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Pion-Pion Scattering Below 850 MeV. A Unique Solution*

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Forward dispersion relations are used as the basis of an analysis of the information on pion-pion scattering available up to 850 MeV. These are shown to result in a model-independent relation between the scattering lengths a_0 and a_2, valid for all possible behaviors of δ_0^0 in the ρ-region, and taking the value $a_0/a_2=-3.2\pm1.0$ from recent low-energy work, the values $a_0=0.16\pm0.04$, $a_2=-0.05\pm0.01$ uniquely result. The ambiguity in δ_0^0 in the ρ region is also studied, by four different methods, and uniquely resolved, allowing a detailed picture of all the phases up to 850 MeV to be given.

I. INTRODUCTION

CONSIDERABLE effort has been devoted in the past few years to the determination from experiment of the $\pi\pi$ scattering amplitude. Progress has been such that in the ρ region, the outstanding uncertainty has been narrowed down to the detailed behaviors of the s waves, especially the $I=0$ wave. Further information on these phases is required in many areas of hadron physics, both in the phenomenology of various processes (e.g., πN, NN scattering, K_{l4} decay, $K_{S,L}\to 2\pi$, etc.) and as a simple testing ground for theoretical ideas. In particular, there are predictions on the low-energy $\pi\pi$ parameters from current algebra by Weinberg,[1] and extending to higher energies, on the general resonance pattern from the Lovelace-Veneziano model.[2] All of these turn on a knowledge of the $\pi\pi$ s waves. Most evidence on these comes from studies of the s-p interference terms in dipion production over the ρ-mass region. For the $I=0$ wave, this leads for each dipion energy to a two-way ambiguity allowing a number of functional forms for the phase shift over the region 600–900 MeV. The ambiguity can in principle be resolved by experiments on $\pi^0\pi^0$ production and data are accumulating, although the experiments are much more difficult and have poorer energy resolution. It is the object of this paper to involve general and uncontroversial principles in order to resolve this ambiguity further, to furnish a precise model-independent determination of the scattering lengths, and to give a complete picture of the phases over the low-energy region.[3]

Some steps toward these objectives were already taken in a previous publication[4] (hereafter referred to as I). In this work a method was developed, using forward dispersion relations and their derivatives with respect to t, to relate the behavior of the $\pi\pi$ amplitudes in the ρ region to their behavior at threshold in a model-independent way. Results were listed over a wide range of possibilities, and, in addition to predictions of the p and d waves (see below), the following relations were established for σ mesons of smoothly varying width: (i) A σ meson of moderate width (or Down-Up type of solution) implies a negative $I=0$ s-wave scattering length a_0; a very broad σ (Up-Down) implies a positive a_0. (ii) The latter "superbroad" σ cases imply "reasonable" behavior of the high-energy cross sections, while the narrow σ requires "unreasonable" high-energy behavior. (This "subtraction constant" argument is set forth in detail in Sec. III).

Since this work, new information has become available which allows us to use similar methods to arrive at much more specific conclusions. This new information, which is discussed in more detail in Sec. II, comprises first a new set of alternatives for δ_0^0 extracted from $\pi^+\pi^-$ production data by a method of extrapolation of the forward-backward asymmetry.[5] In our view, this work along with other analyses established the existence of a distinct "up" and "down" branch above 700 MeV but only delineates a rather broad band of possibilities below 700 MeV. The second fresh ingegredient consists in information on a_0/a_2 from new studies of the threshold region.[6,7] The dispersion relations provide a bridge between these two pieces of information, leading immediately to a complete determination of the threshold region, and severely restricting the choice of forms consistent with the data in the ρ region. In fact, using the information already discussed, together with the dispersion relations, we

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[1] S. Weinberg, Phys. Rev. Letters 17, 616 (1966).
[2] J. Yellin and J. Shapiro (unpublished); J. Shapiro, Phys. Rev. 179, 1345 (1969); C. Lovelace, Phys. Letters 28B, 264 (1968).
[3] It is interesting to note in this respect the important role played by the imposition of general theoretical restrictions in the phase-shift analysis of pion-nucleon scattering. See, for example, A. Donnachie, in *Particle Interactions at High Energies* (Oliver and Boyd, London, 1967).
[4] D. Morgan and G. Shaw, Nucl. Phys. B10, 261 (1968).
[5] J. H. Scharenguivel *et al.*, Phys. Rev. 186, 1397 (1969).
[6] L. J. Gutay, F. T. Meiere, and J. H. Scharenguivel, Phys. Rev. Letters 23, 431 (1969).
[7] D. Cline, K. J. Braun, and F. R. Scherer, University of Wisconsin Report, 1969 (unpublished).

reduce the possible forms of solution to two, which only differ appreciably above 700 MeV. To provide independent confirmation of this and to discriminate between the two alternatives, we then turn to the information on $\pi^0\pi^0$ production.[8,9]

As we shall see, the resulting solution strikingly confirms the theoretical predictions of the chiral and dual models. It is clearly important that these predictions be confronted with experiment in as objective and transparent a fashion as possible, and that the intricacies of the experimental unravelment, especially severe for the $\pi\pi$ system, where the actual data is for $\pi N \to \pi\pi N$, not be compounded with the freedom of the theoretical models. The present approach employs only rather general and uncontroversial theoretical ideas to assist in the determination of the elastic amplitudes from experiment. In addition to unitarity, we demand that the amplitude satisfy forward dispersion relations and fulfill many of the relations of crossing symmetry. These restrictions, which are of very general validity, by themselves do not by any means uniquely determine the amplitudes. They do severely restrict the possible forms, and by linking together the different energy regions allow a unique solution to be arrived at much more easily.

Thus our aim is to apply the method which we developed in I, and used to demonstrate some general relations between the ρ and threshold regions, to tie together the various pieces of information which have become available since that time, arriving from these, on general grounds, at a unique and quite detailed picture of the $\pi\pi$ phases over the whole region up to 850 MeV. We shall in fact extend our discussions in part to higher energies to discuss the possibility of a zero and second resonance in the $I=0$ s wave in this region, as suggested by some authors.[10]

In Sec. II we discuss the empirical framework of our calculations, state the problem, and discuss the new information in more detail. In Sec. III we go on briefly to describe our method, and in Sec. IV we present the results. These have been divided into three parts—the determination of the threshold parameters, the resolution of the ρ-region ambiguities, and the final set of phases which results, including the important question of the values of the K mass.

II. POSSIBLE FORMS FOR $\pi\pi$ SCATTERING FROM EXPERIMENT. STATEMENT OF PROBLEM

The pion-pion scattering amplitudes have so far been most thoroughly studied in the region of the ρ resonance ($600 < M_{\pi\pi} < 900$ MeV), and a number of results have been established.[11] First, inelasticity is extremely small,[12] and we shall neglect it throughout. We shall also neglect f waves and above,[13] so that the scattering is described by the five phase shifts $\delta_J^I \equiv \delta_0^0$, δ_0^2, δ_1^1, δ_2^0, and δ_2^2. Further, we shall assume that the p wave is dominated by the ρ resonance (we take $M_\rho=765$, $\Gamma_\rho=120\pm20$ MeV) and the $I=2$ s wave is rather small and negative over this region. (The magnitude of δ_0^2 is discussed further below.)[11]

This leaves the behavior of the $I=J=0$ s wave δ_0^0, which constitutes the major uncertainty over this energy range ($600<M_{\pi\pi}<900$ MeV). Studies of the s-p interference term in $\pi^+\pi^-$ production lead via the well-known Up-Down[14,15] ambiguity to the possibilities Up-Up, Up-Down, Down-Up, and Down-Down for the functional form of δ_0^0 in this region. Attempts to resolve the ambiguity from the peripheral $\pi^+\pi^-$ data alone have so far led to disagreement, various methods having led to the Down-Up,[5,16] Up-Up,[17] and Up-Down[18] cases. In fact, all these results rest on attempts to isolate the s-wave part from the isotropic term in this process, and since this is obscured by a large contribution from depolarization of the ρ, it is not surprising that different attempts have led to contradictory conclusions. However, a simple evasive Chew-Low extrapolation in $\pi^0\pi^0$ production indicates rather strongly an Up-Down type of solution.[8] It is the object of this paper to resolve this ambiguity further, to furnish a precise model-independent determination of the scattering lengths, and to give a complete picture of the phases over the whole low-energy region.

As we have already mentioned in the Introduction, a number of steps toward these objectives were already taken in our previous work.[4] Since then the results of both new experiments and novel methods of analysis have enabled us to use our method to arrive at a complete solution to the stated problem. Before going on to describe this in the following sections, let us first consider these new pieces of information in turn.

1. Alternatives for δ_0^0 in the ρ region. The situation is now much clearer owing to the extrapolation of the forward-backward asymmetry to the pion pole in $\pi^- p \to \pi^+\pi^- n$,[5] the results of which are shown in Fig. 1. This is the most model-independent analysis yet available. In particular, it is insensitive to d waves (δ_2^0 is predicted to be about 2° at the ρ—see Sec. IV),

[8] W. Deinet *et al.*, Phys. Letters **30B**, 359 (1969).

[9] D. Cline, K. J. Braun, and V. R. Scherer, in Proceedings of the Conference on $\pi\pi$ and πK Interactions, Argonne National Laboratory, 1969 (unpublished).

[10] See, e.g., R. Bizzarri *et al.*, Nucl. Phys. **B14**, 169 (1969).

[11] For a survey of the information available, see Proceedings of the Conference on $\pi\pi$ and πK Interactions, Argonne National Laboratories, 1969 (unpublished).

[12] S. U. Chung *et al.*, Phys. Rev. **165**, 1491 (1968).

[13] For a discussion of this, and the determination of the d waves, see I (Ref. 4); see also Sec. IV C of this paper.

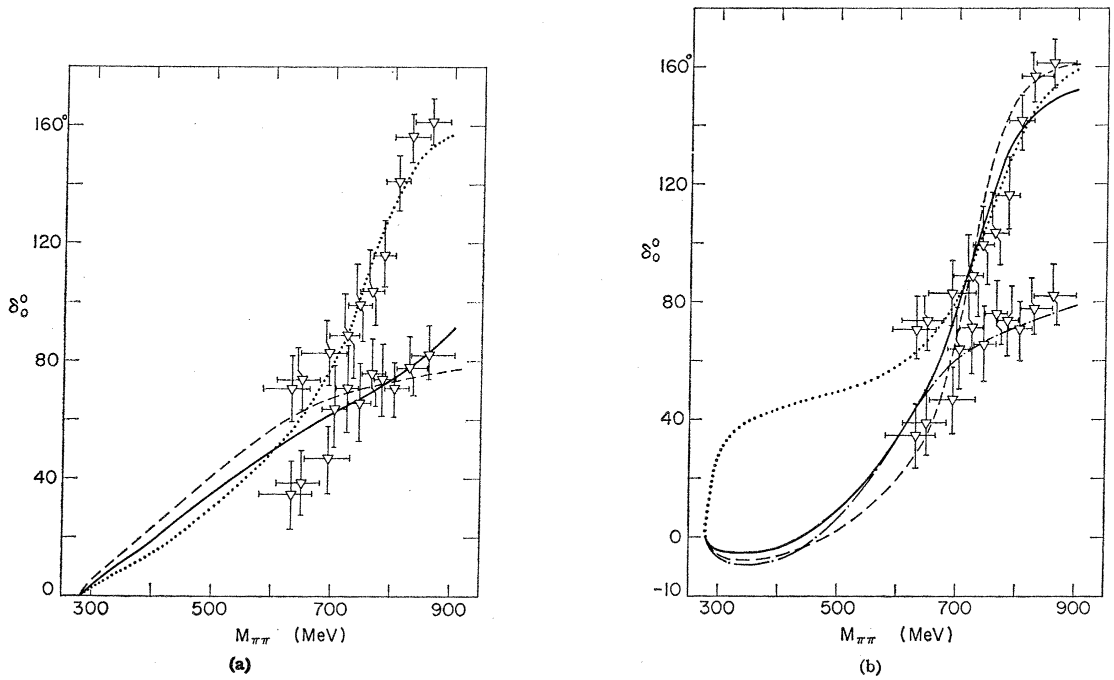
[14] E. Malamud and P. E. Schlein, Phys. Rev. Letters **19**, 1056 (1967).

[15] δ_0^0 enters this term in the form $\sin(2\delta_0^0-\delta_1^1)$, which is unchanged by the substitution $\delta_0^0 \to \frac{1}{2}\pi+\delta_1^1-\delta_0^0$. Hence the ambiguity.

[16] S. Marateck *et al.*, Phys. Rev. Letters **21**, 1613 (1968).

[17] E. Malamud and P. E. Schlein in Ref. 11.

[18] W. D. Walker, J. Carroll, A. Garfinkel, and B. Oh, Phys. Rev. Letters **18**, 630 (1967).



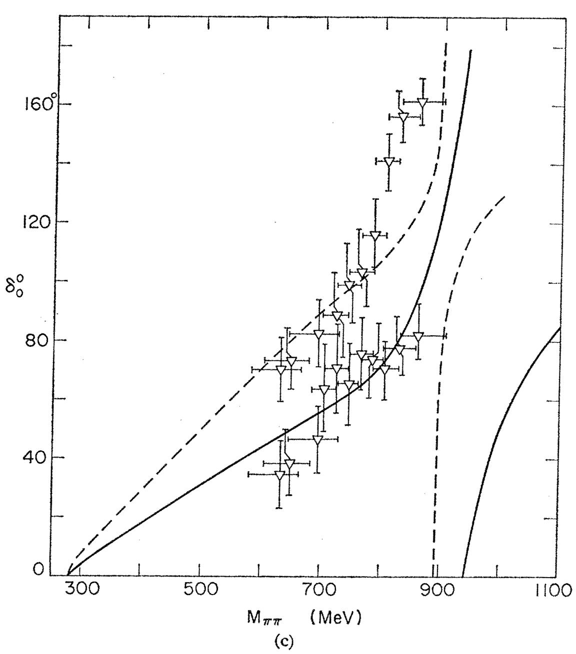


FIG. 1. Comparison of alternative computed forms for δ_0^0 with the results of Ref. 5. (a) BDI (full curve), UD (dashed curve), and BU (dotted curve) solutions. (b) UU (dotted curve), DD (dash-dot curve), and DUI (solid curve) and DUII (dashed curve) solutions. (c) BDII (solid curve) and LV (dashed curve) solutions.

and makes less stringent assumptions on the behavior at $\Delta^2=0$ than the usual extrapolation of the differential cross section. Inputs to this analysis are the forms of the p-wave phase shift δ_1^1 and the $I=2$ s wave δ_0^2, for which the phases of Baton *et al.*[19] were used. The latter are somewhat smaller in magnitude than some of the cases considered below. However, the effects of small changes in δ_0^2 on the result for δ_0^0 can be easily calculated, and are essentially negligible compared to the other errors, except for the low-energy points on the Up branch, which may be lowered several degrees,[20] bringing the two branches closer. Further, when the results of the above analysis are plotted together with those from other analyses—e.g., those of Ref. 17—below 700 MeV, the two branches are no longer distinct,

[19] J. P. Baton, G. Laurens, and J. Reignier, Nucl. Phys. **B3**, 349 (1967).

[20] With our finally preferred δ_0^2, about 4°, whereas the Down branch is raised by about 2°. At 760 and 855 MeV, the changes in both branches are about 1°.

although at higher energies they are still clearly separated by several standard deviations. Thus, since it is not clear that the two branches below 700 MeV correspond to clear and distinct solutions, in addition to the four classes of solution defined by the points in Fig. 1 (Up-Up, Up-Down, etc.), we shall also consider two extra classes of solution which are intermediate in the 600–700-MeV region—the "Between-Up" and "Between-Down" solutions.

2. Determination of a_0/a_2. There have recently been two determinations of this quantity. First, Gutay *et al.*[6] have studied the relation between Δ^2 and s at the point $\Delta_0{}^2$ where the forward-backward asymmetry vanishes in peripheral $\pi^+\pi^-$ production. Assuming an exact linear form in s, t, and u for the real part of the amplitude and imposing the Adler consistency condition leads to the relation

$$\Delta_0{}^2(s)=\frac{2a_0-5a_2}{10a_0+23a_2}(s-\mu^2).$$

They find that the experimental values of $\Delta_0{}^2(s)$ for various values of the dipion mass $\sqrt{s}$ do conform to the above relation yielding a value $a_0/a_2=-3.2\pm0.1$. Making allowance for possible quadratic terms,[21] something like $a_0/a_2=-3.2\pm1.0$ seems to be strongly indicated.

Also, Cline *et al.*[7] have deduced this ratio from a study of the charge branching ratios $R_1=\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^+)$ and $R_2=\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^-)$ near threshold. Noting that the two ratios depended essentially on only one parameter, the ratio of the $I=0$ and 2 s waves, and correcting for p and d waves using the results of I, they showed that the ratio demanded was the same in each process, arriving at the value $a_0/a_2=-3.2\pm1.1$, in agreement with the result of the very different method of Gutay *et al.*[6]

In this work we shall adopt the value $a_0/a_2=-3.2\pm1.0$. As we shall see, the dispersion relations provide a bridge between this and the previously discussed information, leading immediately to a complete determination of the threshold region, and severely restricting the choice of forms consistent with the information in the ρ region.

3. Neutral pion production; method of ratios. We shall find that the information already discussed, together with the dispersion relations, will reduce the possible choices to two, which only differ noticeably above 700 MeV. To confirm this independently, and to discriminate between them, we shall turn to information on $\pi^0\pi^0$ production. There have been two steps forward in this area. First, a much-higher-statistics spark-chamber experiment has enabled Deinet *et al.*[8] to make a simple evasive Chew-Low extrapolation to the total cross section over the ρ region, obtaining results consistent only with a large, slowly varying $\delta_0{}^0$ in this region. Secondly, Cline *et al.*[9] have clearly demonstrated the usefulness of the "method of ratios" in interpreting the bubble-chamber data, noting that the broad and narrow resonance solutions of I give very strikingly different predictions for the ratios R_1 and R_2. We shall reapply their method using the much better alternative fits to the ρ-region data obtained in this paper, as well as comparing to the extrapolated cross section of Deinet *et al.*[8] In addition, we shall find that the ratio R_1 is also particularly useful in the discussion of the value of $\delta_0{}^2$ at the ρ mass.

[21] It is sufficient for deducing a value of a_0/a_2 to regard this method as giving a correct extrapolation to the on-shell zero of $A_0{}^{+-}$, since $A_0{}^{+-}(s=4\mu^2)=\frac{1}{3}a_0+\frac{1}{6}a_2$, and $A_0{}^{+-}(s=0)=\frac{1}{6}a_0+(7/12)\times a_2+$small corrections from nonlinearity. Considerations of this type lead to the stated result.

III. METHOD

The ingredients of our method, for a detailed discussion of which we refer to I, are the fixed-t dispersion relations and their first derivatives with respect to momentum transfer, both evaluated in the forward direction ($t=0$), and crossing symmetry and unitarity, where the latter is used in its elastic form. g waves and above are assumed negligible, and the f wave shown to be negligible. The aim is—given the behavior of the phase $\delta_0{}^0$ in the ρ region, the mass and width of the ρ itself, and the value of $\delta_0{}^2$ at the ρ mass—to construct the low-energy amplitudes which follow from the above assumptions. Although we are concerned with partial-wave amplitudes, we nonetheless prefer the use of fixed-variable rather than partial-wave dispersion relations for three reasons: (1) The former involve only integrals over physical amplitudes in the physical region, (2) the distant contributions are very directly related to the high-energy scattering amplitude, and (3) the existence of partial-wave relations has been called into question by models with infinitely rising trajectories, especially the Veneziano model.[22]

To illustrate our actual procedure, let us consider just a single amplitude—the forward amplitude with $I=0$ in the t channel. This will satisfy a once-subtracted dispersion relation which can be written in terms of the crossing-symmetric variable[23] $z=(s-u)/4\mu^2$ in the form

$$T^0(z)=\frac{2}{\pi}\int_1^{z_2}\frac{z'dz'\,\mathrm{Im}T^0(z')}{z'^2-z^2}+e_0{}^0+\frac{2z^2}{\pi}\int_{z_2}^{\infty}\frac{dz'\,\mathrm{Im}T^0(z')}{z'(z'^2-z^2)},\quad(1)$$

where

$$e_0{}^0=T^0(z=0)-\frac{2}{\pi}\int_1^{z_2}\frac{dz'\,\mathrm{Im}T^0(z')}{z'}.$$

For $0\leqslant z\leqslant z_r<z_2$ this is approximated by

$$T^0(z)=\frac{2}{\pi}\int_1^{z_2}\frac{z'dz'\,\mathrm{Im}T^0(z')}{z'^2-z^2}+e_0{}^0+e_2{}^0\left(\frac{z}{z_r}\right)^2,\quad(2)$$

[22] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[23] Our notation throughout follows that of I, Ref. 4.

where z_r corresponds to the ρ mass and z_2 to a cutoff taken above the f_0. There are analogous equations for the $T^{1,2}$ and the derivatives with respect to t, and we have divided the energy region of the integral into two parts; the low-energy region $0 \leqslant z \leqslant z_1$ where amplitudes are to be determined, and an intermediate region $z_1 \leqslant z \leqslant z_2$ where $\delta_0{}^0$ is assumed known, and including the point $z = z_r$ where the conditions on $\delta_1{}^1$ and $\delta_0{}^2$ are imposed. The equations are solved iteratively using the dispersion relations to pass from $\mathrm{Im}T^I(z)$ to $\mathrm{Re}T^I(z)$ and unitarity to pass back from $\mathrm{Re}T^I(z)$ to $\mathrm{Im}T^I(z)$. At each stage the subtraction constants are adjusted to reproduce the assumed behavior of $\mathrm{Re}T^I(z)$ in the ρ region (plus threshold conditions from Bose statistics) and the system cycled to self-consistency in the low-energy region.

In this way, amplitudes can be constructed over the whole region corresponding to a given range of s-wave inputs in the ρ region. Further, we find that a study of the subtraction constants, in particular, $e_2{}^0$, leads to a means of discriminating between different input forms. First, it follows trivially from the above equations and the optical theorem that $e_2{}^0 > 0$. Secondly, it can be shown for any reasonable form for the high-energy scattering that whereas $e_0{}^0$ may be quite large, $e_2{}^0$ is always small, simply because distant singularities can give only comparatively slowly varying contributions in the low-energy region. Thus if we evaluate the explicit Regge-pole model of Rarita *et al.*[24] above 1.5 GeV/c, we obtain contributions to $e_0{}^0$ and $e_2{}^0$ of -0.90 and 0.04, respectively. Additional sources of contributions are the resonances which we have not explicitly included, e.g., $\rho_N(1650)$. Assuming this to have spin-parity 3^- and width 120 MeV and to decay mainly into the 2π system, we obtain a contribution to $e_2{}^0$ of 0.04. Even if we had neglected the f_0, which is explicitly included, it would only give a contribution of 0.04. (The higher terms in the expansion are indeed negligible in all the above cases, the corresponding contributions to the next term $e_4{}^0$ being 0.4×10^{-3}, 2×10^{-3}, and 5×10^{-4}, respectively.) We therefore conclude that values of $e_2{}^0$ much larger than 0.2 cannot be understood in terms of reasonable effects, and this furnishes one means of discriminating between input forms.

Another possibility arises from crossing symmetry. Our solutions have exact $s \leftrightarrow u$ crossing by construction and, as discussed in I, a good deal of $s \leftrightarrow t$ symmetry indirectly imposed. However, it has not been exactly imposed, so that the rigorous consistency conditions dependent on this,[25] and those which also incorporate positivity,[26] can in principle be used to discriminate between different forms. In fact, as we shall see in Sec. IV, these are not nearly so useful as the subtraction criteria.

[24] W. Rarita, R. J. Riddell, C. B. Chiu, and R. J. N. Phillips, Phys. Rev. **165**, 1615 (1968).

[25] R. Roskies, Phys. Letters **30B**, 42 (1969); Nuovo Cimento **65A**, 467 (1970).

[26] These are conveniently summarized in G. Auberson, O. Piguet, and G. Wanders, Phys. Letters **28B**, 41 (1968).

IV. APPLICATION AND RESULTS

Before going on to apply the above techniques, we now need only to settle the question of the input to be used. This consists in each case of the mass and width of the ρ, the value of $\delta_0{}^2$ at the ρ mass, and the behavior of $\delta_0{}^0$ over the range $600 < M_{\pi\pi} < 900$ MeV. Except when otherwise stated, we take here the values $m_\rho = 765$ MeV, $\Gamma_\rho = 120$ MeV, and $\delta_0{}^2 = -20°$. The ρ parameters are now fairly uncontroversial; but in the case of $\delta_0{}^2$, whereas the qualitative nature of the phase is fairly well agreed on, there remains doubt[27] as to whether $\delta_0{}^2(M_\rho)$ is around $-10°$ [19] or $-20°$.[18] However, our previous calculation showed that results in the threshold region are insensitive to reasonable changes in Γ_ρ and $\delta_0{}^2(M_\rho)$ [see also Sec. IV C] and can be tabulated essentially solely in terms of the s-wave input $\delta_0{}^0$. For $\delta_0{}^0$ itself we consider a variety of forms consistent with the information from the extrapolation of the forward-backward asymmetry. Thus, input forms are presented corresponding to Up-Down (designated UD), Between-Down (BDI), Between-Up (BU), Up-Up (UU), Down-Down (DD), and two Down-Up (DUI,DUII) cases [cf. Figs. 1(a) and 1(b)]. We have also considered two more complicated forms [Fig. 1(c)]. The first of these (LV) is chosen to agree with Lovelace's two-channel unitarized Veneziano model[28] in the ρ region. This resonates at 700 MeV and has a second narrow resonance at 900 MeV, and it does not in fact give a very good fit to the experimental points. However, the possibility of a second resonance somewhat above the region considered has been suggested on empirical grounds by, for example, Bizzarri *et al.*,[10] who suggest what is essentially a Between-Down form going through 90° at 853 MeV, and again at 1115 MeV with a zero at 940 MeV. In order to investigate the effects of this type of structure, this case is also considered (BDII).

These forms, and the resulting extrapolations to low energies of the phase $\delta_0{}^0$, are shown in Figs. 1(a)–1(c), and some of the more important parameters which result are given in Table I. We stress again that the various categories (Down, Between, Up) are defined for this paper by the "experimental" points of Fig. 1, and care should be taken before identifying with the same

[27] The use of the $\pi^\pm\pi^0$ channels to study the $I=2$ phase shift has been criticized by Malamud and Schlein (Ref. 17) on the grounds that below the ρ, ω exchange may become increasingly important. We note that extrapolation methods such as that used in Ref. 19 should handle this problem best. Further, at the ρ mass (which is precisely where we need to know $\delta_0{}^2$ in this work) the extrapolated cross section is in excellent agreement with the unitarity limit. This is a useful consistency check on both the handling of the ω exchange (which occurs, if at all, in the $I=1$ production which is dominant at this energy) and the use of the "evasive" assumption at $t=0$. [See C. D. Froggatt and D. Morgan, Phys. Rev. **187**, 2044 (1969).] We therefore regard this determination (Ref. 19) as the best available at present.

[28] C. Lovelace, Ref. 11.

TABLE I. Values of some important parameters for the various trial forms of Fig. 1, and for the final solution.

Input forms	Threshold parameter s wave a_0	a_2	p wave a_1	d wave c_0	c_2	$\delta_0{}^0(M_K)$ (degrees)	Subtraction parameter $e_2{}^0$
DD	−0.31	−0.22	0.032	0.0016	0.0001	5	0.61
DUI	−0.21	−0.17	0.031	0.0014	0.0001	8	0.43
DUII	−0.27	−0.19	0.030	0.0013	−0.0001	2	0.47
BU	0.12	−0.05	0.033	0.0014	0.0001	29	0.21
BDI	0.16	−0.05	0.035	0.0015	0.0002	34	0.25
BDII	0.16	−0.04	0.034	0.0015	0.0002	30	0.22
UD	0.24	−0.03	0.037	0.0017	0.0004	40	0.27
UU	1.49	0.19	0.047	0.0022	0.0017	49	−0.53
LV	0.29	−0.02	0.037	0.0018	0.0006	49	0.23
Final solution	0.16±0.04	−0.05±0.01	0.035±0.002	0.0016±0.0002	0.0003±0.0001	33±5	

solution types in other analyses—for example, Ref. 6. Let us now go on to apply these results to the various problems we have set out to answer.

A. Threshold Region—Universal Curve

In Fig. 2 we plot the value of the isospin-zero scattering length a_0 against the isospin-two scattering length a_2 for each of our possible input forms. We also show the plots for roughly symmetrical σ's of varying widths with the mass fixed at values of 600, 765, and 900 MeV, the range of widths considered running from about 150 to 800 MeV. The striking fact emerges that for this very wide range of input, which more than embraces all possible peripheral solutions, a_0 and a_2 are related by a universal curve to a very good approximation. Taking the value $a_0/a_2=-3.2\pm1.0$ from the determinations discussed in the Introduction, and plotting the corresponding segment on Fig. 2, the scattering length values can be read off immediately, giving $a_0=0.16\pm0.04$ and $a_2=-0.05\pm0.01$. These values are in excellent agreement with the current-algebra model of Weinberg.[1] The p- and d-wave scattering lengths can similarly be extracted and are given in Table I. The quoted errors include the effect of reasonable changes in Γ_ρ and $\delta_0{}^2(m_\rho)$—thus, for example, changes of ±10 MeV in Γ_ρ and $\pm5°$ in $\delta_0{}^2(m_\rho)$ both result in changes in a_1 of about ±0.0006, and this is included in the error.

Let us now comment briefly on this universal curve, since it is a rather striking effect. The combination of scattering lengths relevant to the Adler-Weisberger theorem is $6L=2a_0-5a_2$, for which the values 0.54 and 0.69 are predicted depending on whether the partially conserved axial-vector current (PCAC) constant is evaluated from the pion decay rate or the Goldberger-Treiman relation. In fact, for narrow-σ solutions ($a_0<0$), our universal curve corresponds to constant L. Over the region $a_0>0$, $a_2<0$, it roughly corresponds to $2a_0-8a_2$ constant, but since a_2 is small, L is still fairly slowly varying. The value of L obtained from Fig. 2, again using $a_0/a_2=-3.2\pm1.0$, is

$$6L=2a_0-5a_2=0.58\pm0.06.$$

There have also been other attempts to determine this parameter. For example, Olsson[29] noted in 1967 that there could be constructed a forward sum rule for the p-wave scattering length a_1 which is fairly insensitive to the s-wave input and the high-energy behavior, and obtained the value $a_1=0.040\pm0.005$ by considering a range of inputs. By assuming the amplitude to be exactly linear, one could then arrive at an estimate of L. We note that the Olsson sum rule, and all the other sum rules at $t=0$, are automatically satisfied in both this and our previous work (including, e.g., DUI with $a_1=0.031$). Another possible approach is to assume the current-algebra results for the linear terms and calculate corrections by the so-called "hard-pion" method. The results of such an approach are quoted by Cline *et al.*[7] to give, together with their a_0/a_2 value, results for a_0/a_2 essentially in agreement with ours. We consider, however, that since the main interest in finding out the low-energy amplitude is to establish the success or otherwise of the current-algebra–PCAC

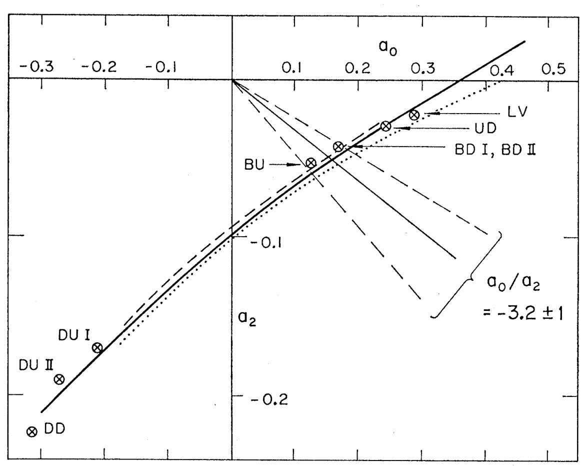


FIG. 2. Plot of a_2 versus a_0 from the tabulation of I (Ref. 4). Curves are for $M_\sigma=900$ (dashed line), $M_\sigma=765$ (solid line), and $M_\sigma=600$ (dotted line). The alternatives considered here are also shown, and the region admitted by the new determinations (Refs. 6 and 7) of a_0/a_2 also indicated.

[29] M. G. Olsson, Phys. Rev. 162, 1338 (1967).

picture, it is important not to use any current-algebra or PCAC (i.e., approximate linearity) assumptions in the determination. In this determination of the threshold parameters this has been completely avoided, and the picture of the amplitude which emerges (cf. Table I) gives striking confirmation of the Weinberg picture.

B. Resolution of ρ-Region Ambiguities

There are four possible tools to bring to bear on this problem—the rigorous conditions listed in Refs. 25 and 26, the subtraction argument, the value of a_0/a_2, and the data on other charge states. The first of these turns out to be not very useful, presumably because the assumptions on which the conditions are based are to a large degree built into the method. The nine conditions are, in fact, satisfied in all cases. This is true even for the Up-Up solution, where the negative e_2^0 value implies negative total cross-section contributions.

The subtraction argument discussed in Sec. II is a much more useful criterion. As we have seen, the crucial parameter is e_2^0, which must be positive and not much larger than about 0.2. Reference to Table I shows that whereas the Up-Down, Between-Down, and Between-Up forms are satisfactory in this respect, the Down-Up are much less so, and the Down-Down and Up-Up are clearly ruled out. (We will deal with LV separately.) These results are independently confirmed, and markedly refined, by the condition that the form should be compatible with the ratio $a_0/a_2=-3.2\pm1.0$. Inspection of Fig. 2 or Table I leads to the exclusion of the same solutions as the subtraction argument and, in addition, the UD solution is also excluded ($a_0/a_2 \simeq -8$).

Thus we have been led rather simply to an essentially unique choice below the ρ (the Between branch), with either alternative above the ρ allowed. The former result is independently confirmed, and the ambiguity above the ρ removed, by a discussion of the data on the charge states $\pi^0\pi^0$ and $\pi^+\pi^+$ to which we now turn. A useful method for doing this is the method of ratios. This has been stressed by Cline *et al.*,[9] who pointed out that the broad- and narrow-resonance solutions of I lead to qualitatively very different behaviors for the charge ratios $R_1=\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^+)$ and $R_2=\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^-)$ in the low-energy region. Comparing our solutions for σ's at the ρ mass of widths 200 and 600 MeV with the experimental ratios for the production processes, using one-pion exchange, they concluded that the broad resonance case was clearly favored. However, the solutions used are not compatible with the forward-backward asymmetry determination (which was not then available), so we therefore repeat their work using the better trial forms given in this paper. We also extend the comparison to somewhat higher energies, since the main open question now concerns the behavior above the ρ, and comment further on δ_0^2. The predicted ratios are compared with the bubble-chamber data in Fig. 3 for the BDI, UD, and DU cases. The results for the two cases are consistent over the whole range, and provide clear evidence for the Down rather than the Up branch above the ρ. In contrast, the DU solution is in gross disagreement over the whole range, confirming our previous result. We also note here that whereas the shape of the curve for R_1 (and R_2) depends critically on the shape of δ_0^0, its magnitude is very sensitive to δ_0^2 since only $I=2$ contributes to $\pi^+\pi^+$, and in Fig. 4 we plot this for the values $\delta_0^2(M_\rho) = -10°, -15°, -20°$, and $-25°$ for the final BDI solution. As can be seen, the second value is favored—we return to this in Sec. IV C.

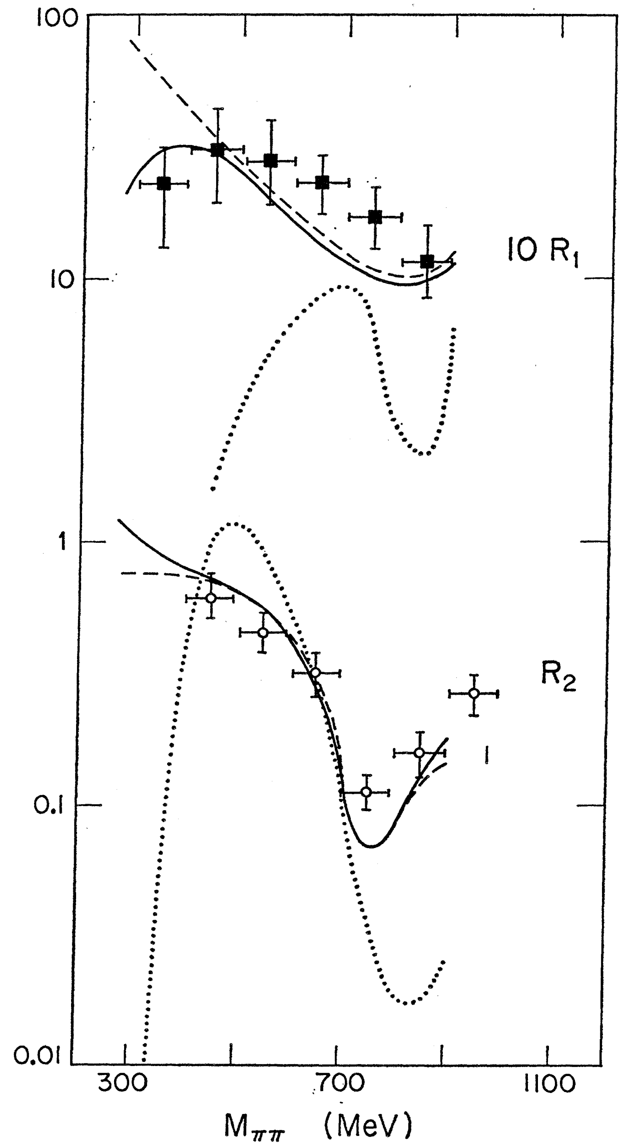


FIG. 3. Alternative predictions for $R_1=\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^+)$ and $R_2=\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^-)$ with data taken from Ref. 7. The curves correspond to our solutions BDI (full line), DUI (dotted line), and UD (dashed line).

We now have one further piece of information mentioned in the Introduction to take into account—the spark chamber data of Deinet *et al.*[8] on the process $\pi^- p \to \pi^0\pi^0 n$. The high statistics of this experiment enabled the authors to make a Chew-Low extrapolation to the total cross section for the process $\pi^-\pi^+ \to \pi^0\pi^0$, obtaining results which on comparison with the alternatives of Malamud and Schlein[14] are claimed to be compatible with the Up-Down solution only. In Fig. 5 we compare their results with the predicted cross sections for our forms BDI, BDII, and BU. Again the Down branch above the ρ is clearly favored, selecting the Between-Down form as the final solution in confirmation of the above argument based on the bubble-chamber data.

We have thus, by applying the various tests discussed in this section, arrived at a clear case for the selection

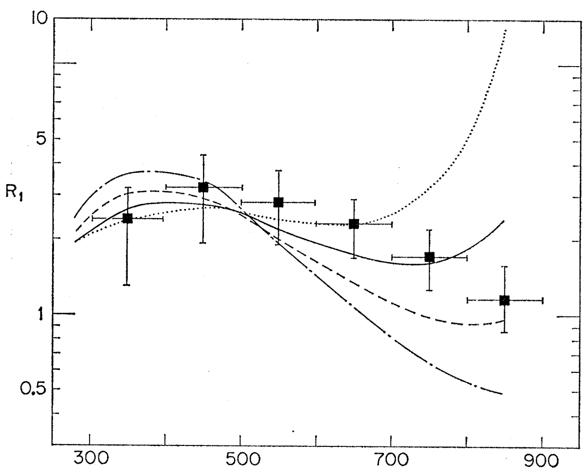


FIG. 4. Alternative predictions for $R_1=\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^+)$ with data taken from Ref. 7, for the BDI case with the $\delta_0^2(M_\rho)$ values taken as $-10°$ (dotted line), $-15°$ (full line), $-20°$ (dashed line), and $-25°$ (dash-dot line).

of the Between-Down form as the physical solution. We have tried to summarize the evidence for this in Table II.

It now finally remains in this subsection to comment briefly on the more complex forms shown in Fig. 1(c). The first of these (BDII) was included to investigate whether, if a second resonance at around 1100 MeV exists in conjuction with the Between-Down form as suggested by some authors,[10] this has any effects on the argument. In fact, all the arguments for BDI (with no second resonance) apply equally well with the BDII form, where the zero and resonance positions are those suggested by Bizzarri *et al.*[10] Thus, although the choice between these alternatives is of the great interest in itself, it has little effect on the discussion below 850 MeV. The second of these forms (LV) was chosen to agree above 700 MeV with Lovelace's two-channel unitarization[28] of the one-term Lovelace-Veneziano

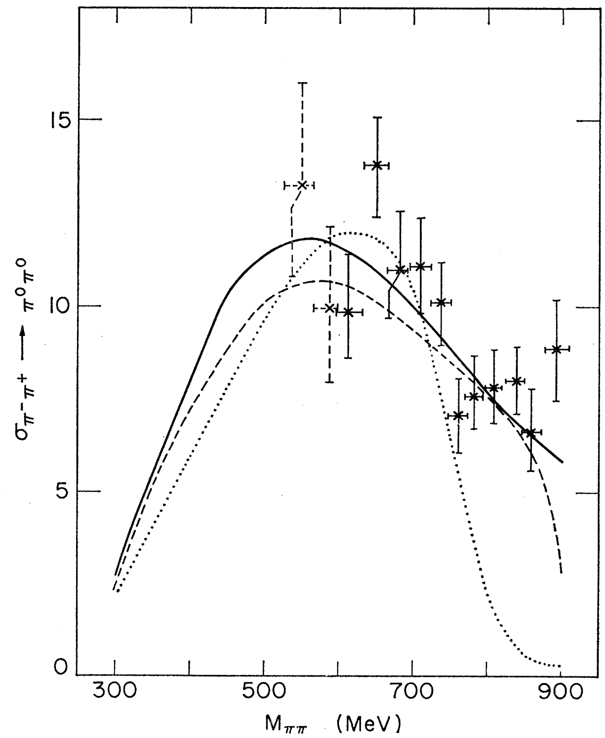


FIG. 5. Comparison of the predictions for $\sigma(\pi^+\pi^- \to \pi^0\pi^0)$ with the results of Deinet *et al.* (Ref. 8) for the forms BDI (full line), BDII (dashed line), and BU (dotted line).

formula.[2] Although this form does not agree with the $\pi^+\pi^-$ forward-backward asymmetry data, we nonetheless consider it since it has aroused great interest. However, there are other serious objections to this form in detail which allow it to be rather easily excluded. For example, when crossing symmetry is imposed, it leads to an unacceptable value for a_0/a_2 (Table I), and it also will not be compatible with the $\pi^0\pi^0$ data above the ρ, since, like the Up branch, it rises rapidly towards 180° instead of remaining in the region of 90° as required by the data.

C. ππ Phases

We now summarize the results on each phase shift, and investigate the chosen class of solutions in more detail.

TABLE II. Schematic rating of alternative forms for δ_0^0 according to the criteria considered (see text for finer points).

Input case	Subtraction argument	a_0/a_2	Method of ratios	$\sigma_\epsilon(\pi^+\pi^- \to \pi^0\pi^0)$
DD	no	no	no	no
DUI	no	no	no	no
DUII	no	no	no	no
BU	yes	yes	no	no
BDI	yes	yes	yes	yes
BDII	yes	yes	yes	yes
UD	yes	no	yes	yes
UU	no	no	no	no

p and d waves. The results on the p and d waves, and also the confirmation of the negligible size of the f wave, are unchanged from I, where it was shown that although the isospin-two d wave was negligible ($\delta_2^2 \lesssim 0.2°$), The isospin-zero d wave was not. This conclusion has since received some experimental support,[30] and we stress that the effects of this phase shift, which is plotted in Fig. 6, should be included in experimental analyses. Also in Fig. 6 we show the p-wave phase shift, including for comparison the results obtained with $\Gamma_\rho = 100$, 140 MeV also. It is perhaps also useful to note that these two waves are well approximated over this region by the simple formulas

$$\left(\frac{\nu}{\nu+1}\right)^{1/2}\cot\delta_1^1=\frac{(1-0.1536\nu)(1+0.028\nu)}{0.035\nu},$$

$$\left(\frac{\nu}{\nu+1}\right)^{1/2}\cot\delta_2^0=\frac{(1-0.0524\nu)(1+0.204\nu+0.0015\nu^2)}{0.0015\nu^2}.$$

s waves. As we noted in the Introduction, estimated values of $\delta_0^2(M_\rho)$ from the peripheral $\pi^\pm\pi^0$ data lie in the range $-10°$–$20°$, and in Fig. 5 the ratio R_1, which is particularly sensitive to δ_0^2, was plotted for the values $-10°$, $-15°$, and $-20°$, the central value being favored. In Fig. 7 we show the actual δ_0^2 phase shifts corresponding to these three cases (with $a_2 = -0.05$). We also plot a band of solutions for δ_0^0 in agreement with the Down branch above 700 MeV, again with the threshold parameters fixed at their final values, to give an estimate of the $I=0$ s-wave uncertainties also.

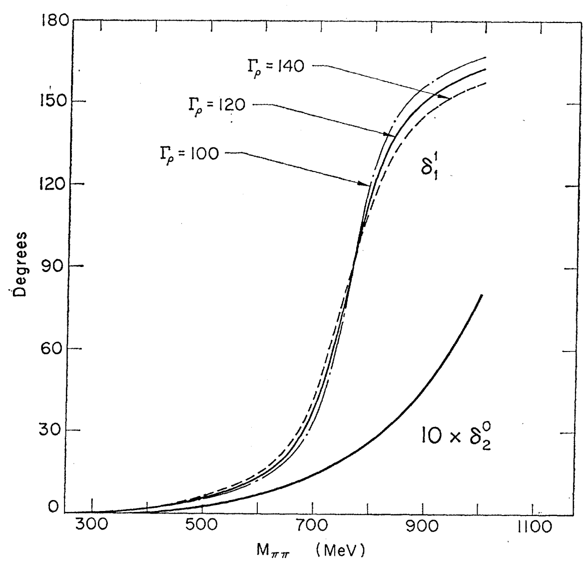


FIG. 6. d-wave phase shift δ_2^0 and p-wave δ_1^1 for $\Gamma_\rho = 100$, 120, and 140 MeV.

[30] N. N. Biswas *et al.*, Phys. Rev. D 1, 2705 (1970).

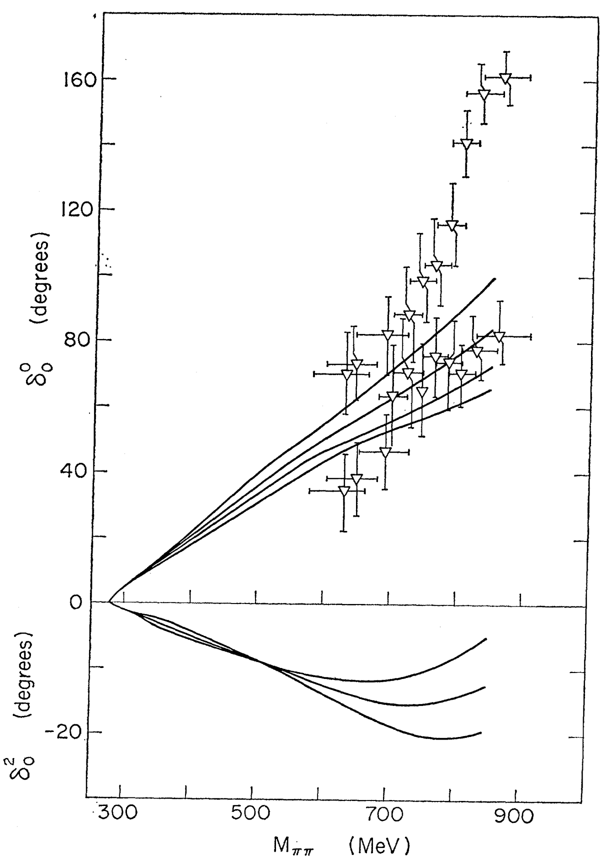


FIG. 7. Final band of solutions for the s-wave phase shift δ_0^0, and the $I=2$ s-wave δ_0^2 corresponding to the values $\delta_0^2(M_\rho) = -10°$, $-15°$, and $-20°$. The points are those of Ref. 5.

Finally, we note that these two sets of curves immediately allow us to read off the phases at the K mass, obtaining for the important combination $\delta_0^0 - \delta_0^2$ the value $(41 \pm 6)°$.

We thus arrive at a fairly complete picture of the $\pi\pi$ phases up to 850 MeV. Further confirmation may be looked for in future dipion production experiments with higher statistics and new approaches, for example, the Coulomb interference method.[31] It is very desirable to tie down further the behavior of the $I=2$ s-wave phase shift and to achieve a more quantitative picture of the onset of inelasticity. The results on the dipion energy region considered here are in remarkable agreement with the theoretical predictions from chiral and dual models. The accumulation of data on the next region up in dipion energy, say 850–1700 MeV, is now of the utmost importance.

ACKNOWLEDGMENT

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[31] See N. N. Biswas, Ref. 11.