + |B|^2). \tag{3.11}$$

For the nonradiative process, $p_\Lambda = p_p + p_\pi$, which enables us to write $(p_p \cdot p_\Lambda)/m_p m_\Lambda = -(m_\Lambda{}^2 + m_p{}^2 - m_\pi{}^2)/2m_p m_\Lambda$. Thus, for the nonradiative process, $|T|^2$ is a constant. For the radiative process, however, we must substitute Eq. (3.11) directly into Eq. (3.8), considering $|T|^2$ as a function of the (independent) radiative variables p_p and p_Λ. As discussed earlier, since $N=3$, the A and B are still constants, and no derivatives of A and B enter. The term $\sum_i Q_i D_\lambda(p_i) \partial(p_p \cdot p_\Lambda)/\partial p_{i\lambda}$ does produce a contribution, however, and must be included. Finally, we note that the operator part of Eq. (3.8) is independent of whether radiation was emitted by particles or antiparticles.

Given the result (3.8), the sum over photon polarizations can be carried out immediately using Eq. (3.1). To illustrate the application of Eq. (3.8) to the present problem, we remember that the particular form chosen for the nonradiative amplitude makes no difference to order k^{-2} and k^{-1} in $\sum |\mathfrak{M}|^2$, so we choose the expression given in Eq. (2.13b), since it involves explicitly only the neutral momentum P. Then, writing

$$\sum_{\text{spins}} |T(\bar{K}_{l3}{}^0)|^2 = \sum_{\text{spins}} |\bar{u}(p)[2f_+(t) i\gamma \cdot P + m f_2(t)](1+\gamma_5) v(q)|^2$$
$$= (2/mm_\nu)\{4f_+{}^2(t)[2P\cdot pP\cdot q + M^2 p\cdot q] - f_2{}^2(t) m^2 p\cdot q + 4m^2 f_+(t) f_2(t) P\cdot q\} \tag{3.12}$$

and using Eqs. (3.8) and (3.1), we find for the sum over spins and polarizations

$$\sum_{\text{spins, pol.}} |T(\bar{K}^0 \to \pi l \nu \gamma)|^2 = \left\{ \left(\frac{Q_\mu}{Q\cdot k} - \frac{p_\mu}{p\cdot k}\right)^2 + \left(\frac{Q_\mu}{Q\cdot k} - \frac{p_\mu}{p\cdot k}\right) \times \left[-\left(\frac{p_\mu k_\lambda}{p\cdot k} - \delta_{\mu\lambda}\right)\frac{\partial}{\partial p_\lambda} + \left(\frac{Q_\mu k_\lambda}{Q\cdot k} - \delta_{\mu\lambda}\right)\frac{\partial}{\partial Q_\lambda}\right]\right\} \sum_{\text{spins}} |T(\bar{K}_{l3}{}^0)|^2 + O(k^0). \tag{3.13}$$

Carrying out the explicit differentiations indicated, we find

$$\tfrac{1}{2} mm_\nu \sum_{\text{spins, pol.}} |T(\bar{K}^0 \to \pi^+ l^- \bar{\nu}\gamma)|^2 = -\left[\frac{\mu^2}{(Q\cdot k)^2} + \frac{m^2}{(p\cdot k)^2} + \frac{2p\cdot Q}{p\cdot k Q\cdot k}\right][4f_+{}^2(t)(2P\cdot pP\cdot q + M^2 p\cdot q)$$
$$- f_2{}^2(t) m^2 p\cdot q + 4m^2 f_+(t) f_2(t) P\cdot q] - 8f_+{}^2(t) P\cdot q \left\{\left[\frac{p\cdot Q}{p\cdot kQ\cdot k} + \frac{m^2}{(p\cdot k)^2}\right]P\cdot k - \frac{P\cdot Q}{Q\cdot k} + \frac{P\cdot p}{p\cdot k}\right\} - [4M^2 f_+{}^2(t) - m^2 f_2{}^2(t)]$$
$$\times \left\{\left[\frac{p\cdot Q}{p\cdot kQ\cdot k} + \frac{m^2}{(p\cdot k)^2}\right]q\cdot k - \frac{q\cdot Q}{Q\cdot k} + \frac{p\cdot q}{p\cdot k}\right\} - 2\left\{\left[\frac{p\cdot Q}{p\cdot kQ\cdot k} + \frac{\mu^2}{(Q\cdot k)^2}\right]P\cdot k - \frac{P\cdot p}{p\cdot k} + \frac{P\cdot Q}{Q\cdot k}\right\}$$
$$\times \left\{8f_+(t)\frac{\partial f_+(t)}{\partial t}(2P\cdot pP\cdot q + M^2 p\cdot q) - 2m^2 f_2(t)\frac{\partial f_2(t)}{\partial t} p\cdot q + 4m^2\left[f_+(t)\frac{\partial f_2(t)}{\partial t} + f_2(t)\frac{\partial f_+(t)}{\partial t}\right]P\cdot q\right\} + O(k^0). \tag{3.14}$$

The utility of the Burnett-Kroll theorem as a computational tool should now be evident, as the procedure for obtaining this result from Eq. (3.8) was certainly much simpler than the explicit evaluation of the square of Eq. (2.17). In Appendix A we list the result for the complete square of the main term of the matrix element in Eq. (2.24) together with the terms arising from interference between the main term and the other terms after having eliminated q using four-momentum conservation. One can check that the $O(k^{-2})$ and $O(k^{-1})$ terms in Eqs. (A2) and (A3) are identical to those in Eq. (3.14).

IV. EVALUATION OF STRUCTURE-DEPENDENT FORM FACTORS

Let us now estimate values for the coefficients A, B, C, and D in Eq. (2.24). We assume partial conservation of axial-vector current (PCAC) in the form

$$\partial_\lambda \mathfrak{F}_{5\lambda}{}^j(x) = \frac{\mu^3 a_\pi}{\sqrt{2}}\phi^j(x) = \frac{\mu^2 f_\pi}{\sqrt{2}}\phi^j(x)\,,\quad j=1,2,3 \quad (4.1)$$

where μ is the pion mass, $a_\pi = 0.94$, and $\phi^j(x)$ is the field operator which creates a pion with isospin j. By a Lehmann-Symanzik-Zimmermann (LSZ) reduction of the charged pion field, we can express the matrix element as

$$\begin{aligned}(2Q_0V)^{1/2}&\langle\pi^+\gamma|J_\nu{}^{4+i5}(0)|\bar{K}^0\rangle \\ &= \frac{-i}{\sqrt{2}}\int d^4y\, e^{-iQ\cdot y}(-\Box_y+\mu^2) \\ &\qquad \times\langle\gamma|T(\phi^{1-i2}(y)J_\nu{}^{4+i5}(0))|\bar{K}^0\rangle \\ &= \frac{Q^2+\mu^2}{\mu^3 a_\pi}\int d^4y\, e^{-iQ\cdot y}Q_\lambda\langle\gamma|T(\mathfrak{F}_{5\lambda}{}^{1-i2}(y)J_\nu{}^{4+i5}(0))|\bar{K}^0\rangle \\ &\quad +\frac{i(Q^2+\mu^2)}{\mu^3 a_\pi}\int d^4y\, e^{-iQ\cdot y}\delta(y_0) \\ &\qquad \times\langle\gamma|[\mathfrak{F}_{50}{}^{1-i2}(y),J_\nu{}^{4+i5}(0)]|\bar{K}^0\rangle\,, \quad (4.2)\end{aligned}$$

where J_ν is either V_ν or A_ν.[18] We drop the surface terms arising from the partial integration in deriving Eq. (4.2). Taking the limit $Q_\lambda \to 0$, we note that the first term has no pole because we have extracted the bremsstrahlung contribution.[19] Thus we find

$$\begin{aligned}(2Q_0V)^{1/2}&\langle\pi^+\gamma|J_\nu{}^{4+i5}(0)|\bar{K}^0\rangle \\ &= (i/\mu a_\pi)\langle\gamma|[F_5{}^{1-i2}(0), J_\nu{}^{4+i5}(0)]|\bar{K}^0\rangle\,, \quad (4.3)\end{aligned}$$

where

$$F_5{}^j(y_0) = \int d^3y\, \mathfrak{F}_{50}{}^j(y)\,.$$

The usual commutation rules of Gell-Mann[13] now yield

$$\begin{aligned}(2Q_0V)^{1/2}&\langle\pi^+\gamma|V_\nu{}^{4+i5}(0)|\bar{K}^0\rangle \\ &= (i/\mu a_\pi)\langle\gamma|A_\nu{}^{6+i7}(0)|\bar{K}^0\rangle\,, \\ (2Q_0V)^{1/2}&\langle\pi^+\gamma|A_\nu{}^{4+i5}(0)|\bar{K}^0\rangle \\ &= (i/\mu a_\pi)\langle\gamma|V_\nu{}^{6+i7}(0)|\bar{K}^0\rangle\,,\end{aligned} \quad (4.4)$$

i.e., a relation between radiative K_{l3} and radiative "K_{l2}" decays.[20] These equations are the analog of Eqs. (3.3) of I. Unfortunately, the photon does not have well-defined isospin, so we cannot make an isospin rotation in Eq. (4.4) to relate the neutral-K matrix element to that of the charged K. If, however, we assume that the photon is a U-spin scalar, use charge conjugation and $SU(3)$ invariance, and dominate the neutral currents $A_\nu{}^{6+i7}$, $V_\nu{}^{6+i7}$ by K^{*0} resonances, then the coupling constants $G(K^{*0}\bar{K}^0\gamma)$ and $G(K^{*+}K^-\gamma)$ are related by a Clebsch-Gordan coefficient of -2.[21] We can therefore take the values for A and B from the pole dominance of the $K^+\to l^+\nu\gamma$ matrix element given in I. This yields [see I, Eq. (3.21)]

$$|B(0)|\cong(2M/\mu a_\pi)|\tilde{b}(0)|\,,$$

i.e.,

$$0.5\lesssim|B(0)|\lesssim 6\,. \quad (4.5)$$

The ratio of A to B is not determined very accurately through $K_A(1320)$ dominance as discussed in I. We derived there that $0.1\lesssim|A(0)/B(0)|\lesssim 0.9$, which we also assume to hold in this paper.

We now attempt to calculate the form factors C and D, which vanish in the soft-pion limit, by making a soft-kaon approximation. Although it is widely recognized that PCAC is not as good an approximation for kaons as it is for pions, we are only using it to estimate form factors whose net contributions to the decay rate are small. This justifies the rather poor theoretical model. We assume PCAC for kaons in the form

$$\partial_\lambda \mathfrak{F}_\lambda{}^j(x) = (M^2 f_K/\sqrt{2})\phi^j(x)\,,\quad j=4,5,6,7 \quad (4.6)$$

where f_K is defined by

$$\langle 0|A_\lambda{}^{4\pm i5}(0)|K^\mp\rangle = [i/(2P_0V)^{1/2}]f_K P_\lambda$$

for charged K decays. By an isospin rotation, the same f_K describes neutral K decays, because

$$\langle 0|A_\lambda{}^{4+i5}(0)|K^-\rangle = -\langle 0|A_\lambda{}^{6+i7}(0)|\bar{K}^0\rangle\,. \quad (4.7)$$

Next we make an LSZ reduction of the matrix element

$$\begin{aligned}(2P_0V)^{1/2}\langle\pi^+\gamma|J_\nu{}^{4+i5}(0)|\bar{K}^0\rangle &= -\frac{P^2+M^2}{\sqrt{2}C_K}\int d^4x\, e^{iP\cdot x} \\ &\quad \times P_\lambda\langle\pi^+\gamma|T(\mathfrak{F}_{5\lambda}{}^{6-i7}(x)J_\nu{}^{4+i5}(0))|0\rangle \\ &\quad +\frac{i(P^2+M^2)}{\sqrt{2}C_K}\int d^4x\, e^{iP\cdot x}\delta(x_0) \\ &\qquad \times\langle\pi^+\gamma|[\mathfrak{F}_{50}{}^{6-i7}(x),J_\nu{}^{4+i5}(0)]|0\rangle\,, \quad (4.8)\end{aligned}$$

where

$$C_K = -M^2 f_K/\sqrt{2}\,.$$

Now we take a soft-kaon limit $P_\lambda\to 0$, so that[18,19]

$$\begin{aligned}(2P_0V)^{1/2}&\langle\pi^+\gamma|J_\nu{}^{4+i5}(0)|\bar{K}^0\rangle \\ &= \frac{iM^2}{\sqrt{2}C_K}\langle\pi^+\gamma|[F_5{}^{6-i7}(0),J_\nu{}^{4+i5}(0)]|0\rangle\,. \quad (4.9)\end{aligned}$$

[18] Most authors would write $\langle\pi^+\gamma|J_\nu{}^{4+i5}(0)|\bar{K}^0\rangle$ as the matrix element for $\bar{K}^0\to\pi^+l^-\nu\gamma$ decay after the lepton bremsstrahlung has been extracted. We have already extracted more than this, i.e., the pion pole term and the seagull term necessary to make the k^{-1} and k^0 terms gauge invariant. Hence we identify $\langle\pi^+\gamma|J_\nu{}^{4+i5}(0)|\bar{K}^0\rangle$ as only the unknown structure-dependent part of the decay amplitude.

[19] The bremsstrahlung parts of the amplitudes for radiative $K_{l3}{}^+$ and $K_{l2}{}^+$ decays depend only upon the ordinary $K_{l3}{}^+$ and $K_{l2}{}^+$ decay amplitudes and are related in the soft-pion limit by the usual Callan-Treiman relation; see C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966). There is a corresponding relation for the bremsstrahlung terms in $\bar{K}_{l3}{}^0$ decay and $\pi_{l2}{}^+$ decay in the soft-kaon limit which was first derived by R. Oehme, *ibid.* **16**, 215 (1966).

[20] If one generalizes CVC to strangeness-changing processes, then the right-hand side of Eq. (4.4) can be related to the decay $\bar{K}^0\to 2\gamma$.

[21] M. Gourdin, *Unitary Symmetry* (North-Holland, Amsterdam, 1967), p. 100.

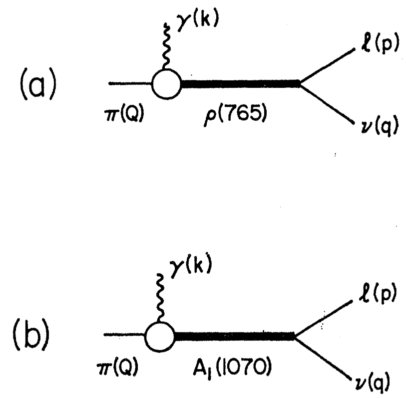

FIG. 5. (a) ρ-pole contribution to the structure-dependent vector matrix element in $\pi^- \to l^-\bar{\nu}\gamma$ decay. (b) A_1-pole contribution to the structure-dependent axial-vector matrix element in $\pi^- \to l^-\bar{\nu}\gamma$ decay.

The commutation relations of Gell-Mann[13] now yield

$$\begin{aligned}(2P_0V)^{1/2}\langle\pi^+\gamma|V_\nu^{4+i5}(0)|\bar{K}^0\rangle &= \frac{iM^2}{\sqrt{2}C_K}\langle\pi^+\gamma|A_\nu^{1+i2}(0)|0\rangle\,,\\ (2P_0V)^{1/2}\langle\pi^+\gamma|A_\nu^{4+i5}(0)|\bar{K}^0\rangle &= -\frac{iM^2}{\sqrt{2}C_K}\langle\pi^+\gamma|V_\nu^{1+i2}(0)|0\rangle\,.\end{aligned} \tag{4.10}$$

The matrix elements on the right-hand side of these equations are the structure-dependent terms in the $\pi^- \to l^-\bar{\nu}\gamma$ decay, which we define by

$$\mathfrak{M} = {}_{\rm out}\langle l^-\bar{\nu}\gamma|\pi^-\rangle_{\rm in} = -i(2\pi)^4\delta^4(Q-p-q-k)\times\left(\frac{mm_\nu}{4Q_0p_0q_0k_0V^4}\right)^{1/2}\frac{eG\cos\theta}{\sqrt{2}}T\,, \tag{4.11}$$

with

$$T = imf_\pi\bar{u}(p)\left(\frac{\epsilon\cdot p}{p\cdot k}-\frac{\epsilon\cdot Q}{Q\cdot k}+\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}\right)(1+\gamma_5)v(q) + i(\tilde{c}/\mu)(\epsilon\cdot lQ\cdot k-\epsilon\cdot Qk\cdot l) + i(\tilde{d}/\mu)\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu l_\nu Q_\alpha k_\beta\,. \tag{4.12}$$

Obviously, in the soft-kaon limit the terms involving A and B in Eq. (2.24) vanish and the terms involving C and D are related to the $\tilde{c}$ and $\tilde{d}$ terms in Eq. (4.12) by Eqs. (4.10). Hence

$$\left|\frac{C}{M^2}\sin\theta\right| = \left|\frac{M^2}{\sqrt{2}C_K}\frac{\tilde{c}}{\mu}\cos\theta\right| = \left|\frac{\tilde{c}\cos\theta}{f_K\mu}\right|.$$

(Note that we do not know the sign of $\tilde{c}$.) Hence

$$|C| = \left|\frac{\tilde{c}M^2\cot\theta}{f_K\mu}\right|. \tag{4.13}$$

Similarly, we find the corresponding formula for D,

$$|D| = \left|\frac{\tilde{d}M^2\cot\theta}{f_K\mu}\right|. \tag{4.14}$$

The estimation of $\tilde{c}$ and $\tilde{d}$ is now made by assuming ρ and A_1 dominance of the vector and axial-vector matrix elements describing $\pi^- \to l^-\bar{\nu}\gamma$ (see Fig. 5). It is not necessary to describe these calculations in detail because they parallel the corresponding calculations for $K^+ \to l^+\nu\gamma$ in I. A treatment of radiative π_{l2} decay has also been given in the recent book of Marshak, Riazuddin, and Ryan.[22] We find

$$\left|\frac{\tilde{d}(0)}{\mu}\right| = \left|\frac{f_\rho G_{\rho\pi\gamma}}{M_\rho^2-\mu^2}\right|, \tag{4.15}$$

where the numerical values of the coupling constants are

$$f_\rho = \sqrt{2}(M_\rho^2/f_{\rho\pi\pi}) \cong 0.26M_\rho^2\,,\qquad G_{\rho\pi\gamma} = G_{K^*K\gamma} \cong 0.37/M\,.$$

The only unknown is now f_K, for which we accept the usual value[23]

$$f_K = 1.28f_\pi\,.$$

Substituting these values into Eqs. (4.14) and (4.15) we find

$$|D| = 1.37\,. \tag{4.16}$$

Similar considerations hold for the value of C. If we assume A_1 dominance and follow the discussion given in Marshak, Riazuddin, and Ryan,[22] then we find

$$\left|\frac{\tilde{c}}{\mu}\right| = \left|\frac{2f_{A_1}f_{A_1\pi\gamma}}{(M_{A_1}^2-\mu^2)^2}\right|, \tag{4.17}$$

where from Fayyazuddin and Riazuddin[24] we have

$$f_{A_1}f_{A_1\pi\gamma}/(M_{A_1}^2-\mu^2) = f_\pi\,.$$

Hence, using $M_{A_1} = \sqrt{2}M_\rho$,[25] we find

$$\tilde{c}/\mu \cong f_\pi/M_\rho^2\,. \tag{4.18}$$

This yields the result

$$|C| = (M^2f_\pi/M_\rho^2f_K)\cot\theta = 1.55\,. \tag{4.19}$$

The value given in Eq. (4.19) depends on specific analyticity properties for form factors and can vary

[22] R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley-Interscience, New York, 1969), pp. 359–363. We note that the $f_{A_1\pi\gamma}$ coupling has a kinematic zero in its definition, whereas the $h_{K_AK\gamma}$ coupling defined in I was free of kinematic singularities. This accounts for the extra power of $M_{A_1}^2-\mu^2$ in the denominator of Eq. (4.17) as compared with Eq. (3.23) of I.

[23] J. Bernstein, *Elementary Particles and Their Currents* (W. H. Freeman and Co., San Francisco, 1968), p. 272. Slightly smaller values of this ratio have been derived using spectral function sum rules; see H. T. Nieh, Phys. Rev. Letters **19**, 43 (1967); and S. L. Glashow, H. J. Schnitzer, and S. Weinberg, *ibid.* **19**, 139 (1967).

[24] Fayyazuddin and Riazuddin, Phys. Rev. Letters **18**, 715 (1967).

[25] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

appreciably. Specific models are discussed in Ref. 22, all of which yield smaller values for $|C|$.

We would like to stress that the values for A, B, C, and D derived in this section are not intended to be taken literally, but only as a rough order-of-magnitude estimate. To give some idea of the influence of these terms, we shall plot the photon spectrum in the next section for a typical set of values $A=B=2.5$, $C=D=1.0$. The signs are taken positive to yield constructive interference with the other terms in the matrix element.

V. CONCLUSIONS

Let us first give the final numbers which determine the rates. Our procedure is to evaluate the direct square of the main term in the radiative decay amplitude, Eq. (2.24), up to and including terms linear in Λ. This gives seven terms, i.e., $f_+{}^2(0)$, $f_+(0)f_2(0)$, $f_2{}^2(0)$, $f_+{}^2(0)\Lambda_+$, $f_2{}^2(0)\Lambda_2$, $f_+(0)f_2(0)\Lambda_+$, and $f_+(0)f_2(0)\Lambda_2$. The square of the derivative term of order k^0 (as well as the square of the structure-dependent terms) was found to be so small that it could be safely neglected. The remaining terms come from the interference between the main term and the other terms. Here we are justified in keeping only $f_+(0)$ and $f_2(0)$ in the main term, and we thus generate the terms

$$f_+(t)\frac{\partial}{\partial t}f_+(t) \to f_+{}^2(0)\Lambda_+\,,$$

$$f_+(t)\frac{\partial}{\partial t}f_2(t) \to f_+(0)f_2(0)\Lambda_2\,,$$

$$f_2(t)\frac{\partial}{\partial t}f_+(t) \to f_+(0)f_2(0)\Lambda_+\,,$$

$$f_2(t)\frac{\partial}{\partial t}f_2(t) \to f_2{}^2(0)\Lambda_2\,,$$

$$f_+(0)A,\quad f_2(0)A,\quad f_+(0)B,\quad f_2(0)B,\quad f_+(0)C,\quad f_2(0)C,\quad f_+(0)D,\quad \text{and}\quad f_2(0)D.$$

In summary, we have kept all terms in $T_L{}^2$ except those of order Λ^2 and those terms in $T_S\times T_L$ (which is already at least of order k^0) which are of order zero in Λ. The rate therefore is a double integral over the sum of these nineteen terms. Details of the necessary phase-space integrals are given in Appendix B. In practice, the decay rate for the electron mode has only seven terms because all the coefficients of the form factor $f_2(0)$ are suppressed by an extra power of $(m/M)^2$. The results are as follows:

$$\Gamma(\bar{K}^0\to\pi^+e^-\bar{\nu}\gamma, E_\gamma>30\text{ MeV})=\frac{G^2\sin^2\theta M^5}{64\pi^3}\times10^{-3}[1.1152f_+{}^2(0)+0.3646f_+{}^2(0)\Lambda_+-0.0390f_+{}^2(0)\Lambda_+ +0.0037f_+(0)A+0.0012f_+(0)B+0.0028f_+(0)C+0.0012f_+(0)D]. \quad (5.1)$$

As noted above, the two terms proportional to $f_+{}^2(0)\Lambda_+$ in Eq. (5.1) have different origins and thus have not been combined in order to exhibit their relative magnitudes. The first term in $f_+{}^2(0)\Lambda_+$ comes from the t expansion of the $f_+{}^2$ term and the second $f_+{}^2(0)\Lambda_+$ comes from interference between the main term and the derivative term. We follow the same procedure for the muon decay mode:

$$\Gamma(\bar{K}^0\to\pi^+\mu^-\bar{\nu}\gamma,\ E_\gamma>30\text{ MeV})=\frac{G^2\sin^2\theta M^5}{64\pi^3}\times10^{-5}[7.5452f_+{}^2(0)+0.1406f_2{}^2(0)-1.0346f_+(0)f_2(0)$$
$$+4.2380f_+{}^2(0)\Lambda_++0.0973f_2{}^2(0)\Lambda_2-0.3261f_+(0)f_2(0)(\Lambda_++\Lambda_2)-1.3354f_+{}^2(0)\Lambda_+-0.0162f_2{}^2(0)\Lambda_2$$
$$+0.0832f_+(0)f_2(0)\Lambda_++0.1015f_+(0)f_2(0)\Lambda_2+0.0840f_+(0)A+0.0055f_2(0)A+0.0341f_+(0)B-0.0054f_2(0)B$$
$$+0.0589f_+(0)C+0.0011f_2(0)C+0.0224f_+(0)D-0.0024f_2(0)D]. \quad (5.2)$$

Taking now the rate for $\bar{K}^0\to\pi^+e^-\bar{\nu}$ from Appendix B, Eq. (B9), we find

$$R_1=\frac{(\bar{K}^0\to\pi^+e^-\bar{\nu}\gamma,\ E_\gamma>30\text{ MeV})}{\Gamma(\bar{K}^0\to\pi^+e^-\bar{\nu})}$$
$$\times\left(\frac{1.1152+0.3646\Lambda_+-0.0390\Lambda_++0.0037A/f_+(0)+0.0012B/f_+(0)+0.0028C/f_+(0)+0.0012D/f_+(0)}{4(1.1738+0.3191\Lambda_+)}\right)\times10^{-1}. \quad (5.3)$$

The usual model of $K^*(890)$ dominance of the vector form factor yields $\Lambda_+=M^2/M_{K^*}{}^2=0.31$. If we assume $\sin\theta=0.21$, then $f_+(0)=1.04$, from the $K_L{}^0$ rate[26]

$$\Gamma(K_L{}^0\to\pi^\pm e^\mp\nu)=(7.65\pm0.30)\times10^6\text{ sec}^{-1}.$$

The corresponding number in I came from the charged K decay rate and gave $f_+(0)=0.76$. Our new value of $f_+(0)$ differs slightly from $f_+(0)=0.76\sqrt{2}$, which it would be if the $|\Delta I|=\frac{1}{2}$ rule were exact, primarily be-

[26] See the article by J. W. Cronin in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 284.

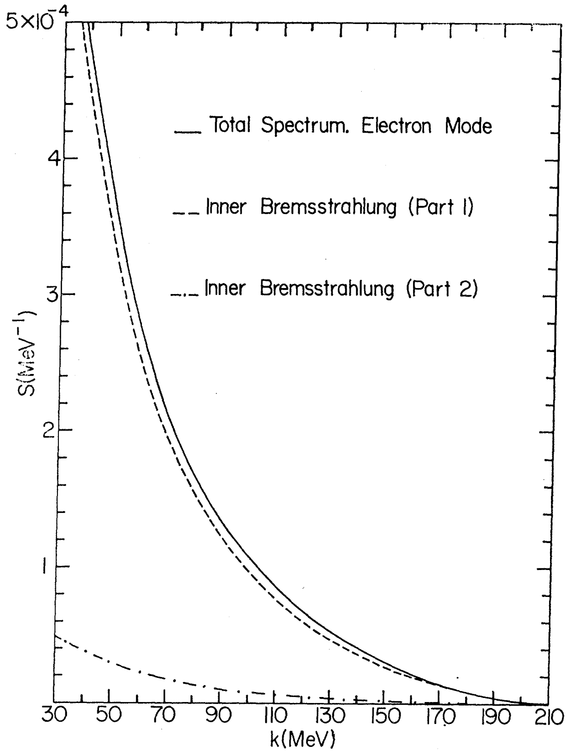

FIG. 6. Photon spectrum $\bar{K}^0 \to \pi^+ e^- \bar{\nu}\gamma$, with $\Lambda_+ = 0.31$ and $A = B = C = D = 0$, normalized by dividing by the $\bar{K}^0 \to \pi^+ e^- \bar{\nu}$ decay rate with the same value of Λ_+. The significance of the separation into parts 1 and 2 is explained in the text.

cause the particular experimental values for the rate used here and in I are not quite in the ratio 2 :1. Note that in R_1 and R_2, $f_+(0)$ enters explicitly in the terms involving A, B, C, and D. With these values, we find

$$R_1(E_\gamma > 30 \text{ MeV}) = (2.389 + 0.007A + 0.002B + 0.005C + 0.002D) \times 10^{-2}. \quad (5.4)$$

The experimental number quoted in Eq. (1.1) cannot be directly compared with this because we do not know the value of the photon cutoff. However, our result for this mode will yield a rough check on the presence of the A, B, C, and D terms once R_1 is better known.

Now let us turn to the branching ratio involving muons. Using the rate for $\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}$ decay from Appendix B, Eq. (B10), and Eq. (5.2), we obtain the branching ratio

$$R_2 = (\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}\gamma,\ E_\gamma > 30 \text{ MeV}) / \Gamma(\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}) \quad (5.5)$$

as a rather complicated function of many parameters. Assuming, for example, that $f_+(0) = 1.04$ and $\Lambda_+ = \Lambda_2 = 0.31$, we find

$$R_2(E_\gamma > 30 \text{ MeV},\ \xi = 0) = (2.163 + 0.026A + 0.008B + 0.017C + 0.006D) \times 10^{-3},$$

$$(5.6)$$

$$R_2(E_\gamma > 30 \text{ MeV},\ \xi = -1) = (2.356 + 0.033A + 0.008B + 0.021C + 0.006D) \times 10^{-3}.$$

In principle, a knowledge of the branching ratio R^2 could be used to set limits on the other parameters but, as no experimental data are available, we only quote these two results.

We complete our analysis by plotting the photon spectra for different values of the coupling constants. Figure 6 shows the photon spectrum $S = [d\Gamma(K_{l3\gamma})/dk]/\Gamma(K_{l3})$ for the decay $\bar{K}^0 \to \pi^+ e^- \bar{\nu}\gamma$ with $A = B = C = D = 0$. The correction due to a small finite value of these constants is almost unobservable and has not been included. By "inner bremsstrahlung part 1" we mean the contribution from the square of the main term in Eq. (2.24) with $f_+(t) = f_+(0)$ and $f_2(t) = f_2(0)$. "Inner bremsstrahlung part 2" refers to all the terms proportional to Λ_+ or Λ_2 in the square of the matrix element. This curve includes an infrared-divergent part, which is suppressed by the extra power of t, but still divergent at the lower end of the photon spectrum. The finite, order-k^0 terms proportional to Λ_+ or Λ_2 are of the same order as the corrections due to the inclusion of the structure-dependent terms, so no effort was made to plot part 2 as a finite part and a divergent part. For the muon mode the inner bremsstrahlung is suppressed and thus the contribution of the smaller terms is relatively enhanced. In Fig. 7 we plot the photon spectrum for $\zeta = 2.0$ ($\xi = -1.0$), $\Lambda_+ = \Lambda_2 = 0.31$, and $A = B = C = D = 0$.

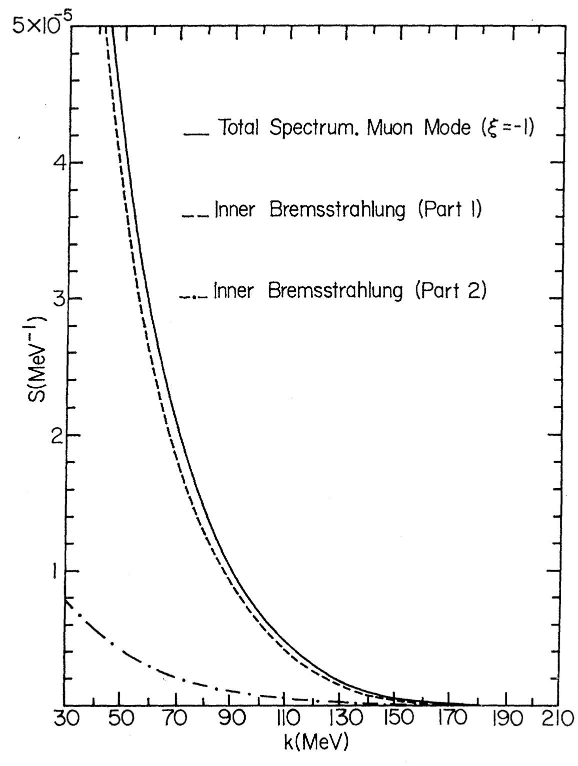

FIG. 7. Photon spectrum in $\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}\gamma$, with $\zeta = 2$ ($\xi = -1$), $\Lambda_+ = \Lambda_2 = 0.31$, and $A = B = C = D = 0$, normalized by dividing by the $\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}$ decay rate with the same values for the form factors. The significance of the separation into parts 1 and 2 is explained in the text.

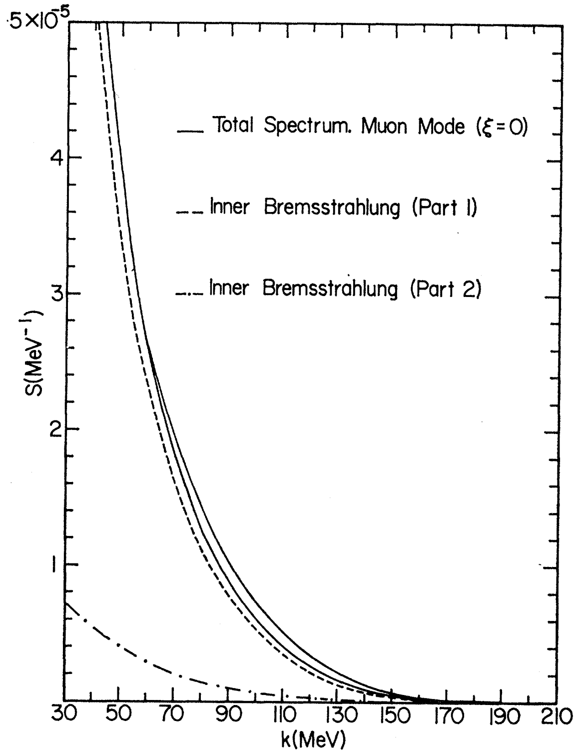

FIG. 8. Photon spectrum in $\bar{K}^0\to\pi^+\mu^-\bar{\nu}\gamma$, with $\zeta=1$ ($\xi=0$), $\Lambda_+=\Lambda_2=0.31$, and $A=B=C=D=0$, normalized by dividing by the $\bar{K}^0\to\pi^+\mu^-\bar{\nu}$ decay rate with the same values for the form factors. The significance of the separation into parts 1 and 2 is explained in the text. The upper solid curve was calculated with $A=B=2.5$, $C=D=1.0$, and $f_+(0)=1.04$ and thus shows the effect of including structure-dependent contributions.

Again the separation into parts 1 and 2 refers to the square of the main term and the terms proportional to Λ_+ and Λ_2. Figure 8 shows the same photon spectrum for $\zeta=1.0$ ($\xi=0$), $\Lambda_+=\Lambda_2=0.31$, and $A=B=2.5$, $C=D=1.0$. The effects due to the presence of the structure-dependent terms are small and we would need to know more accurately the values of the $\bar{K}_{l3}{}^0$ form factors before their presence could be precisely determined.

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APPENDIX A

In this appendix we give the results of the evaluation of the sum over spins and polarizations for the square of the $\bar{K}_{l3\gamma}{}^0$ matrix element given in Eq. (2.24). Basic identities and conventions are the same as in Paper I. The final answer is very complicated and we have split it up into several parts. First we take the direct square of the main term in Eq. (2.24); then we add the interference between the derivative term and the main term. These terms are listed according to their power in k. Next we add the interference between the structure-dependent terms and the main term. We call these terms $O_S(k^0)$ even though they include a few terms with higher powers of k which arise from the substitution $q=P-Q-p-k$ or from interference with the $\gamma\cdot\epsilon\gamma\cdot k/2p\cdot k$ term which, as a matter of convenience, we included as part of the main term. Note that the $O(k)^0$ part also includes a few terms of order k. Thus

$$\tfrac{1}{2}mm_\nu\sum_{\text{spins, pol.}}|T(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)|^2=O(k^{-2})+O(k^{-1})+O(k^0)+O_S(k^0)\,,\qquad\text{(A1)}$$

where

$$O(k^{-2})=\left[\frac{\mu^2}{(Q\cdot k)^2}+\frac{m^2}{(p\cdot k)^2}+\frac{2Q\cdot p}{p\cdot kQ\cdot k}\right]\{4f_+{}^2(t)[2P\cdot p(M^2+P\cdot Q+P\cdot p)+M^2(-P\cdot p+p\cdot Q-m^2)]$$

$$+m^2f_2{}^2(t)(P\cdot p-p\cdot Q+m^2)+4m^2f_+(t)f_2(t)(M^2+P\cdot Q+P\cdot p)\}\,,\qquad\text{(A2)}$$

$$O(k^{-1})=4f_+{}^2(t)\left[\frac{2(P\cdot p)^2}{p\cdot k}-\frac{2(P\cdot Q)^2}{Q\cdot k}+\frac{M^2P\cdot p}{p\cdot k}-\frac{M^2P\cdot Q}{Q\cdot k}+\frac{2P\cdot pP\cdot Q}{p\cdot k}-\frac{2P\cdot pP\cdot Q}{Q\cdot k}+\frac{M^2m^2}{p\cdot k}+\frac{2M^2Q\cdot p}{p\cdot k}\right.$$

$$+\frac{2M^2Q\cdot p}{Q\cdot k}+\frac{M^2\mu^2}{Q\cdot k}+\frac{M^2m^2P\cdot k}{(p\cdot k)^2}+\frac{4m^2P\cdot kP\cdot p}{(p\cdot k)^2}+\frac{2m^2P\cdot kP\cdot Q}{(p\cdot k)^2}+\frac{2\mu^2P\cdot kP\cdot p}{(Q\cdot k)^2}+\frac{M^2\mu^2p\cdot k}{(Q\cdot k)^2}+\frac{M^2P\cdot kQ\cdot p}{p\cdot kQ\cdot k}$$

$$\left.+\frac{6P\cdot kP\cdot pQ\cdot p}{p\cdot kQ\cdot k}+\frac{2P\cdot kP\cdot QQ\cdot p}{p\cdot kQ\cdot k}+\frac{m^2M^2Q\cdot k}{(p\cdot k)^2}\right]+4m^2f_+(t)f_2(t)\left[\frac{m^2P\cdot k}{(p\cdot k)^2}+\frac{\mu^2P\cdot k}{(Q\cdot k)^2}+\frac{2P\cdot kQ\cdot p}{p\cdot kQ\cdot k}\right]$$

$$+f_2{}^2(t)m^2\left[\frac{P\cdot p}{p\cdot k}-\frac{P\cdot Q}{Q\cdot k}-\frac{m^2}{p\cdot k}-\frac{2Q\cdot p}{p\cdot k}-\frac{2Q\cdot p}{Q\cdot k}-\frac{\mu^2}{Q\cdot k}+\frac{m^2P\cdot k}{(p\cdot k)^2}-\frac{\mu^2p\cdot k}{(Q\cdot k)^2}+\frac{P\cdot kQ\cdot p}{p\cdot kQ\cdot k}-\frac{m^2Q\cdot k}{(p\cdot k)^2}\right]$$

$$-\frac{16}{M^2}f_+(t)\frac{\partial f_+(t)}{\partial t}\left[\frac{2(P\cdot p)^3}{p\cdot k}+\frac{M^2(P\cdot p)^2}{p\cdot k}-\frac{2P\cdot p(P\cdot Q)^2}{Q\cdot k}+\frac{2(P\cdot p)^2P\cdot Q}{p\cdot k}-\frac{2(P\cdot p)^2P\cdot Q}{Q\cdot k}-\frac{M^2P\cdot pP\cdot Q}{Q\cdot k}\right.$$

$$-\frac{M^2m^2P\cdot p}{p\cdot k}+\frac{M^2P\cdot pQ\cdot p}{p\cdot k}+\frac{M^2m^2P\cdot Q}{Q\cdot k}-\frac{M^2P\cdot QQ\cdot p}{Q\cdot k}-\frac{2\mu^2P\cdot k(P\cdot p)^2}{(Q\cdot k)^2}-\frac{\mu^2M^2P\cdot kP\cdot p}{(Q\cdot k)^2}$$

$$-\frac{2\mu^2P\cdot kP\cdot pP\cdot Q}{(Q\cdot k)^2}-\frac{M^2P\cdot kP\cdot pQ\cdot p}{Q\cdot kp\cdot k}-\frac{2P\cdot kP\cdot pP\cdot QQ\cdot p}{p\cdot kQ\cdot k}+\frac{M^2m^2\mu^2P\cdot k}{(Q\cdot k)^2}-\frac{M^2\mu^2P\cdot kQ\cdot p}{(Q\cdot k)^2}$$

$$-\frac{M^2P\cdot k(Q\cdot p)^2}{p\cdot kQ\cdot k}-\frac{2P\cdot k(P\cdot p)^2Q\cdot p}{p\cdot kQ\cdot k}+\frac{M^2m^2P\cdot kQ\cdot p}{p\cdot kQ\cdot k}\Big]-4m^2f_2(t)\frac{\partial f_2(t)}{\partial t}\Big[\frac{(P\cdot p)^2}{p\cdot k}-\frac{P\cdot pP\cdot Q}{Q\cdot k}+\frac{m^2P\cdot p}{p\cdot k}$$

$$-\frac{P\cdot pQ\cdot p}{p\cdot k}-\frac{m^2P\cdot Q}{Q\cdot k}+\frac{P\cdot QQ\cdot p}{Q\cdot k}-\frac{\mu^2P\cdot kP\cdot p}{(Q\cdot k)^2}-\frac{P\cdot kP\cdot pQ\cdot p}{p\cdot kQ\cdot k}-\frac{m^2\mu^2P\cdot k}{(Q\cdot k)^2}+\frac{\mu^2P\cdot kQ\cdot p}{(Q\cdot k)^2}+\frac{P\cdot k(Q\cdot p)^2}{p\cdot kQ\cdot k}$$

$$-\frac{m^2P\cdot kQ\cdot p}{p\cdot kQ\cdot k}\Big]-8m^2\Big[f_+(t)\frac{\partial f_2(t)}{\partial t}+f_2(t)\frac{\partial f_+(t)}{\partial t}\Big]\Big[\frac{(P\cdot p)^2}{p\cdot k}-\frac{(P\cdot Q)^2}{Q\cdot k}+\frac{M^2P\cdot p}{p\cdot k}-\frac{M^2P\cdot Q}{Q\cdot k}+\frac{P\cdot pP\cdot Q}{p\cdot k}$$

$$-\frac{P\cdot pP\cdot Q}{Q\cdot k}-\frac{\mu^2M^2P\cdot k}{(Q\cdot k)^2}-\frac{\mu^2P\cdot kP\cdot p}{(Q\cdot k)^2}-\frac{\mu^2P\cdot kP\cdot Q}{(Q\cdot k)^2}-\frac{M^2P\cdot kQ\cdot p}{p\cdot kQ\cdot k}-\frac{P\cdot kP\cdot pQ\cdot p}{p\cdot kQ\cdot k}-\frac{P\cdot kQ\cdot pP\cdot Q}{p\cdot kQ\cdot k}\Big],\quad \text{(A3)}$$

$$O(k^0)=4f_+^2(t)\Big[M^2+\frac{M^2P\cdot k}{p\cdot k}+\frac{4P\cdot kP\cdot p}{p\cdot k}+\frac{2P\cdot kP\cdot Q}{p\cdot k}-\frac{2P\cdot kP\cdot Q}{Q\cdot kp}+\frac{M^2Q\cdot k}{\cdot k}+\frac{2m^2(P\cdot k)^2}{(p\cdot k)^2}+\frac{2(P\cdot k)^2p\cdot Q}{p\cdot kQ\cdot k}$$

$$+\frac{2(P\cdot k)^2}{p\cdot k}\Big]+m^2f_2^2(t)\Big[\frac{P\cdot k}{p\cdot k}-\frac{Q\cdot k}{p\cdot k}-1\Big]-8f_+(t)\frac{\partial f_+(t)}{\partial t}\Big[M^4+4M^2P\cdot p+M^2P\cdot Q+\frac{6P\cdot k(P\cdot p)^2}{p\cdot k}$$

$$+\frac{2P\cdot k(P\cdot Q)^2}{Q\cdot k}+\frac{M^2P\cdot kP\cdot p}{p\cdot k}+\frac{M^2P\cdot kP\cdot Q}{Q\cdot k}+\frac{2P\cdot kP\cdot QP\cdot p}{p\cdot k}-\frac{2P\cdot kP\cdot pP\cdot Q}{Q\cdot k}-\frac{2M^2P\cdot Qp\cdot k}{Q\cdot k}$$

$$+\frac{M^2P\cdot pQ\cdot k}{p\cdot k}-\frac{M^2P\cdot kQ\cdot p}{p\cdot k}-\frac{2M^2P\cdot kQ\cdot p}{Q\cdot k}-\frac{M^2\mu^2P\cdot k}{Q\cdot k}-\frac{4\mu^2(P\cdot k)^2P\cdot p}{(Q\cdot k)^2}-\frac{2M^2\mu^2P\cdot kp\cdot k}{(Q\cdot k)^2}$$

$$-\frac{M^2(P\cdot k)^2Q\cdot p}{p\cdot kQ\cdot k}-\frac{6(P\cdot k)^2P\cdot pQ\cdot p}{p\cdot kQ\cdot k}-\frac{2(P\cdot k)^2P\cdot QQ\cdot p}{p\cdot kQ\cdot k}+2M^2P\cdot k+\frac{2(P\cdot k)^2P\cdot Q}{Q\cdot k}-\frac{2(P\cdot k)^3Q\cdot p}{p\cdot kQ\cdot k}$$

$$+\frac{2(P\cdot k)^2P\cdot p}{p\cdot k}\Big]+4m^2f_2(t)\frac{\partial f_+(t)}{\partial t}\Big[M^2-\frac{P\cdot kP\cdot p}{p\cdot k}+\frac{2P\cdot kP\cdot Q}{p\cdot k}+\frac{3P\cdot kP\cdot Q}{Q\cdot k}+\frac{M^2Q\cdot k}{p\cdot k}+\frac{2\mu^2(P\cdot k)^2}{(Q\cdot k)^2}$$

$$+\frac{(P\cdot k)^2Q\cdot p}{p\cdot kQ\cdot k}+\frac{\mu^2(P\cdot k)^2}{p\cdot kQ\cdot k}\Big]-4m^2f_+(t)\frac{\partial f_2(t)}{\partial t}\Big[M^2+\frac{3P\cdot kP\cdot p}{p\cdot k}+\frac{2P\cdot kP\cdot Q}{p\cdot k}-\frac{P\cdot kP\cdot Q}{Q\cdot k}+\frac{M^2Q\cdot k}{p\cdot k}$$

$$-\frac{2\mu^2(P\cdot k)^2}{(Q\cdot k)^2}-\frac{3(P\cdot k)^2Q\cdot p}{p\cdot kQ\cdot k}+\frac{\mu^2(P\cdot k)^2}{p\cdot kQ\cdot k}\Big]-2m^2f_2(t)\frac{\partial f_2(t)}{\partial t}\Big[M^2-2P\cdot p+P\cdot Q+\frac{P\cdot kP\cdot p}{p\cdot k}+\frac{P\cdot kP\cdot Q}{Q\cdot k}$$

$$+\frac{2P\cdot Qp\cdot k}{Q\cdot k}-\frac{P\cdot pQ\cdot k}{p\cdot k}+\frac{P\cdot kp\cdot Q}{p\cdot k}+\frac{2P\cdot kQ\cdot p}{Q\cdot k}+\frac{\mu^2P\cdot k}{Q\cdot k}+\frac{2\mu^2P\cdot kp\cdot k}{(Q\cdot k)^2}-\frac{(P\cdot k)^2p\cdot Q}{p\cdot kQ\cdot k}\Big].\quad \text{(A4)}$$

We expand $f(t)$ in the above equations as $f(t)=f(0)(1+\Lambda t/M^2)$ with $t=M^2+\mu^2+2P\cdot Q$ and retain all terms through first order in Λ. As mentioned in the text, this expansion generates an additional set of terms proportional to Λ from the square of the main term, but for the derivative terms, which are already proportional to Λ, amounts to the replacement $f(t)\to f(0)$.

Interference between the structure-dependent terms and the main term yields the result

$$O_S(k^0)=-4f_+(t)\frac{A}{M^2}\Big[2(P\cdot p)^2+M^2P\cdot p-\frac{(P\cdot Q)^2p\cdot k}{Q\cdot k}+\frac{(P\cdot p)^2Q\cdot k}{p\cdot k}-\frac{M^2P\cdot Qp\cdot k}{Q\cdot k}-\frac{2P\cdot pP\cdot Qp\cdot k}{Q\cdot k}+\frac{M^2m^2P\cdot k}{p\cdot k}$$

$$+\frac{M^2P\cdot kp\cdot Q}{Q\cdot k}+\frac{2m^2P\cdot kP\cdot p}{p\cdot k}-\frac{P\cdot kP\cdot pp\cdot Q}{p\cdot k}+\frac{2P\cdot kP\cdot pp\cdot Q}{Q\cdot k}+\frac{m^2P\cdot kP\cdot Q}{p\cdot k}+\frac{P\cdot kP\cdot Qp\cdot Q}{Q\cdot k}$$

$$-\frac{\mu^2P\cdot kP\cdot p}{Q\cdot k}+M^2P\cdot k+3P\cdot kP\cdot p+2P\cdot kP\cdot Q+M^2p\cdot k+M^2Q\cdot k+\frac{m^2(P\cdot k)^2}{p\cdot k}+\frac{(P\cdot k)^2\cdot pQ}{Q\cdot k}$$

$$-\frac{P\cdot kP\cdot Qp\cdot k}{Q\cdot k}+2(P\cdot k)^2\Big]-2f_2(t)m^2\frac{A}{M^2}\Big[P\cdot p-P\cdot Q-\frac{P\cdot Qp\cdot k}{Q\cdot k}+\frac{P\cdot pQ\cdot k}{p\cdot k}+\frac{m^2P\cdot k}{p\cdot k}-\frac{P\cdot kp\cdot Q}{p\cdot k}$$

$$+\frac{P\cdot kp\cdot Q}{Q\cdot k}-\frac{\mu^2P\cdot k}{Q\cdot k}-P\cdot k+\frac{(P\cdot k)^2}{p\cdot k}-\frac{P\cdot kQ\cdot k}{p\cdot k}\Big]-4f_+(t)\frac{B}{M^2}\Big[2P\cdot pP\cdot Q+2M^2p\cdot Q-\frac{(P\cdot Q)^2p\cdot k}{Q\cdot k}$$

$$-\frac{(P\cdot p)^2Q\cdot k}{p\cdot k}+\frac{P\cdot kP\cdot pp\cdot Q}{p\cdot k}+\frac{m^2P\cdot kP\cdot Q}{p\cdot k}+\frac{P\cdot kP\cdot Qp\cdot Q}{Q\cdot k}+\frac{\mu^2P\cdot kP\cdot p}{Q\cdot k}+\frac{\mu^2M^2p\cdot k}{Q\cdot k}+\frac{m^2M^2Q\cdot k}{p\cdot k}$$

$$+M^2P\cdot k+3P\cdot kP\cdot p+2P\cdot kP\cdot Q+M^2p\cdot k+M^2Q\cdot k+\frac{m^2(P\cdot k)^2}{p\cdot k}+\frac{(P\cdot k)^2p\cdot Q}{Q\cdot k}-\frac{P\cdot kP\cdot Qp\cdot k}{Q\cdot k}+2(P\cdot k)^2\Big]$$

$$-2f_2(t)m^2\frac{B}{M^2}\Big[P\cdot k-\frac{(P\cdot k)^2}{p\cdot k}+\frac{P\cdot kQ\cdot k}{p\cdot k}\Big]-4f_+(t)\frac{C}{M^2}\Big[2M^2p\cdot Q+4P\cdot pp\cdot Q+m^2P\cdot Q+P\cdot Qp\cdot Q$$

$$-\frac{P\cdot k(p\cdot Q)^2}{p\cdot k}+\frac{M^2\mu^2p\cdot k}{Q\cdot k}+\frac{2\mu^2P\cdot pp\cdot k}{Q\cdot k}+\frac{\mu^2P\cdot Qp\cdot k}{Q\cdot k}+\frac{M^2m^2Q\cdot k}{p\cdot k}+\frac{m^2P\cdot pQ\cdot k}{p\cdot k}+\frac{P\cdot pQ\cdot kp\cdot Q}{p\cdot k}$$

$$+\frac{m^2P\cdot kp\cdot Q}{p\cdot k}+\frac{m^2P\cdot Qk\cdot Q}{p\cdot k}+\frac{\mu^2m^2P\cdot k}{Q\cdot k}-\frac{\mu^2P\cdot kp\cdot Q}{Q\cdot k}+M^2Q\cdot k+2P\cdot pQ\cdot k-2P\cdot Qk\cdot p+2P\cdot kp\cdot Q$$

$$-\mu^2P\cdot k+\frac{m^2P\cdot kQ\cdot k}{p\cdot k}+2P\cdot kQ\cdot k\Big]-2f_2(t)m^2\frac{C}{M^2}\Big[2p\cdot Q-P\cdot Q+\frac{P\cdot kQ\cdot k}{p\cdot k}+\frac{P\cdot pQ\cdot k}{p\cdot k}-\frac{P\cdot kp\cdot Q}{p\cdot k}$$

$$-\frac{\mu^2P\cdot k}{Q\cdot k}+\frac{\mu^2p\cdot k}{Q\cdot k}+\frac{m^2Q\cdot k}{p\cdot k}-Q\cdot k-\frac{(Q\cdot k)^2}{p\cdot k}\Big]-4f_+(t)\frac{D}{M^2}\Big[-\mu^2P\cdot p-P\cdot Qp\cdot Q+\frac{P\cdot k(p\cdot Q)^2}{p\cdot k}$$

$$-\frac{P\cdot pp\cdot QQ\cdot k}{p\cdot k}-\frac{m^2P\cdot QQ\cdot k}{p\cdot k}-\frac{\mu^2m^2P\cdot k}{p\cdot k}+M^2Q\cdot k+2P\cdot pQ\cdot k-2P\cdot Qp\cdot k+2P\cdot kp\cdot Q-\mu^2P\cdot k$$

$$+\frac{m^2P\cdot kQ\cdot k}{p\cdot k}+2P\cdot kQ\cdot k\Big]-2f_2(t)m^2\frac{D}{M^2}\Big[Q\cdot k+\frac{(Q\cdot k)^2}{p\cdot k}-\frac{P\cdot kQ\cdot k}{p\cdot k}\Big]. \qquad \text{(A5)}$$

APPENDIX B

In this appendix we discuss the steps leading from the expression for the square of the matrix element given in Eq. (A1) to the final numerical results of Eqs. (5.1) and (5.2) and, in particular, the procedure used to evaluate the integrals over the four-body phase space. We also give numerical values for the nonradiative decay rate in terms of the $\bar{K}_{l3}{}^0$ form factors.

The radiative decay rate is given by

$$\Gamma(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)=\left(\frac{G\sin\theta}{\sqrt{2}}\right)^2\frac{1}{4M\pi^6}\int d^4k\;\delta(k^2)\theta(k_0)$$
$$\times\int d^4p\;\delta(p^2+m^2)\theta(p_0)\int d^4Q\;\delta(Q^2+\mu^2)\theta(Q_0)$$
$$\times\int d^4q\;\delta(q^2)\theta(q_0)\delta^4(P-Q-p-q-k)\hat{T}^2, \qquad \text{(B1)}$$

where

$$\hat{T}^2=\frac{mm_\nu\alpha}{8\pi}\sum_{\text{spin, pol.}}|T(\bar{K}^0\to\pi l\nu\gamma)|^2,$$

with $\sum|T|^2$ defined by Eq. (A1). We have used the standard trick for obtaining a four-dimensional integral to replace d^3p/p_0 by $2d^4p\delta(p^2+m^2)\theta(p_0)$. First we use the four-dimensional δ function to eliminate q completely. Now define the basic Q integral $I_Q[f]$ over an arbitrary function f of Q by

$$\int d^4Q\;\delta(Q^2+\mu^2)\theta(Q_0)\delta[(A-Q)^2]\theta[(A-Q)_0]f(Q)$$
$$=\tfrac{1}{2}\pi\theta(A_0)\int dx\;\delta(A^2+x)\theta(x-\mu^2)I_Q[f], \qquad \text{(B2)}$$

where $A=P-k-p$, and where the definition $x=-A^2$ has been incorporated via the δ function $\delta(A^2+x)$.

Similarly, define the basic p integral I_p by

$$\int d^4p\, \delta(p^2+m^2)\theta(p_0)\delta[(B-p)^2+x]\theta[(B-p)_0]g(p)$$
$$=\tfrac{1}{2}\pi\theta(B_0)\int dy\, \delta(B^2+y)\theta[y-(m+\surd x)^2]I_p[g], \quad \text{(B3)}$$

where $B=P-k$, $y=-B^2$, and g is some function of p. When we explicitly evaluate these integrals, we shall justify the particular θ functions which have been factored out. Using these definitions, we obtain

$$\Gamma(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)=\frac{G^2\sin^2\theta}{32M\pi^4}\int dy\int dx\,\theta(x-\mu^2)$$
$$\times\theta[y-(m+\surd x)^2]\int d^4k\,\delta(k^2)\theta(k_0)$$
$$\times\delta[(P-k)^2+y]\theta[(P-k)_0]I_p[I_Q[\hat{T}^2]]. \quad \text{(B4)}$$

The integral on k has the same form as the integral defining $I_p[1]$ and so can be obtained from results below by using some simple substitutions. Thus

$$\int d^4k\,\delta(k^2)\theta(k_0)\delta[(P-k)^2+y]\theta[(P-k)_0]$$
$$=\tfrac{1}{2}\pi\theta(M^2-y)\frac{\lambda^{1/2}(M^2,0,y)}{M^2}, \quad \text{(B5)}$$

where $\lambda(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz$. Finally, putting all of this together, and noting that the remaining θ functions define the region of integration, we obtain for the rate

$$\Gamma(\bar{K}^0\to\pi^+l^-\bar{\nu}\gamma)$$
$$=\frac{G^2\sin^2\theta}{64M^3\pi^3}\int_{(m+\mu)^2}^{M^2-2ME_0}dy\int_{\mu^2}^{(\surd y-m)^2}dx\,\lambda^{1/2}(M^2,0,y)$$
$$\times I_p[I_Q[\hat{T}^2]]. \quad \text{(B6)}$$

Thus to obtain the rate we calculate I_p and I_Q algebraically for each of the some 350 terms in $\hat{T}^2$ and then use Gaussian quadratures to evaluate the two integrals over x and y. Note that $y=M^2-2Mk$ in the rest system of the $\bar{K}^0$ and thus the final integral is essentially an integral over the photon spectrum. Note also that the infrared divergence forces us to cut the integral off at a minimum energy E_0, which means that we restrict the y integration to $y\le M^2-2ME_0$.

Now that we have outlined the general procedure used to obtain the rate, we proceed to the evaluation of I_p and I_Q for the particular functions appearing in $\hat{T}^2$. Consider first the Q integral defined in Eq. (B2). We use the standard trick for evaluating invariant integrals, i.e., evaluate the integral in a particular frame, in this case the rest frame of A, and then generalize the result to an arbitrary frame. First we use the two δ functions to eliminate Q_0 and $|\mathbf{Q}|$. This leads, after a slight rearrangement, to the two θ functions factored out in the definition of I_Q. The remaining angular integrations, which give up to constant factors just $I_Q[f]$, can be calculated explicitly for particular functions f. Alternatively, as proved simpler in some of the more complicated cases, we can expand the result in its most general tensor form and evaluate the coefficients by examining traces or contractions with A_μ, etc. The results for the particular I_Q's we need are listed below:

$$I_Q[1]=Q_1,\quad I_Q[Q_\mu]=Q_2A_\mu,$$
$$I_Q[Q_\mu Q_\nu]=Q_3A_\mu A_\nu+Q_4\delta_{\mu\nu},\quad I_Q[Q_\mu Q_\nu Q_\tau]=Q_5A_\mu A_\nu A_\tau+Q_6(\delta_{\mu\nu}A_\tau+\delta_{\mu\tau}A_\nu+\delta_{\nu\tau}A_\mu),$$
$$I_Q\left[\frac{1}{Q\cdot k}\right]=\frac{Q_7}{A\cdot k},\quad I_Q\left[\frac{Q_\mu}{Q\cdot k}\right]=Q_8\frac{A_\mu}{A\cdot k}+Q_9\frac{k_\mu}{(A\cdot k)^2},$$
$$I_Q\left[\frac{Q_\mu Q_\nu}{Q\cdot k}\right]=Q_{10}\frac{\delta_{\mu\nu}}{A\cdot k}+Q_{11}\frac{A_\mu A_\nu}{A\cdot k}+Q_{12}\frac{k_\mu k_\nu}{(A\cdot k)^3}+Q_{13}\frac{k_\mu A_\nu+A_\mu k_\nu}{(A\cdot k)^2},$$
$$I_Q\left[\frac{Q_\mu Q_\nu Q_\tau}{Q\cdot k}\right]=Q_{14}\frac{A_\mu A_\nu A_\tau}{A\cdot k}+Q_{15}\frac{k_\mu k_\nu k_\tau}{(A\cdot k)^4}+Q_{16}\left(\frac{\delta_{\mu\nu}A_\tau+\delta_{\mu\tau}A_\nu+\delta_{\nu\tau}A_\mu}{A\cdot k}-\frac{k_\mu A_\nu A_\tau+A_\mu k_\nu A_\tau+A_\mu A_\nu k_\tau}{(A\cdot k)^2}\right)$$
$$+Q_{17}\left(\frac{\delta_{\mu\nu}k_\tau+\delta_{\mu\tau}k_\nu+\delta_{\nu\tau}k_\mu}{(A\cdot k)^2}-\frac{2(k_\mu k_\nu A_\tau+k_\mu A_\nu k_\tau+A_\mu k_\nu k_\tau)}{(A\cdot k)^3}\right),$$
$$I_Q\left[\frac{1}{(Q\cdot k)^2}\right]=\frac{Q_{18}}{(A\cdot k)^2},\quad I_Q\left[\frac{Q_\mu}{(Q\cdot k)^2}\right]=Q_{19}\frac{A_\mu}{(A\cdot k)^2}+Q_{20}\frac{k_\mu}{(A\cdot k)^3},$$
$$I_Q\left[\frac{Q_\mu Q_\nu}{(Q\cdot k)^2}\right]=Q_{21}\frac{\delta_{\mu\nu}}{(A\cdot k)^2}+Q_{22}\frac{A_\mu A_\nu}{(A\cdot k)^2}+Q_{23}\frac{k_\mu k_\nu}{(A\cdot k)^4}+Q_{24}\frac{k_\mu A_\nu+A_\mu k_\nu}{(A\cdot k)^3},$$

where

$$Q_1=Q_8=Q_{22}=(\mu^2/x)Q_{18}=(1-\mu^2/x)\,,$$
$$Q_2=Q_{11}=\tfrac{1}{2}(1-\mu^4/x^2)\,,$$
$$Q_3=Q_{14}=\tfrac{1}{3}Q_1(1+\mu^2/x+\mu^4/x^2)\,,$$
$$Q_4=Q_{16}=\tfrac{1}{12}xQ_1{}^3\,,$$
$$Q_5=\tfrac{1}{4}(1-\mu^8/x^4)\,,$$
$$Q_6=\tfrac{1}{24}x(1+\mu^2/x)Q_1{}^3\,,$$
$$Q_7=Q_{19}=\ln(x/\mu^2)\,,$$
$$Q_9=xQ_1-\tfrac{1}{2}x(1+\mu^2/x)Q_7\,,$$
$$Q_{10}=-Q_{13}=\tfrac{1}{4}x(1+\mu^2/x)Q_1-\tfrac{1}{2}\mu^2Q_7\,,$$
$$Q_{12}=\tfrac{1}{4}(x^2+4x\mu^2+\mu^4)Q_7-\tfrac{3}{4}x(x+\mu^2)Q_1\,,$$
$$Q_{15}=-\tfrac{1}{8}(x^3+9x^2\mu^2+9x\mu^4+\mu^6)Q_7+\tfrac{1}{24}x(11x^2+38x\mu^2+11\mu^4)Q_1\,,$$
$$Q_{17}=\tfrac{1}{4}\mu^2x(1+\mu^2/x)Q_7-\tfrac{1}{24}(x^2+10x\mu^2+\mu^4)Q_1\,,$$
$$Q_{20}=xQ_7-[x(x+\mu^2)/2\mu^2]Q_1\,,$$
$$Q_{21}=-\tfrac{1}{2}Q_{24}=-xQ_1+\tfrac{1}{2}(x+\mu^2)Q_7\,,$$
$$Q_{23}=(x/4\mu^2)(x^2+10x\mu^2+\mu^4)Q_1-\tfrac{3}{2}x(x+\mu^2)Q_7\,.$$

The terms in Q_5 and Q_6 are included for completeness, although we did not need them in this calculation.

To evaluate the p integrals, we first define some auxiliary functions corresponding to $I_p[g]$ for the various types of functions g appearing in $I_Q[\hat{T}^2]$:

$$P_{mn}=I_p\left[\frac{(P\cdot p)^n}{(p\cdot k)^m}\right],$$

$$T_{mn}=I_p\left[\frac{(P\cdot p)^n}{(A\cdot k)^m}\right],$$

$$TK_{mn}=I_p\left[\frac{(p\cdot k)^n}{(A\cdot k)^m}\right],$$

$$TK_{mnr}=I_p\left[\frac{(p\cdot k)^n(P\cdot p)^r}{(A\cdot k)^m}\right].$$

About fifty different cases of the above functions are needed. They are, however, related by a number of recursion relations which can be derived by using the relation $\omega+2p\cdot P-2k\cdot p=0$, with $\omega=y-x+m^2$ (implied by the δ functions in the definition of I_p) to eliminate $p\cdot P$ or $k\cdot p$ in favor of the other. Thus we obtain

$$P_{mn}=\sum_{j=0}^{n}(-\tfrac{1}{2}\omega)^{n-j}B_j{}^nP_{m-j,0}\,,$$

$$T_{mn}=\sum_{j=0}^{n}(-1)^j(P\cdot B)^{n-j}B_j{}^nAP_{mj}\,,$$

$$TK_{mn}=\sum_{j=0}^{n}(-1)^j(k\cdot B)^{n-j}B_j{}^nAP_{m-j,0}\,,$$

$$TK_{mnr}=\sum_{j=0}^{n}(\tfrac{1}{2}\omega)^{n-j}B_j{}^nT_{m,r+j}\,,$$

where $B_j{}^n$ is the binomial coefficient $n![j!(n-j)!]^{-1}$, and where AP_{mn} is just P_{mn} with x and m^2 interchanged. Thus we need to calculate only the P_{m0} which are obtained in exactly the same way as the I_Q, i.e., the δ functions are used to eliminate p_0 and $|\mathbf{p}|$, thus giving the correct θ functions, and then the angular integrations are performed explicitly. It turns out that we need only the seven P_{m0} tabulated below:

$$P_{00}=\lambda^{1/2}(y,m^2,x)/y\,,$$
$$P_{10}=\frac{1}{B\cdot k}\ln\left|\frac{\omega+\lambda^{1/2}(y,m^2,x)}{\omega-\lambda^{1/2}(y,m^2,x)}\right|\,,$$
$$P_{20}=[y/m^2(B\cdot k)^2]P_{00}\,,$$
$$P_{30}=[\omega y/2m^4(B\cdot k)^3]P_{00}\,,$$
$$P_{40}=[y/12m^6(B\cdot k)^4][3\omega^2+\lambda(y,m^2,x)]P_{00}\,,$$
$$P_{-1,0}=(\omega/2y)(B\cdot k)P_{00}\,,$$
$$P_{-2,0}=[(B\cdot k)^2/3y](\omega^2/y-m^2)P_{00}\,,$$
$$P_{-3,0}=(B\cdot k)^3(\omega/2y^2)(\omega^2/2y-m^2)P_{00}\;.$$

With these ingredients we obtain an algebraic expression, much too complicated to be reproduced here, for $I_p[I_Q[\hat{T}^2]]$, which can then be integrated over x and y to obtain the numerical results given in Sec. V.

The rate for the nonradiative process $\bar{K}^0\to\pi l\nu$ can be obtained in an analogous fashion. Using the expression for $T(\bar{K}_{l3}{}^0)$ in Eq. (2.13b) and that for

$$\sum_{\text{spins}}|T(\bar{K}^0\to\pi l\nu)|^2$$

given in Eq. (3.12),

$$\Gamma(\bar{K}^0\to\pi l\nu)=\frac{G^2\sin^2\theta}{4M\pi^5}\int d^4p\;\delta(p^2+m^2)\theta(p_0)\times\int d^4Q\;\delta(Q^2+\mu^2)\theta(Q_0)\int d^4q\;\delta(q^2)\theta(q_0)\times\delta^4(P-Q-p-q)\hat{T}^2,\quad\text{(B7)}$$

where now

$$\hat{T}^2=\tfrac{1}{8}mm_\nu\sum_{\text{spins}}|T(\bar{K}^0\to\pi l\nu)|^2.$$

One proceeds in exactly the same way as for the radiative process by defining I_Q and I_p by Eqs. (B2) and (B3) with $k=0$, i.e., with $A=P-p$ and $B=P$. Thus

$$\Gamma(\bar{K}^0\to\pi l\nu)=\frac{G^2\sin^2\theta}{16M\pi^3}\int_{\mu^2}^{(M-m)^2}dx\, I_p[I_Q[\hat{T}^2]], \quad \text{(B8)}$$

which gives the numerical results

$$\Gamma(\bar{K}^0\to\pi^+e^-\bar{\nu})=\frac{G^2\sin^2\theta\, M^5}{16\pi^3}f_+^2(0)\times(1.1738+0.3191\Lambda_+)\times10^{-2}, \quad \text{(B9)}$$

$$\Gamma(\bar{K}^0\to\pi^+\mu^-\bar{\nu})=\frac{G^2\sin^2\theta\, M^5}{16\pi^3}f_+^2(0)[0.9255+0.4221\Lambda_+ -0.1900\zeta-0.0544\zeta(\Lambda_++\Lambda_2)+0.0219\zeta^2 +0.0141\zeta^2\Lambda_2]\times10^{-2}, \quad \text{(B10)}$$

where $\zeta=f_2(0)/f_+(0)$. Note that we have factored out M^6 so that the numerical coefficients are dimensionless and that we explicitly use exact masses in all calculations.

Note added in manuscript. In view of the fact that experimenters are more interested in the $K_{\mu3}/K_{e3}$ branching ratio than in the individual rates, we give here the results for the charged and neutral branching ratios in terms of the conventional parameters. These numbers were obtained directly from Eqs. (2)–(5) of the Letter in Ref. 1, which follow from Eq. (A7) of the paper of Ref. 1 and from Eqs. (B9) and (B10) above by converting the parameters Λ_+, Λ_-, η, ζ to the conventional set λ_+, λ_-, and ξ.

$$\frac{\Gamma(K^-\to\pi^0\mu^-\bar{\nu})}{\Gamma(K^-\to\pi^0e^-\bar{\nu})}=0.6457+0.1264\xi+0.0192\xi^2 +1.4115\lambda_++0.0080\xi\lambda_+-0.0710\xi^2\lambda_+ +0.4754\xi\lambda_-+0.1684\xi^2\lambda_-, \quad \text{(B11)}$$

$$\frac{\Gamma(\bar{K}^0\to\pi^+\mu^-\bar{\nu})}{\Gamma(\bar{K}^0\to\pi^+e^-\bar{\nu})}=0.6452+0.1246\xi+0.0186\xi^2 +1.3162\lambda_++0.0064\xi\lambda_+-0.0644\xi^2\lambda_+ +0.4370\xi\lambda_-+0.1526\xi^2\lambda_-. \quad \text{(B12)}$$