Equivalence of Perturbation-Theory Techniques and the Bethe-Salpeter Equation for Summing Feynman Diagrams*

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Polkinghorne's exact perturbation-theory equation for the leading Regge trajectory resulting from an infinite sum of ladder diagrams is shown to be mathematically equivalent to the partial-wave Bethe-Salpeter equation. Thus, the perturbation-theory equation can be used not only for the leading Regge trajectory, but for all secondary trajectories as well.

THERE exist a number of theoretical models which are used to investigate various properties of Regge poles. One of the most useful has been the description of a Regge pole as an infinite sum of Feynman ladder diagrams in a scalar field theory. Among its other virtues, the ladder-diagram model is the most direct relativistic extension of potential theory, yet it contains many new features such as cuts and daughter poles. There are two basic methods of studying the ladder-diagram model. In one the ladder diagrams are summed into an off-mass-shell integral equation, the ladder approximation to the Bethe-Salpeter equation.[1,2] This equation is expanded in partial waves and continued into the complex angular momentum plane. Regge trajectories are obtained by locating the singularities of the partial-wave amplitude. The integral equation for the partial-wave amplitude has been solved numerically below threshold for arbitrarily strong coupling.[2] It can also be solved formally with Fredholm theory.[1] The zeros of the Fredholm denominator in the weak-coupling limit lead to analytic expressions for the Regge trajectories. Generally speaking, investigations involving the Bethe-Salpeter equation have been characterized by a mathematical preciseness lacking in the perturbation-theory approach to the study of ladder diagrams.

Originally perturbation theory involved calculation of the asymptotic behavior ($t\rightarrow\infty$) of the N-rung ladder diagram and then summation over N to obtain an expression of the form $(-t)^{\alpha(s)}$.[3] It has proved more efficient, however, to begin by taking the Mellin transform with respect to $\tau=-t$ of the N-rung ladder diagram:[4]

$$L_N(\alpha,s)=\int_0^\infty \tau^{-\alpha-1}L(s,\,-\tau)d\tau\,.$$

In its Feynman-parametrized form, this amplitude is seen to have an Nth-order pole at $\alpha=-1$. This pole is isolated and summed to give an expression of the form

$$L(\alpha,s)=\Gamma(-\alpha)\frac{[G(\alpha,s)]^2}{\alpha+1-F(\alpha,s)}. \tag{1}$$

Regge trajectories are given by the zeros of the denominator of (1); and, to lowest order in the coupling constant, their position agrees with that obtained from the zeros of the Fredholm determinant of the Bethe-Salpeter equation. Polkinghorne[5] has isolated the complete set of poles at $\alpha=-1$ and summed them into the form (1) to give an exact perturbation-theory expression for the leading trajectory. The steps leading to (1) lack mathematical rigor. No one has investigated the convergence properties of the series or the behavior of the neglected terms. The solutions of the equation

$$\alpha+1=F(\alpha,s)\,, \tag{2}$$

when the exact expression for $F(\alpha,s)$ is used, should be identical to those obtained from the Bethe-Salpeter equation. However, until recently (2) has been investigated only in the weak-coupling limit.

In this note we use a recently developed reformulation of (2) to prove that the leading Regge trajectory obtained from (2) is indeed identical to that obtained from the Bethe-Salpeter to all orders in the coupling constant.[6] The mathematical rigor of the proof is commensurate with that used in deriving (2). This result establishes the perturbation-theory approach on a firmer foundation. In addition, we show that, although (2) is an equation obtained by summing poles at $\alpha=-1$, where α is the Mellin transform variable, it contains not only the leading Regge trajecotry, but all secondary trajectories. In other words (2), or its reformulation, is mathematically equivalent to the partial-wave Bethe-Salpeter equation when it comes to solving for Regge trajectories if α is identified as the complex angular momentum. The solutions of (2) are Regge trajectories, not Mellin trajectories. The leading Regge pole and Mellin pole necessarily have the same position, but the leading Regge pole generates an infinite sequence of

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[5] J. C. Polkinghorne, J. Math. Phys. **5**, 431 (1964).

[6] A. R. Swift and R. W. Tucker, Phys. Rev. D **1**, 2894 (1970), hereafter referred to as I.

parallel secondary Mellin poles which are not present in (2).[7]

To prove our assertion we start with the reformulation of (2) developed in I. Briefly, it is obtained by writing

$$F(\alpha,s)=\frac{\bar{F}(\alpha,s)}{1+\bar{F}(\alpha,s)/(\alpha+1)}. \tag{3}$$

Values of α and s which generate poles in $\bar{F}(\alpha,s)$ are solutions of (2). The expression for $\bar{F}(\alpha,s)$ in terms of an infinite sum of Feynman integrals is recast into an integral equation by noting that the $\bar{F}(\alpha,s)$ is just an infinite sum of ladder diagrams with contracted ends if we use a special set of Feynman rules. The propagator for the exchanged particle is

$$P(k^2)=\Gamma(\alpha+1)/(k^2+\lambda^2)^{\alpha+1}, \tag{4}$$

and the dimensionality of the momentum-space integrals is $L=2\alpha+4$ rather than just $L=4$. (There are one time dimension and $2\alpha+3$ space dimensions.) All integrals are carried out in L dimensions, L an arbitrary positive integer, and then we set $L=2\alpha+4$. Thus, we can write

$$\bar{F}(\alpha,s)=\frac{G^2}{\pi^{\alpha+2}}\int\frac{d^{2\alpha+4}k\ V(\alpha,s,k)}{[(k+iE)^2+\mu^2][(k-iE)^2+\mu^2]}, \tag{5}$$

where $V(\alpha,s,k)$ satisfies the integral equation

$$V(\alpha,s,k)=1+\frac{G^2}{\pi^{\alpha+2}}\times\int\frac{dq_0d^{N+2}qP((k-q)^2)V(\alpha,s,q)}{[(q+iE)^2+\mu^2][(q-iE)^2+\mu^2]}. \tag{6}$$

N is to be set equal to $2\alpha+1$, and $E=\frac{1}{2}s^{1/2}$.[8] If $V(\alpha,s,k)$ is expanded in normalized hyperspherical harmonics for $N+2$ dimensions,[9]

$$V(\alpha,s,k)=\sum_{n=0}^{\infty}V_n(\alpha,s,k)Y_N(n,0,\Omega_k), \tag{7}$$

we obtain the following integral equation for $V_n(\alpha,s,k)$:

$$V_n(\alpha,s,k)=V_n{}^0+\frac{G^2}{\pi^{\alpha+2}}\frac{N\omega}{(2n+N)}\times\int_{-\infty}^{\infty}dq_0\int_0^{\infty}\frac{q^{N+1}dq\,P_n(k,q)V_n(\alpha,s,q)}{[(q+iE)^2+\mu^2][(q-iE)^2+\mu^2]}, \tag{8}$$

where $\omega=2\pi^{1+N/2}/\Gamma(1+\frac{1}{2}N)$ is the area of a hypersphere of unit radius. We have used

$$P((k-q)^2)=\sum_{n=0}^{\infty}P_n(k,q)C^{N/2}{}_n(z). \tag{9}$$

$C^{N/2}{}_n(z)$ is a Gegenbauer function, $z=\cos\phi$, and ϕ is the angle between the spatial parts of the vectors q and k ($\mathbf{q}\cdot\mathbf{k}=qk\cos\phi$).

$$P_n(k,q)=\frac{2^{N-2}n!(N+2n)[\Gamma(N/2)]^2}{\pi(n+N)}\times\int_{-1}^{1}dz(1-z^2)^{(N-1)/2}C^{N/2}{}_n(z)P((k-q)^2). \tag{10}$$

$C^{N/2}{}_n(z)$ obeys the addition theorem:

$$C^{N/2}{}_n(\cos\phi)=\frac{N\omega}{2n+N}\sum_{m_i}Y_N(n,m_i,\Omega_k)Y_N{}^*(n,m_i,\Omega_q).$$

Since the inhomogeneous term in (6) is a constant, $V_n{}^0(\alpha,s,k)=0$ for $n\neq0$. Inasmuch as we are interested in the poles of $V(\alpha,s,k)$, we set $V_n(\alpha,s,k)=0$, $n\neq0$, and use (8) only for $n=0$:

$$V_0(\alpha,s,k)=V_0{}^0+\frac{2G^2}{(\sqrt{\pi})\Gamma(\alpha+\frac{3}{2})}\times\int\frac{dq_0q^{2\alpha+2}dq\,P_0(k,q)V_0(\alpha,s,q)}{[(q+iE)^2+\mu^2][(q-iE)^2+\mu^2]}, \tag{11}$$

where $P_0(k,q)$ is given by

$$P_0(k,q)=\frac{\Gamma(\alpha+\frac{3}{2})}{(\sqrt{\pi})\Gamma(\alpha+1)}\int_{-1}^{1}dz(1-z^2)^{\alpha}\times P((k_0-q_0)^2+k^2+q^2-2kqz), \tag{12}$$

and $V^0{}_0(\alpha,s,k)=\omega^{1/2}$. If (4) is used for $P(k^2)$, we find[10]

$$P_0(k,q)=\frac{\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}}\frac{Q_\alpha(x)}{(kq)^{\alpha+1}}. \tag{13}$$

$Q_\alpha(x)$ is a Legendre function of the second kind and $x=[(k_0-q_0)^2+k^2+q^2+\lambda^2]/(2qk)$. Thus, when $V_0(\alpha,s,k)$

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[8] We do not attempt to give a rigorous discussion of the analytic continuation involved in setting $N=2\alpha+1$ in (6). The calculations using this unorthodox approach reported in Ref. 6 justify this continuation *a posteriori*. Solutions of Eq. (6) for potential scattering agree well with those obtained by other, more rigorous methods.

[9] The formulas associated with the hyperspherical harmonics can be found in *Higher Transcendental Functions*, edited by Erdélyi (McGraw-Hill, New York, 1953), Vol. II, Chap. 11. We have changed the notation for the hyperspherical harmonic. $Y_n(n,m_i,\Omega)$ is equal to $Y(m_i,\theta,\phi)/[N(m_i)]^{1/2}$ in $p=N$ dimensions. $Y(m_i,\theta,\phi)$ is defined in Eq. (11.3.3) on p. 240; $N(m_i)$ is the normalization constant; $m_0=n$; $\Omega=(\theta_1,\theta_2,\cdots,\theta_N,\phi)$.

[10] See Ref. 9, Vol. I, Chap. 3, p. 155.

is redefined by

$$V_0(\alpha,s,k)=(\omega^{1/2}/k^\alpha)\bar{V}(\alpha,s,k)\,, \tag{14}$$

we obtain

$$\bar{V}(\alpha,s,k)=k^\alpha+\frac{2G^2}{\pi}\times\int\frac{dq_0qdq\,Q_\alpha(x)\bar{V}(\alpha,s,q)}{[(q+iE)^2+\mu^2][(q-iE)^2+\mu^2]}. \tag{15}$$

Except for the inhomogeneous term, (15) is the partial-wave Bethe-Salpeter equation.[1] Since the positions of the poles in $\bar{V}(\alpha,s,k)$ do not depend on the inhomogeneous term, we have our desired result. The Regge trajectories, secondary as well as leading, obtained from (2) by means of (6) are identical to those found by solving the Bethe-Salpeter equation, as expected.

Whether (6) is to be preferred to the Bethe-Salpeter equation depends on the questions being investigated. The kernel of (6) is easier to handle than that in (15), but a price is paid in terms of the continuous-dimensional integration. A simple separable approximation to (6) gives quantitatively good results which can be continued above the elastic threshold,[6] while (15) is to be preferred if exact numerical solutions are desired.[2] As a method of deriving weak-coupling solutions for secondary trajectories, a sequence of separable approximations to (6) proves to be simpler than either perturbation theory applied directly to ladder diagrams to isolate poles at $\alpha=-N$,[11] or weak-coupling approximations to the Bethe-Salpeter equation.[12]

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[12] M. Fontannaz, Nuovo Cimento **59A**, 215 (1969).

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Asymptotic Symmetry, Current Algebra, and the Veneziano Ansatz

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The universality of the slopes of meson trajectories is established in the context of the Veneziano model from the postulate of asymptotic $SU(3)\times SU(3)$ symmetry and without reference to the Adler consistency conditions.

It has been suggested by Mandelstam[1] that a Regge-pole model with linearly rising $\alpha(s)$ must have all trajectories parallel: $\alpha_i(s)=a_i+bs$, where b is a universal constant. This conjecture is fairly well supported by experiment, as is evident from an inspection of the Chew-Frautschi plot for meson and baryon trajectories. Recently, Ademollo, Veneziano, and Weinberg[2] (AVW) have successfully employed this idea of a universal slope in conjunction with the Veneziano representation[3] and the Adler partially conserved axial-vector current (PCAC) condition[4] for a soft pion to predict several mass relations between hadrons. The work of AVW and others[5] was motivated by the work of Lovelace,[6] who first pointed out the importance of the Veneziano model and its possible connection with chiral symmetry. The equality of slopes of various Regge trajectories (assumed linear) of either normality can be derived within the Veneziano framework by appealing to the Adler partial-conservation conditions for π, K, and κ mesons.[7] There is also the closely related question of the universality of coupling of the ρ meson to other hadrons such as π, K, A_1, etc. It has been shown[8] that a universal ρ coupling is a consequence of the requirement that the minimal Veneziano forms for various amplitudes involving the pion be consistent with the low-energy theorems of Adler and Weisberger[9] (AW). The concepts of a universal slope of trajectories and a universal ρ coupling have emerged, therefore, as consistency conditions imposed on the Veneziano amplitudes by PCAC (PCVC) and charge algebra, respectively.

In the present note, we wish to make an exploratory study of the possible high-energy constraints, if any, on the Veneziano amplitudes for meson systems. Since the structure of the Veneziano amplitude is motivated to a large extent by asymptotic considerations, it appears quite natural to look for constraints (on the amplitude)

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