Chiral Symmetry and the Kuo Transformation*

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The various implications of a transformation recently suggested by Kuo in the model of Gell-Mann, Oakes, and Renner have been investigated.

IN a recent paper, Kuo[1] made a very interesting observation that in the model of Gell-Mann, Oakes, and Renner,[2] the Hamiltonian

$$H(x,\epsilon)=H_0(x)+\epsilon_0 S^{(0)}(x)+\epsilon_8 S^{(8)}(x) \tag{1}$$

can be transformed by an operator U into a new form

$$\bar{H}(x,\epsilon)\equiv UH(x,\epsilon)U^{-1}=H_0(x)+\bar{\epsilon}_0 S^{(0)}(x)+\bar{\epsilon}_8 S^{(8)}(x)=H(x,\bar{\epsilon}), \tag{2}$$

with

$$\bar{\epsilon}_0=-\tfrac{1}{3}\epsilon_0-\tfrac{2}{3}\sqrt{2}\epsilon_8, \quad \bar{\epsilon}_8=-\tfrac{2}{3}\sqrt{2}\epsilon_0+\tfrac{1}{3}\epsilon_8. \tag{3}$$

In Eq. (1), $H_0(x)$ is the $SW(3)$ [or $W(3)$] invariant part, while $S^{(0)}(x)$ and $S^{(8)}(x)$ represent the scalar portion of an $(3,3^*)\oplus(3^*,3)$-type violation of the $SW(3)$ [or $W(3)$] group. The operator U is formally unitary and given by[3]

$$U=\exp[\pm\tfrac{3}{2}\pi i(Y-Y_5)], \tag{4}$$

where Y and Y_5 are the hypercharge operator and its chiral counterpart, respectively, and where one can choose either sign in the exponent with the same consequence [see Eq. (12)].

In this paper, we investigate various consequences of this transformation, especially in connection with results obtained in our previous papers.[4,5]

Before going into details, it may be worthwhile to point out that transformation (4) has a very simple physical meaning if we work within the framework of the quark model. In that case, parameters ϵ_0 and ϵ_8 are expressed in terms of bare quark masses m_1, m_2 ($=m_1$), and m_3 simply by

$$\epsilon_0=(\tfrac{2}{3})^{1/2}(2m_1+m_3), \quad \epsilon_8=(\tfrac{4}{3})^{1/2}(m_1-m_3). \tag{5}$$

Then, the Kuo transformation is nothing but the mass-reversal operation[6]

$$m_1\to -m_1, \quad m_2\to -m_2, \quad m_3\to +m_3, \tag{6}$$

induced by the chiral transformation[3]

$$q_1(x)\to\gamma_5 q_1(x), \quad q_2(x)\to\gamma_5 q_2(x), \quad q_3(x)\to q_3(x) \tag{7}$$

when the operator U is applied on the three quark fields.

If we define ξ_0 and ξ_8 by

$$\xi_0=\langle S^{(0)}(0)\rangle_0, \quad \xi_8=\langle S^{(8)}(0)\rangle_0 \tag{8}$$

as in (A), we find that the corresponding quantities $\bar{\xi}_0$ and $\bar{\xi}_8$ defined with respect to the new Hamiltonian $\bar{H}$ of Eq. (2) now transform exactly in the same fashion as in Eq. (3). Setting, as in (A),

$$a=\epsilon_8/\sqrt{2}\epsilon_0, \quad b=\xi_8/\sqrt{2}\xi_0, \quad \gamma=-\tfrac{2}{3}\epsilon_0\xi_0, \tag{9}$$

we see that transformation (4) takes the set ϵ_0, a, b, and γ to a new one $\bar{\epsilon}_0$, $\bar{a}$, $\bar{b}$, and $\bar{\gamma}$, such that

$$\bar{\epsilon}_0=-\tfrac{1}{3}(1+4a)\epsilon_0, \quad \bar{a}=\frac{2-a}{1+4a}, \quad \bar{b}=\frac{2-b}{1+4b}, \quad \bar{\gamma}=\tfrac{1}{9}(1+4a)(1+4b)\gamma. \tag{10}$$

Notice that these relations remain invariant in form if we solve for ϵ_0, a, b, and γ in terms of $\bar{\epsilon}_0$, $\bar{a}$, $\bar{b}$, and $\bar{\gamma}$. We emphasize the fact that quantities ϵ_0, a, b, and γ are defined in the theory with the Hamiltonian H of Eq. (1), while $\bar{\epsilon}_0$, $\bar{a}$, $\bar{b}$, and $\bar{\gamma}$ correspond to the theory with the transformed Hamiltonian $\bar{H}$ of Eq. (2).

It is important to note that transformation (4) changes also the definition of the parity operator P into

$$\bar{P}=UPU^{-1}=e^{\pm 3\pi iY}P, \tag{11}$$

where we have utilized the identity[7]

$$e^{\pm 3\pi iY}=e^{\pm 3\pi iY_5}. \tag{12}$$

A simple way to show this is to observe that the operator $\frac{3}{2}(Y-Y_5)$ has only integral eigenvalues $0, \pm1, \pm2, \ldots$ in the $SW(3)$ theory. Note that $\bar{P}$ changes the definition of the parity of quantities with odd hypercharge in contrast to P. In the quark model, this is transparent from transformation (7). Thus, the vector

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[1] T. K. Kuo, this issue, Phys. Rev. D **2**, 349 (1970).

[2] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968); see also S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968); R. E. Marshak, N. Mukunda, and S. Okubo, Phys. Rev. **137**, B698 (1965).

[3] This transformation is slightly different from that given by Kuo, which corresponds to a multiplication of phase factors i, i, and -1 in Eq. (7).

[4] S. Okubo and V. S. Mathur, Phys. Rev. Letters **23**, 1412 (1969); Phys. Rev. D **1**, 2046 (1970). These papers are concerned with the $W(3)$ group and will be referred to as (A) hereafter.

[5] V. S. Mathur and S. Okubo, Phys. Rev. D **1**, 3468 (1970). This paper deals with consequences of the $SW(3)$ group and will be referred to as (B) hereafter.

[6] J. Tiomno, Nuovo Cimento **1**, 226 (1955). For its application to the weak interaction, see S. Hori and A. Wakasa, *ibid.* **6**, 302 (1957); J. J. Sakurai, *ibid.* **7**, 649 (1958).

[7] This can be proved in a direct way by a method given in J. Schechter, Y. Ueda, and S. Okubo, Ann. Phys. (N.Y.) **32**, 424 (1965).

current $V_\mu^{(\alpha)}(x)$ ($\alpha=4, 5, 6, 7$) will interchange with the axial-vector $A_\mu^{(\alpha)}(x)$ ($\alpha=4, 5, 6, 7$), while the hypercharge-preserving vector and axial-vector currents remain unchanged under (4). This fact implies that the operator U interchanges the ordinary $SU(3)$ and the chimeral[4] $SU(3)$ symmetries. Essentially, transformation (11) underlines the well-known arbitrariness[8] of defining the relative parity of particles with nonzero hypercharge from those with zero hypercharge. Since this relative parity cannot be determined[8] experimentally and is only a matter of convention, transformation (4), which alters the parity convention, should not change the physical description of the system.

We may now ask the important question whether physical quantities should exhibit invariance under the transformation (4) *manifestly* as suggested by Kuo. By this we mean whether the physical description should be *explicitly* symmetric under the transformation. First note that the physical quantities derived from the Hamiltonian [Eq. (1)] will depend on the parameters ϵ_0 and a, whereas those obtained from Eq. (2) will depend upon $\bar{\epsilon}_0$ and $\bar{a}$, which are given in terms of ϵ_0 and a by Eq. (10). Since $\xi=(1/\sqrt{6})\epsilon_0(1-2a)$ remains invariant under the transformation, we consider only the variable $\eta\equiv(1/\sqrt{6})\epsilon_0(1+a)$ which reverses its sign under the transformation. In the simple quark model, note that $\xi=m_3$ and $\eta=m_1$. If we now regard the physical quantities as described in terms of suitable real or complex analytic functions of the parameter η, we have two possibilities: (i) The physical quantities may be analytic everywhere in the η plane, or (ii) the domain of analyticity may be restricted to a region D in this plane. In case (i), Eq. (10) represents a well-defined one-to-one mapping $\eta\rightarrow\bar{\eta}=-\eta$ in the η plane, so that manifest covariance of the physical quantities under the transformation can be built in. However in case (ii), if $\bar{D}$ is the analyticity domain for the transformed physical quantities as functions of $\bar{\eta}\equiv(1/\sqrt{6})\bar{\epsilon}_0(1+\bar{a})\equiv-\eta$ ($\bar{\eta}=-m_1$ for the quark model), manifest covariance of the physical description requires that either $\bar{D}\equiv D$ or one should be able to continue analytically from one domain to the other. It should be noted that if the domains D and $\bar{D}$ are disjoint and one cannot continue from one to the other, one does not infer noninvariance under the discrete operation (4). In this case, however, there will be two distinct worlds, both physically equivalent with the interchange of the role of a and $\bar{a}$ and of ϵ_0 and $\bar{\epsilon}_0$, but an explicitly symmetric description would not be possible.

Now we would like to point out that possibilities (i) and (ii) discussed above are closely related to whether the fundamental symmetry $SW(3)$ or $W(3)$ is realizable in the usual manner with a unique vacuum transforming as a scalar under this symmetry group or through the emergence[2,9] of zero-mass Goldstone bosons with the vacuum invariant only under the smaller $SU(3)$ symmetry. In fact, it was shown in (A) and (B) that if the latter is the case, the points $a=-1$ and $a=2$ are points of discontinuities and the theory is nonanalytic at these points. These may be essential singularities or branch cuts. In any case, the theory [Eq. (1)] is restricted to $-1\leq a\leq 2$, and it is not in general possible to continue physical quantities beyond this region. Of special interest to us is the physical region near $a=-1$. The point $a=-1+\delta$, for an infinitesimal $\delta(>0)$, maps to the point $a=-1-\delta$ under the transformation [Eq. (10)] which lies in the disjoint unphysical region. Thus, an analytic continuation is not possible. We may describe this situation also by realizing the transformation $H(x,\epsilon)\rightarrow H(x,\bar{\epsilon})$ not through the discrete operation (4), but as a result of a more general continuous transformation in which we affect $(\epsilon_0,a)\rightarrow(\bar{\epsilon}_0,\bar{a})$ continuously. Such general transformations can be easily constructed, but we will not go into details here. As is evident from simple considerations of the quark model, any continuous transition $m_1\rightarrow-m_1$ (or $\eta\rightarrow-\eta$, $\xi\rightarrow\xi$ in the general case) will then have to go through $m_1=0$ (or $\eta=0$), i.e., through the point $a=-1$ where the Hamiltonian is $SU^{(+)}(2)\otimes SU^{(-)}(2)$ symmetric. Thus, if $a=-1$ is an essential singularity where one realizes zero-mass Goldstone pions, the transition will not be well defined beyond this piont. On the other hand, in the case when $SW(3)$ or $W(3)$ symmetry is realizable in the usual manner without Goldstone bosons, there is no reason to believe that $a=-1$ is a singular point so that in this case the theory will be manifestly symmetric under the transformation (4).

We would like to mention a rather simple argument which also suggests that an explicitly symmetric theory may not be possible if the fundamental symmetry $SW(3)$ or $W(3)$ is realized through the emergence of zero-mass Goldstone bosons. If $|n\rangle$ represents an eigenstate for the Hamiltonian [Eq. (1)], under transformation (4), let $|n\rangle\rightarrow|\bar{n}\rangle=U|n\rangle$. Note that if $|n\rangle$ is a single-particle state with odd hypercharge, $|n\rangle$ and $|\bar{n}\rangle$ have opposite parities. However, both states have the same eigenvalues with the Hamiltonians H and $\bar{H}$, respectively. An explicitly symmetric description would then require that if a state $|n\rangle$ exists with odd hypercharge, there must also exist simultaneously the opposite-parity state $|\bar{n}\rangle$ in the *same* Hilbert space, since in this case one can continuously change the variables ϵ_0, a into $\bar{\epsilon}_0$, $\bar{a}$ without encountering any singularity. Now, if the $SW(3)$ or $W(3)$ symmetry is realized through the emergence of the zero-mass octet (π,K,η) (adopting the usual convention that K is pseudoscalar) with the vacuum symmetric only under

[8] See, e.g., P. T. Matthews, Nuovo Cimento 6, 642 (1957).

[9] R. F. Dashen, Phys. Rev. **183**, 1245 (1969); R. F. Dashen and M. Weinstein, *ibid.* **183**, 1261 (1969).

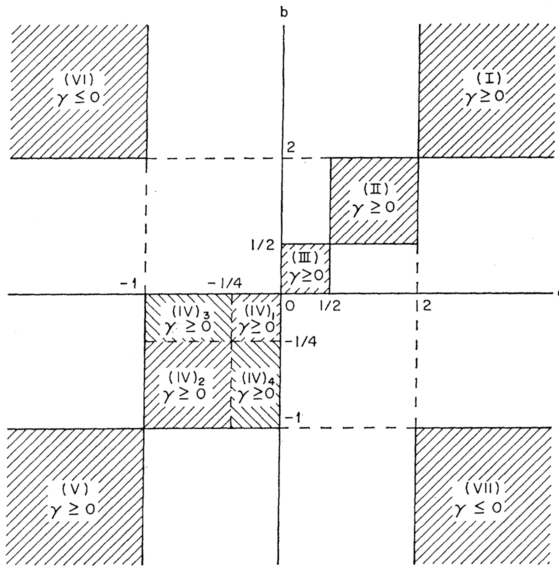

FIG. 1. Allowed domains in the a-b plane for the $W(3)$ group.

$SU(3)$, we know[10] that the scalar meson κ need not exist. This description would then not be explicitly symmetric. The symmetry, which, however, must be respected, is then restored by realizing that an equivalent description of this system is possible under the opposite-parity convention, where the $SW(3)$ or $W(3)$ symmetry is now realized with (π,κ,η) belonging to a massless octet with the vacuum invariant under the chimeral $SU(3)$ symmetry. In this case no K meson exists. Another example which illustrates the same point is the nonlinear realization[11] of the chiral $SW(3)$-invariant Hamiltonian with $(3,3^*)\oplus(3^*,3)$-type violation, where we have neither the scalar mesons nor the negative-parity hyperons.

It may be instructive to see the effect of the transformation in Eq. (10) explicitly in a special case. In (A) and (B), we have determined the maximally allowable domain for a and b from rather general considerations. Here, for simplicity, let us consider the case where H_0 in Eq. (1) is invariant under $W(3)$ symmetry. The case when the fundamental symmetry is $SW(3)$ rather than $W(3)$ is quite similar, with only slight modification. The allowed values of a and b are represented in the a-b plane as in Fig. 1. To see the effect of transformation (10) on this plane, we divide the domain IV of (A) into four subdomains, as shown in Fig. 1. It is easy to check that transformation (10) interchanges these allowed domains among themselves as follows:

$$\text{(I)}\leftrightarrow\text{(IV)}_1,\quad \text{(II)}\leftrightarrow\text{(III)},\quad \text{(IV)}_2\leftrightarrow\text{(V)},\quad \text{(IV)}_3\leftrightarrow\text{(VI)},\quad \text{(IV)}_4\leftrightarrow\text{(VII)}. \tag{13}$$

Now, if $a=-1$ and $a=2$ are the only essential singularities, the physical region contains only the domains II, III, and IV. Note in particular that this physical region is not invariant under transformation (10). It can also be seen that whereas the points $a=-1$ and $a=\frac{1}{2}$ are invariant under the transformation, the points $a=0$ and $a=2$ go into each other, interchanging the ordinary and chimeral $SU(3)$'s.

We would like to comment briefly that the transformation U may actually not be unitarily implementable in the mathematical sense. In the model of Nambu and Jona-Lasinio,[12] one knows for instance that for the "superconducting" solution, a general chiral transformation takes one from a given Hilbert space to another disjoint one. Thus, formal arguments used by Kuo may also be dangerous from this point of view.

We may also mention that if the fundamental symmetry group is $W(3)$ rather than $SW(3)$, we can have another mass-reversal transformation

$$m_1\to -m_1,\quad m_2\to -m_2,\quad m_3\to -m_3 \tag{14}$$

generated by

$$q_1(x)\to\gamma_5 q_1(x),\ q_2(x)\to\gamma_5 q_2(x),\ q_3(x)\to\gamma_5 q_3(x). \tag{15}$$

This corresponds to $\epsilon_0\to-\epsilon_0$, a and b remaining unchanged, and implies a complete reversal of the parity convention.

In conclusion, we believe that the Kuo transformation does not conflict with the results of our previous papers. It is presumably relevant only in the case when the $SW(3)$ symmetry can be realized without Goldstone bosons, with the vacuum symmetric under the full $SW(3)$ group.

[10] In (A), we gave an argument favoring the existence of the κ meson. However, for the case of the $SW(3)$ theory, the same argument does not apply any longer, as we emphasized in (B).

[11] See, e.g., Y. M. P. Lam and Y. Y. Lee, Phys. Rev. Letters **23**, 734 (1969); K. Yoshida, University of Durham Report, 1969 (unpublished); A. M. Harun-Ar Rashid, Trieste Report, 1969 (unpublished); L. M. Brown and H. Munczek, University of Kansas report, 1969 (unpublished).

[12] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).