The consequences of the proposed model for Pomeranchuk exchange were investigated in some detail. The model predicts that elastic differential cross sections shrink with increasing energy at a rate corresponding approximately to an effective Pomeranchuk pole having a slope $\alpha_P'=0.5$ GeV^{-2} in agreement with the recent Serpukhov measurements. If a Pomeranchuk theorem holds, the asymptotic limit of total cross sections are predicted to be approached in a logarithmic fashion from below. Finally, the crossover phenomenon was investigated, which is in this model due to the vanishing of a Regge-pole contribution corrected for absorption and being odd under charge conjugation. The absorptive corrections to conventional Regge-pole expressions predicted by the model are given in terms of quantities characterizing the elastic scattering in the asymptotic region. It was pointed out that the analysis of the crossover condition provides information about total cross sections at asymptotic energies. We conclude by noting that the proposed K-matrix model is not limited to small values of momentum transfers. However, for large values of t it probably becomes essential to take the spin of the external particles into account.

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$K \to 3\pi$ Decays in Dual Models*

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The dual representation of $K \to 3\pi$ decay amplitudes is studied. It is found that $K \to 3\pi$ decay amplitudes in a generalized Veneziano model are incompatible with current-algebra relations. For example, pion poles and kaon poles are not dual to other poles. An example of realistic $K \to 3\pi$ decay amplitudes which contain pion and kaon poles, which contain both $|\Delta \mathbf{I}|=\frac{1}{2}$ and $\frac{3}{2}$ parts, and which are compatible with current-algebra relations is obtained. Our results seem to suggest that we should prefer the charged-current$\times$charged-current nonleptonic weak Hamiltonian to Hamiltonians with pure $|\Delta \mathbf{I}|=\frac{1}{2}$.

I. INTRODUCTION

DUAL representation of reaction amplitudes (Veneziano model[1] and generalized Veneziano model[2–7]) has been discovered. Amplitudes in this representation have resonance poles at low energy, have Regge behavior at high energy, satisfy the crossing relations, and give relations among Regge trajectories such as the exchange degeneracy. In this representation, poles in various channels of a reaction are related so closely that, for example, a sum of all s-channel poles of the amplitude is equal to a sum of poles in other channels. This property of the amplitudes is the so-called (full) duality. However, the above definition of duality is not practical for our purpose. Since the generalized Veneziano amplitude for N-point function is the only amplitude with full duality, we define the full duality of an amplitude as follows: An amplitude of an N-prong reaction is completely dual (has full duality) if and only if it is expressed as a sum of a finite number of generalized Veneziano amplitudes for N-point function.

A purpose of this article is to study whether a weak amplitude has full duality. Dual representations for K_{l4} decay amplitudes have been studied by the present author,[8] and it has been found that the kaon pole is not dual to any other poles if we impose conditions required by current algebra at soft-pion limits. However, it has been found that all poles except for the kaon pole can be dual if the relation among trajectories,

$$\alpha_{K^*}(t)-\alpha_K(t)=1-\alpha_\rho(M_\pi^2)\quad \text{for } \alpha_K(t)=\text{positive integers}, \qquad (1.1)$$

is satisfied.[9] In this article we consider whether it is

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968).

[3] M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969).

[4] M. H. Chan, Phys. Letters **28B**, 425 (1968); M. H. Chan and S. T. Tsou, *ibid.* **28B**, 485 (1969).

[5] Z. Koba and H. D. Nielsen, Nucl. Phys. **B10**, 633 (1969).

[6] C. J. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969).

[7] Z. Koba and H. D. Nielsen, Nucl. Phys. **B12**, 517 (1969).

[8] Y. Hara, Phys. Rev. D **1**, 874 (1970).

[9] By applying the method used in Ref. 8 to the $\pi+\pi\to\pi+l+\nu$ processes, we find a relation

$$\alpha_\rho(t)-\alpha_\pi(t)=1-\alpha_\rho(M_\pi^2)\quad \text{for } \alpha_\pi(t)=\text{positive integers}. \qquad (A)$$

If we assume that all trajectories are linear and parallel, we find the relations $\alpha_\rho(M_\pi^2)=\frac{1}{2}$ and $M_\rho^2-M_\pi^2=M_{K^*}^2-M_K^2$ from the relations (1.1) and (A). No other relations among Regge trajectories are obtained by studying similar leptonic processes such as $\eta+\pi^+\to\eta+l+\nu$.

possible for $K \to 3\pi$ decay amplitudes to have full duality when current-algebra relations in soft-pion limits are imposed. There is a significant difference between K_{l4} decays and $K \to 3\pi$ decays. Leptons are involved in K_{l4} decays and there are no Regge trajectories in lepton-meson channels. On the other hand, only hadrons are involved in $K \to 3\pi$ decays, and we have to introduce a spurion which is responsible for transitions due to the strangeness-changing weak nonleptonic interaction.

Experimental results for $K \to 3\pi$ decays ($K^+ \to \pi^+\pi^+\pi^-$, $K^+ \to \pi^0\pi^0\pi^+$, $K_L{}^0 \to \pi^+\pi^-\pi^0$, and $K_L{}^0 \to \pi^0\pi^0\pi^0$) are conventionally parametrized as

$$|A(\pi_1\pi_2\pi_3)|^2 \approx |A_c(\pi_1\pi_2\pi_3)|^2[1+2a(s_0-s_3)/M_\pi{}^2], \quad (1.2)$$

where $s_i = -[q(K)-q(\pi_i)]^2$ (the third pion is the asymmetric pion), $3s_0 = s_1+s_2+s_3$, and M_π is the mass of a charged pion. The $A_c(\pi_1\pi_2\pi_3)$ is the magnitude of the amplitude at the center of the Dalitz plot,[10] $s_1=s_2=s_3$, and $a(\pi_1\pi_2\pi_3)$ is proportional to the slope of the decay spectrum. According to recent experimental results,[11–14]

$$\begin{aligned} \tfrac{1}{2}|A_c(+\,+\,-)| &= (0.96\pm0.01)\times10^{-6}, \\ |A_c(0\,0\,+)| &= (0.97\pm0.02)\times10^{-6}, \\ |A_c(+\,-\,0)| &= (0.86\pm0.02)\times10^{-6}, \\ \tfrac{1}{3}|A_c(0\,0\,0)| &= (0.85\pm0.02)\times10^{-6}, \\ a(\pm\,\pm\,\mp) &= 0.1000\pm0.0036, \\ a(0\,0\,+) &= -0.258\pm0.010, \end{aligned} \quad (1.3)$$

and

$$a(+\,-\,0) = -0.300\pm0.018. \quad (1.4)$$

These results violate the relations obtained by assuming the $|\Delta\mathbf{I}| = \frac{1}{2}$ rule for the decays,[10]

$$\tfrac{1}{2}A_c(+\,+\,-) = A_c(0\,0\,+) = A_c(+\,-\,0) = \tfrac{1}{3}A_c(0\,0\,0) \quad (1.5)$$

and[15]

$$2a(\pm\,\pm\,\mp) = -a(+\,-\,0) = -a(0\,0\,+), \quad (1.6)$$

by at least about 10%, and clearly indicate the existence of the $|\Delta\mathbf{I}| = \frac{3}{2}$ parts of the amplitudes. However, the existence of the $|\Delta\mathbf{I}| \geq \frac{5}{2}$ parts of the amplitudes are not required by these experiments since the relations

$$\tfrac{1}{2}A_c(+\,+\,-) = A_c(0\,0\,+)$$

and

$$A_c(+\,-\,0) = \tfrac{1}{3}A_c(0\,0\,0), \quad (1.7)$$

which are obtained by assuming the nonexistence of the $|\Delta\mathbf{I}| \geq \frac{5}{2}$ parts of the amplitudes ,are satisfied by these experimental results within experimental errors.

Since it is rather difficult to expect electromagnetic corrections to the $|\Delta\mathbf{I}| = \frac{1}{2}$ rule to be about 10%, it is natural to assume that the weak nonleptonic Hamiltonian is a product of a charged current and its Hermitian conjugate,[16]

$$H_w = (G/\sqrt{2})J_\alpha J_\alpha{}^\dagger, \quad (1.8)$$

where

$$J_\alpha = \cos\theta[i\bar{q}\gamma_\alpha(1+\gamma_5)\tfrac{1}{2}(\lambda_1-i\lambda_2)q] + \sin\theta[i\bar{q}\gamma_\alpha(1+\gamma_5)\tfrac{1}{2}(\lambda_4-i\lambda_5)q]. \quad (1.9)$$

If we assume the Hamiltonian (1.8), the dominance of the $|\Delta\mathbf{I}| = \frac{1}{2}$ part over the $|\Delta\mathbf{I}| = \frac{3}{2}$ part of the amplitude may be explained by a dynamical enhancement of the $|\Delta\mathbf{I}| = \frac{1}{2}$ part of the amplitude.[17]

The $|\Delta\mathbf{I}| = \frac{3}{2}$ parts of the $K \to 3\pi$ decay amplitudes are not expressed by Veneziano amplitudes[18] since there are no exotic resonances with $I = \frac{3}{2}$. Some authors[19] have argued that the $|\Delta\mathbf{I}| = \frac{1}{2}$ rule is a consequence of the fact that the $|\Delta\mathbf{I}| = \frac{3}{2}$ parts of the amplitudes are not expressed in terms of dual amplitudes. However, they have to show that it is possible for the $|\Delta\mathbf{I}| = \frac{1}{2}$

[10] The center of the Dalitz plot of $K^+ \to \pi^0\pi^0\pi^+$ decay and that of $K_L{}^0 \to \pi^+\pi^-\pi^0$ decay is not uniquely defined. If we assume the validity of current algebra, then the $K \to 3\pi$ decay amplitudes in various dynamical models satisfy the relations (1.5) and/or (1.7) (except for negligible corrections) at the point $\omega_1=\omega_2=\omega_3=\frac{1}{3}M_K$. At $\omega_1=\omega_2=\omega_3=\frac{1}{3}M_K$, the magnitudes of the amplitudes are

$$A_c(00+) = (0.98\pm0.02)\times10^{-6}$$

and

$$A_c(+-0) = (0.85\pm0.02)\times10^{-6}.$$

[11] In deriving the magnitudes of the amplitudes (1.3), we have made use of the method proposed by Devlin by assuming a linear spectrum (1.2) and slopes (1.4). T. J. Devlin, Phys. Rev. Letters **20**, 683 (1968).

[12] B. Aubert, in Proceedings of Topical Conference on Weak Interactions, Geneva, 1969 (unpublished), p. 205.

[13] D. Davison *et al.*, Phys. Rev. **180**, 1333 (1969) [$a(00+)$].

[14] R. C. Smith, Ph.D. thesis, University of Maryland, 1970 (unpublished) [$a(+-0)$]. See also, R. C. Smith, L. Wang, M. C. Whatley, G. T. Zorn, and J. Hornbostel, University of Maryland report (unpublished). There are experiments with $a(+-0) \approx -0.21$ (Ref. 12).

[15] In order to derive relations (1.6), we have to assume that the decay matrix elements are linear in s_3. The linear approximation may be justified for the real parts of the matrix elements. If the imaginary parts are proportional to the linear momenta in various channels, the imaginary parts of the $K^+ \to \pi^0\pi^0\pi^+$ and $K_L{}^0 \to \pi^+\pi^-\pi^0$ decay amplitudes are proportional to $(s_3-4M_\pi{}^2)^{1/2}$ and that of the $K^+ \to \pi^+\pi^+\pi^-$ decay amplitude is proportional to $(s_1-4M_\pi{}^2)^{1/2}+(s_2-4M_\pi{}^2)^{1/2}$. Therefore, the linear spectrum approximation for $K^+ \to \pi^0\pi^0\pi^+$ and $K_L{}^0 \to \pi^+\pi^-\pi^0$ decays and the relation $a(+-0) = a(00+)$ may hold even if the amplitudes have imaginary parts. On the other hand, the contribution of the imaginary part to the spectrum of the $K^+ \to \pi^+\pi^+\pi^-$ decay (as a function of s_3) has a cusp at $\omega_3 \approx M_\pi + \frac{1}{6}Q$. However, the imaginary parts of the $K \to 3\pi$ decay amplitudes may be negligible in the physical region.

[16] We neglect the small PC-violating nonleptonic interaction.

[17] See, for example, R. F. Dashen, S. C. Frautschi, and D. Sharp, Phys. Rev. Letters **13**, 777 (1964); Y. Hara, Progr. Theoret. Phys. (Kyoto) **37**, 710 (1967).

[18] For simplicity, let us consider the reaction $\pi+\pi \to \bar{K}+\kappa'$ (κ': $Y=1$, $I=\frac{3}{2}$). The isospin of the reaction is 1 or 2 and the isospin of the crossed reaction $\pi+K \to \pi+\kappa'$ is $\frac{1}{2}$ or $\frac{3}{2}$. It is impossible for the first reaction to be pure $I=1$ and for the second one to be pure $\frac{1}{2}$ simultaneously.

[19] K. Kawarabayashi and S. Kitakado, Phys. Rev. Letters **23**, 440 (1969).

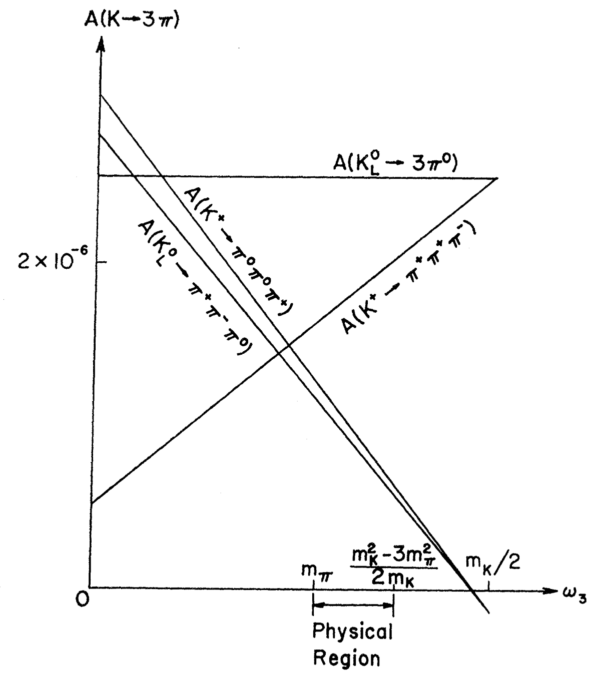

FIG. 1. $K \to 3\pi$ decay amplitudes in linear approximation. Amplitudes (2.5) with $A_2/A_0=0.044$.

parts of the amplitudes to have full duality when current-algebra relations[20–23] are imposed upon them.

In Secs. III and IV of this article we study whether the $|\Delta\mathbf{I}|=\frac{1}{2}$ parts of the $K\to 3\pi$ decay amplitudes are expressed by generalized Veneziano amplitudes for five-point functions,[2,3] and we shall find that kaon poles and pion poles are not dual to other poles just as the kaon pole is not dual to other poles in K_{l4} decays.[8] Since the $|\Delta\mathbf{I}|=\frac{3}{2}$ parts of $K\to 3\pi$ decay amplitudes are not expressed by completely dual amplitudes and since kaon and pion poles are not dual to other poles when current-algebra relations are imposed, in Sec. V we try to find an example of amplitudes which have both kaon and pion poles, which contain both a $|\Delta\mathbf{I}|=\frac{1}{2}$ part and a $|\Delta\mathbf{I}|=\frac{3}{2}$ part, and which reproduce the current-algebra relations.

Through current-algebra relations, the soft-pion limits of $K\to 3\pi$ decay amplitudes are related to $K\to 2\pi$ decay amplitudes.[20–23] However, these current-algebra relations are conditions for off-the-mass-shell $K\to 3\pi$ decay amplitudes and there is an ambiguity in extrapolating amplitudes from off the mass shell to on the mass shell. For simplicity, the $K\to 3\pi$ decay amplitudes were assumed to be bilinear functions of the four-momenta of the mesons,[20] $q(K)$ and $q(\pi_i)$. However, if we consider both the $|\Delta\mathbf{I}|=\frac{1}{2}$ amplitude and the $|\Delta\mathbf{I}|=\frac{3}{2}$ amplitude, this approximation is not justified. The phase difference between these amplitudes is not zero, and the root-type singularities of the imaginary parts of the amplitudes are not well approximated by this approximation.[15] Therefore, the purpose of Secs. III–V of this paper is to find a nontrivial example of $K\to 3\pi$ decay amplitudes which are consistent with current-algebra relations and to compare these amplitudes with experimental results (Sec. V).

Though very unlikely, it is still possible that the weak nonleptonic Hamiltonian has only a $|\Delta\mathbf{I}|=\frac{1}{2}$ part and that the $|\Delta\mathbf{I}|=\frac{3}{2}$ parts of the decay amplitudes are of electromagnetic origin.[24] We discuss this possibility in Sec. VI. An example of the $K\to 2\pi$ decay amplitude in the Veneziano model is given in the Appendix. In Sec. II we discuss the current-algebra relations and the linear approximation of $K\to 3\pi$ decay amplitudes.[25]

II. CURRENT-ALGEBRA RELATIONS AND LINEAR APPROXIMATION OF $K\to 3\pi$ DECAY AMPLITUDES

By making use of the partially conserved axial-vector current (PCAC) condition,

$$2\partial_\mu A_\mu{}^{(i)}=\sqrt{2}f_\pi M_\pi{}^2\phi^{(i)} =-(2M_N M_\pi{}^2 G_A/G_V g_{\pi N})\phi^{(i)}\,, \quad (2.1)$$

we obtain the following relation for a $K\to 3\pi$ decay amplitude in the soft-pion limit:

$$\lim_{q(\pi^a)\to 0}\,[2q_0(\pi^a)]^{1/2}\langle\pi^a\pi^b\pi^c|H_w(0)|K\rangle =-\frac{\sqrt{2}}{f_\pi}\langle\pi^b\pi^c|\left[\int d^3x\, A_0{}^{(a)}(\mathbf{x},0),H_w(0)\right]|K\rangle \quad (2.2)$$

for $q^2(K)=-M_K{}^2$, $q^2(\pi^b)=q^2(\pi^c)=-M_\pi{}^2$. If we assume the current×current weak interaction (1.8) and current algebra,

$$[A_0{}^{(i)}(\mathbf{x},t),J_\mu{}^{(j)}(\mathbf{x}',t)]=if_{ijk}J_\mu{}^{(k)}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{x}') +(c\text{-number Schwinger term})\,, \quad (2.3)$$

from (2.2) we find the following relations[20–23]:

$$\begin{aligned}
&A(K^+\to\pi^+\pi^+\pi^-;\,q(\pi_1{}^+)=0)=-A_0+\sqrt{2}A_2\,,\\
&A(K^+\to\pi^+\pi^+\pi^-;\,q(\pi^-)=0)=-3\sqrt{2}A_2\,,\\
&A(K^+\to\pi^0\pi^0\pi^+;\,q(\pi_1{}^0)=0)\\
&\quad=+(3/2\sqrt{2})A_2=(1/\sqrt{2}f_\pi)A(K^+\to\pi^+\pi^0)\,,\\
&A(K^+\to\pi^0\pi^0\pi^+;\,q(\pi^+)=0)=-A_0-2\sqrt{2}A_2\,,\\
&A(K_L{}^0\to\pi^+\pi^-\pi^0;\,q(\pi^\pm)=0)=-(3/2\sqrt{2})A_2\,,\\
&A(K_L{}^0\to\pi^+\pi^-\pi^0;\,q(\pi^0)=0)\\
&\quad=A_0+(1/\sqrt{2})A_2=(1/\sqrt{2}f_\pi)A(K_S{}^0\to\pi^+\pi^-)\,,\\
&A(K_L{}^0\to\pi^0\pi^0\pi^0;\,q(\pi_1{}^0)=0)\\
&\quad=A_0-\sqrt{2}A_2=(1/\sqrt{2}f_\pi)A(K_S{}^0\to\pi^0\pi^0)\,,\\
&A(K_S{}^0\to\pi^+\pi^-\pi^0;\,q(\pi^\pm)=0)=-(9/2\sqrt{2})A_2\,,
\end{aligned}\quad (2.4)$$

[20] Y. Hara and Y. Nambu, Phys. Rev. Letters **16**, 875 (1966).
[21] D. K. Elias and J. C. Taylor, Nuovo Cimento **44A**, 518 (1966).
[22] C. Bouchiat and P. Meyer, Phys. Letters **25B**, 282 (1967).
[23] A. D. Dolgov and V. I. Zacharov, Yadern. Fiz. **7**, 352 (1968) [Soviet J. Nucl. Phys. 7, 232 (1968)].
[24] See, for example, Y. Hara, Progr. Theoret. Phys. (Kyoto) **37**, 470 (1967).
[25] Much of the content of Sec. II is already discussed in Ref. 22 and especially in Ref. 23.

and

$$A(K_S{}^0 \to \pi^+\pi^-\pi^0;\, q(\pi^0)=0)=0\,.$$

There are electromagnetic corrections to these relations. However, since we have chosen the Hamiltonian (1.8), we assume that these corrections are much smaller than the $|\Delta \mathbf{I}|=\frac{3}{2}$ amplitude A_2.

These relations are those for off-mass-shell $K\to 3\pi$ decay amplitudes, and we have to extrapolate $K\to 3\pi$ decay amplitudes from on the mass shell to off the mass shell. For simplicity the amplitudes were assumed to be linear functions of the energy of asymmetric pions,[20,22,23] $\omega(\pi_3)$ (Fig. 1). Therefore, in this approximation we find

$$\begin{aligned}
A(K^+\to\pi^+\pi^+\pi^-) &= -3\sqrt{2}A_2 \\
&\quad -2(A_0-4\sqrt{2}A_2)\omega(\pi^-)/M_{K^+}\,, \\
A(K^+\to\pi^0\pi^0\pi^+) &= -A_0-2\sqrt{2}A_2 \\
&\quad +2\left(A_0+\frac{11}{2\sqrt{2}}A_2\right)\frac{\omega(\pi^+)}{M_{K^+}}\,, \\
A(K_L{}^0\to\pi^+\pi^-\pi^0) &= A_0+\frac{1}{\sqrt{2}}A_2 \\
&\quad -2\left(A_0+\frac{5}{2\sqrt{2}}A_2\right)\frac{\omega(\pi^0)}{M_{K^0}}\,, \\
A(K_L{}^0\to\pi^0\pi^0\pi^0) &= A_0-\sqrt{2}A_2\,,
\end{aligned} \tag{2.5}$$

and

$$A(K_S{}^0\to\pi^+\pi^-\pi^0) = -\frac{9}{\sqrt{2}}A_2\frac{\omega(\pi^0)}{M_{K^0}}\,.$$

From these amplitudes, we obtain the following predictions for $K\to 3\pi$ decays[10,26]:

$$\begin{aligned}
\tfrac{1}{2}|A_c(++-)| &= |A_c(00+)| = |-\tfrac{1}{3}A_0-(1/3\sqrt{2})A_2| \\
&= (1/3\sqrt{2}f_\pi)\,|A(K_S{}^0\to\pi^+\pi^-)| \\
&\approx 0.76\times 10^{-6} \\
&\quad (\text{left-hand side}=0.97\pm 0.03)\,,
\end{aligned} \tag{2.6a}$$

$$\begin{aligned}
\tfrac{1}{3}|A_c(000)| &= |A_c(+-0)| = \tfrac{1}{3}|A_0-\sqrt{2}A_2| \\
&= (1/3\sqrt{2}f_\pi)\,|A(K_S{}^0\to\pi^0\pi^0)| \\
&\approx 0.71\times 10^{-6} \\
&\quad (\text{left-hand side}=0.85\pm 0.02)\,,
\end{aligned} \tag{2.6b}$$

$$\begin{aligned}
&a(\pm\pm\mp)=0.09 \quad [a(\pm\pm\mp)_{\text{exp}}=0.1000\pm 0.0036]\,, \\
&a(0\,0+) \\
&\quad =-0.27 \quad [a(0\,0+)_{\text{exp}}=-0.258\pm 0.010]\,,
\end{aligned} \tag{2.7}$$

and

$$a(+-0)=-0.27 \quad [a(+-0)_{\text{exp}}=-0.300\pm 0.018]\,,$$

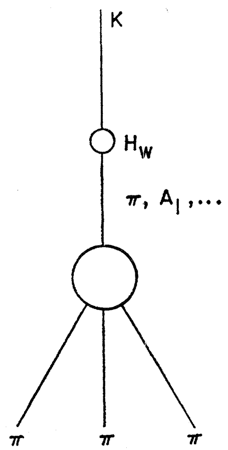

FIG. 2. Pion A_1, . . . , pole diagram.

by making use of the results of $K\to 2\pi$ decay experiments.[27,28]

The agreement between our predictions (2.6) and (2.7) and the experimental results (1.3) and (1.4) is excellent except for the decay rates. However, if we check the predicted relative rate,

$$\frac{\tfrac{1}{2}|A_c(++-)|\approx|A_c(00+)|}{|A_c(+-0)|\approx\tfrac{1}{3}|A_c(000)|} = \frac{|A(K_S{}^0\to\pi^+\pi^-)|}{|A(K_S{}^0\to\pi^0\pi^0)|}\,, \tag{2.8}$$

by experiments, the agreement is satisfactory:

$$\begin{aligned}
&\text{left-hand side}=(1.14\pm 0.05)\times 10^{-6} \\
\text{and}& \\
&\text{right-hand side}=(1.07\pm 0.01)\times 10^{-6}\,.
\end{aligned} \tag{2.9}$$

The discrepancy between experimental absolute rates and predicted absolute rates may be due to off-shell effects on the over-all factor of the amplitudes.

In spite of the above excellent agreements between predictions and experimental results, we cannot justify the use of the linear approximation of the amplitudes (2.5) since there is evidence[27] that the relative phase of A_0 and A_2 is about 40°. The imaginary parts of the decay amplitudes have root-type singularities at $s_i=[M(\pi_j)+M(\pi_k)]^2$ and they are not approximated well by the linear approximation.[15] Therefore, we have to approximate the $K\to 3\pi$ decay amplitudes by realistic functions with dynamical singularities.[29] In Sec. III we deal with the leading generalized Veneziano amplitude.[30]

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[26] We have used $\sqrt{2}M_N g_A/g_{\pi N}$ instead of f_π.

[27] B. Gobbi, D. Green, W. Hakel, R. Moffett, and J. Rosen, Phys. Rev. Letters **22**, 682 (1969).

[28] Particle Data Group, Rev. Mod. Phys. **41**, 109 (1969).

[29] However, the effect of dynamical singularities may be small in the physical region of $K\to 3\pi$ decays since the positions of ρ, K^*, etc. are rather remote.

[30] The content of Secs. III and IV has been reported in Yasuo Hara, International Centre for Theoretical Physics Report No. IC/69/65 (unpublished).

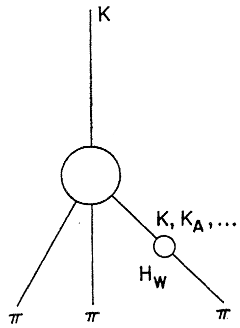

FIG. 3. Kaon, K_A, ..., pole diagram.

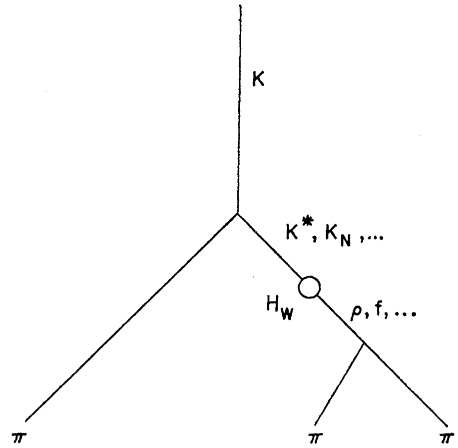

FIG. 4. ρ, K^*, ..., pole diagram.

III. LEADING GENERALIZED VENEZIANO AMPLITUDE

Let us assume the $|\Delta \mathbf{I}|=\frac{1}{2}$ rule, since the $|\Delta \mathbf{I}|=\frac{3}{2}$ parts of the decay amplitudes cannot have full duality.[18] The $K^m\rightarrow \pi^i+\pi^j+\pi^k$ decay amplitudes with full duality are obtained by making use of the generalized Veneziano model of the five-point amplitudes[2,3] $[K^m, \pi^i, \pi^j, \pi^k$, and the spurion $\kappa(I=\frac{1}{2})]$. The five-point amplitude in the generalized Veneziano model is not unambiguous, since trajectories with $\alpha(0)>0$ ($\alpha_\rho=\alpha_f$ and $\alpha_{K^*}=\alpha_{K_N}$) are involved. The leading amplitude, which has pion and kaon poles with Veneziano $\pi\pi$ and $K\pi$ scattering amplitudes,[31,32] which has all other poles shown in Figs. 2–4, and which satisfies a condition imposed by current algebra (the amplitude should vanish at some soft-pion limits), is unique and is given by

$$\begin{aligned}
&\beta[\mathrm{Tr}(M_{\bar{K}}M_6M_iM_jM_k)+\mathrm{Tr}(M_{\bar{K}}M_kM_jM_iM_6)]\Big\{[1-\alpha(s_{ij})-\alpha(s_{jk})]\int_0^1 du_1\int_0^1 du_4(1-u_1u_4)^{-1}u_1^{-\alpha(s_{jk})}\\
&\quad\times\left(\frac{1-u_1}{1-u_1u_4}\right)^{-\alpha_{K^*}(s_{km})}\left(\frac{1-u_4}{1-u_1u_4}\right)^{-\alpha_\pi(k)-1}u_4^{-\alpha_K(p_i)}(1-u_1u_4)^{-\alpha(s_{ij})}+[1+\alpha_K(p_i)-\alpha_{K^*}(s_{km})-\alpha(s_{jk})]\\
&\quad\times\int_0^1 du_1\int_0^1 du_4(1-u_1u_4)^{-1}u_1^{-\alpha(s_{jk})}\left(\frac{1-u_1}{1-u_1u_4}\right)^{-\alpha(s_{ij})}\left(\frac{1-u_4}{1-u_1u_4}\right)^{-\alpha_K(p_i)-1}u_4^{-\alpha_\pi(k)}(1-u_1u_4)^{-\alpha_{K^*}(s_{km})}\Big\}\\
&\qquad\qquad+(\text{terms obtained by permutations of } i,\ j,\text{ and } k)\,, \qquad (3.1)
\end{aligned}$$

where $s_{ij}=-[q(\pi^i)+q(\pi^j)]^2$, $s_{km}=-[q(\pi^k)-q(K^m)]^2$, $k=-[q(K^m)+q(\kappa)]^2$, and $p_i=-[q(\kappa)-q(\pi^i)]^2$ [here $q(K^m)+q(\kappa)=q(\pi^i)+q(\pi^j)+q(\pi^k)$ and $q^2(\kappa)=0$] (see Fig. 5). In Eq. (3.1) the M's are 3×3 matrices,[33]

$$M_i=\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\tau_i} & 0\\ & & 0\\ \hline 0 & 0 & 0\end{array}\right) \quad \text{for} \quad i=1,\ 2,\text{ and } 3\,,$$

$$M_{K_L^0}=M_6=\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0\\ & & 1\\ \hline 0 & 1 & 0\end{array}\right)\,, \qquad (3.2)$$

and

$$M_{K^-}=\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \sqrt{2}\\ & & 0\\ \hline 0 & 0 & 0\end{array}\right).$$

We assume that the π-A_1 trajectory α_π, the K-K_A trajectory α_K, the ρ-f trajectory α, and the K^*-K_N trajectory α_{K^*} are all parallel and that they satisfy the relations[34]

$$\alpha(t)-\alpha_\pi(t)=\alpha_{K^*}(t)-\alpha_K(t)=\tfrac{1}{2}\,. \qquad (3.3)$$

By making use of the relations (3.3), the amplitude (3.1) can be expressed as

$$\begin{aligned}
&\beta[\mathrm{Tr}(M_{\bar{K}}M_6M_iM_jM_k)+\mathrm{Tr}(M_{\bar{K}}M_kM_jM_iM_6)]\\
&\quad\times B(-\alpha_\pi(M_K^2),\,1-\alpha_K(M_\pi^2))[1-\alpha(s_{ij})-\alpha(s_{jk})]\\
&\quad\times[B(1-\alpha_{K^*}(s_{km}),1-\alpha(s_{jk}))-B(1-\alpha(s_{ij}),1-\alpha(s_{jk}))]\\
&\quad+(\text{terms obtained by permutations of } i,\ j,\text{ and } k)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.4)
\end{aligned}$$

for real decays $[q(\kappa)=0]$. Because of the relations (3.3),

[31] C. Lovelace, Phys. Letters **28B**, 265 (1968).
[32] K. Kawarabayashi, S. Kitakado, and H. Yabuki, Phys. Letters **28**, 432 (1969).
[33] J. E. Paton and Chan Hong Mo, Nucl. Phys. **B10**, 516 (1969).
[34] The relation (3.3) is derived from the relations (1.1) and (A) of Ref. (9) if trajectories are assumed to be linear and parallel.

poles at $\alpha(s_{ij})$ = positive integers (α_{K^*} = positive integers) disappear from the first (second) term in the last bracket of the amplitude (3.1). It is interesting to notice that the first (second) integral of (3.1) is proportional to the $K\pi$ ($\pi\pi$) scattering amplitude though the first (second) integral has no kaon (pion) pole.

There is another condition imposed by current algebra at some soft-pion limits. That is, the $K\rightarrow 3\pi$ decay amplitudes should be proportional to the $K\rightarrow 2\pi$ decay amplitudes at these limits. For example,

$$\sqrt{2}f_\pi A(K_L{}^0\rightarrow\pi^+ +\pi^- +\pi^0;\ q(\pi^0)=0) = A(K_S{}^0\rightarrow\pi^+ +\pi^-)\,. \quad (3.5)$$

Our amplitude (3.1) does not satisfy the requirement (3.5). We may regard the left-hand side of the relation (3.5) as the $K\rightarrow 2\pi$ decay amplitude if it has no poles at α_K, α_π=integers. In the above limit, our $K\rightarrow 3\pi$ decay amplitude (3.1) has no pion pole and no kaon poles since the $\pi\pi$ and $K\pi$ scattering amplitudes associated with the pion and kaon poles satisfy the Adler condition. However, this amplitude has undesirable A_1, K_A, etc. poles. This is because their residues do not satisfy the Adler condition.

Therefore, we cannot accept the leading generalized Veneziano amplitude (3.1) as a $K\rightarrow 3\pi$ decay amplitude.

IV. NONLEADING GENERALIZED VENEZIANO AMPLITUDES

Since the leading generalized Veneziano amplitude of $K\rightarrow 3\pi$ decays (3.1) has been found not to satisfy current-algebra relations, we look for nonleading generalized Veneziano amplitudes[30] of $K\rightarrow 3\pi$ decays which satisfy conditions required by current algebra. This is easily done if we notice the relations

$$a[\alpha_K(p_i)-\alpha_{K^*}(s_{km})+\tfrac{1}{2}] + b[1-\lambda\alpha(s_{ij})-(2-\lambda)\alpha(s_{jk})]=0 \quad \text{for}\quad q(\pi^j)=0 \quad (4.1)$$

and

$$c[\alpha(s_{ij})-\alpha_\pi(k)-\tfrac{1}{2}] + d[1-\mu\alpha(s_{jk})-(2-\mu)\alpha_{K^*}(s_{km})]=0 \quad \text{for}\quad q(\pi^k)=0 \quad (4.2)$$

and the fact that[34]

$$\alpha(s_{ij})+\alpha(s_{jk})=n+1 \text{ (integer)} \quad (4.3)$$

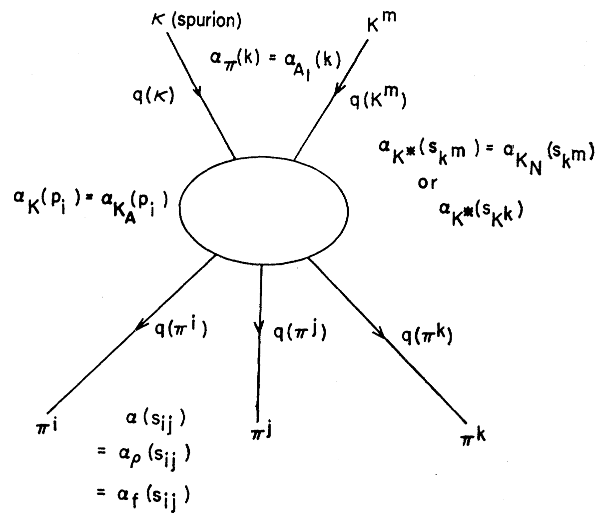

FIG. 5. Lorentz scalars and Regge trajectories.

for $\alpha_\pi(k)=n$ and $q(\pi^i)$ or $q(\pi^k)=0$, where a, b, c, d, λ, and μ are arbitrary parameters.

One example is

$$\begin{aligned}&\beta[\mathrm{Tr}(M_{\bar K}M_6M_iM_jM_k)+\mathrm{Tr}(M_{\bar K}M_kM_jM_iM_6)]\\ &\times[1-2\alpha(s_{jk})]\int_0^1 du_1\int_0^1 du_4(1-u_1u_4)^{-1}u_1{}^{-\alpha_\pi(k)}\\ &\left(\frac{1-u_1}{1-u_1u_4}\right)^{n-\alpha_K(p_i)-1}\left(\frac{1-u_4}{1-u_1u_4}\right)^{-\alpha(s_{ij})}u_4{}^{-\alpha(s_{jk})}\\ &\times(1-u_1u_4)^{-\alpha_{K^*}(s_{km})}+(\text{terms obtained by}\\ &\qquad\text{permutations of } i,\ j, \text{ and } k)\,, \quad (4.4)\end{aligned}$$

where n is a positive integer. This amplitude does not have a pion pole, A_1 pole, etc., because $-\alpha_\pi$ appears in the integrand instead of $-\alpha_\pi-1$. Only their daughter poles appear. If $n=0$, the above amplitude reduces to the Lovelace amplitude.[31] However, the amplitude with $n=0$ is not acceptable since it has a kaon pole though $\langle K|H_w|\pi\rangle=0$. The amplitude with $n=1$ is acceptable.[35]

Another example is

$$\begin{aligned}&[\mathrm{Tr}(M_{\bar K}M_6M_iM_jM_k)+\mathrm{Tr}(M_{\bar K}M_kM_jM_iM_6)]\{a[\alpha_K(p_i)-\alpha_{K^*}(s_{km})+\tfrac{1}{2}]+b[1-\lambda\alpha(s_{ij})-(2-\lambda)\alpha(s_{jk})]\}\\ &\times\{c[\alpha(s_{ij})-\alpha_\pi(k)-\tfrac{1}{2}]+d[1-\mu\alpha(s_{jk})-(2-\mu)\alpha(s_{km})]\}\int_0^1 du_1\int_0^1 du_4(1-u_1u_4)^{-1}u_1{}^{2-\alpha_\pi(k)}\left(\frac{1-u_1}{1-u_1u_4}\right)^{1-\alpha_K(p_i)}\\ &\times\left(\frac{1-u_4}{1-u_1u_4}\right)^{1-\alpha(s_{ij})}u_4{}^{1-\alpha(s_{jk})}(1-u_1u_4)^{1-\alpha_{K^*}(s_{km})}+(\text{terms obtained by permutations of } i,\ j, \text{ and } k)\,, \quad (4.5)\end{aligned}$$

[35] The amplitude (4.4) with $n=1$ gives the following results:

$$\tfrac{1}{2}|A_c(++-)|=|A_c(00+)|=|A_c(+-0)|=\tfrac{1}{3}|A_c(000)|\approx 0.7\times10^{-6}$$

and

$$2a(\pm\pm\mp)=-a(00+)=-a(+-0)=0.29.$$

where a, b, c, d, λ, and μ are arbitrary parameters.[36] To this amplitude the pion trajectory does not contribute. Only daughter trajectories of α_π contribute to this amplitude.

We have found examples of nonleading generalized Veneziano amplitudes for $K\to 3\pi$ decays. However, it is impossible to obtain examples with poles due to the mesons on the pion Regge trajectory. It is not probable that matrix elements of the weak nonleptonic Hamiltonian between the kaon and all mesons on α_π are all zero. Hence, we conclude that amplitudes for weak processes do not have full (or maximal) duality. For example, the pion pole or kaon pole of a $K\to 3\pi$ decay amplitude is not dual to other poles. This conclusion is identical to the one we have reached by studying K_{l4} decays,[8] though no leptons and no fixed poles in the J plane are involved in $K\to 3\pi$ decays. However, we have to notice that the existence of pion and kaon poles in $K\to 3\pi$ decay amplitudes is not always necessary, though the existence of the kaon pole in K_{l4} decay amplitudes is required by the existence of K_{l2} decays.

V. PION- AND KAON-POLE-DOMINANCE MODEL OF $K\to 3\pi$ DECAYS

In previous sections we have proved that the leading term in the generalized Veneziano model of $K\to 3\pi$ decays is not compatible with the current-algebra relations (2.4) and found that nonleading generalized Veneziano amplitudes of decays compatible with current-algebra relations have no pion poles nor kaon poles. Therefore, pion poles and kaon poles of $K\to 3\pi$ decays are found not to be dual to other poles. If we consider both the $|\Delta\mathbf{I}|=\frac{1}{2}$ and $\frac{3}{2}$ parts of the $K\to 3\pi$ decay amplitudes, we have to introduce imaginary parts to decay amplitudes. Thus, we have to consider dynamical singularities of amplitudes.

In this section we find an example of a $K\to 3\pi$ decay amplitude which reproduces current-algebra relations, which has both $|\Delta\mathbf{I}|=\frac{1}{2}$ and $\frac{3}{2}$ parts, and which has a pion pole and kaon poles. The pion-pole-dominance model of $K\to 3\pi$ decays with Veneziano $\pi\pi$ scattering amplitude was first proposed by Lovelace.[31] However, his model has a difficulty. It does not have contributions from kaon poles. The $\pi\pi$ scattering amplitude in the Veneziano model ($\pi^\alpha+\pi^\beta\to\pi^\gamma+\pi^\delta$) is expressed as[31,32]

$$T(s,t,u)=\delta_{\alpha\beta}\delta_{\gamma\delta}A(s,t,u)+\delta_{\alpha\gamma}\delta_{\beta\delta}A(t,u,s)+\delta_{\alpha\delta}\delta_{\beta\gamma}A(u,s,t)\,,\quad(5.1)$$

where

$$A(s,t,u)=f^2[V(s,t)+V(s,u)-V(t,u)]\quad(5.2)$$

and

$$V(s,t)=\alpha'(2M_\pi^2-s-t)\mathfrak{B}(1-\alpha_\rho(s),1-\alpha_\rho(t))\,.\quad(5.3)$$

[36] By adjusting parameters, various slopes of the pion spectrum can be obtained from the amplitude (4.5). However, it seems that we should prefer (4.4) to (4.5) since the amplitude (4.5) is very artificial.

The $K\pi$ scattering amplitude in the Veneziano model is expressed as[32]

$$T(s,t,u)=\delta_{\alpha\beta}A^{(+)}(s,t,u)+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]A^{(-)}(s,t,u)\,,\quad(5.4)$$

where

$$A^{(\pm)}(s,t,u)=\tfrac{1}{2}f^2[V^*(s,t)\pm V^*(u,t)]\quad(5.5)$$

and

$$V^*(s,t)=\alpha'(M_\pi^2+M_K^2-s-t)\times\mathfrak{B}(1-\alpha_{K^*}(s),1-\alpha_\rho(t))\,.\quad(5.6)$$

The f is the $\rho\pi\pi$ coupling constant, $f^2/4\pi=2.10\pm0.11$, and α' is the slope of the ρ trajectory[9,31] $d\alpha_\rho(t)/dt$ at $t=M_\rho^2$,

$$\alpha'\approx[2(M_\rho^2-M_\pi^2)]^{-1}.\quad(5.7)$$

The function $\mathfrak{B}(1-\alpha(s),1-\alpha(t))$ is an example of a crossing-symmetric Regge-behaved amplitude for nonlinear trajectories discovered by Suzuki.[37] An explicit form of $\mathfrak{B}$ is given by

$$\mathfrak{B}(1-\alpha_1(s),1-\alpha_2(t))=\int_0^1 dz\,z^{-\alpha_1(s)+\Delta\alpha_1(s)f(z)}(1-z)^{-\alpha_2(t)+\Delta\alpha_2(t)f(1-z)}\,,\quad(5.8)$$

where

$$\alpha_i(s)=a_is+b_i+\frac{s}{\pi}\int ds'\frac{\operatorname{Im}\alpha_i(s')}{s'(s'-s)}\equiv a_is+b_i+\Delta\alpha_i(s)\quad(5.9)$$

and $f(z)$ is a function with the following properties:

$$f(0)=0\quad\text{and}\quad f(1)=1\,,$$
$$\frac{d^Mf(z)}{dz^M}=0\quad\text{at}\quad z=0\quad\text{and}\quad z=1\quad(5.10)$$

for an arbitrary integer M.

The contribution of the pion pole and kaon poles to $K^+\to\pi_1^++\pi_2^++\pi^-$ decay is expressed as

$$f^2\langle K^+|H_w|\pi^+\rangle\left[\frac{2}{k-M_\pi^2}V(s_{1-},s_{2-})+\frac{1}{p_1-M_K^2}V^*(s_{K2},s_{2-})+\frac{1}{p_2-M_K^2}V^*(s_{K1},s_{-1})\right],\quad(5.11)$$

where

$$k=-[q(K)+q(\kappa)]^2,\quad p_i=-[q(\pi_i^+)-q(\kappa)]^2,$$
$$s_{i-}=-[q(\pi_i^+)+q(\pi^-)]^2,\quad s_{Ki}=-[q(K)-q(\pi_i)]^2$$

(see Fig. 5). The $q(\kappa)$ is the four-momentum of the spurion κ and satisfies $q(K)+q(\kappa)=q(\pi_1^+)+q(\pi_2^+)+q(\pi^-)$. If the $|\Delta\mathbf{I}|=\frac{1}{2}$ rule is satisfied, the following relation must be satisfied:

$$\langle K^+|H_w|\pi^+\rangle=-\langle K_L^0|H_w|\pi^0\rangle\,.\quad(5.12)$$

[37] M. Suzuki, Phys. Rev. Letters **23**, 205 (1969).

We regard the expectation value $\langle K|H_w|\pi\rangle$ as a constant evaluated at $[q(K)+q(\kappa)]^2=-M_\pi^2$, $q^2(\kappa)=0$, and $[q(\pi)-q(\kappa)]^2=-M_K^2$.

The amplitude (5.11) of the pion- and kaon-pole-dominance model does not have a zero at the soft-pion limit $q(\pi^-)=0$, required by current algebra when the $|\Delta \mathbf{I}|=\frac{1}{2}$ rule is valid.[38–40] This is because

$$V^*(M_\pi^2,M_\pi^2)\neq 0.$$

Thus, in order to avoid this difficulty, we have to consider contributions from poles due to higher meson resonances and subtraction terms. Let us assume that the pole residues of these poles are of Veneziano type and let us neglect the k and p_i dependence of these terms. An example of a sum of a pion pole, kaon poles, and extra terms due to higher meson resonances belonging to α_π and α_K families is given in the following[41]:

$$A(K^+\to\pi_1^+ +\pi_2^+ +\pi^-)$$
$$=f^2H_+\left[\frac{2}{k-M_\pi^2}V(s_{1-},s_{2-})+\frac{1}{p_1-M_K^2}V^*(s_{K2},s_{2-})+\frac{1}{p_2-M_K^2}V^*(s_{K1},s_{1-})\right]-\alpha' f^2H_0[W^*(s_{K1},s_{1-})+W^*(s_{K2},s_{2-})]$$
$$+2(H_0+H_+)f^2\alpha' W(s_{1-},s_{2-})+2f^2\alpha'(H_0+H_+)[2\mathcal{B}(1-\alpha_{K^*}(M_K^2),1-\alpha_\rho(M_\pi^2))$$
$$-\mathcal{B}(1-\alpha_\rho(M_\pi^2),1-\alpha_\rho(M_\pi^2))](s_{12}-M_\pi^2)/(M_K^2-M_\pi^2),\quad(5.13\text{a})$$

$$A(K^+\to\pi_1^0+\pi_2^0+\pi^+)$$
$$=H_+\left[\frac{1}{k-M_\pi^2}A(s_{12},s_{1+},s_{2+})+\frac{1}{p_+-M_K^2}A^{(+)}(s_{K1},s_{12},s_{K2})\right]+H_0\left[\frac{1}{p_1-M_K^2}A^{(-)}(s_{K2},s_{2+},s_{K+})\right.$$
$$\left.+\frac{1}{p_2-M_K^2}A^{(-)}(s_{K1},s_{1+},s_{K+})\right]-\tfrac{1}{2}\alpha' f^2H_0[W^*(s_{K1},s_{12})+W^*(s_{K2},s_{12})]+\tfrac{1}{2}\alpha' f^2(2H_++H_0)[W^*(s_{K+},s_{2+})$$
$$-W^*(s_{K2},s_{2+})+W^*(s_{K+},s_{1+})-W^*(s_{K1},s_{1+})]+f^2\alpha'(H_0+H_+)\{[\mathcal{B}(1-\alpha_\rho(M_\pi^2),1-\alpha_\rho(M_\pi^2))$$
$$+2\mathcal{B}(1-\alpha_{K^*}(M_K^2),1-\alpha_\rho(M_\pi^2))](s_{12}-M_\pi^2)-\mathcal{B}(1-\alpha_\rho(M_\pi^2),1-\alpha_\rho(M_\pi^2))(s_{1+}+s_{2+}-2M_\pi^2)\}/(M_K^2-M_\pi^2)$$
$$+(H_0+H_+)f^2\alpha'[W(s_{12},s_{1+})+W(s_{12},s_{2+})-W(s_{1+},s_{2+})],\quad(5.13\text{b})$$

$$A(K_L^0\to\pi_1^0+\pi_2^0+\pi_3^0)$$
$$=H_0\left[\frac{f^2}{k-M_\pi^2}[V(s_{12},s_{23})+V(s_{23},s_{31})+V(s_{31},s_{12})]+\frac{1}{p_1-M_K^2}A^{(+)}(s_{K2},s_{23},s_{K3})+\frac{1}{p_2-M_K^2}A^{(+)}(s_{K3},s_{13},s_{K1})\right.$$
$$\left.+\frac{1}{p_3-M_K^2}A^{(+)}(s_{K1},s_{12},s_{K2})\right]+\tfrac{1}{2}\alpha' f^2H_0[W^*(s_{K1},s_{12})+W^*(s_{K1},s_{13})+W^*(s_{K2},s_{12})$$
$$+W^*(s_{K2},s_{23})+W^*(s_{K3},s_{13})+W^*(s_{K3},s_{23})],\quad(5.13\text{c})$$

$$A(K_L^0\to\pi^++\pi^-+\pi^0)$$
$$=H_0\left[\frac{1}{k-M_\pi^2}A(s_{+-},s_{0+},s_{0-})+\frac{1}{p_0-M_K^2}A^{(+)}(s_{K+},s_{+-},s_{K-})\right]+H_+\left[\frac{1}{p_+-M_K^2}A^{(-)}(s_{K-},s_{-0},s_{K0})\right.$$
$$\left.+\frac{1}{p_--M_K^2}A^{(-)}(s_{K+},s_{+0},s_{K0})\right]+\tfrac{1}{2}\alpha' f^2H_0[W^*(s_{K+},s_{+-})+W^*(s_{K-},s_{+-})]-\tfrac{1}{2}\alpha' f^2(2H_++H_0)$$
$$\times[W^*(s_{K0},s_{0-})+W^*(s_{K0},s_{+0})-W^*(s_{K-},s_{-0})-W^*(s_{K+},s_{+0})],\quad(5.13\text{d})$$

[38] M. Jacob, C. H. Llewellyn Smith, and S. Pokorski, Nuovo Cimento **63A**, 574 (1969).

[39] D. G. Sutherland, Nucl. Phys. **B13**, 45 (1969).

[40] This difficulty can be solved if we regard (5.11) as the contribution of the Feynman diagram drawn in Fig. 2 (pion pole only) and Fig. 3 (kaon pole only) and if we assume that $\lim_{q^2\to 0}\langle K(q)|H_w|\pi(q)\rangle=0$ as was suggested in Ref. 20. The current-algebra relations (2.4) with nonzero A_2 are not satisfied by this model. This model predicts

$$\tfrac{1}{2}|A_c(++-)|=|A_c(00+)|=0.74(1\pm 0.3\lambda),\quad \tfrac{1}{3}|A_c(000)|=|A_c(+-0)|=0.66(1+0.3\lambda),$$

and

$$2a(\pm\pm\mp)=-a(+-0)=-a(00+)=0.29-0.25\lambda,$$

where

$$\lambda=\langle K(q)|H_w|\pi(q)\rangle|_{q^2=-M_\pi^2}/\langle K(q)|H_w|\pi(q)\rangle|_{q^2=-M_K^2}.$$

[41] We have looked for an example of an amplitude which is a sum of pion pole, kaon pole, $W(s_i,s_j)$, and $W^*(s_i,s_j)$ (extra terms due to higher meson resonances) and terms linear in s_i. The amplitude (5.13) is not a unique amplitude which satisfies the above condition. We may add the amplitude (2.5) with arbitrary A_0 and A_2 to (5.13).

and

$$A(K_S{}^0\to\pi^0+\pi^++\pi^-)=H_0\frac{1}{p_0-M_K{}^2}A^{(-)}(s_{K-},s_{+-},s_{K+})$$

$$+H_+\left[\frac{1}{p_+-M_K{}^2}A^{(-)}(s_{K-},s_{-0},s_{K0})-\frac{1}{p_--M_K{}^2}A^{(-)}(s_{K+},s_{+0},s_{K0})\right]-\tfrac{1}{2}\alpha' f^2(4H_++3H_0)$$

$$\times[W^*(s_{K-},s_{+-})-W^*(s_{K+},s_{+-})]-\tfrac{1}{2}\alpha' f^2(2H_++H_0)[W^*(s_{K0},s_{-0})-W^*(s_{K-},s_{-0})+W^*(s_{K+},s_{+0})-W^*(s_{K0},s_{+0})]$$

$$+2\mathcal{B}(1-\alpha_{K^*}(M_K{}^2),\,1-\alpha_\rho(M_\pi{}^2))f^2\alpha'(H_0+H_+)(s_{0+}-s_{0-})/(M_\pi{}^2-M_K{}^2)\,,\quad(5.13\text{e})$$

where

$$H_0=\langle\pi^0|H_w|K_L{}^0\rangle,\quad H_+=\langle\pi^+|H_w|K^+\rangle,$$
$$W(s,t)=\mathcal{B}(1-\alpha(s),\,1-\alpha(t)),$$

and

$$W^*(s,t)=\mathcal{B}(1-\alpha_{K^*}(s),\,1-\alpha(t)).$$

The amplitude (5.13) is one of the simplest amplitudes which satisfy current-algebra relations (2.4), which have pion and kaon poles and have both $|\Delta\mathbf{I}|=\frac{1}{2}$ and $\frac{3}{2}$ parts.

By making use of the current-algebra relations (2.4), the expectation values of the weak nonleptonic interaction H_+ and H_0 are related to the $K\to3\pi$ decay amplitudes:

$$\begin{aligned}A(K_S{}^0\to2\pi^0)&=\sqrt{2}f_\pi A(K_L{}^0\to\pi_1{}^0+\pi_2{}^0+\pi_3{}^0;\,q(\pi_3{}^0)=0)\\&=-\sqrt{2}f_\pi\alpha' f^2H_0[2\mathcal{B}(1-\alpha_\rho(M_\pi{}^2),\,1-\alpha_\rho(M_K{}^2))\\&\quad-\mathcal{B}(1-\alpha_{K^*}(M_K{}^2),\,1-\alpha_\rho(M_\pi{}^2))\\&\quad-\mathcal{B}(1-\alpha_{K^*}(M_\pi{}^2),\,1-\alpha_\rho(M_K{}^2))],\end{aligned}\quad(5.14)$$

$$\begin{aligned}A(K^+\to\pi^+\pi^0)&=\sqrt{2}f_\pi A(K^+\to\pi_1{}^0+\pi_2{}^0+\pi^+;\,q(\pi_1{}^0)=0)\\&=\sqrt{2}f_\pi\alpha' f^2(H_0+H_+)[\tfrac{1}{2}\mathcal{B}(1-\alpha_{K^*}(M_\pi{}^2),\,1-\alpha_\rho(M_\pi{}^2))\\&\quad-\mathcal{B}(1-\alpha_{K^*}(M_K{}^2),\,1-\alpha_\rho(M_\pi{}^2))],\end{aligned}$$

and

$$A(K_S{}^0\to\pi^+\pi^-)-A(K_S{}^0\to\pi^0\pi^0)=2A(K^+\to\pi^+\pi^0).$$

According to recent experimental results,[27,28]

$$\frac{2|A(K_S{}^0\to\pi^+\pi^-)|^2}{|A(K_S{}^0\to\pi^0\pi^0)|^2}=2.285\pm0.055\,,$$
$$\frac{|A(K^+\to\pi^+\pi^-)|^2}{|A(K_S{}^0\to\pi^0\pi^0)|^2}=0.00212\,,\quad(5.15)$$

and

$$|A(K_S{}^0\to\pi^+\pi^-)|=396\times10^{-6}\,\text{MeV}.$$

The phase of $A(K^+\to\pi^+\pi^0)$ is assumed to be zero and the phases of $A(K_S{}^0\to\pi^+\pi^-)$ and $A(K_S{}^0\to\pi^0\pi^0)$ are assumed to be about 40°.

Therefore, the $K\to3\pi$ decay amplitudes are expressed in terms of $K\to2\pi$ decay amplitudes if the explicit form of α, α_{K^*}, and $f(z)$ (which appears in $\mathcal{B}$) are known. For simplicity let us assume that

$$f(z)\approx\tfrac{1}{2}\quad\text{for}\quad0<z<1.\quad(5.16)$$

Then, we are allowed to use $B(1-\alpha^*(s),1-\alpha^*(t))$ and $B(1-\alpha_{K^*}{}^*(s),1-\alpha^*(t))$ instead of $\mathcal{B}(1-\alpha(s),1-\alpha(t))$ and $\mathcal{B}(1-\alpha_{K^*}(s),\,1-\alpha(t))$ if we use $\alpha^*(s)=\alpha(s)-\frac{1}{2}\Delta\alpha(s)$ and $\alpha_{K^*}{}^*(s)=\alpha_{K^*}(s)-\frac{1}{2}\Delta\alpha_{K^*}(s)$. Since we have no detailed information on $\alpha(s)$ and $\alpha_{K^*}(s)$, we shall take the example used in Ref. 31:

$$\alpha^*(s)=\alpha(s)-\tfrac{1}{2}\Delta\alpha(s)=\tfrac{1}{2}+(s-M_\pi{}^2)/[2(M_\rho{}^2-M_\pi{}^2)]+0.14i(s-4M_\pi{}^2)^{1/2}\,\text{GeV}^{-1}$$

and $\quad(5.17)$

$$\alpha_{K^*}{}^*(s)=\tfrac{1}{2}+(s-M_K{}^2)/[2(M_{K^*}{}^2-M_K{}^2)]$$

for $M_\pi{}^2<s<M_K{}^2$. We obtain the following predictions[42] from the amplitudes (5.13):

$$\tfrac{1}{2}|A_c(+\,+\,-)|\approx|A_c(0\,0\,+)|=0.68\,,$$
$$\tfrac{1}{3}|A_c(0\,0\,0)|\approx|A_c(+\,-\,0)|=0.60\,,\quad(5.18)$$
$$a(\pm\,\pm\,\mp)=0.12\,,\quad a(0\,0\,+)=-0.31\,,$$

and

$$a(+\,-\,0)=-0.30.\quad(5.19)$$

The agreement between the above results (5.18) and (5.19) and the experimental results (1.3) and (1.4) is fair. However, as in Sec. II, the agreement between our

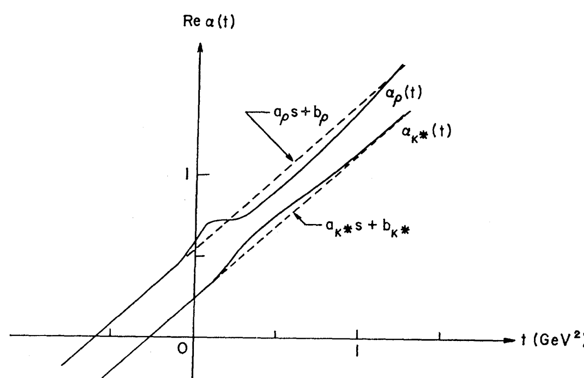

FIG. 6. Nonlinear trajectories for (5.20) and (5.21).

[42] In this case, $|H_0|=5.0\times10^{-5}$ MeV and $H_+=-0.947H_0$. It is interesting to notice that $|H_0|>|H_+|$, though $|A(K_S{}^0\to\pi^+\pi^-)|>|A(K_S{}^0\to\pi^0\pi^0)|$.

TABLE I. Comparison of experimental and theoretical results.

Experiments		Theories[a] Sec. II	Sec. V(a)	Sec. V(b)	Sec. VI
$A\begin{cases}\frac{1}{2}\lvert A_c(++-)\rvert\times 10^6 \\ \lvert A_c(00+)\rvert\times 10^6\end{cases}$	$\left.\begin{matrix}0.96\pm0.01\\0.98\pm0.02\end{matrix}\right\}$	0.76	0.68	0.93	?
$B\begin{cases}\lvert A_c(+-0)\rvert\times 10^6 \\ \frac{1}{3}\lvert A_c(000)\rvert\times 10^6\end{cases}$	$\left.\begin{matrix}0.85\pm0.02\\0.85\pm0.02\end{matrix}\right\}$	0.71	0.60	0.85	0.71
A/B	1.14 ± 0.05	1.07	1.13	1.12	?
$a(\pm\pm\mp)$	0.1000 ± 0.0036	0.09	0.12	0.10	?
$a(00+)$	$-0.258+0.010$	-0.27	-0.31	-0.26	?
$a(+-0)$	-0.300 ± 0.018	-0.27	-0.30	-0.27	-0.27
$-a(00+)/2a(\pm\pm\mp)$	1.29 ± 0.02	1.5	1.3	1.3	?
$-a(+-0)/2a(\pm\pm\mp)$	1.50 ± 0.03	1.5	1.3	1.4	?

[a] Section II: Eqs. (2.6) and (2.7); H_w with $|\Delta\mathbf{I}|=\frac{1}{2}$ and $\frac{3}{2}$; matrix element is linear in $\omega(\pi_3)$. Section V(a): (5.18) and (5.19); H_w with $|\Delta\mathbf{I}|=\frac{1}{2}$ and $\frac{3}{2}$; with pion and kaon poles; $\mathrm{Re}\,\alpha$ is linear. Section V(b): (5.20) and (5.21); H_w with $|\Delta\mathbf{I}|=\frac{1}{2}$ and $\frac{3}{2}$; with pion and kaon poles; $\mathrm{Re}\,\alpha$ is nonlinear. Section VI: (6.3) and (6.4); H_w with $|\Delta\mathbf{I}|=\frac{1}{2}$ only; matrix element is linear in $\omega(\pi_3)$.

predictions and experimental results becomes excellent if we compare only relative rates and relative magnitudes of slopes. The inclusion of dynamical singularities such as a ρ pole and a K^* pole has made the slopes a little steeper [compare (5.19) and (2.7)]. Numerically the contribution of the imaginary part of the amplitude to the decay spectrum is negligible (about 3%).

The discrepancy between the predicted absolute decay rates and the experimental absolute decay rates may be explained as off-mass-shell effects. However, this discrepancy is also removed by considering highly nonlinear trajectories. For example, if we assume that the real parts of the α and α_{K^*} trajectories are those given in Fig. 6 and $\frac{1}{2}\,\mathrm{Im}\alpha(s)=0.2(s-4M_\pi{}^2)^{1/2}$ GeV^{-1}, we find[43]

$$\begin{aligned}\tfrac{1}{2}|A_c(++-)&\approx|A_c(0\,0\,+)|=0.93,\\ |A_c(+-0)|&\approx\tfrac{1}{3}|A_c(0\,0\,0)|=0.85,\\ a(++-)=0.10,&\quad a(0\,0\,+)=-0.26,\end{aligned}\tag{5.20}$$

and

$$a(+-0)=-0.27.\tag{5.21}$$

The agreement between our results and the experimental results is excellent.

VI. DISCUSSION AND CONCLUSION

Though it is very unlikely, let us assume that the weak Hamiltonian has only a $|\Delta\mathbf{I}|=\frac{1}{2}$ part (such as a current$\times$current interaction with neutral currents or[20] $H_w\propto\bar{q}\lambda_6q+i\bar{q}\lambda_7\gamma_5q$) and that the $|\Delta\mathbf{I}|=\frac{3}{2}$ parts of the decay amplitudes are of electromagnetic origin. In this case we have to assume that there are electromagnetic corrections of $O(\frac{1}{10}A_0)$ to the current algebra relations (2.4) with $A_2=0$. However, the electromagnetic corrections to the relations

$$\begin{aligned}&A(K^+\to\pi^0\pi^0\pi^+;\,q(\pi_1{}^0)=0)\\&\qquad=(1/\sqrt{2}f_\pi)A(K^+\to\pi^+\pi^0),\\&A(K_L{}^0\to\pi^+\pi^-\pi^0;\,q(\pi^0)=0)\\&\qquad=(1/\sqrt{2}f_\pi)A(K_s{}^0\to\pi^+\pi^-),\end{aligned}\tag{6.1}$$

and

$$\begin{aligned}&A(K_L{}^0\to\pi^0\pi^0\pi^0;\,q(\pi_1{}^0)=0)\\&\qquad=(1/\sqrt{2}f)A(K_s{}^0\to\pi^0\pi^0)\end{aligned}$$

are $O(\alpha^2)$ and negligible if we assume the following PCAC condition in the presence of the electromagnetic interaction[44]:

$$\partial_\mu A_\mu{}^{(3)}=(1/\sqrt{2})f_\pi M_\pi{}^2\phi^{(3)}+S(\alpha/4\pi)F_{\xi\sigma}F_{\tau\rho}\epsilon_{\xi\sigma\tau\rho},\tag{6.2}$$

where S is a constant and α is the fine-structure constant.

If we assume that decay amplitudes are linear in $\omega(\pi_3)$ as was assumed in Sec. II, we find

$$a(+-0)=-0.27\tag{6.3}$$

and

$$\tfrac{1}{3}|A_c(0\,0\,0)|=|A_c(+-0)|=0.70$$

from (1.7) and (6.1). The slopes $a(\pm\pm\mp)$ and $a(0\,0\,+)$ can be arbitrary.

If we choose the H_w with pure $|\Delta\mathbf{I}|=\frac{1}{2}$, the trouble is the fact that we do not know how to obtain large electromagnetic corrections such as

$$[|A_c(++-)|/2|A_c(+-0)|]-1\approx0.14.$$

This trouble, together with the excellent agreement between the predictions of models based on the current $\times$current Hamiltonian (1.8) and experimental results, seems to suggest that we should prefer the current $\times$current Hamiltonian (1.8) to H_w with pure $|\Delta\mathbf{I}|=\frac{1}{2}$.

If the weak nonleptonic Hamiltonian is the current $\times$current interaction (1.8), the $K\to3\pi$ decay amplitude will be a sum of generalized Veneziano amplitudes such as (4.4) and amplitudes which contain pion and kaon poles such as (5.13).

For convenience, the results obtained in this article are tabulated in Table I.

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[43] In this case $|H_0|=2.7\times10^{-5}$ MeV and $H_+=-0.924H_0$.

[44] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

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APPENDIX: $K\to 2\pi$ DECAY AMPLITUDES IN THE VENEZIANO MODEL

An example of $K\to 2\pi$ decay amplitudes suggested by the Veneziano model is as follows:

$$A(K_s^0\to\pi_1^0\pi_2^0)=(\sqrt{2}/\pi f_\pi)\langle\pi^0|H_w|K_L^0\rangle\biggl(2\mathcal{B}(1-\alpha(M_\pi^2),1-\alpha(s_{12}))-\mathcal{B}(1-\alpha_K(M_K^2),1-\alpha(M_\pi^2))$$
$$+\frac{M_\pi^2-s_{12}-s_{K1}}{2M_K^2}\mathcal{B}(1-\alpha_{K^*}(s_{K1}),1-\alpha(s_{12}))+\frac{M_\pi^2-s_{12}-s_{K2}}{2M_K^2}\mathcal{B}(1-\alpha_{K^*}(s_{K2}),1-\alpha(s_{12}))\biggr),\quad \text{(A1)}$$

$$A(K_s^0\to\pi^+\pi^-)=(\sqrt{2}/\pi f_\pi)\langle\pi^0|H_w|K_L^0\rangle\biggl(2\mathcal{B}(1-\alpha(M_\pi^2),1-\alpha(s_{+-}))-\mathcal{B}(1-\alpha_{K^*}(M_K^2),1-\alpha(M_\pi^2))$$
$$+\frac{M_\pi^2-s_{+-}-s_{K+}}{2M_K^2}\mathcal{B}(1-\alpha_{K^*}(s_{K+}),1-\alpha(s_{+-}))+\frac{M_\pi^2-s_{+-}-s_{K-}}{2M_K^2}\mathcal{B}(1-\alpha_{K^*}(s_{K-}),1-\alpha(s_{+-}))\biggr)$$
$$+(1/\sqrt{2}\pi f_\pi)[\langle\pi^0|H_w|K_L^0\rangle+\langle\pi^+|H_w|K^+\rangle][\mathcal{B}(1-\alpha_{K^*}(s_{K+}),1-\alpha(M_\pi^2))+\mathcal{B}(1-\alpha_{K^*}(s_{K-}),1-\alpha(M_\pi^2))],\quad \text{(A2)}$$

and

$$A(K^+\to\pi^+\pi^0)=(1/\sqrt{2}\pi f_\pi)\langle\pi^+|H_w|K^+\rangle\mathcal{B}(1-\alpha(M_\pi^2),1-\alpha_{K^*}(s_{K0}))$$
$$+(1/\sqrt{2}\pi f_\pi)\langle\pi^0|H_w|K_L^0\rangle\biggl(\mathcal{B}(1-\alpha(s_{+0}),1-\alpha_{K^*}(s_{K+}))\frac{s_{K+}+s_{0+}-M_\pi^2}{M_K^2}$$
$$+\mathcal{B}(1-\alpha(s_{+0}),1-\alpha_{K^*}(s_{K0}))\frac{M_\pi^2-s_{K0}-s_{0+}}{M_K^2}+\mathcal{B}(1-\alpha(M_\pi^2),1-\alpha_{K^*}(s_{K0}))\biggr),\quad \text{(A3)}$$

where $s_{ij}=-[q(\pi^i)+q(\pi^j)]^2=-[q(K)+q(\kappa)]^2$ and $s_{Ki}=-[q(K)-q(\pi^i)]^2$. These $K\to 2\pi$ decay amplitudes satisfy current-algebra relations:

$$\sqrt{2}f_\pi A(K_S^0\to\pi_1^0\pi_2^0;\ q(\pi_1^0)=0)=\langle\pi^0|H_w|K_L^0\rangle$$

and

$$\sqrt{2}f_\pi A(K^+\to\pi^+\pi^0;\ q(\pi^0)=0)=\langle\pi^+|H_w|K^+\rangle,\quad \text{(A4)}$$

when the α and α_{K^*} trajectories are real and are given by the real parts of the trajetcories in (5.17).