K-Matrix Model for Pomeranchuk Exchange*

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Starting from the assumption that the inelastic states in the unitarity relation can effectively be represented by a set of quasi-two-particle states, a K-matrix formalism is set up for high-energy elastic scattering and diffraction-dissociation processes. Using arguments similar to those of Freund, it is shown that the Pomeranchuk contribution to elastic scattering and diffraction dissociation can be generated by multiple exchange of an exchange-degenerate quantum number carrying Regge trajectory R, by considering at the same time a formation of a sequence of excited intermediate states of the colliding particles between the individual R exchanges. This unitarization procedure leads to an imaginary as well as a real contribution for vacuum exchange, corresponding basically to sums of double and triple R-exchange contributions, respectively. At the same time, the K-matrix formalism produces an absorptive correction to the input Born terms. The consequences of the proposed model are worked out, particularly as regards the asymptotic behavior of total cross sections and the interpretation of the crossover phenomenon.

I. INTRODUCTION

It is well known that Regge theory provides a reasonably good description of inelastic processes at high energies. Elastic scattering, however, is still less well understood. This is related to the fact that the true nature of the Pomeranchuk trajectory is unknown. Originally introduced as an ordinary Regge pole carrying the quantum numbers of the vacuum and possessing the largest intercept allowed by unitarity, it soon became clear that this trajectory had very special properties: (i) Its slope turned out to be smaller than that of quantum-number-carrying trajectories which have a slope of order 1 GeV^{-2}, and (ii) there seem to be no particles related to the Pomeranchuk trajectory. Furthermore, the following well-known conceptual difficulty appears: Iterating a Pomeranchuk pole with intercept one in an elastic scattering amplitude produces cuts in the angular-momentum plane which accumulate at $j=1$ for vanishing momentum transfer t and dominate each other for increasing order of iteration at negative t. This seems to indicate that the full Pomeranchuk contribution is basically a more complicated object. As an ansatz to a more refined theory for elastic scattering, various phenomenological multiple-scattering models have been discussed, describing the Pomeranchuk contribution effectively as a superposition of Regge cuts. It has been found useful, in order to incorporate this multiple-scattering aspect into the theory, to treat elastic scattering in the Glauber-eikonal type of approximation.[1–3] However, it is still unclear to what extent the Glauber multiple-scattering picture,[4,5] which was originally intended to describe the scattering of composite objects at energies where particle creation and annihilation are negligible, can in fact be regarded as a satisfactory description in the relativistic domain. In order to be able to include inelastic states in the multiple-scattering chain, we will not use the Glauber model here. The main reason is that the implications of unitarity in relativistic particle scattering are not easily incorporated into that model. Our aim is to satisfy s-channel unitarity at least in a certain approximation to be discussed in detail below, and to describe what is conventionally called "Pomeranchuk exchange" as a unitarity effect—an s-channel phenomenon—which is complicated when analyzed in terms of t-channel exchanges. We therefore choose as our starting point a K-matrix type of parametrization for the scattering amplitudes in the way first discussed by Blankenbecler and Goldberger[6] and by Baker and Blankenbecler[7] in connection with the Fourier-Bessel representation of scattering amplitudes.[8–11] Our approach has some resemblance to recent investigations of the multiperipheral model,[12] although in detail it is quite different. The main distinction is that we introduce here a *two-cluster hypothesis* for the description of the intermediate states in the unitarity relations. In the multiperipheral model, on the other hand, s-channel unitarity is satisfied *exactly* at the expense of having to face the difficult problem of handling the multiparticle phase space.

Furthermore, we will not assume a Pomeranchuk trajectory as an input term, i.e., as a "driving term" in this formalism. Instead, we shall investigate under what conditions a vacuum exchange contribution can be generated from multiple exchange of lower-lying trajectories, allowing for a whole set of excited inter-

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[1] S. Frautschi and B. Margolis, Nuovo Cimento **56A**, 1155 (1968).
[2] C. B. Chiu and J. Finkelstein, Nuovo Cimento **57A**, 649 (1968); **59A**, 92 (1968).
[3] N. W. Dean, Phys. Rev. **182**, 1695 (1969).
[4] R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. Brittin *et al.* (Interscience, New York, 1959), Vol. I, p. 315.
[5] V. Franco and R. J. Glauber, Phys. Rev. **142**, 1195 (1966).
[6] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).
[7] M. Baker and R. Blankenbecler, Phys. Rev. **128**, 415 (1962).
[8] Compare in this context also Refs. 9–11.
[9] A. P. Contogouris, Phys. Letters **23**, 698 (1966).
[10] H. D. I. Abarbanel, S. D. Drell, and F. Gilman, Phys. Rev. **177**, 2458 (1969).
[11] G. Cohen-Tannoudji, A. Morel, and Ph. Salin, CERN Report No. TH.1003, 1969 (unpublished).
[12] J. W. Dash, J. R. Fulco, and A. Pignotti, Report No. RLO-1388-571, 1969 (unpublished).

mediate states. The basic diagrams producing a Pomeranchuk contribution in this formalism will be those shown in Fig. 1, where the transfer of the vacuum quantum numbers in the t-channel corresponds to a back-and-forth exchange of quantum numbers carried by a trajectory R, together with an excitation of all possible "resonances" in the intermediate state produced by the incoming particles a and b at a particular c.m. energy squared $s=(p_a+p_b)^2$. Our basic statement will be that, although the Regge cuts corresponding to a double R-exchange and a certain well-defined quasi-two-particle intermediate state in Fig. 1 are asymptotically suppressed in the near-forward direction compared to single R-exchange, the consideration between the R-exchanges of *excited states, which grow in number as the energy increases*, can in fact compensate this suppression of the individual terms in the sum. Thereby, indeed, a vacuum contribution can be generated which dominates elastic scattering asymptotically without having to postulate in the theory a Pomeranchuk pole in the beginning.

The plan of our presentation will be as follows. In Sec. II we introduce the multichannel K-matrix description of scattering amplitudes in the impact-parameter language and discuss various approximations inherent in our approach. In this section we state our main results for particle-particle scattering and extend them in Sec. III to the case of particle-antiparticle scattering where additional annihilation channels are open. In Sec. IV the implications of the proposed model regarding the real part of high-energy forward-scattering amplitudes and the existence or nonexistence of a Pomeranchuk limit as well as a Pomeranchuk theorem are studied. Section V is devoted to a discussion of the crossover phenomenon, and Sec. VI to some final remarks.

II. K-MATRIX FORMALISM

Our starting point will be the impact-parameter representation for the scattering amplitude at high energies.[13] Neglecting complications due to spin and isospin, the elastic s-channel scattering amplitude is, in our normalization, given by

$$f^{(1)}(s,t)=2\pi s\int_0^\infty bdb\,\eta^{(1)}(b,s)J_0(b\sqrt{-t})\,. \qquad (1)$$

The elastic differential cross section and the optical theorem read[14]

$$\frac{d\sigma}{dt}=\frac{1}{4\pi q^2 s}|f^{(1)}(s,t)|^2 \qquad (2)$$

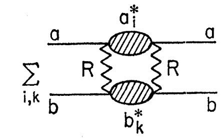
and

$$\operatorname{Im}f^{(1)}(s,t=0)=\tfrac{1}{2}q(\sqrt{s})\sigma_{\text{tot}}(s)\,,$$

[13] R. Henzi, Nuovo Cimento **46A**, 370 (1966).

[14] We label elastic amplitudes with a superscript (1) to distinguish them from certain inelastic amplitudes to be introduced below.

FIG. 1. Double-Regge-exchange diagrams contributing to elastic scattering.



where q is the relative momentum in the c.m. frame, given at high energies by $q\approx\frac{1}{2}\sqrt{s}$.

It was shown by Blankenbecler and Goldberger[6] and by Cottingham and Perierls[15] that unitarity is expressed at high energies in a simple manner in terms of the Fourier-Bessel coefficients $\eta^{(1)}(b,s)$. To leading order in s, the unitarity relations for the $\eta^{(1)}(b,s)$ correspond to the ones for the partial-wave amplitudes, i.e., one has

$$(1/2i)[\eta^{(1)}(b,s)-\eta^{(1)}(b,s)^*]=\eta^{(1)}(b,s)\eta^{(1)}(b,s)^* + \sum_{j=1}^{n(s)}\eta_j^{(2)}(b,s)\eta_j^{(2)}(b,s)^*+h^{(1)}(b,s)\,. \qquad (3)$$

Here the amplitudes $\eta_j^{(2)}(b,s)$, $j=1,2,\ldots,n(s)$, are the amplitudes for transitions to the open two-body channels, and $h^{(1)}(b,s)$ represents the overlap function[16,17] in the impact-parameter representation, describing the effect that the transitions to multiparticle intermediate states have on the elastic amplitude. Conventionally the sum over the two-body "quasielastic" *and* the true inelastic transitions in Eq. (3) are called the overlap function, being denoted by $g^{(1)}(b,s)$.

To satisfy s-channel unitarity, there are now in principle two possibilities open. On the one hand, one can try to parametrize elastic scattering globally in terms of the combined effect of *all* inelastic states appearing in the unitarity relations, i.e., make a suitable ansatz for $g^{(1)}(b,s)$. On the other hand, one can try to find a set of quasi-two-body states $(2)_j$, $j=1,2,\ldots,n(s)$, such that Eq. (3) is approximately satisfied with $h^{(1)}(b,s)\approx 0$. The first alternative was advocated in a number of papers[16–18] following the original suggestion by Van Hove,[19] and most recently in Ref. 20. We shall follow here the second alternative and investigate the consequences which result from it.

Introducing a matrix $\boldsymbol{\eta}(b,s)$ of dimension $[1+n(s)]$ for the transitions between all the coupled quasi-two-body channels, we write Eq. (3) for $h^{(1)}(b,s)\approx 0$ as

$$(1/2i)[\boldsymbol{\eta}(b,s)-\boldsymbol{\eta}(b,s)^\dagger]=\boldsymbol{\eta}(b,s)\boldsymbol{\eta}(b,s)^\dagger\,. \qquad (4)$$

To satisfy this relation, we now introduce a K-matrix parametrization[21,22] for the impact-parameter matrix

[15] W. N. Cottingham and R. F. Peierls, Phys. Rev. **137**, B147 (1965).

[16] A. Białas and L. Van Hove, Nuovo Cimento **38**, 1385 (1965).

[17] A. Białas, Th. W. Ruijgrok, and L. Van Hove, Nuovo Cimento **37**, 608 (1965).

[18] R. Henzi, Nuovo Cimento **52A**, 772 (1967); **53A**, 301 (1968).

[19] L. Van Hove, Rev. Mod. Phys. **36**, 655 (1964).

[20] R. Henzi, A. Kotanski, D. Morgan, and L. Van Hove, CERN Report No. TH.1086, 1969 (unpublished).

[21] R. C. Arnold, Phys. Rev. **136**, B1388 (1964).

[22] K. Dietz and H. Pilkuhn, Nuovo Cimento **37**, 1561 (1965); **39**, 928 (1965).

$\boldsymbol{\eta}(b,s)$, and write

$$\boldsymbol{\eta}(b,s)=\mathbf{N}(b,s)[1-i\mathbf{N}(b,s)]^{-1} =[1-i\mathbf{N}(b,s)]^{-1}\mathbf{N}(b,s). \quad (5)$$

Here the matrix $\mathbf{N}(b,s)$—the "Born matrix"—contains the driving terms which we will relate below to single Regge-pole exchanges. The full unitary amplitudes, which constitute the matrix $\boldsymbol{\eta}(b,s)$, will then automatically contain the iterated Born terms describing multi-Regge-pole exchanges. In particular, the unitarized amplitudes will develop a piece which can be identified with the exchange of the vacuum quantum numbers and hence can be interpreted as the Pomeranchuk contribution.

The matrix $\mathbf{N}(b,s)$ will be represented as

$$\mathbf{N}(b,s)=\begin{bmatrix} N^{(1)}(b,s) & N_1{}^{(2)}(b,s) & N_2{}^{(2)}(b,s) & \cdots & N_{n(s)}{}^{(2)}(b,s) \\ N_1{}^{(2)}(b,s) & N_1{}^{(3)}(b,s) & B_{12}(b,s) & \cdots & B_{1n(s)}(b,s) \\ N_2{}^{(2)}(b,s) & B_{12}(b,s) & N_2{}^{(3)}(b,s) & \cdots & B_{2n(s)}(b,s) \\ \vdots & & & & \vdots \\ N_{n(s)}{}^{(2)}(b,s) & & & \cdots & N_{n(s)}{}^{(3)}(b,s) \end{bmatrix}. \quad (6)$$

A corresponding labeling of rows and columns is assumed for the matrix $\boldsymbol{\eta}(b,s)$. $N^{(1)}(b,s)$ is the single-Regge-exchange term for the elastic scattering in the impact-parameter representation, and $N_j{}^{(2)}(b,s)$ is the corresponding quantity for the transition from the initial state $a+b$ to the quasi-two-body state labeled $(2)_j$ containing one or two resonances denoted by a_i^* and/or b_k^*. Finally, the $N_j{}^{(3)}(b,s)$ describe elastic scattering of these resonances, and the $B_{jj'}(b,s)$ represent the transitions between the various excited two-particle channels.

We point out that, since the individual Born terms entering in (6) are real and $\mathbf{N}(b,s)$ is symmetric because of time-reversal invariance, the form (5) for the matrix $\boldsymbol{\eta}(b,s)$ indeed implies $\mathbf{h}(b,s)=0$, where $\mathbf{h}(b,s)$ is now a $[n(s)+1]\times[n(s)+1]$ overlap matrix. This relation holds because in the K-matrix language one has

$$\mathbf{h}(b,s)=[1-i\mathbf{N}(b,s)]^{-1}(1/2i)[\mathbf{N}(b,s)-\mathbf{N}^{\dagger}(b,s)] \times[1+i\mathbf{N}^{\dagger}(b,s)]^{-1}. \quad (7)$$

We are aiming at a description of high-energy elastic scattering and diffraction dissociation and shall construct the individual Born terms by considering only the dominating exchanges for large s, i.e., Regge trajectories having intercepts of order $\alpha(0)\approx 0.5$. Remember that we do not regard the conventional Pomeranchuk pole as a possible input here, quite independently of the fact that it does not satisfy the required reality condition. As mentioned above, a vacuum contribution will, under certain conditions, come out automatically as a result of the unitarization procedure implied by the form (5) of the impact-parameter matrix.

To construct the matrix (6) explicitly we take as the single-scattering contributions to the amplitudes the terms corresponding to the exchange of an *exchange-degenerate* Regge trajectory R in the t channel, having trajectory $\alpha(t)=\alpha(0)+\alpha' t$, with intercept $\alpha(0)=0.5$ and slope $\alpha'=1$ GeV^{-2}, i.e., we take a trajectory corresponding to ρ and A_2 or ω and f^0 ($f^0=P'$). We therefore write

$$\mathbf{f}^{\text{single scat.}}(s,t)=\mathbf{g}_R(s,t)=(s/s_0)^{\alpha(t)}\boldsymbol{\beta}, \quad (8)$$

where $\boldsymbol{\beta}$ is a real-symmetric constant matrix constructed in analogy to Eq. (6), with matrix elements $\beta^{(1)}$, $\beta_j{}^{(2)}$, $\beta_j{}^{(3)}$, $j=1,2,\ldots,n(s)$, and $\beta_{jj'}$, $j,j'=1,2,\ldots,n(s)$. s_0 is a scaling energy taken as usual to be $s_0=1$ GeV^2. To be specific, we consider the elastic channel to be pp, π^+p, or K^+p scattering. The corresponding antiparticle reactions, where additional charge or hypercharge annihilation channels are contributing, will be considered in Sec. III.

Notice that we have assumed a certain ghost-eliminating mechanism operating in Eq. (8). The factor $1/\sin\pi\alpha(t)$ contained originally in the signature factor of the Regge-pole contribution is assumed to be canceled by a corresponding factor in the conventional Regge residue $\beta(t)$, i.e., we put $\beta(t)=\beta\,\sin\pi\alpha(t)$, with β taken to be constant. One could call this "maximal ghost elimination" for an exchange-degenerate trajectory in contrast to a weaker ghost-eliminating mechanism operating possibly for non-exchange-degenerate trajectories which we will discuss in Sec. IV.

The Fourier-Bessel transformation of Eq. (8), together with the above assumption of a linear Regge trajectory, now yields the Born matrix $\mathbf{N}(b,s)$ according to

$$\mathbf{N}(b,s)=\mathbf{N}_R(b,s)=\frac{1}{2\pi s}\int_0^\infty xdx\,\mathbf{g}_R(s,-x^2)J_0(bx) =\frac{1}{(s/s_0)^{1/2}}\boldsymbol{\beta}\tilde{I}_0(b,s), \quad (9)$$

with $\tilde{I}_0(b,s)$ given by

$$\tilde{I}_0(b,s)=\frac{1}{2\pi s_0}\frac{\exp(-b^2/4\bar{\rho})}{2\bar{\rho}}, \quad \bar{\rho}=\alpha'\ln(s/s_0). \quad (10)$$

Let us now carry out the matrix inversion implied by Eq. (5), first without using the information provided by Eq. (9). In order to be able to proceed one has to introduce a simplifying assumption. We are going to suppose that the coupling between *different* excited two-body channels [the $B_{jj'}$ in Eq. (6)] are small compared to the other elements of the matrix $\mathbf{N}(b,s)$, i.e., $(B_{jj'})^2 \ll N_j^{(2)}N_{j'}^{(2)}$. We shall find below that the $B_{jj'}$ (or, correspondingly, the $\beta_{jj'}$) are related in our description to diffraction-dissociation processes which are experimentally known to be about a factor of $\frac{1}{6}$ to $\frac{1}{8}$ smaller than the corresponding elastic scattering.[23,24] Neglecting, therefore, quadratic terms in the $B_{jj'}$ and using the abbreviations $A^{(1)}(b,s)=1-iN^{(1)}(b,s)$ and $A_j^{(3)}(b,s)=1-iN_j^{(3)}(b,s)$, $j=1,2,\ldots,n(s)$, one obtains for the elements in the first column of the matrix $\boldsymbol{\eta}(b,s)$, i.e., for elastic scattering of particles a and b, described by $\eta^{(1)}(b,s)$, and for resonance production by the same incoming particles, described by $\eta_j^{(2)}(b,s)$, $j=1,2,\ldots,n(s)$,

$$\eta^{(1)}=\left[\frac{N^{(1)}}{A^{(1)}}+i\sum_{j=1}^{n(s)}\frac{(N_j^{(2)})^2}{A^{(1)}A_j^{(3)}}-\sum_{\substack{j,j'\\ j\neq j'}}^{n(s)}\frac{N_j^{(2)}N_{j'}^{(2)}B_{jj'}}{A^{(1)}A_j^{(3)}A_{j'}^{(3)}}\right]\Bigg/\left[1+\sum_{j=1}^{n(s)}\frac{(N_j^{(2)})^2}{A^{(1)}A_j^{(3)}}+i\sum_{\substack{j,j'\\ j\neq j'}}^{n(s)}\frac{N_j^{(2)}N_{j'}^{(2)}B_{jj'}}{A^{(1)}A_j^{(3)}A_{j'}^{(3)}}\right], \qquad (10')$$

$$\eta_j^{(2)}=\left[\frac{N_j^{(2)}}{A_j^{(3)}}+i\frac{N^{(1)}N_j^{(2)}}{A^{(1)}A_j^{(3)}}+i\sum_{j'=1}^{n(s)}\frac{N_{j'}^{(2)}B_{j'j}}{A_{j'}^{(3)}A_j^{(3)}}-\sum_{\substack{j'=1\\ j'\neq j}}^{n(s)}\frac{N^{(1)}N_{j'}^{(2)}B_{j'j}}{A^{(1)}A_{j'}^{(3)}A_j^{(3)}}\right]\Bigg/$$

$$\left[1+\sum_{j'=1}^{n(s)}\frac{(N_{j'}^{(2)})^2}{A^{(1)}A_{j'}^{(3)}}+i\sum_{\substack{j',j''\\ j'\neq j''}}^{n(s)}\frac{N_{j'}^{(2)}N_{j''}^{(2)}B_{j'j''}}{A^{(1)}A_{j'}^{(3)}A_{j''}^{(3)}}\right]. \qquad (10'')$$

We now introduce the further assumption that $A^{(1)}=A_j^{(3)}$, $j=1,2,\ldots,n(s)$, which, loosely speaking, means that the initial- and final-state interactions in the process $a+b\to(2)_j=a_i^*+b_k^*$ are equal. The next step now is the evaluation of the sums appearing in Eqs. (10). Our claim is that although the individual Born terms are of order $1/\sqrt{s}$ [compare Eq. (9)], the *s-dependent sums* of squares or third powers of such contributions may asymptotically be of considerably larger size compared to the contributions of the individual terms in the sums which are of order s^{-1} and $s^{-3/2}$, respectively.

We define with the help of Eqs. (9) and (10)

$$C(b,s)=\sum_{j=1}^{n(s)}[N_j^{(2)}(b,s)]^2=\left(\frac{s}{s_0}\right)^{-1}[\tilde{I}_0(b,s)]^2\sum_{j=1}^{n(s)}(\beta_j^{(2)})^2, \qquad (11a)$$

$$D_j(b,s)=\sum_{\substack{j'=1\\ j'\neq j}}^{n(s)}N_{j'}^{(2)}(b,s)B_{j'j}(b,s)=\left(\frac{s}{s_0}\right)^{-1}[\tilde{I}_0(b,s)]^2\sum_{\substack{j'=1\\ j'\neq j}}^{n(s)}\beta_{j'}^{(2)}\beta_{j'j}, \qquad (11b)$$

$$D(b,s)=\sum_{j=1}^{n(s)}D_j(b,s)N_j^{(2)}(b,s)=\sum_{\substack{j,j'\\ j\neq j'}}^{n(s)}N_j^{(2)}(b,s)N_{j'}^{(2)}(b,s)B_{jj'}(b,s)=\left(\frac{s}{s_0}\right)^{-3/2}[\tilde{I}_0(b,s)]^3\sum_{\substack{j,j'\\ j\neq j'}}^{n(s)}\beta_j^{(2)}\beta_{j'}^{(2)}\beta_{jj'}. \qquad (11c)$$

Writing each sum over j as a double sum over the individual excited states contained in the two-particle intermediate state which can be produced by R-exchange, one sees that the quantities $C(b,s)$, $D_j(b,s)$, and $D(b,s)$ correspond to the diagrams shown in Figs. 1, 2, and 3, respectively. The blobs in these diagrams contain all possible excited states (resonances) of variable mass up to a highest one with mass depending on s.

Let us first treat the sum appearing in Eq. (11a). Following Freund,[25] we write it as

$$\sum_{j=1}^{n(s)}[\beta_j^{(2)}]^2=\sum_{i=1}^{\bar{\alpha}'s}\sum_{k=1}^{\bar{\alpha}'^2\mu^2 s/i}[\beta_{aa_i^*,R}^{(2)}\beta_{bb_k^*,R}^{(2)}]^2=(\beta^{(2)})^2\sum_{i=1}^{\bar{\alpha}'s}\sum_{k=1}^{\bar{\alpha}'^2\mu^2 s/i}(ik)^\kappa. \qquad (12)$$

First a comment on the limits appearing in the summation over i and k is in order. A particular intermediate state (i,k) in Fig. 1 will only contribute appreciably to forward scattering if it can be produced with small

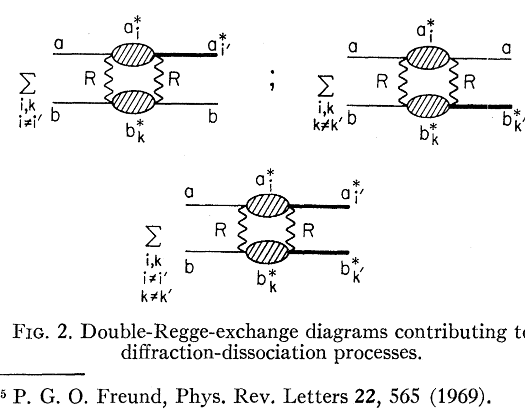


FIG. 2. Double-Regge-exchange diagrams contributing to diffraction-dissociation processes.

[23] E. W. Anderson *et al.*, Phys. Rev. Letters **16**, 855 (1966).
[24] I. M. Blair *et al.*, Phys. Rev. Letters **17**, 789 (1966).

[25] P. G. O. Freund, Phys. Rev. Letters **22**, 565 (1969).

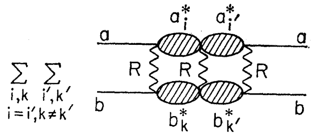


FIG. 3. Triple-Regge-exchange diagrams contributing to elastic scattering.

momentum transfer $t_{\min}\approx 0$. Now $t_{\min}$ is given by

$$|t_{\min}|\approx m_{a_i^*}{}^2 m_{b_k^*}{}^2/s. \tag{13}$$

It is therefore required that the masses of the intermediate particles appearing in Eq. (12) have to obey the relation

$$m_{a_i^*}{}^2 m_{b_k^*}{}^2 \lesssim \mu^2 s, \tag{14}$$

where μ^2 is some small constant mass. Assuming, moreover, the a_i^* and b_k^* to lie on a linear Regge trajectory of slope $\bar{\alpha}'\approx 1$ GeV^{-2}, the masses of the intermediate excited states are given by

$$\begin{aligned} m_{a_i^*}{}^2 &= m_i^2 = m_a^2 + i/\bar{\alpha}', \quad i=1,2,\ldots,n_i(s) \\ m_{b_k^*}{}^2 &= m_k^2 = m_b^2 + k/\bar{\alpha}', \quad k=1,2,\ldots,n_k(s). \end{aligned} \tag{15}$$

Taking Eq. (14) into account, the highest possible values of $n_i(s)$ and $n_k(s)$ are obtained as indicated in Eq. (12).

Let us now justify the last step in Eq. (12). Following again Freund,[25] we assumed the coupling strength between the excited intermediate state labeled (i,k) and the "elastic state" to be proportional to $(ik)^{\kappa/2}$. Here κ is some constant determining the relative coupling strength of the various higher excited states $|a_i^*, b_k^*\rangle$ to the elastic or ground state $|a,b\rangle$.

One can now investigate different assumptions concerning the value of κ which determines the behavior for large s of the quantity $C(b,s)$. The most natural choice seems to assume that all excited intermediate quasi-two-particle states are coupled equally strongly to the elastic channel independently of the masses of the particles produced. This corresponds to $\kappa\approx 0$ and leads to the following behavior of the sum (12) for large energies:

$$\sum_{j=1}^{n(s)} [\beta_j^{(2)}]^2 = C\left(\frac{s}{s_0}\right)\ln\frac{s}{s_0}. \tag{12$'$}$$

Here C is a positive constant which we shall show below to be related to Pomeranchuk exchange. With the result (12′), one obtains finally for $C(b,s)$

$$C(b,c) = C\ln(s/s_0)[\tilde{I}_0(b,s)]^2. \tag{16}$$

In a completely analogous fashion, making the same assumptions for the sum $\sum_{j'}\beta_{j'}{}^{(2)}\beta_{j'j}$ which were made for the sum (12), one derives for the quantities $D_j(b,s)$ corresponding to Fig. 2 the result

$$D_j(b,s) = D_j\ln(s/s_0)[\tilde{I}_0(b,s)]^2, \tag{17}$$

where the constant D_j will be related below to diffraction-dissociation processes, i.e., the production, via Pomeranchuk exchange, of quasi-two-body states containing excited baryons and/or mesons. Familiar examples of such processes are, for instance, N^* and/or vector-meson production in πN collisions.

For consistency with the derivation of Eqs. (10′) and (10″), one must require that $D_j < C$ in analogy to $(\beta_{jj'})^2 \ll \beta_j^{(2)}\beta_{j'}{}^{(2)}$ for $j,j'=1,2,\ldots,n(s)$. We shall return to this point below when we discuss the implications of the assumptions made so far and, in particular, study the mechanism which will give rise to a vacuum-exchange contribution in this K-matrix description.

To conclude our discussion of Eqs. (11), we finally have to determine the large-s behavior of the sum (11c) corresponding to Fig. 3. With the help of Eq. (17), Eq. (11c) can be written

$$D(b,s) = \frac{1}{(s/s_0)^{1/2}}\ln\frac{s_0}{s_0}[\tilde{I}_0(b,s)]^3\sum_{j=1}^{n(s)} D_j\beta_j^{(2)}. \tag{18}$$

Applying here the same argument which led to Eqs. (16) and (17), i.e., assuming again equal strength of all terms in the sum

$$\sum_{j=1}^{n(s)} D_j\beta_j^{(2)},$$

would result in a large-s behavior $D(b,s)\sim(s/s_0)^{1/2} \times[\ln(s/s_0)]^{-1}F(b)$, which can be shown to violate Frossart's bound. At most, positive powers of $\ln(s/s_0)$ are allowed to appear on the right-hand side of Eq. (18) in order to yield an elastic forward-scattering amplitude bounded by $(s/s_0)[\ln(s/s_0)]^2$ as s goes to infinity. We therefore conclude that the contributions of the higher excited states appearing in the sum $\sum D_j\beta_j^{(2)}$ are more strongly damped as compared to the sum $\sum(\beta_j^{(2)})^2$. Without offering a deeper justification, we assume that the sum on the right-hand side of Eq. (18) behaves for large energies as the largest possible power in s consistent with the Froissart bound. In particular, we assume that

$$\sum_{j=1}^{n(s)} D_j\beta_j^{(2)} = D\left(\frac{s}{s_0}\right)^{1/2}\left(\ln\frac{s}{s_0}\right)^m, \tag{19}$$

with D being a constant and m an arbitrary positive integer. This leads to

$$D(b,s) = D\left(\ln\frac{s}{s_0}\right)^{m+1}[\tilde{I}_0(b,s)]^3. \tag{20}$$

We are aware of the fact that the assumptions made to arrive at Eqs. (19) and (20) are more difficult to justify theoretically than those which lead to Eqs. (16) and (17). Moreover, the power m of the factor $\ln(s/s_0)$ in Eq. (19) is unknown. We shall explore the consequences of various possible values for m in this

K-matrix approach in more detail in Sec. IV, where we investigate its connection to the real parts of forward elastic scattering amplitudes at very large s as well as the existence of a Pomeranchuk limit. For definiteness, we shall assume in most of the following discussion that $m=1$, which will be shown in Sec. IV to imply a Pomeranchuk theorem in this formalism.

Having obtained the high-energy behavior of the sums appearing in Eqs. (11a)–(11c), we can now, after insertion of the results given by Eqs. (16), (17), and (20) into Eqs. (10′) and (10″), make an expansion of the right-hand side of these equations in powers of $1/(s/s_0)^{1/2}$. Remembering that the unitarity relations (3) in the impact-parameter language were only valid to leading order in s, we neglect in Eqs. (10′) and (10″) all terms of order $1/s$ and smaller. With $C(b,s)$, $D_j(b,s)$, and $D(b,s)$ as given by Eqs. (16), (17), and (20), respectively, the results for the unitarized elastic scattering as well as resonance production amplitudes are now given by

$$\eta^{(1)}(b,s)=\eta_P{}^{(1)}(b,s)+\eta_R{}^{(1)}(b,s)=\frac{iC(b,s)-D(b,s)}{1+C(b,s)+iD(b,s)}+N^{(1)}(b,s)\frac{1-C(b,s)-2iD(b,s)}{[1+C(b,s)+iD(b,s)]^2}, \quad (21a)$$

$$\eta_j{}^{(2)}(b,s)=\eta_{j,P}{}^{(2)}(b,s)+\eta_{j,R}{}^{(2)}(b,s)=\frac{iD_j(b,s)}{1+C(b,s)+iD(b,s)}+\frac{N_j{}^{(2)}(b,s)}{1+C(b,s)+iD(b,s)}-\frac{3N^{(1)}(b,s)D_j(b,s)[1-C(b,s)]}{[1+C(b,s)+iD(b,s)]^2}. \quad (21b)$$

Considering Eq. (21a), we see that the unitarization procedure has generated from the driving terms $N^{(1)}$ and $N_j{}^{(2)}$ of order $1/(s/s_0)^{1/2}$ not only an "absorptive correction" to $N^{(1)}$ represented by the factor multiplying the Born term in Eq. (21a), but also a contribution

$$\eta_P{}^{(1)}(b,s)=\frac{iC(b,s)-D(b,s)}{1+C(b,s)+iD(b,s)} \quad (22)$$

which behaves like a constant at large s (apart from logarithmic factors). After Fourier-Bessel transformation [compare Eq. (1)] the contribution (22) gives rise to a term proportional to s, which can be interpreted as representing the Pomeranchuk exchange contribution since it corresponds to no net quantum-number exchange. For the imaginary part of $\eta_P{}^{(1)}(b,s)$, which is determined by the double-R-exchange contributions represented by $C(b,s)$ and shown in Fig. 1, the latter is evidently true since the two-step processes can proceed by twice the exchange of quantum numbers (ρ, A_2, or ω component of R), or twice the exchange of vacuum quantum numbers (P' component of R). For the (supposedly small) real part of $\eta_P{}^{(1)}(b,s)$ which is basically determined by a threefold exchange of the trajectory R and represented in Eq. (22) by $D(b,s)$, the requirement that no net quantum numbers are transferred corresponds to a restriction on *one* of the R-exchanges in Fig. 3, i.e., only the P'-component of R is allowed to be effective.

The contribution originating from Eq. (22) has a number of interesting properties. First of all it represents a superposition of cuts in the angular-momentum plane, since—as is clear from Eq. (22)—it can be written for large s as a power series in $[C(b,s)+iD(b,s)]$ corresponding to an *iteration* of graphs of the type shown in Figs. 1 and 3. We point out, however, that the multiple-scattering series obtained by expanding the denominator in Eq. (22) can in general only be assumed to converge for very large s since $C(b,s)$ and $D(b,s)$ are of order const/$\ln(s/s_0)$.[26] For low values of s, the Fourier-Bessel transform of the right-hand side of Eq. (22) must in general be performed as it stands in order to yield the Pomeranchuk contribution to $f^{(1)}(s,t)$. It is, however, interesting to determine the "effective" contribution provided by Eq. (22) for large s and small t (corresponding to large impact parameters), which is given by

$$\eta_P{}^{(1)}(b,s)\approx iC(b,s)-D(b,s)=iC\ln\frac{s}{s_0}[\tilde{I}_0(b,s)]^2-D\left(\ln\frac{s}{s_0}\right)^2[\tilde{I}_0(b,s)]^3. \quad (23)$$

Equation (23) leads after Fourier-Bessel transformation to

$$f_P{}^{(1)}(s,t)\sim i\frac{1}{2}\frac{C}{4\pi\alpha' s_0}\left(\frac{s}{s_0}\right)^{1+\frac{1}{2}\alpha' t}-\frac{1}{3}\frac{D}{(4\pi\alpha' s_0)^2}\left(\frac{s}{s_0}\right)^{1+\frac{1}{3}\alpha' t}. \quad (24)$$

This equation shows that for a purely imaginary high-energy elastic scattering amplitude in the near-forward direction, i.e., for $D/4\pi s_0\alpha'$ small compared to C, the "effective Pomeranchuk-pole trajectory" at large s and small t is given in this model by $[\alpha_P(t)]_{\text{eff}}=1+\frac{1}{2}\alpha' t$, which means that the slope of the effective P-trajectory is one-half of the generating exchange-degenerate trajectory $\alpha(t)$. If the real part of the elastic scattering amplitude at large s, reflecting the contributions of the diffraction dissociation channels, is small but non-negligible, one gets a further contribution having an effective slope $\frac{1}{3}\alpha'$.

We remark that a slope of the order $\frac{1}{2}\alpha'$ is of the right magnitude to explain the shrinkage found in elastic pp collisions up to the highest available energies.[27] This shrinkage corresponds to a slope of an effective Pomer-

[26] For $D(b,s)$ this is true only for $m=1$. For $m>1$ see our discussion in Sec. IV.

[27] G. G. Beznogikh *et al.*, Phys. Letters **30B**, 2741 (1969).

anchuk pole given by $\alpha_P' = 0.40 \pm 0.09$ GeV^{-2}. A similar shrinkage seems to be observed in K^+p scattering corresponding to a somewhat larger effective Pomeranchuk slope.[28] We are inclined to take the result that our model predicts the forward peak in particle-particle elastic scattering to continue to shrink at approximately the correct rate as energy increases as a support for the described picture for the Pomeranchuk contribution.

In contrast to the Glauber-eikonal type of description of elastic scattering proceeding via multiple exchange of a supposedly existing single Pomeranchuk pole of natural parity, it turns out that in our description the Pomeranchuk term—even in lowest-order rescattering —does *not* have a definite natural or unnatural parity. The reason is that already the lowest-order term given by Eq. (24) and corresponding to the graphs shown in Figs. 1 and 3 represents a Regge cut (double or triple R-exchange), which cannot be associated with a definite parity being exchanged in the t channel.[29] This is true despite the fact that the leading quantum-number-changing trajectories going into R, i.e., $f^0(P')$, A_2, ω, or ρ, are all of natural parity.

Similar arguments apply to the Pomeranchuk contribution to resonance production, which is represented by the first term on the right-hand side of Eq. (21b) corresponding basically to the diagrams shown in Fig. 2. The full diffraction-dissociation amplitude can, analogously to the elastic case, be depictured as consisting of a sum of terms corresponding to a chain of graphs of the type shown in Figs. 1 and 3 together with a final link of the type shown in Fig. 2. As for elastic scattering, this multiple-scattering series corresponds to the expansion of the denominator in the expression for $\eta_{j,P}^{(2)}(b,s)$ in Eq. (21b).

From experiments on N^* production in pp collisions, one concludes that the constant D_j appearing in the amplitude $\eta_{j,P}^{(2)}(b,s)$ for diffraction dissociation is smaller than the constant C (describing predominatly the elastic scattering) by a factor of $\frac{1}{6}$ to $\frac{1}{8}$ at incident laboratory momentum between 6 and 30 GeV/c,[23,24] giving thus *a posteriori* a justification for having neglected quadratic terms in the $\beta_{jj'}$ in deriving Eqs. (10) above.

It is obvious from the above discussion that the Pomeranchuk contribution to elastic scattering as well as to diffraction-dissociation processes does not factorize. Furthermore, the phase of the vacuum contribution to elastic scattering is given by the relative strengths of the diagrams shown in Figs. 1 and 3. A similar statement applies to diffraction dissociation. We shall come back to this point in Sec. IV.

We finally mention in connection with Eqs. (21a) and (21b) that the K-matrix formalism described above leads automatically to a damping of the input Born terms. This "absorptive correction" to the quantum-number-changing contributions contained in the second term, called $\eta_R^{(1)}(b,s)$ in Eq. (21a), and in the second and third terms, called $\eta_{j,R}^{(2)}(b,s)$ in Eq. (21b), is given in terms of the quantities which determine the high-energy elastic scattering. We shall see in Sec. V, in treating the crossover phenomenon, how this absorptive correction to single Regge exchanges can in principle be used to determine properties of the elastic scattering.

In summing up, let us list the assumptions made in the course of the derivation of Eqs. (21).

(i) The inelastic states contributing in the unitarity relations can be represented by a set of quasi-two-particle states (resonances).

(ii) The coupling strength for the transition between different excited two-particle channels *in the Born matrix* is smaller than the corresponding coupling to the elastic channel, i.e., $(\beta_{jj'})^2 \ll \beta_j^{(2)}\beta_{j'}^{(2)}$, $j,j'=1,2,\ldots,n(s)$, such that terms quadratic in the $\beta_{jj'}$ are negligible.

(iii) The coupling strength for producing a certain resonance from the elastic channel is independent of the mass of the produced resonance, and the density of the excited states is that provided by a linearly rising Regge trajectory.

(iv) The diagonal elements in the Born matrix are all approximately equal, i.e., $\beta^{(1)} \approx \beta_j^{(3)}$, $j=1,2,\ldots,n(s)$.

As mentioned at the beginning of this section, (i) is our main assumption. The assumptions (ii)–(iv) could in principle be altered and the consequences for the vacuum contribution be worked out in essentially the same framework.

In concluding this section we would like to stress that we do not pretend to have shown that indeed Pomeranchuk exchange in particle-particle collisions is necessarily being generated by multiple-Regge-pole exchanges combined with the excitation of resonances in the intermediate state. In deriving Eqs. (21a) and (21b), the above-stated assumptions had to be made which are difficult to justify *a priori*. However, we do believe to have demonstrated that the described interpretation of the Pomeranchuk contribution is possible and in fact plausible. We shall, therefore, derive in the following sections some consequences of this picture for the Pomeranchuk contribution. In particular, we shall study the total cross sections and the crossover of elastic differential cross sections in this model. Before we can do this, however, we have to investigate whether or not the presented approach to particle-particle collisions can also be applied to particle-antiparticle collisions and what the implications are in this case.

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[28] T. Lasinsky, R. Levi-Setti, and E. Predazzi, Phys. Rev. **179**, 1426 (1969).

[29] In the approach of Ref. 1 only the single-P-exchange contribution, which is dominant at low values of t, has definite (natural) parity. All higher-order rescattering cut contributions—and consequently the full vacuum-exchange contribution to elastic scattering—represent a mixture of natural and unnatural parity components. It seems difficult to obtain a Pomeranchuk term of purely natural parity in any of these multiple-scattering models.

III. PARTICLE-ANTIPARTICLE SCATTERING AND CONTRIBUTION OF ANNIHILATION CHANNELS

Up to now we have treated elastic processes like pp, π^+p, or K^+p scattering and the associated diffraction-dissociation processes. We have pointed out that the Pomeranchuk contribution to such processes can in an average sense be generated from an exchange-degenerate Regge trajectory, considering at the same time all the possible quasi-two-particle inelastic intermediate states which can be produced at a certain energy. Now the question immediately poses itself: Can $p\bar{p}$, π^-p, or K^-p collisions be understood in a similar way? How do the charge or hypercharge annihilation channels, which can in addition contribute here, influence the description?

We shall introduce in this K-matrix model an *identical* coupling of the exchange-degenerate Regge trajectory R in pp and in $p\bar{p}$ collisions. This assumption has to be made in our approach in order to guarantee the Born terms representing the exchange of the trajectory R to be *real* in pp as well as in $p\bar{p}$ collisions (and correspondingly in the other pairs $\pi^{\pm}p$ and $K^{\pm}p$). We thus take the view that the additional annihilation channels, which can contribute to $p\bar{p}$ collisions compared to pp collisions, are in fact *negligible at high energies*.[30] At low energies these annilhilation channels, of course, affect the amplitude for $p\bar{p}$ scattering, forcing it to be different from the amplitude for pp scattering. We shall attribute, for instance, the fact that $\sigma_{\text{tot}}{}^{p\bar{p}}(s)$ is larger than $\sigma_{\text{tot}}{}^{pp}(s)$ at present energies in the familiar way to a different coupling of the *lower-lying trajectories* in pp and in $p\bar{p}$ collisions. More specifically, we will atrribute it to the fact that the ω- and ρ-exchange contribution is odd under charge conjugation and hence enters with a different sign in the pp amplitude compared to the $p\bar{p}$ amplitude. Having generated the vacuum contribution in an average sense from the lower-lying trajectories, we therefore set up a model by considering *afterwards* for the Regge exchange described by $N^{(1)}(b,s)$ in Eq. (21a) the *correct* individual Regge-pole contributions, allowing, furthermore, for a breaking of exchange degeneracy.

The above remark—that we will consider the effect of the annihilation channels as negligible at high energies—now implies that we have in Eqs. (16), (20), and (21)

$$C_{pp}=C_{p\bar{p}}, \quad D_{pp}=D_{p\bar{p}}, \tag{25}$$

and correspondingly for the other pairs $\pi^{\pm}p$ and $K^{\pm}p$. The content of Eq. (25) is equivalent to the statement that the exchange-degenerate trajectory R, which is supposed to give rise to a *real* Born term for pp as well as for $p\bar{p}$ scattering, is *even* under charge conjugation.

We remark that it would be aesthetically more attractive to generate the Pomeranchuk contribution in pp and $p\bar{p}$ collisions from the exchange of an object having mixed properties under the C operation. This, however, destroys the reality requirements and therefore, by Eq. (7), our basic assumption that a set of two-particle channels are able to represent the true inelastic channels open at a certain (large) value of s. If we were to give up this idea, we have essentially returned to the overlap-function approach or a combination thereof with our present parametrization. Our aim, however, was to explore the other extreme and assume that a quasi-two-particle description of the intermediate states in the unitarity relations is in fact possible. Having now stated all the assumptions involved in our approach, we proceed to work out the consequences.

IV. REAL PART OF ELASTIC FORWARD-SCATTERING AMPLITUDES AND TOTAL CROSS SECTIONS

From Eq. (21a), or more directly from Eq. (24), it follows that for $m=1$ the ratio of the real to imaginary part of the elastic forward-scattering amplitude at high energies is given by[31]

$$\frac{\operatorname{Re} f^{(1)}(s,t=0)}{\operatorname{Im} f^{(1)}(s,j=0)}=\xi(s)=\frac{-\frac{2}{3}(D/4\pi\alpha' s_0)}{C}. \tag{26}$$

We have neglected in Eq. (26) the contribution of the lower-lying trajectories which are of order $(s/s_0)^{-1/2}$ compared to the leading term. While in the conventional Regge-pole theory $\xi(s)$ is predicted to approach zero, in this model $\xi(s)$ is predicted to go to a constant, provided, of course, that the value of m is taken to be 1. The magnitude of this constant depends on the contribution of the diffraction-dissociation channels on the elastic transition (compare Fig. 3) and its sign on the sign of D. In Sec. V we shall give some evidence which indicates that D is likely to be positive.

From forward dispersion relations one knows that the fact that $\xi(s)$ is bounded by a constant implies the Pomeranchuk theorem which states that particle-particle and particle-antiparticle total cross sections approach the same constant limiting value at asymptotic energies.[32] Assuming quantities of order $(D/4\pi\alpha' s_0)^2$ to be small compared to those involving C in confirmity with our assumption (ii) in Sec. II, one obtains for the Pomeranchuk contribution to the total cross section

[30] Remember that this is consistent with the conventional description in which the assumed Pomeranchuk-pole trajectory is coupled with identical residues in particle-particle and particle-antiparticle interactions.

[31] For short, we again drop the labels on C and D and reinsert them whenever necessary.

[32] I. Ya. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **34**, 725 (1958) [Soviet Phys. JETP **34**, 499 (1958)].

from Eqs. (2) and (21a)

$$\sigma_{\text{tot}}{}^P(s)=\frac{2}{q\sqrt{s}}\,\text{Im}f_P{}^{(1)}(s,t=0)$$
$$=\frac{4\pi\sqrt{s}}{q}\int_0^\infty bdb\,\frac{Q(s)e^{-(b^2/2\bar{\rho})}}{1+Q(s)e^{-(b^2/2\bar{\rho})}}$$
$$=\frac{4\pi\sqrt{s}}{q}\alpha'\ln\frac{s}{s_0}\ln[1+Q(s)],\qquad(27)$$

with

$$Q(s)=\frac{C}{(4\pi s_0\alpha')^2}\frac{1}{\ln(s/s_0)}.$$

At very high energies one can expand the logarithm in Eq. (27), set $q\approx\frac{1}{2}\sqrt{s}$, and show that the total cross section approaches its asymptotic limit in this model in a logarithmic way from below, i.e.,

$$\sigma_{\text{tot}}{}^P(s)\sim\sigma_{\text{tot}}(\infty)\left(1-\frac{\sigma_{\text{tot}}(\infty)}{16\pi\alpha'\ln(s/s_0)}\right),\qquad(28)$$

with

$$\sigma_{\text{tot}}(\infty)=C/2\pi\alpha' s_0{}^2.\qquad(29)$$

An asymptotic behavior of the kind (28) is typical for cut models for the Pomeranchuk contribution. It can also be obtained in the Glauber-eikonal type of approach by iterating an ordinary Pomeranchuk pole of slope of order 1 GeV^{-2} as was shown in Ref. 1.

We finally write the Pomeranchuk contribution to the total cross section compactly as

$$\sigma_{\text{tot}}{}^P(s)=\frac{4\pi\alpha'\sqrt{s}}{q}\ln\frac{s}{s_0}\ln\left[1+\frac{\sigma_{\text{tot}}(\infty)}{8\pi\alpha'\ln(s/s_0)}\right].\qquad(30)$$

Apart from the slope α' of the generating trajectory, which is assumed to be 1 GeV^{-2}, there enters only one parameter into Eq. (30), namely, $\sigma_{\text{tot}}(\infty)$ for the process in question. Notice, however, that one has still some freedom in adjusting s_0 which is conventionally taken to be 1 GeV^2. In the numerical analysis to be discussed below we shall adhere to this value for s_0.

In the conventional Regge-pole description the total cross sections at nonasymptotic energies, for instance for pp and $p\bar{p}$ collisions, are given by a *constant* piece, identical for pp and $p\bar{p}$, originating from an assumed Pomeranchuk pole, plus various contributions of order $(s/s_0)^{-1/2}$ coming from the lower-lying trajectories which differ for pp and $p\bar{p}$ according to the C parity exchanged, i.e., $P'\pm\omega$ in the chosen example. In the present model the total cross section is, at finite energies, given by a logarithmically rising Pomeranchuk contribution given by Eq. (30), plus Regge-pole contributions decreasing like $(s/s_0)^{-1/2}$ originating from the Fourier-Bessel transform of the term $\eta_R{}^{(1)}(b,s)$ in Eq. (21a). This term describes the true Regge-pole contribution including the absorptive correction. Remember that for $N^{(1)}(b,s)$ in Eq. (21a) we take in our model as described in Sec. III the terms corresponding to the true non-exchange-degenerate Regge-pole contributions[33] with their known transformation properties under charge conjugation. Neglecting again terms quadratic in D, one obtains for the total cross sections for pp and $p\bar{p}$ collisions

$$\sigma_{\text{tot}}(s)=\sigma_{\text{tot}}{}^P(s)+\frac{4\pi\sqrt{s}}{q}\times\text{Im}\left\{\int_0^\infty bdb\,[N_{P'}{}^{(1)}(b,s)\pm N_\omega{}^{(1)}(b,s)]\times\left[\frac{1-C(b,s)}{[1+C(b,s)]^2}-4i\frac{D(b,s)}{[1+C(b,s)]^3}\right]\right\}.\qquad(31)$$

Here $N_{P'}{}^{(1)}(b,s)$ and $N_\omega{}^{(1)}(b,s)$ are the Fourier-Bessel transforms divided by s [compare Eq. (9)] of the conventional P'- and ω-Regge-pole contributions which are given by Eq. (32) below. $C(b,s)$ and $D(b,s)$ were defined in Eqs. (16) and (20) [compare also Eq. (29)]. The positive sign under the integral in Eq. (31) corresponds to pp scattering and the negative sign corresponds to $p\bar{p}$ scattering, where we have assumed the sign convention of Barger *et al.*[34] for the Regge-pole contributions, i.e.,

$$P':\quad -\frac{1}{2}\frac{1+e^{-i\pi\alpha_{P'}(t)}}{\sin\pi\alpha_{P'}(t)}\beta_{P'}(t)\left(\frac{s}{s_0}\right)^{\alpha_{P'}(t)}=-\tfrac{1}{2}[\cot\tfrac{1}{2}\pi\alpha_{P'}(t)-i]\beta_{P'}\left(\frac{s}{s_0}\right)^{\alpha_{P'}(t)},$$
$$\omega:\quad -\frac{1}{2}\frac{1-e^{-i\pi\alpha_\omega(t)}}{\sin\pi\alpha_\omega(t)}\beta_\omega(t)\left(\frac{s}{s_0}\right)^{\alpha_\omega(t)}=-\tfrac{1}{2}[\tan\tfrac{1}{2}\pi\alpha_\omega(t)+i]\beta_\omega(t)\left(\frac{s}{s_0}\right)^{\alpha_\omega(t)}.\qquad(32)$$

To be able to compute the Fourier-Bessel transforms of these expressions and obtain $N_{P',\omega}{}^{(1)}(b,s)$, a certain ghost-eliminating mechanism has to be operative. In Sec. II we assumed for the exchange-degenerate trajectory R that its residue contained a factor $\sin\pi\alpha(t)$, which was called there the maximal ghost-eliminating mechanism. However, for a non-exchange-degenerate trajectory such a factor induces additional zeros in the amplitude in addition to those which are required

[33] This corresponds to the conventional description of, for example, the pp and $p\bar{p}$ total cross sections where exchange degeneracy has to be violated in order to account for the observed variation in s of $\sigma_{\text{tot}}{}^{pp}(s)$ at present energies. Exact exchange degeneracy would in the conventional model predict a constant pp total cross section.

[34] V. Barger, M. Olsson, and D. D. Reeder, Nucl. Phys. **B5**, 411 (1968).

for the ghost elimination. In the non-exchange-degenerate case it is sufficient to assume that the residues in Eq. (32) contain for the positive-signature P' pole a factor $\sin\frac{1}{2}\pi\alpha_{P'}(t)$, and correspondingly for the negative-signature ω pole a factor $\cos\frac{1}{2}\pi\alpha_\omega(t)$. This situation could be called minimal ghost elimination. We shall not discuss here further ghost-eliminating mechanisms and their influence on the total cross sections in this model. We only remark that a definite mechanism has to be adopted for all the poles appearing for arbitrary negative t in Eq. (32) before the Fourier-Bessel transform can be computed and Eq. (31) be applied. A more detailed comparison with the experimental data on total cross sections using the theoretical ideas outlined above will be presented in a later publication. In this paper we want to study first the various theoretical possibilities contained in the described K-matrix formalism.

Up to now, we assumed $m=1$ and considered the coefficient D measuring the diffraction-dissociation contribution to elastic scattering to be small compared to C such that quadratic terms in D could be neglected. Let us now assume $m=2$ in Eq. (19) and discuss the implications in this case.

Although the constant D is still considered to be small, the additional factor $\ln(s/s_0)$ appearing now in $D(b,s)$ will eventually force the D contribution to dominate such that the Pomeranchuk contribution to $\eta^{(1)}(b,s)$ is, at very high energies, given by

$$\eta_P{}^{(1)}(b,s)\approx-\frac{D(b,s)}{1+iD(b,s)},\qquad D(b,s)=D\left(\ln\frac{s}{s_0}\tilde{I}_0(b,s)\right)^3. \tag{33}$$

It is easy to show that the real as well as the imaginary part of the elastic forward-scattering amplitude now behaves for large s like $(s/s_0)\ln(s/s_0)$, and that the total cross section is given by

$$\sigma_{\mathrm{tot}}{}^P(s)\approx(8/3)\pi\alpha'\ln\frac{s}{s_0}\ln\left[1+\frac{D^2}{(4\pi\alpha' s_0)^6}\right],\quad m=2. \tag{34}$$

Considering finally arbitrary positive values of m bigger than 2 in Eq. (19) changes the diverging asymptotic behavior (34) by an additional factor $\ln[\ln(s/s_0)]$. In detail, one obtains the large-s behavior

$$\sigma_{\mathrm{tot}}{}^P(s)\sim(m-2)\gamma\ln\frac{s}{s_0}\ln\left(\ln\frac{s}{s_0}\right)\quad\text{for}\quad m\geq3, \tag{35}$$

where γ is a constant. It thus results from Eq. (35) that an arbitrary power of $\ln(s/s_0)$ in Eq. (19) is still in agreement with the Froissart limit for total cross sections.

Having investigated the consequences of the possible values of m in Eq. (19), we do not pursue the possibility of logarithmically diverging total cross sections any further here. Instead, we ask the more interesting question: Does the proposed K-matrix model provide an example for the behavior

$$\mathrm{Re}f^{(1)}(s,t=0)\sim(s/s_0)\ln(s/s_0),\qquad \mathrm{Im}f^{(1)}(s,t=0)\sim s/s_0\,? \tag{36}$$

According to the usual arguments involved in the proof of the Pomeranchuk theorem, Eqs. (36) imply that although total cross sections become constant asymptotically, they in fact approach *different* constant values for particle-particle and particle-antiparticle scattering, i.e., the Pomeranchuk theorem is violated. It would be illuminating to have a relativistic model for elastic scattering satisfying Eqs. (36) explicitly without having to derive this property from a forward dispersion relation under the above-stated assumption regarding the particle-particle and particle-antiparticle cross sections at infinity. In particular, after the total cross-section measurements from Serpukhov have appeared, it would be interesting to investigate relativistic theories in which Eqs. (36) are true. Unfortunately the model proposed in this paper is not of this category. It is impossible to obtain the behavior (36) starting from an expression for $\eta_P{}^{(1)}(b,s)$ having the structure of the right-hand side of Eq. (22). At most, one can obtain from Eq. (22) the behavior $\mathrm{Re}f^{(1)}(s,t=0)\sim(s/s_0)[\ln(s/s_0)]^{1/2}$, $\mathrm{Im}f^{(1)}(s,t=0)\sim s/s_0$ for $C(b,s)$ as given by Eq. (16) and $D(b,s)$ as given by Eq. (20) with $m=\frac{3}{2}$.

V. CROSSOVER PHENOMENON

Since the results on total cross sections from Serpukhov have appeared, a number of theoretical models have been investigated[35–37,1] which predict a logarithmic approach to asymptotic conditions similar to the behavior obtained in Eq. (28) above. Moreover, the question has been raised whether the Pomeranchuk theorem in fact holds or whether total cross sections approach different values for particle-particle and particle-antiparticle collisions, or even grow logarithmically. Even if the latter two possibilities were rendered unlikely by new experimental data as, for instance, precise determinations of the phases of elastic forward-scattering amplitudes at high energies, one would still have to conclude—assuming now a Pomeranchuk theorem to hold—that asymptotic conditions are approached only at extremely high energies. In such a situation it would be interesting to see whether there are further measurable quantities related to the limiting values $\sigma_{\mathrm{tot}}(\infty)$ for various processes. In the K-matrix model presented in this paper this is true in principle for the crossover point. We briefly recall

[35] N. W. Dean, Phys. Rev. D **1**, 2703 (1970).
[36] V. Barger and R. J. N. Phillips, Phys. Rev. Letters **24**, 291 (1970).
[37] J. M. Kaplan and L. Schiff, Nuovo Cimento Letters **3**, 19 (1970).

that the crossover (c.o.) point is the momentum-transfer value $t=t_{c.o.}$ of order -0.2 to -0.3 GeV^{-2} where the differential cross sections for pp and $p\bar{p}$, π^+p and π^-p, and K^+p and K^-p intersect, respectively.

For a long time this crossover phenomenon presented a difficulty in the framework of the Regge-pole theory and could only be accounted for by the insertion of *ad hoc* zeros into the residue functions of certain lower-lying trajectories. This prescription, however, was in disagreement with factorization. On the other hand, the crossover phenomenon can be understood in models which include absorptive corrections to Regge exchanges, avoiding at the same time the contradiction with the factorization principle for Regge poles.[38,39]

We have seen in Sec. III that the K-matrix model produces, besides a vacuum-exchange contribution, an absorptive correction to the Regge-pole exchanges. Moreover, the damping factor which results from the unitarization procedure leading to Eqs. (21) above is expressible in terms of the same quantities which govern the elastic scattering. We therefore ask the question: What kind of constraints result in this model from the crossover condition? In particular, can one obtain some connection between the constants C and D or—what amounts to the same thing—between $\sigma_{\text{tot}}(\infty)$ and $D/4\pi\alpha' s_0 C$ appearing in Eqs. (21), (26), and (30)? We first derive the crossover condition in this model. Then we turn to its numerical evaluation under a certain assumption regarding the ghost-eliminating mechanism. To be specific, we consider the case of elastic pp and $p\bar{p}$ collisions.

The vanishing of the cross-section difference $(d\sigma/dt)_{pp} - (d\sigma/dt)_{p\bar{p}}$ at the crossover point is usually attributed to the vanishing of the interference term between the (absorptive-corrected) ω contribution and the Pomeranchuk contribution. The ω term is supposed to be the only exchange with C number -1 present in pp and $p\bar{p}$ interactions. A possible ρ contribution is usually neglected. The crossover condition, therefore, reads in our language

$$0=\left[\left(\frac{d\sigma}{dt}\right)_{pp}-\left(\frac{d\sigma}{dt}\right)_{p\bar{p}}\right]_{t=t_{c.o.}}=\frac{1}{4\pi q^2 s}\times\left[f_P^*(s,t_{c.o.})f_\omega(s,t_{c.o.})+f_P(s,t_{c.o.})f_\omega^*(s,t_{c.o.})\right], \quad (37)$$

where $f_P(s,t)$ and $f_\omega(s,t)$ are given by

$$f_P(s,t)=2\pi s\int_0^\infty bdb\frac{iC(b,s)-D(b,s)}{1+C(b,s)+iD(b,s)}J_0(b\sqrt{-t}) \quad (38)$$

and

$$f_\omega(s,t)=2\pi s\int_0^\infty bdb\, N_\omega^{(1)}(b,s)\times\frac{1-C(b,s)-2iD(b,s)}{[1+C(b,s)+iD(b,s)]^2}J_0(b\sqrt{-t}), \quad (39)$$

with $C(b,s)$ and $D(b,s)$ as defined in Eqs. (16) and (20) (the latter with $m=1$). Considering again only linear terms in D, one derives from Eqs. (37)–(39) the following general crossover condition for *one* participating Regge trajectory—here the ω trajectory:

$$0=\left[\int_0^\infty bdb\,\operatorname{Im}\eta_P^{(1)}(b,s)J_0(b\sqrt{-t_{c.o.}})\right]\times\left[\int_0^\infty bdb\,[\operatorname{Re}N_\omega^{(1)}(b,s)A(b,s)-\operatorname{Im}N_\omega^{(1)}(b,s)B(b,s)]\times J_0(b\sqrt{-t_{c.o.}})\right]-\left[\int_0^\infty bdb\,\operatorname{Re}\eta_P^{(1)}(b,s)J_0(b\sqrt{-t_{c.o.}})\right]\times\left[\int_0^\infty bdb\,\operatorname{Re}N_\omega^{(1)}(b,s)B(b,s)J_0(b\sqrt{-t_{c.o.}})\right], \quad (40)$$

where

$$A(b,s)=4\frac{R(s)\exp(-3b^2/4\bar{\rho})}{[1+Q(s)\exp(-b^2/2\bar{\rho})]^3}, \quad (41a)$$

$$B(b,s)=\frac{1-Q(s)\exp(-b^2/2\bar{\rho})}{[1+Q(s)\exp(-b^2/2\bar{\rho})]^2}, \quad (41b)$$

$$\operatorname{Im}\eta_P^{(1)}(b,s)=\frac{Q(s)\exp(-b^2/2\bar{\rho})}{1+Q(s)\exp(-b^2/2\bar{\rho})}, \quad (41c)$$

$$\operatorname{Re}\eta_P^{(1)}(b,s)=-\frac{R(s)\exp(-3b^2/4\bar{\rho})}{[1+Q(s)\exp(-b^2/2\bar{\rho})]^2}. \quad (41d)$$

We have used here as abbreviations the quantities $Q(s)$ defined in Eq. (27) [compare also Eq. (29)], and $R(s)$ defined by

$$R(s)=\frac{D}{(4\pi\alpha' s_0)^3\ln(s/s_0)}. \quad (42)$$

Furthermore, one has

$$\frac{R(s)}{Q(s)}=\frac{D/4\pi\alpha' s_0}{C}=D'. \quad (43)$$

If D were exactly zero and the ghost-eliminating mechanism for the ω trajectory known such that $N_\omega^{(1)}(b,s)$ could be regarded as uniquely given, then Eq. (40) would allow a determination of $\sigma_{\text{tot}}(\infty)$ from the experimental measurement of the crossover point. In practice, however, there are a number of difficulties. First, the crossover points are not known accurately enough. For the pair pp, $p\bar{p}$ the crossover point is, from the data of Foley *et al.*[40] at $p_{\text{lab}}=11.8$ GeV/c, found to be at $t\approx-0.20$ GeV^2. The differential cross section curves for π^+p and π^-p at $p_{\text{lab}}=12.4$ GeV/c[41] intersect

[38] For a more detailed discussion see Ref. 39.
[39] W. Drechsler, Fortschr. Physik (to be published).

[40] K. J. Foley *et al.*, Phys. Rev. Letters **15**, 45 (1965).
[41] D. Harting *et al.*, Nuovo Cimento **38**, 60 (1965).

in a broad region around $t=-0.37$ GeV^{-2}. There seems to be no reasonably accurate determination of the crossover in high-energy $K^{\pm}p$ scattering. Secondly, the assumption of only one contributing Regge trajectory in Eq. (40) might be misleading, in particular since we are studying a situation where the leading Regge-pole contribution [the second integral in each term in Eq. (40)] is very small near the crossover point. It is, therefore, not necessarily safe to neglect other contributions, for example the ρ pole.[42] A further uncertainty is introduced by the particular ghost-eliminating mechanism obeyed by the trajectory (compare the discussion at the end of Sec. IV). Despite these difficulties, we have numerically investigated Eq. (40) on a computer, primarily to get at least some approximate numerical information about the values for[43] $D'=D/4\pi C$ which are involved. Remember that our derivations above and in the preceding sections were based on the assumption that D' is small. Quadratic terms in D' were neglected throughout. It would therefore be interesting to see what values of D' are needed in this formalism to account for the crossover phenomenon.

We do not consider the numerical results given below to be more than a qualitative estimate. Assuming the minimal ghost-eliminating mechanism introduced in Sec. IV and taking $\alpha_\omega(0)=\frac{1}{2}$,[44] we derive from Eq. (40) for $p_{\rm lab}=11.8$ GeV/c and for various assumed values for $t_{\rm c.o.}$ the possible values of $\sigma_{\rm tot}(\infty)$ and D' for pp and $p\bar{p}$ collisions shown in Fig. 4.

We first note that the result is rather sensitive to the actual value of $t_{\rm c.o.}$. The upper three curves in Fig. 4 correspond to the value $t_{\rm c.o.}=-0.20$ GeV^2, having an estimated statistical error $\Delta t_{\rm c.o.}=\pm 0.02$ GeV^2. For increasing positive values of D', the corresponding values for $\sigma_{\rm tot}(\infty)$ are found to fall. The opposite is true for negative D'. Positive values for D' seem, therefore, to be favored. However, this statement has to be checked by a more detailed analysis of the pp and $p\bar{p}$ differential cross-section data in the framework of the K-matrix model.

The value for $\sigma_{\rm tot}{}^{pp}(\infty)$, obtained from the data of Ref. 31, is definitely too large. If at about $p_{\rm lab}=100$ GeV/c the contribution of the lower-lying trajectories to the total pp and $p\bar{p}$ cross sections are supposed to be small and neglected, and if a total cross section of 35.7 mb—corresponding according to Barger *et al.*[34] to the Pomeranchuk limit in a pure Regge-pole model—is identified with the Pomeranchuk contribution given by Eq. (30), one expects a value of 51 mb for $\sigma_{\rm tot}{}^{pp}(\infty)$. [This value is considered to be an upper limit on $\sigma_{\rm tot}{}^{pp}(\infty)$.] In the present model, and with the additional assumptions made in numerically evaluating Eq. (40), a crossover value of about $t_{\rm c.o.}=-0.37$ GeV^2 at $p_{\rm lab}=11.8$ GeV/c incident protons or antiprotons is needed to obtain such a value (compare Fig. 4). A more definite statement about the crossover predicted in our model can evidently only be obtained from a detailed fit to the experimental data on pp and $p\bar{p}$ differential cross sections. We note in passing that in the Glauber type of analysis of pp and $p\bar{p}$ scattering carried out by Chiu and Finkelstein[2,45] the crossover obtained from a fit to the experimental differential cross-section data at this energy is found to be at $t=-0.37$ GeV^2. It would be interesting to have new and accurate experimental information on the crossover points. We finally remark that the K-matrix model predicts the crossover to be shifted to smaller values of $|t|$ when the collision energy increases.

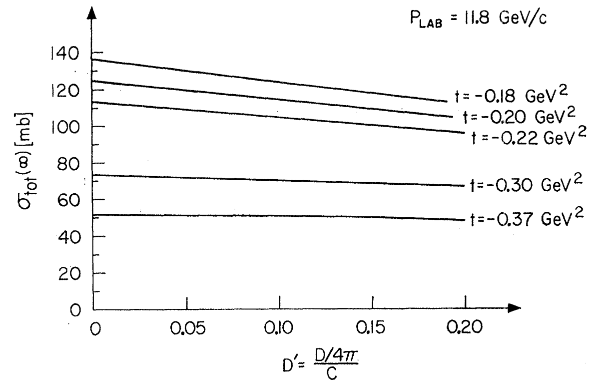

FIG. 4. Dependence of $\sigma_{\rm tot}(\infty)$ on D' for various assumed values of the crossover point in pp and $p\bar{p}$ scattering at $p_{\rm lab}=11.8$ GeV/c.

VI. DISCUSSION

Neglecting complications due to spin and isospin, we started from the assumption that the inelastic states in the unitarity relations can effectively be represented by a set of quasi-two particle states. A K-matrix formalism for high-energy scattering was proposed, using as a framework the impact-parameter representation of scattering amplitudes. It was shown with the help of this unitarization procedure that the Pomeranchuk contribution to high-energy elastic scattering and diffraction-dissociation processes can be interpreted as being due to multiple-Regge-pole exchanges accompanied by the formation of a sequence of excited intermediate states of the colliding particles. In terms of j-plane properties, this interpretation of the vacuum-exchange contribution corresponds to a superposition of cuts in the angular-momentum plane.

[42] This is true only if $\alpha_\rho(t)$ is different from $\alpha_\omega(t)$. If the ω and the ρ trajectory are taken to be equal and, furthermore, obey the same ghost-eliminating mechanism, the crossover condition is again given by Eq. (40).

[43] We take, as before, $\alpha'=1$ GeV^{-2} and $s_0=1$ GeV^2.

[44] We varied $\alpha_\omega(0)$ by about 20% around the value 0.50 and found that this had little effect on the analysis compared to the other uncertainties involved, i.e., the experimental error of the crossover determination. To give, however, an impression, we remark that lowering the ω intercept to 0.40 would lower, for instance, the curve corresponding to $t=-0.37$ GeV^2 in Fig. 4 by about 4 mb. Furthermore, the slope α_ω' has been taken to be 1 GeV^{-2}. The residue β_ω drops out of Eq. (40).

[45] C. B. Chiu, Rev. Mod. Phys. 41, 640 (1969).

The consequences of the proposed model for Pomeranchuk exchange were investigated in some detail. The model predicts that elastic differential cross sections shrink with increasing energy at a rate corresponding approximately to an effective Pomeranchuk pole having a slope $\alpha_P' = 0.5\ \mathrm{GeV}^{-2}$ in agreement with the recent Serpukhov measurements. If a Pomeranchuk theorem holds, the asymptotic limit of total cross sections are predicted to be approached in a logarithmic fashion from below. Finally, the crossover phenomenon was investigated, which is in this model due to the vanishing of a Regge-pole contribution corrected for absorption and being odd under charge conjugation. The absorptive corrections to conventional Regge-pole expressions predicted by the model are given in terms of quantities characterizing the elastic scattering in the asymptotic region. It was pointed out that the analysis of the crossover condition provides information about total cross sections at asymptotic energies. We conclude by noting that the proposed K-matrix model is not limited to small values of momentum transfers. However, for large values of t it probably becomes essential to take the spin of the external particles into account.

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$K \to 3\pi$ Decays in Dual Models*

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The dual representation of $K \to 3\pi$ decay amplitudes is studied. It is found that $K \to 3\pi$ decay amplitudes in a generalized Veneziano model are incompatible with current-algebra relations. For example, pion poles and kaon poles are not dual to other poles. An example of realistic $K \to 3\pi$ decay amplitudes which contain pion and kaon poles, which contain both $|\Delta \mathbf{I}| = \frac{1}{2}$ and $\frac{3}{2}$ parts, and which are compatible with current-algebra relations is obtained. Our results seem to suggest that we should prefer the charged-current$\times$ charged-current nonleptonic weak Hamiltonian to Hamiltonians with pure $|\Delta \mathbf{I}| = \frac{1}{2}$.

I. INTRODUCTION

DUAL representation of reaction amplitudes (Veneziano model[1] and generalized Veneziano model[2–7]) has been discovered. Amplitudes in this representation have resonance poles at low energy, have Regge behavior at high energy, satisfy the crossing relations, and give relations among Regge trajectories such as the exchange degeneracy. In this representation, poles in various channels of a reaction are related so closely that, for example, a sum of all s-channel poles of the amplitude is equal to a sum of poles in other channels. This property of the amplitudes is the so-called (full) duality. However, the above definition of duality is not practical for our purpose. Since the generalized Veneziano amplitude for N-point function is the only amplitude with full duality, we define the full duality of an amplitude as follows: An amplitude of an N-prong reaction is completely dual (has full duality) if and only if it is expressed as a sum of a finite number of generalized Veneziano amplitudes for N-point function.

A purpose of this article is to study whether a weak amplitude has full duality. Dual representations for K_{l4} decay amplitudes have been studied by the present author,[8] and it has been found that the kaon pole is not dual to any other poles if we impose conditions required by current algebra at soft-pion limits. However, it has been found that all poles except for the kaon pole can be dual if the relation among trajectories,

$$\alpha_{K^*}(t) - \alpha_K(t) = 1 - \alpha_\rho(M_\pi^2) \quad \text{for } \alpha_K(t) = \text{positive integers}, \quad (1.1)$$

is satisfied.[9] In this article we consider whether it is

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968).

[3] M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969).

[4] M. H. Chan, Phys. Letters **28B**, 425 (1968); M. H. Chan and S. T. Tsou, *ibid.* **28B**, 485 (1969).

[5] Z. Koba and H. D. Nielsen, Nucl. Phys. **B10**, 633 (1969).

[6] C. J. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969).

[7] Z. Koba and H. D. Nielsen, Nucl. Phys. **B12**, 517 (1969).

[8] Y. Hara, Phys. Rev. D **1**, 874 (1970).

[9] By applying the method used in Ref. 8 to the $\pi + \pi \to \pi + l + \nu$ processes, we find a relation

$$\alpha_\rho(t) - \alpha_\pi(t) = 1 - \alpha_\rho(M_\pi^2) \quad \text{for } \alpha_\pi(t) = \text{positive integers}. \quad (A)$$

If we assume that all trajectories are linear and parallel, we find the relations $\alpha_\rho(M_\pi^2) = \frac{1}{2}$ and $M_\rho^2 - M_\pi^2 = M_{K^*}^2 - M_K^2$ from the relations (1.1) and (A). No other relations among Regge trajectories are obtained by studying similar leptonic processes such as $\eta + \pi^+ \to \eta + l + \nu$.