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Asymptotic $SU(2)$ Symmetry. I. Broken $SU(2)$ Mass Sum Rules

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As a natural extension of the formulation of asymptotic $SU(3)$ symmetry previously presented, a new formulation of asymptotic $SU(2)$ symmetry is proposed. Intuitively speaking, we assume that both the $SU(3)$ and $SU(2)$ symmetries are well realized among particles of extremely high momenta where masses are not important. This point of view is formulated by assuming that, only in the asymptotic limit, the matrix elements of the $SU(2)$ generators V_{π^+} and V_{π^-} and also the $SU(3)$ generator V_K [which, in the symmetry limit, are the isotopic-spin raising and lowering and the $SU(3)$ raising operators, respectively] behave, to a reasonably good approximation, as if the symmetries were not broken. Mass sum rules are obtained by using these asymptotic $SU(2)$ and $SU(3)$ symmetries together with exotic charge commutators which involve the time derivative of $V_{\pi^\pm}$ and which do not depend explicitly on the specific parameters of symmetry breaking. Assuming that the basic (and not effective) $SU(2)$-breaking interaction transforms like an $SU(3)$ octet, the exotic commutators are $[\dot{V}_{\pi^+},V_{\pi^+}]=[V_{K^0},\dot{V}_{\pi^-}]=[V_{K^+},\dot{V}_{\pi^+}]=0$, etc. From them we reproduce almost all the $SU(2)$ sum rules previously obtained on the assumption of effective octet dominance. However, contrary to previous results, the π^+ and π^0, for example, are no longer degenerate in mass. The mass difference is explained in terms of the η^0-π^0 and η'^0-π^0 mixings. A study is also made of the exotic commutators involving the axial-vector charges. The commutator $[\dot{V}_{\pi^+},A_{\pi^+}]=0$ is the least model-dependent one. From this we obtain an intermultiplet baryon mass formula involving the a decuplet and the b octet: $(\delta_a)^2\simeq(p_b)^2-(n_b)^2$ (a and b are arbitrary). (p_b) and (n_b) denote the masses of the proton and neutron of the b octet, respectively. $(\delta_a)^2$ denotes the equal-squared-mass spacing of the a decuplet, i.e., $(\delta_a)^2=(\Delta_a^{++})^2-(\Delta_a^+)^2=(\Delta_a^+)^2-(\Delta_a^0)^2=(\Xi_a^{*0})^2-(\Xi_a^{*-})^2$. The case of $a=\frac{3}{2}^+$ and $b=\frac{1}{2}^+$ coincides with the good sum rule of $SU(6)$. For bosons, the exotic commutators such as $[\dot{V}_{\pi^-},A_{K^0}]=[\dot{V}_{\pi^+},A_{K^+}]=0$, which are more model dependent than the $[\dot{V}_{\pi^+},A_{\pi^+}]=0$, produce the following general intermultiplet mass sum rule: $(K_\alpha^0)^2-(K_\alpha^+)^2=\text{const}\simeq0.004$ (GeV)^2 (α is arbitrary). Here (K_α^0) denotes the mass of the K^0 meson belonging to the α octet. For the 0^- and 1^- octets, it implies $(K^0)^2-(K^+)^2=(K^{*0})^2-(K^{*+})^2$, which is not inconsistent with present experiment. We also show that both the commutators, $[\dot{V}_{\pi^+},V_{\pi^+}]=0$ and $[\dot{V}_{\pi^+},A_{\pi^+}]=0$, give, in general, the same mass sum rule when they are applied to the same $SU(2)$ multiplet. Sum rules for the axial-vector semileptonic hyperon decay couplings in broken $SU(3)$ and $SU(2)$ symmetries are also derived.

I. INTRODUCTION

THE concept of asymptotic $SU(3)$ symmetry has been applied by many authors. The idea was applied particularly to the spectral functions of an appropriately chosen combination of the vector and axial-vector currents.[1] Physically, the hope is that the kinematical term will eventually dominate in the high-energy region over all masses and interaction Hamiltonians.

Recently we have proposed another approach.[2] We single out the charge operator V_K [which is the $SU(3)$ raising or lowering operator in the $SU(3)$ limit] among many other physical quantities. The reason is that, if the application of the notion of asymptotic $SU(3)$ symmetry were indeed to be successful, such symmetry should probably be reflected in the asymptotic behavior of the matrix elements of the charge V_K. As the simplest possibility we have chosen the asymptotic condition as follows:

Even in broken $SU(3)$ symmetry, the operator V_K still behaves, to a good approximation, as if it were an exact $SU(3)$ generator. However, we assume this only in the asymptotic limit, i.e., when we deal with the matrix elements of V_K evaluated *only* in the zero four-momentum transfer limit.

Since, in the presence of $SU(3)$ mass splitting, this limit can be realized only by taking an appropriate infinite-momentum limit, this approach may also be viewed as a kind of asymptotic symmetry. We have shown that this approach not only can reproduce all the good results of spectral functions sum rules,[1] but can also produce many other broken $SU(3)$ sum rules,[2,3] when combined with the $SU(3)\otimes SU(3)$ current algebra.[4] We have also shown that the study of the commutators involving $\dot{V}_K$, the time derivative of V_K, gives informa-

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[1] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967); T. Das, V. S. Mathur, and S. Okubo, *ibid.* **18**, 761 (1967). For review of extensive literature, see S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 253.

[2] For example, S. Matsuda and S. Oneda, Phys. Rev. **174**, 1992 (1968); Nucl. Phys. **B9**, 55 (1969).

[3] S. Matsuda and S. Oneda, Phys. Rev. **158**, 1594 (1967); **165**, 1749 (1968); **169**, 1172 (1968); **171**, 1743 (1968); S. Matsuda, S. Oneda, and P. Desai, *ibid.* **178**, 2129 (1969); G. Fourez and S. Oneda, Nuovo Cimento **59A**, 65 (1969).

[4] M. Gell-Mann, Physics **1**, 63 (1964).

tion about the pattern of mass splitting of hadrons. If we find simple charge commutators involving the $\dot{V}_K$ which do not depend explicitly on the specific parameters of $SU(3)$ breaking, our asymptotic symmetry is able to produce mass formulas in an algebraic way. Vice versa, these mass formulas can also be viewed as the constraints which produce, to a reasonable approximation, the asymptotic symmetry discussed above. We have suggested that such exotic commutators do exist and have derived from them not only the Gell-Mann–Okubo (GMO) mass formulas but also the intermultiplet mass sum rules which include the $SU(6)$ formulas as a special case.[2,5,6] In this paper we wish to extend our concept of asymptotic symmetry to the case of broken $SU(2)$ symmetry and to study the broken $SU(2)$ mass formulas. We find that the argument here is almost parallel to the argument in the case of broken $SU(3)$ symmetry. We assume asymptotic $SU(2)$ symmetry for the isotopic-spin raising or lowering operator, I_+ or I_-, and look for the exotic commutators involving the time derivative of I_+ and I_-. In this way we cannot only reproduce all the good results of the old treatment based on the assumption of octet dominance but also derive new sum rules, including, in particular, the general intermultiplet mass sum rules. In our approach the $\pi^\pm$ and π^0, for example, are no longer degenerate in mass. The $\pi^\pm$-π^0 mass difference can be explained in terms of the η^0-π^0 and η'^0-π^0 mixings. The results seem to be a significant improvement of the previous broken $SU(2)$ sum rules.

II. ASYMPTOTIC $SU(2)$ SYMMETRY

The third component of the isotopic spin, I_3 (in our notation $I_3 \equiv V_{\pi^0}$), is conserved if we deal with only the strong interaction and $SU(2)$-breaking interactions such as the electromagnetic interaction. However, the isotopic-spin raising and lowering operators, I_+ and I_- ($I_+ \equiv V_{\pi^+}$ and $I_- \equiv V_{\pi^-}$ in our notation), are no longer conserved in broken $SU(2)$ symmetry. Our asymptotic $SU(2)$ symmetry can be stated roughly as follows:

Even in broken $SU(2)$ symmetry the charges V_{π^+} and V_{π^-} still act as exact $SU(2)$ generators[7] but *only* in the appropriately chosen infinite-momentum limit.

The spirit of the approximation may be seen as follows. Consider, for example, the following matrix element of V_{π^-}:

$$\langle K^0(\mathbf{p}')|V_{\pi^-}|K^+(\mathbf{p})\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')(2p_0 2p_0')^{1/2} \times[F_+(q^2)(p_0'+p_0)+F_-(q^2)(p_0-p_0')], \quad (1)$$

where $q^2=(p-p')_\mu{}^2$. In exact $SU(2)$ symmetry where $m_{K^+}=m_{K^0}$, q^2 is always zero and $F_+(0)$ and $F_-(0)$ take the $SU(2)$ values $F_+(0)=F_+{}^s(0)=1$ and $F_-(0)=F_-{}^s(0)=0$, i.e.,

$$\langle K^0(\mathbf{p}')|V_{\pi^-}|K^+(\mathbf{p})\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')F_+{}^s(0). \quad (2)$$

Now, in broken $SU(2)$ symmetry, q^2 is, in general, no longer zero, because of the mass difference, and the values of $F_+(q^2)$ and $F_-(q^2)$ will be different from their $SU(2)$ values. However, if we take the infinite-momentum limit $|\mathbf{p}|=\infty$, we can still deal with *only* the F_+-type form factor of the matrix element of V_{π^-} evaluated at the zero-momentum-transfer limit. Namely, we still have in broken symmetry

$$\lim_{|\mathbf{p}|\to\infty}\langle K^0(\mathbf{p}')|V_{\pi^-}|K^+(\mathbf{p})\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')F_+(0). \quad (3)$$

Now our asymptotic symmetry requires that even in broken symmetry the value of $F_+(0)$ is, to a good approximation, not renormalized, i.e., $F_+(0)=F_+{}^s(0)=1$, and that the V_{π^+} and V_{π^-} act as if they were exact $SU(2)$ generators. However, this is assumed only in the infinite-momentum limit. We note that we do not need to impose any condition on the F_- form factor since it is multiplied by a factor (p_0-p_0') which vanishes in our limit.[8] The approximation involved looks reasonable, especially in light of the following proof of the Ademollo-Gatto theorem.[9] Let us symbolically denote the strength of the $SU(2)$-symmetry-breaking interaction by ϵ. Insert the equal-time commutator of the isotopic-spin operators, $[V_{\pi^+},V_{\pi^-}]=2V_{\pi^0}$, between the states $\langle K^+(\mathbf{q})|$ and $|K^+(\mathbf{q}')\rangle$ with $|\mathbf{q}|=\infty$. We then obtain

$$\langle K^+|V_{\pi^+}|K^0\rangle\langle K^0|V_{\pi^-}|K^+\rangle +\sum_n \langle K^+|V_{\pi^+}|n\rangle\langle n|V_{\pi^-}|K^+\rangle -\sum_{n'} \langle K^+|V_{\pi^-}|n'\rangle\langle n'|V_{\pi^+}|K^+\rangle =2\langle K^+|V_{\pi^0}|K^+\rangle, \quad \text{with } |\mathbf{q}|\to\infty\,.$$

The right-hand side of this equation is 1 [apart from the factor $(2\pi)^3\delta^3(\mathbf{q}-\mathbf{q}')$]. On the left-hand side the nondiagonal matrix elements $\langle K^+|V_{\pi^+}|n\rangle$, $\langle K^+|V_{\pi^-}|n'\rangle$, etc., are at least of the order $O(\epsilon)$. Using Eq. (3), we thus obtain $F_+{}^2(0)+O(\epsilon^2)=1$. This gives the Ademollo-Gatto theorem,[9] $F_+(0)=F_+{}^s(0)+O'(\epsilon^2)$. In the above proof it is important to notice that the $F_-(0)$, which is of the order $O(\epsilon)$, does not contribute. This is only possible when we take the limit $|\mathbf{q}|\to\infty$. Thus we have explicitly seen that the effect of symmetry breaking is indeed apparently minimal at the points near $q^2=0$ where our assumption of asymptotic symmetry is

[5] S. Matsuda and S. Oneda, Phys. Rev. **179**, 1301 (1969).

[6] S. Matsuda and S. Oneda, Phys. Rev. D **1**, 944 (1970).

[7] By this we mean that the V_{π^+} and V_{π^-} connect only the members of the same $SU(2)$ multiplet and the values of these matrix elements take the exact $SU(2)$ values.

[8] We assume that the $F_-(q^2)$ does not have a singularity of the form $1/q^2$. This is quite unlikely.

[9] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1964). Our argument here is along the lines first discussed by Fubini and co-workers: S. Fubini and G. Furlan, Physics **1**, 229 (1965); S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965); G. Furlan, F. Lannoy, C. Rossetti, and G. Segrè, *ibid.* **40**, 597 (1965). We may even think of a possibility that the $O(\epsilon^2)$ or $O'(\epsilon^2)$ term is proportional to q^2 so that $F_+(0)\simeq F_+{}^s(0)$ to a very good approximation.

effectively made. The $O'(\epsilon^2)$ term will be exactly zero[9] if the nondiagonal matrix elements of the $V_{\pi^\pm}$ vanish in the asymptotic limit. If this is close to the real situation, the $V_{\pi^\pm}$ act as if they were the exact $SU(2)$ generators but, of course, *only* in this asymptotic limit. We do not offhand know to what extent this approximation is valid. In broken $SU(3)$ symmetry we have demonstrated[2] that the asymptotic $SU(3)$ symmetry for the V_K is compatible with having the GMO mass splitting. This is quite satisfactory. In the above argument the possible $SU(2)$ particle mixing has not been considered. Our asymptotic $SU(3)$ and $SU(2)$ symmetries are essentially equivalent to assuming that the "in" and "out" states of a particle transform linearly according to a definite irreducible representation of the group, but that this is best justified in the asymptotic limit described above. If the symmetry-breaking interaction is able to induce mixing of particles which belong to different irreducible representations in the symmetry limit, the proper "in" and "out" states must be constructed by diagonalization. We can consistently take into account this effect of mixing in the asymptotic limit. For illustration let us consider the η^0-π^0 mixing assuming that the η' [$I=0$, $SU(3)$-singlet P meson] does not exist. (For the inclusion of the η' see Sec. V.) We consider the matrix element

$$\langle\pi^+(\mathbf{q})|[V_{\pi^+},V_{\pi^-}]|\pi^+(\mathbf{q}')\rangle = 2\langle\pi^+(\mathbf{q})|V_{\pi^0}|\pi^+(\mathbf{q}')\rangle \quad \text{with } |\mathbf{q}|\to\infty\,.$$

We obtain by picking up now the π^0 and η^0 intermediate states

$$\langle\pi^+(\mathbf{q})|V_{\pi^+}|\pi^0\rangle\langle\pi^0|V_{\pi^-}|\pi^+(\mathbf{q}')\rangle + \langle\pi^+(\mathbf{q})|V_{\pi^+}|\eta^0\rangle\langle\eta^0|V_{\pi^-}|\pi^+(\mathbf{q}')\rangle + O(\epsilon^2) = 2(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')\,. \quad (4)$$

Formally, the above η^0 term is also of the order $O(\epsilon^2)$. However, this term can no longer be ignored. In broken $SU(2)$ symmetry, $\langle\pi^+|V_{\pi^+}|\eta^0\rangle$ is no longer zero. We write, in the frame $|\mathbf{q}|\to\infty$, the physical $\langle\pi^0|$ and $\langle\eta^0|$ states in terms of the $SU(3)$ states,

$$\langle\pi^0(\mathbf{q})| = \cos\theta\langle\pi_8{}^0(\mathbf{q})| + \sin\theta\langle\eta_8{}^0(\mathbf{q})|$$

and

$$\langle\eta(\mathbf{q})| = -\sin\theta\langle\pi_8{}^0(\mathbf{q})| + \cos\theta\langle\eta_8{}^0(\mathbf{q})|\,.$$

Here $\pi^0\to\pi_8{}^0$ and $\eta^0\to\eta_8{}^0$ when $\epsilon\to 0$, and we assume

$$\lim_{|\mathbf{q}|\to\infty}\langle\eta_8{}^0(\mathbf{q})|V_{\pi^-}|\pi^+(\mathbf{q}')\rangle = 0$$

and

$$\lim_{|\mathbf{q}|\to\infty}\langle\pi_8{}^0(\mathbf{q})|V_{\pi^-}|\pi^+(\mathbf{q}')\rangle = (2\pi)^3\delta^3(\mathbf{q}-\mathbf{q}')G_+(0)\,,$$

where $G_+(q^2)$ is the form factor analogous to $F_+(q^2)$. Corresponding to Eq. (3), we now have

$$\lim_{|\mathbf{q}|\to\infty}\langle\pi^0(\mathbf{q})|V_{\pi^-}|\pi^+(\mathbf{q}')\rangle = (2\pi)^3\delta^3(\mathbf{q}-\mathbf{q}')\cos\theta\, G_+(0)$$

and

$$\lim_{|\mathbf{q}|\to\infty}\langle\eta^0(\mathbf{q})|V_{\pi^-}|\pi^+(\mathbf{q}')\rangle = (2\pi)^3\delta^3(\mathbf{q}-\mathbf{q}')\sin\theta\, G_+(0)\,.$$

Then from Eq. (4) we again obtain $G_+(0)=G_+{}^s(0)+O(\epsilon^2)$, $G_+{}^s(0)$ being the exact $SU(2)$ value, $-\sqrt{2}$. This is the modification of the Ademollo-Gatto theorem when particle mixing takes place. We take into account the $SU(2)$ mixing always in the matrix elements of the vector charges $V_{\pi^\pm}$, and this is always carried out in the infinite-momentum frame where the actual value of the mass of a particle is not relevant.

III. USEFUL EQUAL-TIME COMMUTATORS INVOLVING TIME DERIVATIVE

Previously we have shown[2,5,6] that the combined use of charge commutators involving the time derivative of the $SU(3)$ charge V_K with our asymptotic $SU(3)$ symmetry can yield not only the GMO mass formulas but also the intermultiplet mass formulas, which include the $SU(6)$ formulas as a special case. In our approach, where the infinite-momentum limit is always utilized, the time derivative of a charge operator taken between two states, such as $\lim_{|\mathbf{p}|\to\infty}\langle A(\mathbf{p})|\dot{V}_K|B(\mathbf{p})\rangle$, gives rise to a factor $\lim_{|\mathbf{p}|\to\infty}[(\mathbf{p}^2+m_A{}^2)^{1/2}-(\mathbf{p}^2+m_B{}^2)^{1/2}]$, which produces a factor $(m_A{}^2-m_B{}^2)/|\mathbf{p}|$ in the limit. Therefore, useful mass sum rules may be derived if we find the commutators involving $\dot{V}$ which are not explicitly dependent on the specific parameters of the symmetry breaking. The mass formulas obtained by using such "exotic" commutators are least dependent on the model of symmetry breaking and may be the only sum rules which can be obtained in a purely algebraic manner. We first review the commutators[2,6,10] used for deriving the $SU(3)$ mass sum rules, since the situation of broken $SU(2)$ symmetry will be quite analogous to that of broken $SU(3)$ symmetry. We use here a quark model as a guide though we believe that the result derived here will also be obtained in a more sophisticated model. Once suitable commutators are established, we may forget about their derivations, since only these commutators are relevant to our arguments.

We assume as usual that the $SU(3)$-breaking Hamiltonian density $H'(x)$ transforms like the $I=Y=0$ member of the $SU(3)$ octet. We stress that our $H'(x)$ is not meant to be the effective Hamiltonian density. We then find the following "exotic" commutators as first noted by Fubini and co-workers,[9]

$$[V_{K^0},\dot{V}_{K^0}] = [V_{K^+},\dot{V}_{K^+}] = [V_{\pi^-},\dot{V}_{K^0}] = [V_{\pi^+},\dot{V}_{K^+}] = 0\,, \quad (5)$$

and their conjugate complex equations. We have shown previously[2] that $[V_{K^0},\dot{V}_{K^0}]=[V_{K^+},\dot{V}_{K^+}]=0$ will give rise to the GMO mass formulas, including mixing if it exists. Next we search for the "exotic" commutators of

[10] C. A. Nelson, Phys. Rev. **181**, 1946 (1969).

the form $[A_i,\dot{V}_K]=0$. For this we need to specify the form of $H'(x)$.

Consider the $SU(3)$-breaking Hamiltonian density $H'(x)$, which has the following rather general form[2,6,10] in a quark model under consideration:

$$H'=\alpha S_8(x)+\beta d_{8ij}J_\mu{}^i(x)J_\mu{}^j(x)\,. \qquad (6)$$

Here $S_8(x)=\bar{q}(x)\lambda_8 q(x)$, the simplest mass splitting interaction in the quark model. d_{ijk} is Gell-Mann's d symbol, and $J_\mu{}^i(x)J_\mu{}^j(x)$ can be written in general[11] as

$$J_\mu{}^i(x)J_\mu{}^j(x)=V_\mu{}^i(x)V_\mu{}^j(x)+\gamma A_\mu{}^i(x)A_\mu{}^j(x)\,. \qquad (7)$$

For arbitrary values of the coefficients α, β, and γ, we find the following "exotic" commutators.

Group (A):

$$[\dot{V}_{K^0},A_{K^0}]=[\dot{V}_{K^+},A_{K^+}]=0$$
$$(\alpha,\ \beta,\ \text{and } \gamma \text{ are arbitrary}). \qquad (8)$$

In our model these "exotic" commutators are the safest ones to use. We may consider a more restricted form of symmetry breaking, for example, $\gamma=1$ (chiral invariance for the vector and axial-vector currents). We then find, in addition to the group-(A) commutators,

Group (B):

$$[\dot{V}_{K^0},A_{\pi^-}]=[\dot{V}_{K^0},A_{K^+}]=[\dot{V}_{K^+},A_{\pi^+}]=0$$
$$(\alpha \text{ and } \beta \text{ are arbitrary, but } \gamma=1). \qquad (9)$$

The Hamiltonian of Gell-Mann, Oakes, and Renner[12] (GOR), $H'=-u_0-Cu_8$, is not as general as the one given by Eq. (6). Their model leads not only to the group-(A) commutators, but also to the group-(B) commutators. The use of the group-(A) commutators and the asymptotic $SU(3)$ symmetry gave rise to general intermultiplet mass formulas[6] between a decuplet and an octet of baryons which take the form in the absence of $SU(3)$ particle mixing

$$(\Xi_a)^2-(\Sigma_a)^2=(\delta_b)^2=\text{const}$$
$$(a \text{ and } b \text{ are arbitrary}). \qquad (10)$$

Here (Ξ_a) and (Σ_a) denote the masses of the Ξ and Σ members of the a octet (a specifies the quantum number of the a octet) and $(\delta_b)^2$ is the equal-squared-mass splitting of the b decuplet. The case $a=\frac{1}{2}^+$ and $b=\frac{3}{2}^+$ coincides with the $SU(6)$ formula.[13,14] However, Eq. (10) goes further, i.e., these spacings are universal among any octet and decuplet baryons as long as we neglect mixing. One may now naturally ask: How far can one trust the results obtained by using the group-(B) commutators which are more model dependent? According to our previous work,[2,5,6] the use of the group-(B) commutators and our asymptotic $SU(3)$ symmetry gives rise to the Σ-Λ degeneracy in the baryon mass sum rules. Therefore, in the GOR model our asymptotic $SU(3)$ symmetry encounters the problem of Σ-Λ degeneracy. This reminds us of the similar situation met in the simple $SU(6)$ symmetry in which the $H'(x)$ is simply taken to be $S_8(x)$.[13] Thus, to the extent that we tolerate the Σ-Λ degeneracy, we may use the commutator (B) for baryons. For bosons the use of the group-(B) commutators (which are also valid in the GOR model) gives a seemingly good result, such as $(K)^2-(\pi)^2=(K^*)^2-(\rho)^2=(K^{**})^2-(A_2)^2$, $(\rho)^2=(\omega)^2$, etc. (the particle symbol always denotes the mass of the particle). The group-(A) commutators give, for example, a formula such as $(K)^2-(\pi)^2=(\kappa)^2-(\delta)^2$ (κ and δ denote the $I=\frac{1}{2}$ and $I=1$ 0^+ mesons, respectively) if we disregard the octet-singlet mixing.[5,15] In the boson case, it is not easy to distinguish the result based on the commutators belonging to the group (A) from that based on group (B). This is partly because we do not meet a situation similar to the Σ-Λ degeneracy in the boson case (the Σ boson and the Λ boson have different G parity) and partly because the bosons usually form a nonet, so that the singlet-octet mixing gives more parameters. However, we think that it is a remarkable fact that the group-(B) commutators [which are more model dependent than the group-(A) commutators] give a nice result for bosons if we take into account mixing.[16] As will be shown later, a similar situation also takes place in broken $SU(2)$ symmetry. Further study for the cause of this is certainly desired.

We now wish to study broken $SU(2)$ symmetry. Analogous to Eq. (6), we may consider the $SU(2)$-breaking Hamiltonian of the following general form in addition to the $SU(3)$-breaking one, $H'(x)$:

$$H''(x)=\alpha' S_3(x)+\beta' d_{3ij}J_\mu{}^i(x)J_\mu{}^j(x)\,, \qquad (11)$$

where $S_3(x)=\bar{q}(x)\lambda_3 q(x)$ and $J_\mu{}^i(x)J_\mu{}^j(x)=V_\mu{}^i(x)V_\mu{}^j(x)+\gamma' A_\mu{}^i(x)A_\mu{}^j(x)$. The S_3 term is the simplest $SU(2)$ mass-splitting interaction in the quark model. Under the usual assumption that $H''(x)$ belongs to an $SU(3)$ octet, the β' term may provide the next more sophisticated model of $SU(2)$ breaking in this model.[17] The $SU(2)$ breaking need not be entirely due to the electro-

[11] Here we have assumed that the quark fields are the only basic fields. Instead we may consider a basic system consisting of quarks and octet (or nonet) vector mesons $\phi_\mu{}^i(x)$ and axial-vector meson $\psi_\mu{}^i(x)$. We may then introduce $SU(3)$-breaking interaction of the form $d_{8ij}V_\mu{}^i(x)\phi_\mu{}^j(x)$ and $d_{8ij}A_\mu{}^i(x)\psi_\mu{}^j(x)$. The inclusion of the d_{8ij} term is necessary to distinguish the group-(A) commutator from group (B).

[12] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968). Earlier references will be found here.

[13] See the summary by B. Sakita, *Advance in Particle Physics* (Interscience, New York, 1968), Vol. 1, p. 219.

[14] Also see, for survey of extensive literature on $SU(6)$, A. Pais, Rev. Mod. Phys. **38**, 215 (1966). For a quark model prediction see, for example, S. Ishida, K. Konno, P. Roman, and H. Shimodaira, Nucl. Phys. **B2**, 307 (1967), and papers cited therein.

[15] For the comprehensive study of the use of the commutator $[\dot{V}_{K^0},A_{K^0}]=0$, see G. Fourez, Ph.D. thesis, University of Maryland, 1969 (unpublished).

[16] If we discard mixing we obtain an absurd result. The problem of f-f' and ρ-ω mixing has been treated in Ref. 2. See also Ref. 15.

[17] Such terms will naturally be present if we consider a system of quarks and octet [or nonet] vector and axial-vector mesons (considered in Ref. 11) which will interact with the electromagnetic field.

magnetic effect. We emphasize again that $H''(x)$ is not meant to be the effective Hamiltonian density. For the commutator of the type $[\dot{V},V]=0$, we always find

$$[\dot{V}_{\pi^+},V_{\pi^+}]=[\dot{V}_{\pi^-},V_{K^0}]=[\dot{V}_{\pi^+},V_{K^+}]=0\,, \qquad (12)$$

as long as $H''(x)$ belongs to an $SU(3)$ octet. For the commutators of the form $[\dot{V},A]=0$, we again distinguish two groups analogous to those in the $SU(3)$ case.

Group (A'):

$$[\dot{V}_{\pi^+},A_{\pi^+}]=[\dot{V}_{\pi^-},A_{\pi^-}]=0 \quad (\alpha',\ \beta',\ \text{and}\ \gamma'\ \text{and also}\ \alpha,\ \beta,\ \text{and}\ \gamma\ \text{are arbitrary}). \qquad (13)$$

These are the commutators in our model which can be used with the utmost confidence. We also have some which are more model dependent.

Group (B'):

$$[\dot{V}_{\pi^-},A_{K^0}]=[\dot{V}_{\pi^+},A_{K^+}]=0 \quad (\alpha'\ \text{and}\ \beta'\ \text{and also}\ \alpha\ \text{and}\ \beta\ \text{are arbitrary but}\ \gamma=\gamma'=1). \qquad (14)$$

Note that with $H=H'(x)+H''(x)$ group-(A) commutators always hold. However, for group (B), $[\dot{V}_{K^0},A_{K^+}]=0$ requires $\gamma=1$ and $[\dot{V}_{K^0},A_{\pi^-}]=[\dot{V}_{K^+},A_{\pi^+}]=0$ requires $\gamma=\gamma'=1$, although α and β and also α' and β' are arbitrary.

Minimal quark electromagnetic interactions or simple quark magnetic moment interactions of the form $\bar{q}(x)\sigma_{\mu\nu}q(x)F_{\mu\nu}(x)$ lead not only to the group-(A') commutators but also to the group-(B') commutators. As in the case of group (B) for $SU(3)$, the domain of validity of the group-(B') commutators is smaller than that of the group-(A') commutators, and they are not very useful in the baryon case as will be shown below. However, they seem to be still useful in the case of bosons as was the case for the group-(B) commutators for $SU(3)$.

In the next sections we study predictions obtained from these "exotic" commutators and our asymptotic $SU(2)$ symmetry.

IV. BARYON $SU(2)$ MASS DIFFERENCES

A. $\frac{1}{2}^+$ Baryons

Consider the matrix element

$$\langle\Sigma^+(\mathbf{q})|[V_{\pi^+},\dot{V}_{\pi^+}]|\Sigma^-(\mathbf{q})\rangle=0\,, \quad \text{with } |\mathbf{q}|=\infty\,.$$

With the prescription described in Sec. II, we need to consider only certain intermediate states in our asymptotic symmetry,

$$\begin{aligned}\langle\Sigma^+|V_{\pi^+}|\Sigma^0\rangle\langle\Sigma^0|\dot{V}_{\pi^+}|\Sigma^-\rangle &+\langle\Sigma^+|V_{\pi^+}|\Lambda^0\rangle\langle\Lambda^0|\dot{V}_{\pi^+}|\Sigma^-\rangle\\ &-\langle\Sigma^+|\dot{V}_{\pi^+}|\Sigma^0\rangle\langle\Sigma^0|V_{\pi^+}|\Sigma^-\rangle\\ &-\langle\Sigma^+|\dot{V}_{\pi^+}|\Lambda^0\rangle\langle\Lambda^0|V_{\pi^+}|\Sigma^-\rangle=0\,.\end{aligned}$$

In order to take into account the electromagnetic Σ^0-Λ^0 mixing, we write

$$\langle\Sigma^0(\mathbf{q})|=\cos\theta\langle\Sigma_8{}^0(\mathbf{q})|+\sin\theta\langle\Lambda_8{}^0(\mathbf{q})|$$

and

$$\langle\Lambda^0(\mathbf{q})|=-\sin\theta\langle\Sigma_8{}^0(\mathbf{q})|+\cos\theta\langle\Lambda_8{}^0(\mathbf{q})|\,,$$

in the limit $|\mathbf{q}|\to\infty$. Here $\langle\Sigma^0|\to\langle\Sigma_8{}^0|$ and $\langle\Lambda^0|\to\langle\Lambda_8{}^0|$ in the $SU(2)$ limit. We note, for example, that

$$\langle\Sigma^+(\mathbf{q})|\dot{V}_{\pi^+}|\Lambda(\mathbf{q})\rangle\propto[E(\Sigma^+)-E(\Lambda^0)]\langle\Sigma^+(\mathbf{q})|V_{\pi^+}|\Lambda(\mathbf{q})\rangle\,,$$

where $E(\Sigma^+)=[\mathbf{q}^2+(\Sigma)^2]^{1/2}$, while

$$\lim_{|\mathbf{q}|\to\infty}\langle\Sigma^+(\mathbf{q})|V_{\pi^+}|\Lambda(\mathbf{q})\rangle=\sqrt{2}\sin\theta\,.$$

We then obtain a sum rule

$$[(\Sigma^-)^2-(\Sigma^0)^2]\cos^2\theta=[(\Sigma^0)^2-(\Sigma^+)^2]\cos^2\theta +[(\Sigma^+)^2+(\Sigma^-)^2-2(\Lambda^0)^2]\sin^2\theta\,.$$

Since θ is small and $[(\Sigma^-)^2-(\Sigma^0)^2]$ and $[(\Sigma^0)^2-(\Sigma^+)^2]$ are already of first order in the $SU(2)$ mass difference, the sum rule, Eq. (8), can be written as

$$[(\Sigma^-)^2-(\Sigma^0)^2]=[(\Sigma^0)^2-(\Sigma^+)^2] -[(\Sigma^+)^2+(\Sigma^-)^2-2(\Lambda^0)^2]\theta^2\,. \qquad (15)$$

The term involving θ^2 exhibits the effect of Σ^0-Λ^0 mixing. However, as will be shown below, the effect is small in this case (this is partly due to the small Σ-Λ mass difference) so that to a good approximation (less than 3% error) we obtain

$$(\Sigma^-)^2-(\Sigma^0)^2=(\Sigma^0)^2-(\Sigma^+)^2\,. \qquad (16)$$

In our asymptotic symmetry we get quadratic mass formulas rather than linear mass formulas, even for baryons. Since we are using commutators of the form $[\dot{V},V]=0$, the term neglected in our asymptotic symmetry is at least of the second order in the symmetry breaking. Therefore, the discrepancy between the quadratic mass formula obtained and the true mass relation is of second order in the symmetry breaking.[18] If we use the experimental values[19] $(\Sigma^-)^2=1.434$, $(\Sigma)^2=1.422$, and $(\Sigma^+)^2=1.412$, in GeV^2, then Eq. (10) reads $0.012=0.010$, in GeV^2. The mass sum rule, $(\Sigma^-)-(\Sigma^0)=(\Sigma^0)-(\Sigma^+)$, has been obtained previously in a tadpole model of Coleman and Glashow[20] and also by assuming that the electromagnetic mass differences are dominated by $|\Delta\mathbf{I}|=1$ transitions.[21] We now derive the mass formula analogous to the other Coleman-Glashow formula[20] and also the value of θ. Consider now the following matrix elements in the limit $|\mathbf{q}|=\infty$:

$$\langle\Sigma^+(\mathbf{q})|[V_{K^+},\dot{V}_{\pi^+}]|\Xi^-(\mathbf{q})\rangle=0\,, \qquad (17)$$

[18] In a soluble model one can explicitly demonstrate this. One of us (S. O.) wishes to thank Professor H. Umezawa for the discussion on this point.

[19] N. Barash-Schmidt, A. Barbaro-Galtieri, L. R. Price, A. H. Rosenfeld, P. Söding, C. G. Wohl, M. Roos, and G. Conforto, Rev. Mod. Phys. **41**, 109 (1969).

[20] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1968); Phys. Rev. **134**, B671 (1964). See also, S. Okubo, Phys. Letters **4**, 14 (1963).

[21] R. E. Marshak, S. Okubo, and E. C. G. Sudarshan, Phys. Rev. **106**, 599 (1957).

$$\langle p(\mathbf{q})|[V_{K^+},\dot{V}_{\pi^+}]|\Sigma^-(\mathbf{q})\rangle=0, \quad (18)$$

$$\langle n(\mathbf{q})|[V_{K^0},\dot{V}_{\pi^+}]|\Sigma^+(\mathbf{q})\rangle=0, \quad (19)$$

$$\langle \Sigma^-(\mathbf{q})|[V_{K^0},\dot{V}_{\pi^-}]|\Xi^0(\mathbf{q})\rangle=0. \quad (20)$$

We now use our asymptotic $SU(3)$ symmetry for the operator V_K and the asymptotic $SU(2)$ symmetry for the operators $V_{\pi^\pm}$. We obtain to order θ

$$[(\Xi^-)^2-(\Xi^0)^2]+[(\Sigma^+)^2-(\Sigma^0)^2] =\sqrt{3}\theta[(\Sigma^+)^2-(\Lambda^0)^2], \quad (21)$$

$$[(\Sigma^0)^2-(\Sigma^-)^2]+[(n)^2-(p)^2] =\sqrt{3}\theta[(\Lambda^0)^2-(\Sigma^-)^2], \quad (22)$$

$$[(n^2)-(p^2)]+[(\Sigma^+)^2-(\Sigma^0)^2] =\sqrt{3}\theta[(\Lambda^0)^2-(\Sigma^+)^2], \quad (23)$$

$$[(\Xi^-)^2-(\Xi^0)^2]+[(\Sigma^0)^2-(\Sigma^-)^2] =\sqrt{3}\theta[(\Sigma^-)^2-(\Lambda^0)^2]. \quad (24)$$

By eliminating θ from Eqs. (21) and (23), we obtain

$$[(n)^2-(p)^2]+[(\Xi^-)^2-(\Xi^0)^2]=2[(\Sigma^0)^2-(\Sigma^+)^2], \quad (25)$$

and from Eqs. (22) and (24),

$$[(n)^2-(p)^2]+[(\Xi^-)^2-(\Xi^0)^2]=2[(\Sigma^-)^2-(\Sigma^0)^2]. \quad (26)$$

Equations (25) and (26) are consistent with Eq. (16), and, if we use Eq. (16), they lead to the following sum rule, which corresponds to the Coleman-Glashow mass formula[20]:

$$[(n)^2-(p)^2]+[(\Xi^-)^2-(\Xi^0)^2]=(\Sigma^-)^2-(\Sigma^+)^2. \quad (27)$$

These relations are rather impressively satisfied by experiments.[19] If we use $(\Xi^-)^2=1.746$, $(\Xi^0)^2=1.728$, $(n)^2=0.882$, and $(p)^2=0.880$, in GeV^2, then Eq. (25), for example, reads $0.020=0.020$, in GeV^2, and Eq. (27) reads $0.020=0.022$, in GeV^2. In the above derivation of sum rules we have neglected the possible $SU(2)$ or $SU(3)$ mixing between the $\frac{1}{2}^+$ baryons under consideration and the higher-lying $\frac{1}{2}^+$ baryons the existence of which is now rather well established. From our point of view this will give the most important correction to our sum rules obtained above. One may first attribute the small discrepancy with experiment to this effect. From Eqs. (21)–(24), we can evaluate the value of θ. We obtain a value of $\theta\simeq 0.02$–0.03. Therefore, the θ^2 contribution in the sum rule, Eq. (15), is not important. [The effect of $SU(2)$ breaking on the Cabibbo sum rules for the semileptonic hyperon decays has been discussed by Matsuda, Oneda, and Desai.[3] See Appendix A.] We have shown that our asymptotic condition for the $V_{\pi^\pm}$ (which is the only assumption involved) gives rise to an effective octet enhancement in the mass sum rules. Note that our $SU(2)$-breaking Hamiltonian density given by Eq. (11) is not meant to be an effective Hamiltonian density and that we are not using a perturbation theory.

B. $\frac{3}{2}^+$ Decuplets

The preceding argument can be extended to the case of a $\frac{3}{2}^+$ decuplet. In this case the electromagnetic mixing analogous to the Σ^0-Λ^0 mixing does not arise. Of course, we again neglect other possible types of mixing: the mixing between the $\frac{3}{2}^+$ decuplet under consideration and the higher-lying $\frac{3}{2}^+$ baryons through the $SU(2)$- and the $SU(3)$-breaking interactions. We may attribute the discrepancies between our mass formulas and experiment, if they exist, to the neglect of such mixing, before blaming our asymptotic symmetry. Consider the equation

$$\lim_{|\mathbf{q}|\to\infty}\langle\Delta^{++}(\mathbf{q})|[V_{\pi^+},\dot{V}_{\pi^+}]|\Delta^0(\mathbf{q})\rangle=0.$$

This gives a constraint on the masses, $(\Delta^{++})^2-(\Delta^+)^2=(\Delta^+)^2-(\Delta^0)^2$. Also, the equation

$$\lim_{|\mathbf{q}|\to\infty}\langle\Delta^{+}(\mathbf{q})|[V_{\pi^+},\dot{V}_{\pi^+}]|\Delta^-(\mathbf{q})\rangle=0$$

gives $(\Delta^+)^2-(\Delta^0)^2=(\Delta^0)^2-(\Delta^-)^2$. We thus have

$$(\Delta^{++})^2-(\Delta^+)^2=(\Delta^+)^2-(\Delta^0)^2=(\Delta^0)^2-(\Delta^-)^2. \quad (28)$$

In a similar way, $\langle Y^+(\mathbf{q})|[V_{\pi^+},\dot{V}_{\pi^+}]|Y^-(\mathbf{q})\rangle=0$ with $|\mathbf{q}|=\infty$ gives rise to

$$(Y^+)^2-(Y^0)^2=(Y^0)^2-(Y^-)^2. \quad (29)$$

Now we consider the $SU(3)$ version of the mass formulas. We can consider

$$\begin{aligned}\langle Y^+(\mathbf{q})|[V_{K^+},\dot{V}_{\pi^+}]|\Xi^{-*}(\mathbf{q})\rangle &=\langle\Delta^+(\mathbf{q})|[V_{K^+},\dot{V}_{\pi^+}]|Y^-(\mathbf{q})\rangle\\ &=\langle\Delta^{++}(\mathbf{q})|[V_{K^+},\dot{V}_{\pi^+}]|Y^0(\mathbf{q})\rangle=0,\end{aligned}$$

and also

$$\begin{aligned}\langle Y^-(\mathbf{q})|[V_{K^0},\dot{V}_{\pi^-}]|\Xi^{*0}(\mathbf{q})\rangle &=\langle\Delta^0(\mathbf{q})|[V_{K^0},\dot{V}_{\pi^-}]|Y^+(\mathbf{q})\rangle\\ &=\langle\Delta^-(\mathbf{q})|[V_{K^0},\dot{V}_{\pi^-}]|Y^0(\mathbf{q})\rangle=0,\quad \text{with } |\mathbf{q}|=\infty.\end{aligned}$$

We use asymptotic $SU(3)$ symmetry for the V_K together with the asymptotic $SU(2)$ symmetry applied to the $V_{\pi^\pm}$. The sum rules obtained are all consistent with each other, and combining them with the ones given by Eqs. (28) and (29), we finally obtain a simple prediction

$$\begin{aligned}(\Delta^{++})^2-(\Delta^+)^2=(\Delta^+)^2-(\Delta^0)^2&=(\Delta^0)^2-(\Delta^-)^2\\ &=(Y^+)^2-(Y^0)^2=(Y^0)^2-(Y^-)^2\\ &=(\Xi^{*0})^2-(\Xi^{*-})^2\equiv(\delta)^2. \quad (30)\end{aligned}$$

These results have also been obtained, for example, in the tadpole model of Coleman and Glashow.[20] Present experiments,[19] which have large errors, cannot test the above formulas unambiguously. According to the Rosenfeld table,[19] $(\Delta^0)-(\Delta^{++})=0.45\pm0.85$ MeV, $(\Delta^-)-(\Delta^{++})=7.9\pm6.8$ MeV, $(Y^+)=1382\pm1$ MeV, $(Y^-)=1388\pm3$ MeV, $(\Xi^{*0})=1528.9\pm1.1$ MeV, $(\Xi^{*-})=1533.8\pm1.9$ MeV. The sum rules, Eq. (30), can be said to be consistent with present experiment and, in particular, the sign seems to come out right.

We also note that in the above derivation the spin and parity of the baryons are irrelevant. Therefore, the sum rule of the form of Eq. (24) holds for any decuplet with an arbitrary J^P if we neglect the mixing mentioned before.

C. Intermultiplet Mass Formulas

We now show that the use of commutators, $[\dot{V}_{\pi^+},A_{\pi^+}]=0$ and $[\dot{V}_{\pi^+},A_{\pi^+}]=0$, enables one to derive the $SU(6)$-like intermultiplet mass formulas. Since A_{π^+} is not an $SU(2)$ generator in the $SU(2)$ limit, the truncation of the intermediate states sandwiched between the factors V_{π^+} and A_{π^+} mainly[22] depends on the asymptotic behavior of the V_{π^+}. Therefore, the use of $[\dot{V}_{\pi^+},V_{\pi^+}]=0$ is safer than that of $[\dot{V}_{\pi^+},A_{\pi^+}]=0$ since the selection of the intermediate states is carried out by two V_{π^+}. However, we first note that even if we replace the commutator $[\dot{V}_{\pi^+},V_{\pi^+}]=0$ by $[\dot{V}_{\pi^+},A_{\pi^+}]=0$ in the discussion of the $\frac{3}{2}^+$ decuplet made in Sec. IV B, we again arrive at the same decuplet mass formula, Eq. (30). This certainly lends good support for our asymptotic symmetry for the V_{π^+}.[23] Let us now insert these commutators between the appropriate $\frac{1}{2}^+$ octet baryon state and the $\frac{3}{2}^+$ decuplet state. Consider the equations with $|\mathbf{q}|=\infty$,

$$\langle\Delta^{++}(\mathbf{q})|A_{\pi^+}|p\rangle\langle p|V_{\pi^+}|n(\mathbf{q})\rangle = \langle\Delta^{++}(\mathbf{q})|V_{\pi^+}|\Delta^+\rangle\langle\Delta^+|A_{\pi^+}|n(\mathbf{q})\rangle \quad (31)$$

and

$$\langle\Delta^{++}(\mathbf{q})|A_{\pi^+}|p\rangle\langle p|\dot{V}_{\pi^+}|n(\mathbf{q})\rangle = \langle\Delta^{++}(\mathbf{q})|\dot{V}_{\pi^+}|\Delta^+\rangle\langle\Delta^+|A_{\pi^+}|n(\mathbf{q})\rangle. \quad (32)$$

These are consistent only if the energy relation $E(p)-E(n)=E(\Delta^{++})-E(\Delta^+)$ is satisfied in the limit $|\mathbf{q}|=\infty$, which gives the mass formula

$$(\Delta^{++})^2-(\Delta^+)^2=(p)^2-(n)^2. \quad (33)$$

If we also take these commutators between the states $\langle p(\mathbf{q})|$ and $|\Delta^-(\mathbf{q})\rangle$ with $|\mathbf{q}|=\infty$, we obtain

$$(\Delta^0)^2-(\Delta^-)^2=(p)^2-(n)^2. \quad (34)$$

Equations (33) and (34) are apparently consistent with Eq. (30). We thus predict that the equal $SU(2)$ squared-mass spacing between the decuplet states, $(\delta)^2$, is equal to the spacing between the proton and neutron:

$$(\delta)^2=(p)^2-(n)^2. \quad (35)$$

This is not inconsistent with present experiment. In deriving this sum rule, we did not use any information about the spin and parity of the baryons. Therefore, the formula holds between any $SU(3)$ decuplet baryons and any $SU(3)$ octet baryons. Therefore, the mass formula, Eq. (30), is universal, i.e.,

$$(\delta_a)^2=(\delta_b)^2=\cdots=(p)^2-(n)^2 = (p_l)^2-(n_l)^2=(p_m)^2-(n_m)^2=\cdots, \quad (36)$$

where the subscripts $a,b,\ldots$ and $l,m,\ldots$ denote the kind of the decuplet baryons and octet baryons, respectively. This simple prediction may be tested by future experiment. We think that these intermultiplet mass formulas are the most trustworthy ones which can be obtained from our approach. This is, of course, because the commutator utilized, $[\dot{V}_{\pi^+},A_{\pi^+}]=0$, is most trustworthy. The discrepancy from experiment, if it exists, should first be attributed to the neglect of the $SU(2)$ or $SU(3)$ mixing between the baryons which have the same J^P but belong to different $SU(3)$ multiplets. In the present intermultiplet case where we use the commutator $[\dot{V}_j,A_i]=0$, the sum rules will involve this mixing angle θ through the terms proportional to $\cos\theta$ and $\sin\theta$, whereas in the case of the $SU(2)$ multiplet mass formulas, where one can use the commutator $[\dot{V}_j,V_i]=0$, the dependence of the sum rules on the angle θ is proportional to $\sin^2\theta$ and $\cos^2\theta$. Therefore, the intermultiplet mass formulas will be more affected by the presence of the mixing. As shown before, the validity of the group-(B') commutators, such as $[\dot{V}_{\pi^+},A_{K^+}]=0$ and $[\dot{V}_{\pi^-},A_{K^0}]=0$, is less certain than that of the one used here, $[\dot{V}_{\pi^+},A_{\pi^+}]=0$. If we use these commutators, we obtain not only the formula $(p)^2-(n)^2=(Y^0)^2-(Y^-)^2$, which is consistent with Eq. (35), but also

$$(\Xi^0)^2-(\Xi^-)^2=(Y^+)^2-(Y^0)^2=(Y^0)^2-(Y^-)^2,$$

$$(\Sigma^+)^2-(\Sigma^0)^2=(\Delta^{++})^2-(\Delta^+)^2=(\Sigma^+)^2-(\Lambda^0)^2,$$

and

$$(\Sigma^-)^2-(\Sigma^0)^2=(\Delta^-)^2-(\Delta^0)^2=(\Sigma^-)^2-(\Lambda^0)^2.$$

The last two equations lead to the Σ^0-Λ^0 degeneracy. Formulas of this kind have also been obtained in the $SU(6)$ symmetry theory with an assumption that only considers the charge operator to second order. However, $SU(6)$ mass formulas with the weaker assumption that considers both the charge operator and the magnetic operator to second order are very close to our sum rules given by Eqs. (30), (33), and (34), although our results are more general. (Compare these with the results listed in Ref. 14.) Therefore, we conclude that the exotic commutators of the form $[\dot{V}_{\pi^+},A_{\pi^+}]=0$ and $[\dot{V}_{\pi^-},A_{\pi^-}]=0$ are most trustworthy and they lead to the intermultiplet mass sum rules given by Eq. (36). The use of group-(B') commutators in the baryon case leads to unsatisfactory results. This is understandable since the group-(B') commutators have a smaller domain of validity than the group-(A') commutators. The seemingly good results of $SU(6)$ are reproduced by

[22] Of course, the selection by the chiral charge A_{π^+} will also be important. However, according to our experience the selection is not as good as by the $SU(2)$ or $SU(3)$ charges, V_π and V_K.

[23] In broken $SU(3)$, both the commutators $[\dot{V}_{K^0},V_{K^0}]=0$ and $[\dot{V}_{K^0},A_{K^0}]=0$ also lead to the same GMO mass formula for the decuplet and also for the $\frac{1}{2}^+$ octet. See also Appendix B.

these group-(A') commutators. Of course, we do not assume $SU(6)$ symmetry. Thus baryon intermultiplet mass formulas favor the group-(A') commutators rather than the group-(B') commutators—a situation similar to that of broken $SU(3)$ symmetry.

V. BOSON $SU(2)$ MASS DIFFERENCES

A. 0^- Mesons

It is well known that in a simple model (in the simple tadpole model[20] or when the mass-splitting interaction[21] transforms like a linear combination of scalar and vector terms in isotopic-spin space), the $\pi^\pm$ and π^0 masses are degenerate. In our approach this is also true if we neglect the broken $SU(2)$ η-π and η'-π mixings as seen below. However, the effect of these mixings should not be neglected in our approach, and, as a matter of fact, the correct sign and the magnitude of the $\pi^\pm$-π^0 mass difference can be explained by reasonable values of these mixing angles. A general treatment of this three-particle mixing and its application to the problem of the violation of the $|\Delta\mathbf{I}|=\frac{1}{2}$ rule in the K_{e3} decays will be discussed elsewhere.[24] For completeness and also for application to similar problems for bosons, we give a brief summary in this section. Let us first study the matrix element $\langle\pi^+(\mathbf{q})|[V_{\pi^+},\dot{V}_{\pi^+}]|\pi^-(\mathbf{q})\rangle=0$ in the limit $|\mathbf{q}|=\infty$. According to our asymptotic symmetry we need to consider only the following terms:

$$\begin{aligned}\langle\pi^+|V_{\pi^+}|\pi\rangle\langle\pi|\dot{V}_{\pi^+}|\pi^+\rangle+\langle\pi^+|V_{\pi^+}|\eta\rangle\langle\eta|\dot{V}_{\pi^+}|\pi^+\rangle\\+\langle\pi^+|V_{\pi^+}|\eta'\rangle\langle\eta'|\dot{V}_{\pi^+}|\pi^+\rangle\\-(\text{terms obtained by }\dot{V}_{\pi^+}\rightleftarrows V_{\pi^+})=0.\end{aligned}\quad(37)$$

In order to take into account the mixing, we write the physical π, η, and η' states in the infinite limit, approximately, as follows:

$$\begin{aligned}|\pi\rangle&=[1-\tfrac{1}{2}(\beta^2+\gamma^2)]|\pi_8\rangle+\beta|\eta_8\rangle+\gamma|\eta_1\rangle,\\|\eta\rangle&=-\beta'|\pi_8\rangle+a|\eta_8\rangle+b|\eta_1\rangle,\\|\eta'\rangle&=-\gamma'|\pi_8\rangle+c|\eta_8\rangle+d|\eta_1\rangle.\end{aligned}\quad(38)$$

In the $SU(3)$ and $SU(2)$ limits, $|\pi\rangle\to|\pi_8\rangle$, $|\eta\rangle\to|\eta_8\rangle$, and $|\eta'\rangle\to|\eta_1\rangle$. β, γ, β', and γ' denote the $SU(2)$ mixings under consideration. a, b, c, and d are of the order of $SU(3)$ mixing. Up to the second order in the $SU(2)$ mixing, we have $a^2+b^2+\beta'^2=1$, $c^2+d^2+\gamma'^2=1$, $\beta'=a\beta+b\gamma$, $\gamma'=c\beta+d\gamma$, and $\beta'\gamma'+ac+bd=0$. Then to this order we obtain, from Eq. (37),

$$(\pi^+)^2-(\pi^0)^2=[(\eta)^2-(\pi^+)^2]\beta'^2+[(\eta')^2-(\pi^+)^2]\gamma'^2.\quad(39)$$

This equation already exhibits the importance of the effect of mixing in the 0^+-meson case. Since

$$[(\eta)^2-(\pi^+)^2]\gg[(\pi^+)^2-(\pi^0)^2]$$

and

$$[(\eta')^2-(\pi^+)^2]\gg[(\pi^+)^2-(\pi^0)^2],$$

even small values of the mixing angles β' and γ' should contribute to the right-hand side of Eq. (39). Note that the sign of the π^+-π^0 mass difference is always correctly predicted by these mixings.

Next we consider

$$\lim_{|\mathbf{q}|\to\infty}\langle K^0(\mathbf{q})|[V_{K^0},\dot{V}_{\pi^-}]|\pi^+(\mathbf{q})\rangle=0.\quad(40)$$

By using the asymptotic $SU(3)$ symmetry for the V_{K^0} and the asymptotic $SU(2)$ symmetry for the V_{π^-} we obtain from Eq. (40), to the same order as Eq. (39),

$$\begin{aligned}[(\pi^0)^2-(\pi^+)^2](1-\sqrt{3}\beta)+[(\eta)^2-(\pi^+)^2]\sqrt{3}a\beta'\\+[(\eta)^2-(\pi^+)^2]\beta'^2+[(\eta')^2-(\pi^+)^2]\sqrt{3}c\gamma'\\+[(\eta')^2-(\pi^+)^2]\gamma'^2=(K^0)^2-(K^+)^2.\end{aligned}\quad(41)$$

We have written, according to our approximation,

$$\begin{aligned}[(\pi^0)^2-(\pi^+)^2](1-\beta^2-\gamma^2)(1-\sqrt{3}\beta)\\\simeq[(\pi^0)^2-(\pi^+)^2](1-\sqrt{3}\beta).\end{aligned}$$

We may also consider the equation

$$\lim_{|\mathbf{q}|\to\infty}\langle K^+(\mathbf{q})|[V_{K^+},\dot{V}_{\pi^+}]|\pi^-(\mathbf{q})\rangle=0,\quad(42)$$

and obtain, to the same approximation,

$$\begin{aligned}[(\pi^0)^2-(\pi^-)^2](-1-\sqrt{3}\beta)+[(\eta)^2-(\pi^-)^2]\sqrt{3}a\beta'\\-[(\eta)^2-(\pi^-)^2]\beta'^2+[(\eta')^2-(\pi^-)^2]\sqrt{3}c\gamma'\\-[(\eta')^2-(\pi^+)^2]\gamma'^2=(K^0)^2-(K^+)^2.\end{aligned}\quad(43)$$

From Eqs. (41) and (43), Eq. (39) again follows, which indicates the internal consistency of our calculation. We choose as the two independent sum rules from our approach the one given by Eq. (39) and the following one obtained from Eqs. (39) and (41):

$$\begin{aligned}[(\pi^+)^2-(\pi^0)^2]\sqrt{3}\beta+[(\eta)^2-(\pi^+)^2]\sqrt{3}a\beta'\\+[(\eta')^2-(\pi^+)^2]\sqrt{3}c\gamma'=(K^0)^2-(K^+)^2.\end{aligned}\quad(44)$$

If there is no η' meson, then $\beta=\beta'$, $a=1$, $\gamma=b=\gamma'=c=d=0$. Equation (39) then reduces to

$$(\pi^+)^2-(\pi^0)^2=[(\eta)^2-(\pi^0)^2]\beta^2,\quad(45)$$

while Eq. (44) reduces to

$$[(\pi^+)^2-(\pi^0)^2+(\eta)^2-(\pi^+)^2]\sqrt{3}\beta=(K^0)^2-(K^+)^2.\quad(46)$$

Equation (46) is equivalent to the η-π transition mass given by Okubo and Sakita.[25] If we compute the value of β from Eq. (40), we obtain $|\beta|\simeq0.067$, whereas Eq. (46) gives $\beta\simeq0.0082$. The former value of β seems somewhat large as a value of the electromagnetic mixing angle (we have obtained in Sec. IV A the Σ-Λ mixing angle of the order $\theta\simeq0.02$–0.03) while the latter value is small. Therefore, with the η meson only, we cannot satisfy Eqs. (45) and (46) simultaneously. We now wish to see how the inclusion of the η' meson changes the situation. We here make an approximate

[24] S. Oneda, H. Umezawa, and Seisaku Matsuda (unpublished).

[25] S. Okubo and B. Sakita, Phys. Rev. Letters **11**, 50 (1963).

calculation assuming that the $SU(3)$ η-η' mixing is much larger than the $SU(2)$ mixing. Namely, we take $a\simeq\cos\theta$, $b\simeq\sin\theta$, $c\simeq-\sin\theta$, $d\simeq\cos\theta$, $\beta'\simeq\beta\cos\theta+\gamma\sin\theta$, and $\gamma'=-\beta\sin\theta+\gamma\cos\theta$, where θ is the η-η' mixing angle.[26] Although this should be usually a good approximation, it is not very accurate in this case, since the $SU(3)$ η-η' mixing angle is known to be small. Equations (39) and (44) then read

$$[(\pi^+)^2-(\pi^0)^2]=[(\eta)^2-(\pi^+)^2](\beta\cos\theta+\gamma\sin\theta)^2 +[(\eta')^2-(\pi^+)^2](-\beta\sin\theta+\gamma\cos\theta)^2 \quad (47)$$

and

$$[(\pi^+)^2-(\pi^0)^2]\sqrt{3}\beta +[(\eta)^2-(\pi^+)^2]\sqrt{3}\cos\theta(\beta\cos\theta+\gamma\sin\theta) +[(\eta')^2-(\pi^+)^2]\sqrt{3}(-\sin\theta)(-\beta\sin\theta+\gamma\cos\theta) =(K^0)^2-(K^+)^2. \quad (48)$$

From the GMO mass formula for the pseudoscalar mesons, which can be obtained by using the commutator $[\dot{V}_{K^0},V_{K^0}]=0$, we have $\sin\theta\simeq\pm0.18$. [Actually, by using Eq. (38) we can compute the modification of the GMO formula due to $SU(2)$ breaking.[26]] By solving Eqs. (47) and (48) for this value of θ, we obtain the following values of β and γ:

(I) For $\sin\theta=+0.18$,

(i) $\beta=0.022$ and $\gamma=0.038$

and

(ii) $\beta=-0.0064$ and $\gamma=-0.038$.

(II) For $\sin\theta=-0.18$,

(iii) $\beta=0.019$ and $\gamma=-0.032$

and

(iv) $\beta=-0.0061$ and $\gamma=0.038$.

Because of our approximation only the first figures of the numbers for β and γ may be trusted. These values of β and γ are of reasonable order of magnitude. The magnitude of the η'-π mixing angle $|\gamma|$ is in fact larger than that of η-π mixing angle $|\beta|$, and on the right-hand side of Eq. (47) the η' term indeed gives the dominant contribution. There is no *a priori* reason that $|\beta|>|\gamma|$. In a tadpole model, if the magnitude of the $\delta\eta\pi$ coupling is smaller than that of the $\delta\eta'\pi$ coupling (δ is the isovector 0^+ meson), we obtain $|\beta|<|\gamma|$. The relative sign of β and γ is also not *a priori* fixed. One place where one can test the above result will be the $\eta\to3\pi$ decay. It is easily seen that in our approach the $\eta\to3\pi$ decay amplitude is dominated by the diagrams involving the η-π and η'-π transitions.[27] This will be discussed elsewhere. The other place where one may hope to detect the larger value of β ($\beta\simeq0.02$) is in the violation of the $|\Delta\mathbf{I}|=\frac{1}{2}$ rule in the K_{e3} decay.[24]

B. 1^- Mesons

Results analogous to those obtained above in the case of 0^- mesons also hold for other mesons. We here consider the 1^- mesons. Corresponding to Eq. (38), we write the physical ρ-, ϕ-, and ω-meson states as follows in the infinite-momentum frame:

$$\begin{aligned}|\rho\rangle&=[1-\tfrac{1}{2}(\beta_v{}^2+\gamma_v{}^2)]|\rho_8\rangle+\beta_v|\phi_8\rangle+\gamma_v|\omega_1\rangle,\\ |\phi\rangle&=-\beta_v'|\rho_8\rangle+a_v|\phi_8\rangle+b_v|\omega_1\rangle,\\ |\omega\rangle&=-\gamma_v'|\rho_8\rangle+c_v|\phi_8\rangle+d_v|\omega_1\rangle.\end{aligned} \quad (49)$$

In the $SU(3)$ and $SU(2)$ limit, $|\rho\rangle\to|\rho_8\rangle$, $\phi\to|\phi_8\rangle$, and $|\omega\rangle\to|\omega_1\rangle$. The sum rule corresponding to the one given by Eq. (40) is

$$(\rho^+)^2-(\rho^0)^2=[(\phi^0)^2-(\rho^+)^2]\beta_v'^2 +[(\omega^0)^2-(\rho^+)^2]\gamma_v'^2. \quad (50)$$

Namely, the ρ^+-ρ^0 mass difference comes essentially from the ρ-ϕ and ρ-ω mixing. If $(\omega^0)>(\rho^+)$, as suggested by the Rosenfeld table,[19] Eq. (50) indicates that the ρ^+ is heavier than the ρ^0. Corresponding to Eq. (44), we obtain

$$[(\rho^+)^2-(\rho^0)^2]\sqrt{3}\beta_v+[(\phi^0)^2-(\rho^+)^2]\sqrt{3}a_v\beta_v' +[(\omega^0)^2-(\rho^+)^2]\sqrt{3}c_v\gamma_v'=(K^{*0})^2-(K^{*+})^2. \quad (51)$$

In terms of the usual ω-ϕ mixing angle θ_v, a_v, b_v, c_v, and d_v are expressed (to a good approximation, contrary to the case of the pseudoscalar meson) as $a_v=\cos\theta_v$, $b_v=\sin\theta_v$, $c_v=-\sin\theta_v$, and $d_v=\cos\theta_v$. β_v' and γ_v' are given by

$$\beta_v'=\beta_v\cos\theta_v+\gamma_v\sin\theta_v \quad (52)$$

and

$$\gamma_v'=\beta_v(-\sin\theta_v)+\gamma_v\cos\theta_v. \quad (53)$$

Therefore, if we know the correct masses of the 1^- mesons, we can evaluate the values of β_v' and γ_v' or the values of β_v and γ_v from Eqs. (50) and (51). On the other hand, if we know the values of β_v' and γ_v' or of β_v and γ_v from other processes (such as the $\phi\to\pi^++\pi^-$ and $\omega\to\pi^++\pi^-$ decays), we can test the sum rules (50) and (51). This will be discussed elsewhere. These arguments can, of course, be extended to other mesons of higher spin. Generalization of Eqs. (39) and (50) implies that if the mass of the $I=1$ meson is smaller than that of $I=Y=0$ members of the nonet, then the charged $I=1$ meson is heavier than its neutral counterpart.

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[26] By considering $\langle K^0(\mathbf{q})|[\dot{V}_{K^0},V_{K^0}]|\bar{K}^0(\mathbf{q})\rangle=0$ with $|\mathbf{q}|\to\infty$ and Eq. (38), the following modification of GMO mass formula to order β^2 and γ^2 is obtained:

$$[(K^0)^2-(\pi^0)^2][\tfrac{1}{2}(1-\beta^2-\gamma^2)-\sqrt{3}\beta+\tfrac{3}{2}\beta^2]+[(K^0)^2-(\eta)^2] \times(\tfrac{1}{2}\beta'^2+\sqrt{3}a\beta'+\tfrac{3}{2}a^2)+[(K^0)^2-(\eta')^2] \times(\tfrac{1}{2}\gamma'^2+\sqrt{3}c\gamma'+\tfrac{3}{2}c^2)=0.$$

The effect is around 5% to decrease the absolute value of the angle θ.

[27] It is known that in the pion-pole model of $\eta\to3\pi$ decay, the magnitude of the η-π transition matrix element given by Ref. 25 is too small to explain the magnitude of the $\eta\to3\pi$ branching ratio. The larger value of β found here may be helpful.

C. Intermultiplet Mass Formulas

It is certainly interesting to see whether the use of the exotic commutator $[\dot{V}_{\pi^+},A_{\pi^+}]=0$ leads to intermultiplet mass formulas similar to the sum rules for baryons given, for example, by Eq. (34). The corresponding exotic commutators for $SU(3)$ are the group-(A) commutators such as $[\dot{V}_{K^0},A_{K^0}]=0$. For octet bosons these commutators gave[15] rise either to the intermultiplet sum rules which are complicated by the $I=Y=0$ singlet-octet meson mixings or to the ones which take a form $A=B$, but with $A\simeq 0$ and $B\simeq 0$ owing to the GMO mass formulas derived by the use of $[\dot{V}_{K^0},V_{K^0}]=0$ and asymptotic $SU(3)$ symmetry. On the other hand, the group-(B) commutators such as $[\dot{V}_{K^0},A_{\pi^-}]=0$, which apparently depend more on the model of $SU(3)$ breaking, gave[5] the following general intermultiplet mass formulas free from the parameters of singlet-octet (nonet) mixing[28]:

$$(K_\alpha)^2-(\pi_\alpha)^2\simeq\text{const}\quad(\alpha\text{ is arbitrary}).\tag{54}$$

Here K_α and π_α denote the masses of $I=\frac{1}{2}$ and $I=1$ members of an α octet meson, respectively, with an arbitrary spin and parity.

We shall show below that the situation is very similar even in the case of broken $SU(2)$ symmetry. Let us first study the group-(A') commutator, i.e., consider

$$\langle\rho^+(\mathbf{q})|[\dot{V}_{\pi^+},A_{\pi^+}]|\pi^-(\mathbf{q})\rangle=0,\tag{55}$$

with $|\mathbf{q}|=\infty$ and use asymptotic $SU(2)$ for the V_{π^+}. Matrix elements involving the charge A_{π^+}, such as $\langle\rho^+(\mathbf{q})|A_{\pi^+}|\pi^0(\mathbf{q})\rangle$, can be related to each other by using the charge algebra instead of using exact $SU(2)$ symmetry. [For explicit examples, see, for instance, Eqs. (59), (60), and (64).] We then arrive at the following intermultiplet mass relation:

$$\begin{aligned}&[(\pi^+)^2-(\pi^0)^2]-[(\pi^+)^2-(\eta)^2]\beta'^2\\&\quad-[(\pi^+)^2-(\eta')^2]\gamma'^2=[(\rho^+)^2-(\rho^0)^2]\\&\qquad-[(\rho^+)^2-(\phi)^2]\beta_v'^2-[(\rho^+)^2-(\omega)^2]\gamma_v'^2.\end{aligned}\tag{56}$$

However, from Eqs. (39) and (50), Eq. (56) is certainly valid but takes the form $0\simeq 0$. Thus no new information is obtained. In $SU(3)$, the equation

$$\langle K^{*+}(\mathbf{q})|[\dot{V}_{K^0},V_{K^0}]|K^-(\mathbf{q})\rangle=0\quad\text{with }|\mathbf{q}|=\infty$$

led[15] also to a relation which takes the form $0\simeq 0$. Even in the case when we obtain a nontrivial sum rule by using the group-(A') commutators, the sum rule is always complicated by the $I=Y=0$ singlet-octet [nonet] mixing, and it is not very useful at present.

We now study instead the consequence of the use of group-(B') commutators. Let us consider

$$\langle K^*(\mathbf{q})|[\dot{V}_{\pi^-},A_{K^0}]|\pi^+(\mathbf{q})\rangle=0\quad\text{with}\,|\mathbf{q}|=\infty.$$

[28] If we neglect the nonet mixing, we also arrive at the formula given by Eq. (54) by using the group-(A) commutators. See Ref. 15.

We obtain, using asymptotic symmetry,

$$\begin{aligned}\sum_{\alpha=\pi^0,\eta,\eta'}&\langle K^{*0}(\mathbf{q})|A_{K^0}|\alpha\rangle\langle\alpha|\dot{V}_{\pi^-}|\pi^+\rangle\\&=\langle K^{*0}(\mathbf{q})|\dot{V}_{\pi^-}|K^{*+}\rangle\langle K^{*+}|A_{K^0}|\pi^+\rangle.\end{aligned}\tag{57}$$

We first rewrite $\langle K^{*0}(\mathbf{q})|A_{K^0}|\alpha(\mathbf{q})\rangle$. Using the charge algebra, $A_{K^0}=-2[A_{\pi^0},V_{K^0}]$, we obtain for $|\mathbf{q}|=\infty$ by our asymptotic $SU(3)$

$$\begin{aligned}\langle K^*(\mathbf{q})|A_{K^0}|\alpha\rangle=&-2\langle K^*(\mathbf{q}^0)|A_{\pi^0}|K^0\rangle\langle K^0|V_{K^0}|\alpha\rangle\\&+2\sum_{\beta=\rho,\omega,\phi}\langle K^*(\mathbf{q}^0)|V_{K^0}|\beta\rangle\langle\beta|A_{\pi^0}|\alpha\rangle.\end{aligned}$$

For $\alpha=\pi$, η, and η', the second term of the right-hand side of the above equation vanishes because of charge conjugation invariance. (Throughout this paper we assume this invariance.) The vanishing of this term is essential for the derivation of our formula below. Thus the left-hand side of Eq. (57) can be written as

$$-2\sum_\alpha\langle K^*(\mathbf{q})|A_{\pi^0}|K^0\rangle\langle K^0|V_{K^0}|\alpha\rangle\langle\alpha|\dot{V}_{\pi^-}|\pi^+\rangle,$$

which, after using the relation

$$\langle K^0(\mathbf{q})|[V_{K^0},\dot{V}_{\pi^-}]|\pi^+(\mathbf{q})\rangle=0\quad\text{with }|\mathbf{q}|=\infty,$$

becomes

$$-2\langle K^{*0}(\mathbf{q})|A_{\pi^0}|K^0\rangle\langle K^0|\dot{V}_{\pi^-}|K^+\rangle\langle K^+|V_{K^0}|\pi^+\rangle.\tag{58}$$

On the other hand, the right-hand side of Eq. (57), after using again the commutator $A_{K^0}=-2[A_{\pi^0},V_{K^0}]$, becomes

$$\begin{aligned}\langle K^{*0}(\mathbf{q})|\dot{V}_{\pi^-}|K^{*+}\rangle\{&-2\langle K^{*+}|A_{\pi^0}|K^+\rangle\langle K^+|V_{K^0}|\pi^+\rangle\\&+2\langle K^{*+}|V_{K^0}|\rho^+\rangle\langle\rho^+|A_{\pi^0}|\pi^+\rangle\}.\end{aligned}\tag{59}$$

The nonvanishing of the matrix element $\langle\rho^+|A_{\pi^0}|\pi^+\rangle$ in Eq. (59) under, for example, G invariance is also important for our derivation. We now derive sum rules which relate the matrix elements $\langle K^{*0}(\mathbf{q})|A_{\pi^0}|K^0\rangle$, $\langle K^{*+}(\mathbf{q})|A_{\pi^0}|K^+\rangle$, and $\langle\rho^+(\mathbf{q})|A_{\pi^0}|\pi^+\rangle$ in the asymptotic limit. By using the commutator $2A_{\pi^0}=[V_{\pi^+},A_{\pi^-}]$, we obtain[29]

$$\lim_{|\mathbf{q}|\to\infty}\langle K^{*0}(\mathbf{q})|A_{\pi^0}|K^0(\mathbf{q})\rangle=-\tfrac{1}{2}\lim_{|\mathbf{q}|\to\infty}\langle K^{*0}(\mathbf{q})|A_{\pi^-}|K^+(\mathbf{q})\rangle\tag{60}$$

and

$$\lim_{|\mathbf{q}|\to\infty}\langle K^{*+}(\mathbf{q})|A_{\pi^0}|K^+(\mathbf{q})\rangle=\tfrac{1}{2}\lim_{|\mathbf{q}|\to\infty}\langle K^{*0}(\mathbf{q})|A_{\pi^-}|K^+(\mathbf{q})\rangle.\tag{61}$$

We need one more relation. We consider, with $|\mathbf{q}|\to\infty$,

$$\begin{aligned}\sum_{\alpha=\rho,\omega,\phi}&\langle K^{*0}(\mathbf{q})|V_{K^0}|\alpha\rangle\langle\alpha|A_{\pi^-}|\pi^+\rangle\\&=\langle K^{*0}(\mathbf{q})|A_{\pi^-}|K^+\rangle\langle K^+|V_{K^0}|\pi^+\rangle.\end{aligned}\tag{62}$$

We rewrite the left-hand side of this equation by using again the commutator $A_{\pi^-}=[V_{\pi^-},A_{\pi^0}]$. It becomes

[29] Note that we are not using exact $SU(2)$ symmetry.

equal to

$$\sum_{\alpha=\rho,\omega,\phi} \langle K^{*0}(\mathbf{q})|V_{K^0}|\alpha\rangle\langle\alpha|V_{\pi^-}|\rho^+\rangle\langle\rho^+|A_{\pi^0}|\pi^+\rangle$$

and finally reduces to

$$\langle K^{*0}(\mathbf{q})|V_{K^0}|\rho^0\rangle\langle\rho^0|V_{\pi^-}|\rho^+\rangle\langle\rho^+|A_{\pi^0}|\pi^+\rangle \quad \text{with } |\mathbf{q}|=\infty. \quad (63)$$

In obtaining Eq. (63), we retain only the term $\alpha=\rho^0$ since the contributions of other terms, $\alpha=\omega^0$ and ϕ^0, are much smaller [of the order of $SU(2)$ breaking]. Thus from Eqs. (62) and (63), by noting also

$$\lim_{|\mathbf{q}|\to\infty}\langle K^{*0}(\mathbf{q})|V_{K^0}|\rho^0(\mathbf{q})\rangle=\sqrt{\tfrac{1}{2}},$$

$$\lim_{|\mathbf{q}|\to\infty}\langle\rho^0(\mathbf{q})|V_{\pi^-}|\rho^+(\mathbf{q})\rangle=-\sqrt{2},$$

and

$$\lim_{|\mathbf{q}|\to\infty}\langle K^+(\mathbf{q})|V_{K^0}|\pi^+(\mathbf{q})\rangle=-1,$$

we obtain

$$\lim_{|\mathbf{q}|\to\infty}\langle\rho^+(\mathbf{q})|A_{\pi^0}|\pi^+(\mathbf{q})\rangle \simeq -\lim_{|\mathbf{q}|\to\infty}\langle K^{*0}(\mathbf{q})|A_{\pi^-}|K^+(\mathbf{q})\rangle. \quad (64)$$

Thus by equating Eq. (58) to Eq. (59), utilizing the sum rules Eqs. (60), (61), and (64), and using the asymptotic values of the matrix elements of the charges V_{π^-} and V_{K^0}, we finally obtain an intermultiplet sum rule

$$(K^0)^2-(K^+)^2=(K^{*0})^2-(K^{*+})^2. \quad (65)$$

The use of another commutator belonging to the group (B'), i.e.,

$$\langle K^{*+}(\mathbf{q})|[\dot{V}_{\pi^+},A_{K^+}]|\pi^-(\mathbf{q})\rangle=0 \quad \text{with } |\mathbf{q}|=\infty,$$

leads also to the same sum rule by using a similar argument. By following exactly the same steps we can also derive an intermultiplet sum rule between the $J^{PC}=1^{--}$ and 2^{++} octets. Namely, from

$$\langle K^{**0}(\mathbf{q})|[\dot{V}_{\pi^-},A_{K^0}]|\rho^+(\mathbf{q})\rangle=0 \quad \text{with } |\mathbf{q}|=\infty,$$

we obtain

$$(K^{*0})^2-(K^{*+})^2=(K^{**0})^2-(K^{**+})^2. \quad (66)$$

K^{**} denotes the $K_N(1420)$.[19] In fact, the procedures used to derive these sum rules are quite general and almost independent of the spins and parities of the octets involved. Consider two octet bosons α $(\pi_\alpha,K_\alpha,\eta_\alpha)$ and β $(\pi_\beta,K_\beta,\eta_\beta)$. The conditions that the above procedures of deriving the sum rule, Eq. (65), will go through in the general case are as follows.

(a) The two octet bosons α and β must have opposite charge conjugation parities.

(b) The matrix element $\langle K_\alpha{}^0(\mathbf{q})|A_{\pi^0}|K_\beta{}^0(\mathbf{q})\rangle$ exists, i.e., it does not vanish, for example, by the requirement of parity conservation.

(c) The matrix element $\langle\pi_\alpha{}^+|A_{\pi^0}|\pi_\beta{}^+\rangle$ also exists, i.e., it does not vanish by G invariance.

Therefore, for such a pair of octets α and β which satisfies these conditions (a)–(c), we obtain a general intermultiplet mass formula

$$(K_\alpha{}^0)^2-(K_\alpha{}^+)^2=(K_\beta{}^0)^2-(K_\beta{}^+)^2. \quad (67)$$

If we assume that both the octets with normal and abnormal charge conjugation parity always exist,[30] we then obtain a universal formula for octet bosons

$$(K_\alpha{}^0)^2-(K_\alpha{}^+)^2=\text{const} \quad [\alpha \text{ is arbitrary}]. \quad (68)$$

From the known K^0-K^+ mass difference, we then predict that the neutral member of the octet kaon is always heavier than the charged one and that their quadratic mass spacing is universal and is of the order $\simeq 0.004$ GeV^2. For the 1^{--} octet, present experiment[19] indicates $(K^{*0})^2-(K^{*+})^2=(0.01\pm0.007)$ GeV^2, which is consistent with our prediction, Eq. (65). The above mass formulas have taken into account the effect of the $I=Y=0$ singlet-octet (nonet) mixing. However, one should bear in mind the fact that they are subject to the limitations mentioned before, i.e., the neglect of mixing other than the above nonet mixing and the validity of the commutator utilized. The general $SU(2)$ intermultiplet boson sum rule, Eq. (68), can be compared with the $SU(3)$ one given by Eq. (54).

VI. FINAL REMARKS

We have shown that our asymptotic symmetry [both $SU(2)$ and $SU(3)$] and the use of exotic commutators of the form $[\dot{V},V]=0$ reproduce all the good results of the effective octet dominance models or the Coleman-Glashow tadpole model. We stress the fact that we did not make a perturbation argument, i.e., we did not use the assumption that the *effective* Hamiltonian transforms like an octet. Therefore, without assuming it, the octet dominance, in fact, effectively comes out from our asymptotic conditions. The mass degeneracies between the $I=1$ charged and neutral bosons are removed. In our approach this is achieved by the effect of mixings, which is again the result of effective octet dominance. Therefore, the assumption that the basic (not effective) $SU(3)$- and $SU(2)$-breaking Hamiltonians transform like members of an octet seems to work quite well. We have also obtained general simple intermultiplet mass formulas which include the $SU(6)$ result as a special case. Their validity is not as certain as that of the GMO mass formulas since they are more affected by particle mixings (but not by the nonet mixing). However, they will be useful as a first guide in hadron spectroscopy, and they certainly fix the scales of the hadron mass splittings which are realized in nature.

[30] For bosons already known to us, we do not need to use this assumption in deriving the sum rules Eq. (67) which include them.

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APPENDIX A: SUM RULES FOR SEMILEPTONIC HYPERON DECAY COUPLINGS IN BROKEN SYMMETRY

Define $g_{p\Lambda}(0)$ and $f_{p\Lambda}(0)$ by

$$\lim_{|\mathbf{q}|\to\infty}\langle p(\mathbf{q})|A_{K^+}|\Lambda^0(\mathbf{q}')\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{q}-\mathbf{q}')(m_p/E_p)^{1/2}(m_\Lambda/E_\Lambda)^{1/2}\times g_{p\Lambda}(0)\bar{u}_p(\mathbf{q})\gamma_4\gamma_5 u_\Lambda(\mathbf{q}')$$

and

$$\lim_{|\mathbf{q}|\to\infty}\langle p(\mathbf{q})|V_{K^+}|\Lambda^0(\mathbf{q}')\rangle$$
$$=(2\pi)^3\delta^3(\mathbf{q}-\mathbf{q}')(m_p/E_p)^{1/2}(m_\Lambda/E_\Lambda)^{1/2}\times f_{p\Lambda}(0)\bar{u}_p(\mathbf{q})\gamma_4 u_\Lambda(\mathbf{q}'),$$

respectively. Then $G(\sqrt{\tfrac{1}{2}})g_{p\Lambda}(0)\sin\theta_A$ and $G(\sqrt{\tfrac{1}{2}})f_{p\Lambda}(0)\times\sin\theta_V$ will be the observed axial-vector and vector coupling constants [at zero four-momentum transfer] for the $\Lambda^0\to p+e^-+\bar{\nu}$ decay, respectively. We note that the usual chiral $SU(3)\otimes SU(3)$ charge algebra still holds even in the presence of $SU(3)$-and $SU(2)$-breaking interaction. Therefore, by using our asymptotic $SU(3)$ and $SU(2)$ symmetries for the charges V_K and $V_{\pi^\pm}$, respectively, we can derive sum rules for the g's and f's from the algebra. The sum rules thus obtained are compatible with the $SU(3)$ and $SU(2)$ hadron mass splittings discussed in this paper. For example, we obtain

$$g_{\Lambda\Sigma^-}(0)=(\sqrt{\tfrac{3}{2}})(1+\sqrt{3}\theta)g_{pn}(0)+(1+2\sqrt{3}\theta)g_{p\Lambda}(0)$$

and

$$f_{p\Lambda}(0)=-(\sqrt{\tfrac{3}{2}})[1-(\sqrt{\tfrac{1}{3}})\theta],\quad\text{etc.}$$

Here θ is the Σ^0-Λ^0 mixing angle defined in Sec. IV A. The complete set of sum rules were listed in our previous work.[31] The effect of $SU(2)$ violation appears through the Σ^0-Λ^0 mixing angle in our approach. The value of θ is estimated to be around 0.02–0.03. Although θ is small, the determination of θ_A is, in some cases, affected rather seriously by the effect of $SU(2)$ breaking. For example the determination of the Cabibbo angles from the $\Lambda\to p$, $\Sigma^+\to\Lambda$, and $n\to p$ decays is not very sensitive to the effect of the θ correction whereas that from the $\Sigma^-\to n$, $\Sigma^-\to\Lambda$, and $n\to p$ decays is rather sensitive.[31]

APPENDIX B: $SU(2)$ MASS SUM RULES FOR OCTET BARYONS FROM COMMUTATOR $[\dot{V}_{\pi^+},A_{\pi^+}]=0$

Consider the equation

$$\lim_{|\mathbf{q}|\to\infty}\langle\Sigma^+(\mathbf{q})|[\dot{V}_{\pi^+},A_{\pi^+}]|\Sigma^-(\mathbf{q})\rangle=0.$$

We then obtain from our asymptotic $SU(2)$

$$\sum_{n=\Sigma^0,\Lambda^0}\langle\Sigma^+(\mathbf{q})|\dot{V}_{\pi^+}|n\rangle\langle n|A_{\pi^+}|\Sigma^-\rangle=\sum_{n'=\Sigma^0,\Lambda^0}\langle\Sigma^+(\mathbf{q})|A_{\pi^+}|n\rangle\langle n|\dot{V}_{\pi^+}|\Sigma^-\rangle,$$

with $|\mathbf{q}|=\infty$. We then obtain

$$\sqrt{2}\cos\theta g_{\Sigma^+\Sigma^0}[(\Sigma^0)^2-(\Sigma^-)^2]+\sqrt{2}(-\sin\theta)g_{\Sigma^+\Lambda}[(\Lambda^0)^2-(\Sigma^-)^2]$$
$$=-\sqrt{2}\cos\theta g_{\Sigma^0\Sigma^-}[(\Sigma^+)^2-(\Sigma^0)^2]+\sqrt{2}(\sin\theta)g_{\Lambda\Sigma^-}[(\Sigma^+)^2-(\Sigma^0)^2].$$

Again θ is the Σ^0-Λ^0 mixing angle and the g's are defined in Appendix A. By using the sum rules for the g's, which can be obtained by using the $SU(3)\otimes SU(3)$ charge algebra and the asymptotic symmetries, we can eliminate g's from the above equation and finally obtain a mass sum rule,

$$[(\Sigma^-)^2-(\Sigma^0)^2]\cos^2\theta=[(\Sigma^0)^2-(\Sigma^+)^2]\cos^2\theta+[(\Sigma^+)^2+(\Sigma^-)^2-2(\Lambda^0)^2]\sin^2\theta.$$

This was also obtained in Sec. IV A by using the commutator $[\dot{V}_{\pi^+},V_{\pi^+}]=0$. The same argument holds for any octet baryons. Thus both the commutators, $[\dot{V}_{\pi^+},V_{\pi^+}]=0$ and $[\dot{V}_{\pi^+},A_{\pi^+}]=0$, give the same mass formula when they are applied to the same $SU(2)$ multiplet. The A_{π^+} is not an $SU(2)$ generator in the $SU(2)$ limit. Therefore, the above result indicates that the asymptotic $SU(2)$ symmetry for the V_{π^+} is a very good one. The same arguments also hold[2,3] for the commutators $[\dot{V}_{K^0},V_{K^0}]=0$ and $[\dot{V}_{K^0},A_{K^0}]=0$ and they also suggest that the asymptotic $SU(3)$ symmetry for the V_K is a good one.

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[31] S. Matsuda, S. Oneda, and P. Desai, Phys. Rev. **178**, 2129 (1969). The Σ^0-Λ^0 mixing angle θ'' defined there is related to the present θ by $\theta''=-\theta$. Equation (14) of this reference should be replaced by one of Eqs. (21)–(24) of this paper.