PHYSICAL REVIEW D VOLUME 2, NUMBER 2 15 JULY 1970

Lorentz Transformation Laws of Interacting Radiation-Gauge Fields

CARL M. BENDER*
Institute for Advanced Study, Princeton, New Jersey 08540
AND
DAVID J. GRIFFITHS
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 2 March 1970)

We consider interacting quantum electrodynamics with the electric current j^μ unspecified and compute the homogeneous Lorentz group transformation laws of the "vector" potential A^μ in the radiation gauge. We show that A^μ decomposes uniquely into a direct sum of an infinite number of linear, infinite-dimensional, nonunitary, indecomposable representations of the homogeneous Lorentz group. One of these representations has spin-multiplicity 2, belongs to a class of representations recently analyzed by Gel'fand and Ponomarev, and may be the first recognized physical example of this class of representations. Finally, the noninteracting limit is shown to reduce correctly to the free-field case in which the transformation properties of the "vector" potential are already known.

I. INTRODUCTION

IN the study of interacting quantum electrodynamics one faces an apparent dilemma. In the Lorentz gauge the "vector" potential A^μ has a well-defined (vector) transformation law but the Hilbert space of states is complicated by the lack of a positive-definite metric and thus the space contains unphysical states. On the other hand, in the radiation or Coulomb gauge (the unique gauge in which the Hilbert space metric is positive definite and all fields are completely determined by the independent fields[1]) the infinitesimal transformation law of A^μ [see Eq. (2)] is so complicated that heretofore it was not clear if the radiation gauge was even covariant (that is, if A^μ obeyed a linear Lorentz transformation law).

In this paper we show that the above dilemma is only apparent. In the radiation gauge, A^μ decomposes uniquely into a direct sum of linear, indecomposable representations of the homogeneous Lorentz group. Thus, the radiation gauge is just as covariant as the Lorentz gauge.

Specifically, in the notation of Gel'fand, Minlos, and Shapiro,[2] which we explain in Sec. II, we prove that the Coulomb potential A^0 transforms as the spin-0 component of

$$\sum_{j=0}^{\infty} \oplus (0,2j) \tag{1a}$$

and that the transverse potential $A^T{}_i$ transforms as the spin-1 component of

$$\sum_{j=0}^{\infty} \oplus (0,2j) \oplus \sum_{j=1}^{\infty} \oplus (1,2j-1) \oplus \sum_{j=1}^{\infty} \oplus (1,-2j+1) \oplus (1,1)_{n=2} \oplus (1,-1)_{n=2}. \tag{1b}$$

In Eq. (1), $(0,2j)$ and $(1,\pm 2j\mp 1)$ are infinite-dimensional, nonunitary, operator-irreducible representations. $(1,\pm 1)_{n=2}$ are spin-multiplicity-2 representations belonging to a class of indecomposable representations recently analyzed by Gel'fand and Ponomarev,[3] and, in fact, these may constitute the first recognized physical example of this class of representations.

The question of covariance of the radiation gauge was first raised by Strocchi,[4] who showed that the vector potential in free-field quantum electrodynamics cannot transform as a vector. Bender[5] then discovered that the free field $A^T{}_i$ transforms as the spin-1 component of the $(1,1)\oplus(1,-1)$ representation. This result was later rederived by Frishman and Itzykson.[6] Both Bender and Frishman and Itzykson noted the importance of extending their results to the interacting case and also the difficulty of performing such a task.

To demonstrate why the interacting case is so much more difficult to handle than the noninteracting case, we first give the infinitesimal transformation laws[1,7] of A^0 and $A^T{}_i$ in the presence of a current:

$$-i[A^0(x),J^{0k}] = (x^k\partial^0 - x^0\nabla_k)A^0(x) + \nabla_k\partial_0\nabla^{-2}A^0(x) + \Box^2\nabla^{-2}A^T{}_k(x), \tag{2a}$$

$$-i[A^T{}_i(x),J^{0k}] = (x^k\partial^0 - x^0\nabla_k)A^T{}_i(x) + \nabla_i\partial_0\nabla^{-2}A^T{}_k(x) + (\delta_{ik} - \nabla_i\nabla_k\nabla^{-2})A^0(x). \tag{2b}$$

---

* Research sponsored by the National Science Foundation under Grant No. GP-16147.

[1] J. Schwinger, *Brandeis Lectures, 1964* (Prentice-Hall, Englewood Cliffs, N. J., 1965), p. 147.

[2] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (MacMillan, New York, 1963), pp. 188–197.

[3] I. M. Gel'fand and V. A. Ponomarev, Usp. Mat. Nauk **23**, 3 (1968) [Russian Math. Surveys **23**, 1 (1968)]; see also D. P. Zhelobenko, Dokl. Akad. Nauk SSSR **121**, 586 (1958).

[4] F. Strocchi, Phys. Rev. **162**, 1429 (1967).

[5] C. M. Bender, Phys. Rev. **168**, 1809 (1968). This paper also gives the transformation laws of the transverse potential for noninteracting massless field theories of arbitrary spin.

[6] Y. Frishman and C. Itzykson, Phys. Rev. **180**, 1556 (1969); **183**, 1520 (E) (1969).

[7] We use the metric $(+,+,+,-)$ and define $\nabla^{-2}f(x) \equiv -(4\pi)^{-1} \times \int dy|x-y|^{-1}f(y)$. $J^{\mu\nu}$ is the generator of the homogeneous Lorentz group.

A^0 and $A^T{}_i$ satisfy the field equations[1]

$$A^0 = -\nabla^{-2} j^0 \tag{3a}$$

and

$$\square^2 A^T{}_i = -j^T{}_i\,, \tag{3b}$$

where $j^T{}_i$ is defined by

$$j^T{}_i \equiv (\delta_{ik} - \nabla_i \nabla_k \nabla^{-2}) j_k \tag{3c}$$

and the transverse potential $A^T{}_i$ obeys the radiation-gauge condition

$$\nabla_i A^T{}_i = 0\,. \tag{3d}$$

Next we recover the noninteracting limit of the theory by taking $j^\mu \to 0$. Equation (3) implies that in this limit $A^0 \to 0$ and $\square^2 A^T{}_i \to 0$. Hence, in this limit the transformation laws in Eqs. (2) decouple and Eq. (2a) disappears entirely, leaving

$$-i[A^T{}_i(x), J^{0k}] = (x^k \partial^0 - x^0 \nabla_k) A^T{}_i(x) + \nabla_i \partial_0 \nabla^{-2} A^T{}_k(x)\,. \tag{4}$$

Note that an infinitesimal algebra which closes under Eq. (4) can no longer close under Eq. (2). This unusual behavior (which is a direct result of minimal coupling and is not observed in massive field theories) accounts for the added complexity in the interacting theory: *The transformation properties of the vector potential depend upon the presence of the electric current.*

The correct approach to this problem arises from the observation that repeated commutation of, say, A^0 with J^{0k} gives an infinite sequence of spin-0 components:

$$A^0,\ \partial_0{}^2 \nabla^{-2} A^0,\ \partial_0{}^4 \nabla^{-4} A^0,\ \ldots\,. \tag{5}$$

Thus, A^0 cannot transform as a *finite* direct sum of indecomposable representations. While this observation suggests that the solution to the problem is not simple, it also suggests a technique which works: First, we construct indecomposable representations of the Lorentz group out of infinite linear combinations of terms of the form $\partial_0{}^{2N} \nabla^{-2N} A^0$. Then, we recover A^0 from an infinite linear combination of these indecomposable representations. This technique is used repeatedly to achieve the complete and unique solution given in Eq. (1).

Our paper is organized as follows: Section II gives a quick survey of the properties of the representations encountered in this paper and of the terms used to describe them. (This section is included to make the paper self-contained.) In Sec. III we analyze a simple example to demonstrate the technique described above of taking infinite linear combinations of infinite linear combinations. Then this technique is applied in Secs. IV and V to determine the transformation properties of A^0 and $A^T{}_i$, respectively. Finally, in Sec. VI we discuss the noninteracting limit of Eq. (1), discuss possible continuations of this work, and comment briefly on the intricate structure we have unearthed in our quest for covariance.

II. SURVEY OF INDECOMPOSABLE REPRESENTATIONS

Below, we list very briefly some characteristics of the indecomposable representations of the homogeneous Lorentz group. These representations are described in terms of their spin content, where each spin component is an irreducible representation of the rotation subgroup.

A. Irreducible Representations

Irreducible representations are uniquely specified in the notation of Gel'fand, Minlos, and Shapiro[2] by a pair of numbers (l_0, l_1). l_0 is the lowest spin contained in the representation and is thus a non-negative integer or half-integer. l_1 can be any complex number. The sequence of spins contained in an irreducible representation has the form

$$l_0,\ l_0+1,\ l_0+2,\ l_0+3,\ \ldots\,. \tag{6}$$

This sequence terminates if and only if the representation is finite dimensional, and in that case the highest spin contained in the representation is $|l_1|-1$.

One can picture the infinitesimal algebra of an irreducible representation schematically as

$$\chi_{l_0} \rightleftarrows \chi_{l_0+1} \rightleftarrows \chi_{l_0+2} \rightleftarrows \chi_{l_0+3} \rightleftarrows \cdots\,. \tag{7}$$

In Eq. (7), χ_l is the lth spin component and each arrow indicates one commutation with J^{0k}, the generator of pure Lorentz transformations. In and only in an irreducible representation every χ_l is connected to every other $\chi_{l'}$ by a sequence of arrows going in both directions, as in Eq. (7).

Irreducible representations fall into two classes.

(*i*) *Nonsingular class.* $l_1 - l_0$ is not a nonzero integer. All nonsingular representations are infinite dimensional.

(*ii*) *Singular class.* $l_1 - l_0$ is a nonzero integer. If $|l_1| > l_0$, the representation is finite dimensional and if $|l_1| < l_0$, the representation is infinite dimensional. Singular representations can be organized into pairs by switching l_0 and l_1 (with a sign change in the event l_1 is negative):

$$(l_0, l_1) \text{ is paired with } [|l_1|,\ (\text{sgn } l_1) l_0]\,. \tag{8}$$

One member of the pair is finite dimensional; the other (called the *tail* of the finite-dimensional representation) is infinite dimensional.

B. Indecomposable (Noncompletely Reducible) Representations

Indecomposable but not irreducible representations (representations which have invariant subspaces but which cannot be decomposed into a direct sum of irreducible representations) occur because the Lorentz group is not compact. To construct indecomposable representations one "glues" together several irreducible representations having the same infinitesimal transformation laws.

(*i*) *Nonsingular class.* One can glue together n replicas of a nonsingular representation (l_0, l_1).[3] The resulting representation contains all spins l, $l_0 \leq l \leq \infty$, each spin occurring n times. There is, up to equivalence, only one way that these n (l_0,l_1) replicas can be glued together indecomposably. Thus, three numbers, l_0, l_1, and n, uniquely label these representations. Figure 1 gives a schematic picture of the infinitesimal algebra that results when n (the *spin multiplicity*) is chosen to be 2. In Fig. 1, the $\{R_{0l}\}$ span the invariant subspace which starts at spin 1 and the $\{S_l\}$ comprise the rest of the representation. We will encounter this representation in Sec. V. When we decompose the transverse potential A^T into indecomposable representations [see Eq. (1b)], we find a representation of the type shown in Fig. 1.

(*ii*) *Singular class*. Since the infinitesimal algebra of a representation (l_0,l_1) depends upon l_0 and l_1 symmetrically (that is, it depends on l_0l_1 and $l_0^2+l_1^2$),[2] a representation and its tail have the same infinitesimal transformation law. Thus, one can glue a representation to its tail. In general, one can glue n_0 replicas of the finite-dimensional (l_0,l_1) to n_1 replicas of its tail.[3] The resulting representation contains all spins l, $l_0 \leq l \leq |l_1|-1$, n_0 times, and all spins l, $|l_1| \leq l \leq \infty$, n_1 times. The singular case is more complicated than the nonsingular case because l_0, l_1, n_0, and n_1 are not sufficient to specify up to equivalence the complicated subspace structures which can occur. A string of integers and a complex number μ are also required.

Fortunately, in massless field theory the most complicated representations which have occurred have $n_0=n_1=1$. These representations are called *operator irreducible*.[8] Operator-irreducible representations have just two possible subspace structures. Either $l_0 \leq l \leq |l_1|-1$ or $|l_1| \leq l < \infty$ constitute an invariant subspace. Thus, these representations are completely specified when one states whether it is the finite part or the tail which forms the invariant subspace.[9] We diagram below the infinitesimal algebra of an operator-irreducible representation which has an infinite-dimensional invariant subspace:

$$\chi_{l_0} \rightleftarrows \chi_{l_0+1} \rightleftarrows \cdots \rightleftarrows \chi_{|l_1|-2} \rightleftarrows \chi_{|l_1|-1} \rightarrow \chi_{|l_1|} \rightleftarrows \chi_{|l_1|+1} \rightleftarrows \cdots . \quad (9)$$

III. ILLUSTRATIVE EXAMPLE

Let x^μ be a vector having nonzero norm s:

$$|\mathbf{x}|^2-(x^0)^2 \equiv s \neq 0. \quad (10)$$

To illustrate the techniques used in Secs. IV and V, we will determine how $(x^0)^{-\alpha}$ transforms (α is a complex number).

[8] These representations are discussed in I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. V, Chap. III.

[9] Operator-irreducible representations of both kinds are constructed and discussed in detail by C. M. Bender and D. J. Griffiths, Phys. Rev. D **1**, 2335 (1970). The stress tensor for all free massless field theories of integer spin $L \geq 2$ is the direct sum of two operator-irreducible representations: [(0,1) "glued" to its tail (1,0), with (1,0) the invariant subrepresentation] ⊕ [(0,3) "glued" to its tail (3,0), with (3,0) the invariant subrepresentation].

The infinitesimal transformation laws for a vector x^μ are

$$J^{0k}: x^0 \rightarrow x^k, \qquad J^{0k}: x^i \rightarrow \delta^{ik} x^0. \quad (11)$$

Using Eq. (11), we observe that repeated application of J^{0k} to $(x^0)^{-\alpha}$ gives an infinite sequence of spin-0 components:

$$(x^0)^{-\alpha},\ s(x^0)^{-\alpha-2},\ s^2(x^0)^{-\alpha-4},\ \ldots . \quad (12)$$

This sequence, which is similar to that in Eq. (5), hints that $(x^0)^{-\alpha}$ transforms as the spin-0 component of the direct sum of an infinite number of indecomposable representations.[10]

Equation (12) motivates the definition

$$P_j^0 \equiv (x^0)^{-\alpha} \sum_{N=0}^{\infty} f_j(N)[s(x^0)^{-2}]^{N+j}, \quad j=0,1,2,\ldots \quad (13)$$

where the $f_j(N)$ are numbers and the superscript 0 indicates that P_j^0 carries spin 0.

Using Eqs. (12) and (13), we will show in the calculation which follows that (a) there is a unique expression giving $f_j(N)$ in terms of $f_j(0)$ for which P_j^0 transforms as the spin-0 component of the $[0,l_1(j)]$ representation and (b) there is a unique choice of $f_j(0)$ for which

$$\sum_{j=0}^{\infty} P_j^0 = (x^0)^{-\alpha}.$$

We will then conclude that $(x^0)^{-\alpha}$ transforms as the spin-0 component of

$$\sum_{j=0}^{\infty} \oplus [0,l_1(j)].$$

A. Determination of P_j^0 in terms of $f_j(0)$

Boosting P_j^0 twice with J^{0k} [using Eq. (12)] and extracting the spin-0 part of the result (isolating the term with a nonzero trace) gives

$$(x^0)^{-\alpha} \sum_{N=0}^{\infty} [s(x^0)^{-2}]^{N+j} [-f_j(N)(\alpha+2N+2j) + \tfrac{1}{3} f_j(N)(\alpha+2N+2j)(\alpha+1+2N+2j) + \tfrac{1}{3} f_j(N-1)(\alpha-2+2N+2j)(\alpha-1+2N+2j)], \quad (14)$$

where $f_j(-1)$ is 0.

[10] If s were zero, the sequence in Eq. (12) would terminate after the first entry. Then $(x^0)^{-\alpha}$ would itself be the spin-0 component of an indecomposable representation. [This case is treated in detail in Ref. 9, where $(x^0)^{-\alpha}$ is shown to transform as the $(0, 1-\alpha)$ representation.] Note the beautiful symmetry between (a) the transformation laws of *free* massless fields (which describe particles constrained to the light cone and which transform as single indecomposable representations) and interacting massless fields [which allow virtual particles to leave the light cone and which transform as a direct sum of indecomposable representations as in Eq. (1)] and (b) the transformation properties of $(x^0)^{-\alpha}$ for $s=0$ and $s \neq 0$.

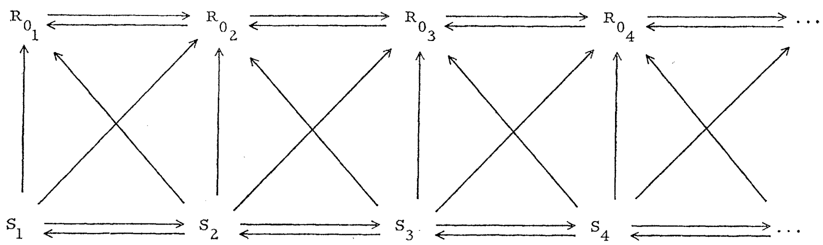

FIG. 1. Schematic picture of the infinitesimal algebra of a spin-multiplicity-2 indecomposable representation. An arrow indicates commutation with J^{0k}. The subscript on R_0 or S is the spin carried by that component. The lowest spin contained in this representation is 1.

We want $P_j{}^0$ to transform as the spin-0 component of the $[0,l_1(j)]$ representation. For this to be true, the expression in Eq. (14) must be equal to $\frac{1}{3}\{[l_1(j)]^2-1\}$ times the expression in Eq. (13).[2] (We are demanding that the infinitesimal algebra close under commutation with J^{0k}.) Thus we obtain a set of difference equations which $f_j(N)$ must satisfy.

The first of these equations (the one corresponding to $N=0$) is an indicial equation [$f_j(0)$ is nonzero, of course] which determines $l_1(j)$:

$$l_1(j)=\pm(\alpha-1+2j). \tag{15}$$

We substitute Eq. (15) into the rest of the difference equations (those corresponding to $N=1, 2, 3, \ldots$) and get a *unique* solution for $f_j(N)$ in terms of $f_j(0)$:

$$f_j(N)=\frac{(-1)^N f_j(0)\Gamma(N+j+\frac{1}{2}\alpha)\Gamma(N+j+\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\alpha+2j)}{N!\Gamma(N+\alpha+2j)\Gamma(j+\frac{1}{2}\alpha)\Gamma(j+\frac{1}{2}\alpha+\frac{1}{2})}. \tag{16}$$

Then, defining

$$z\equiv -s(x^0)^{-2} \tag{17}$$

and substituting Eqs. (16) and (17) into Eq. (13), we obtain

$$P_j{}^0=\frac{f_j(0)(x^0)^{-\alpha}(-z)^j\Gamma(\alpha+2j)}{\Gamma(j+\frac{1}{2}\alpha)\Gamma(j+\frac{1}{2}\alpha+\frac{1}{2})}\times\sum_{N=0}^{\infty}\frac{z^N\Gamma(N+j+\frac{1}{2}\alpha)\Gamma(N+j+\frac{1}{2}\alpha+\frac{1}{2})}{N!\Gamma(N+\alpha+2j)}. \tag{18}$$

The summation in Eq. (18) can be performed explicitly.[11] The result is

$$P_j{}^0=f_j(0)(x^0)^{-\alpha}(-z)^j(1-z)^{-1/2}\times[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{1-\alpha-2j}. \tag{19}$$

Equation (19) gives the final expression for $P_j{}^0$ in terms of $f_j(0)$.

B. Determination of $(x^0)^{-\alpha}$ in terms of $P_j{}^0$

We must now show that there is a unique set $\{f_j(0)\}$ which solves the equation

$$\sum_{j=0}^{\infty} f_j(0)(-z)^j(x^0)^{-\alpha}(1-z)^{-1/2}\times[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{1-\alpha-2j}=(x^0)^{-\alpha} \tag{20}$$

for all z.

Substituting

$$\nu\equiv -\tfrac{1}{4}z[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{-2} \tag{21}$$

into Eq. (20) simplifies Eq. (20) to

$$\sum_{j=0}^{\infty} f_j(0)(4\nu)^j=(1+\nu)(1-\nu)^{-\alpha}. \tag{22}$$

The right-hand side of Eq. (22) has a power-series expansion in ν which has almost the same form as the power series in z in the right-hand side of Eq. (18).[11] Comparing terms having the same power of ν gives a *unique* solution for $f_j(0)$:

$$f_j(0)=\frac{(2j+\alpha-1)\Gamma(j+\alpha-1)}{4^j j!\Gamma(\alpha)}. \tag{23}$$

This completes the calculation for this subsection.

Thus, we have uniquely decomposed $(x^0)^{-\alpha}$ algebraically:

$$(x^0)^{-\alpha}=\sum_{j=0}^{\infty} P_j{}^0, \tag{24a}$$

and group theoretically:

$$(x^0)^{-\alpha}\text{ transforms as }\sum_{j=0}^{\infty}\oplus(0,\alpha+2j-1). \tag{24b}$$

Finally, we note that when $\alpha+2j-1$ is not a nonzero integer the representation $(0,\alpha+2j-1)$ is infinite dimensional and irreducible, and that when $\alpha+2j-1$ is a nonzero integer the representation $(0,\alpha+2j-1)$ is infinite dimensional and operator irreducible with an

[11] See A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1, Chap. II.

infinite-dimensional invariant subspace starting at spin $|\alpha+2j-1|$.

IV. TRANSFORMATION LAW OF COULOMB POTENTIAL A^0

We now use the techniques developed in Sec. III to determine how A^0 transforms. The sequence in Eq. (5) motivates the definition

$$Q_j{}^0=\sum_{N=0}^{\infty} z^{N+j}f_j(N)A^0\,,\quad j=0,1,2,\ldots \tag{25}$$

where

$$z\equiv\Box^2\nabla^{-2}. \tag{26}$$

Following the procedure in Sec. III A, we commute twice with J^{0k} using Eq. (2), and compute that $Q_j{}^0$ transforms indecomposably (the algebra closes) if and only if

$$f_j(N)=f_j(0)\frac{\Gamma(N+j+1)\Gamma(N+j-\frac{1}{2})\Gamma(2j+1)}{N!\Gamma(N+2j+1)\Gamma(j+1)\Gamma(j-\frac{1}{2})}. \tag{27}$$

When Eq. (27) is satisfied, $Q_j{}^0$ transforms as the spin-0 component of the $(0,2j)$ representation. $(0,2j)|_{j=0}$ is infinite dimensional and irreducible. $(0,2j)|_{j=1,2,3,\ldots}$ is operator irreducible and contains an infinite-dimensional invariant subspace starting at spin $2j$.

Next, we plug Eq. (27) into Eq. (25), sum the series,[11] and get

$$Q_j{}^0=f_j(0)z^j[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{-2j}\times\{1-z(2l+1)^{-1}[1+(1-z)^{1/2}]^{-1}\}A^0. \tag{28}$$

Finally, using the substitution in Eq. (21), we calculate the *unique* set of numerical values for $f_j(0)$,

$$f_0(0)=1\,,\quad f_j(0)=-2(2j-1)^{-1}(-4)^{-j},\quad \text{for}\quad j\geq 1\,, \tag{29}$$

which satisfy the equation

$$A^0=\sum_{j=0}^{\infty} Q_j{}^0. \tag{30}$$

We conclude that the Coulomb potential A^0 transforms as the spin-0 component of a direct sum of infinite-dimensional representations:

$$\sum_{j=0}^{\infty}\oplus(0,2j). \tag{31}$$

This verifies the assertion made in Eq. (1a).

V. TRANSFORMATION LAW OF TRANSVERSE POTENTIAL $A^T{}_i$

The transformation law of $A^T{}_i$ [see Eq. (1b)] is more complicated than that of A^0. $A^T{}_i$ transforms as a direct sum of three different types of terms which we call Q, R, and S type. The Q-type terms are the spin-1 components of the Q_j representations derived in Sec. IV. The R-type terms are a new set of indecomposable representations whose lowest-spin component is 1 [they are the $(1,\pm 2j\mp 1)$, $j=1,2,3,\ldots$, operator-irreducible representations with infinite-dimensional invariant subspaces]. The S-type terms are nonsingular, spin-multiplicity-2, indecomposable $(1,\pm 1)_{n=2}$ representations. In this section we find the most general collection of indecomposable representations which (i) have a spin-1 component and (ii) could possibly add up to give $A^T{}_i$. We then show that there is a *unique* way to add all these representations together to get pure $A^T{}_i$.

A. Q-Type Terms

In Sec. IV we introduced the Q-type representations whose sequence of spin components takes the form

$$\{Q_j{}^0,Q_j{}^m,Q_j{}^{mn},Q_j{}^{mnp},\ldots\}\,, \tag{32}$$

where m, n, p, ... are three-space indices. $Q_j{}^0$ was given in Eqs. (25), (28), and (29). The Q-type representations are the most general set of indecomposable representations whose lowest-spin component is 0 and whose spin-1 component could give pure $A^T{}_i$ in our final sum.

We get $Q_j{}^m$, the spin-1 component of the Q representations, from the commutation relation

$$-i[Q_j{}^0,J^{0m}]=(x^m\partial^0-x^0\nabla_m)Q_j{}^0+Q_j{}^m \tag{33}$$

and from Eqs. (2), (25), (28), and (29). The result, after some algebra, is

$$\begin{aligned}Q_j{}^m=E(j)z^j[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{-2j}&\times\{[z-z^2(2l+1)^{-1}[1+(1-z)^{1/2}]^{-1}]A^T{}_m\\&+[2+(2l-1)(1-z)^{-1/2}-2z(2l+1)^{-1}\\&\times[1+(1-z)^{1/2}]^{-1}]\partial_0\nabla_m\nabla^{-2}A^0\}\,,\end{aligned} \tag{34}$$

where z is given in Eq. (26) and $E(j)$ is an undetermined numerical constant which reflects the arbitrary normalization of $Q_j{}^i$ relative to $Q_j{}^0$. We will determine $E(j)$ in Sec. V D.

B. R-Type Terms

Since the most general three-space vector that one can construct from A^0 and $A^T{}_i$ is a linear combination of $\partial_0\nabla_m\nabla^{-2}A^0$, $A^T{}_m$, and $\epsilon_{mpq}\nabla^{-2}\partial_0\nabla_pA^T{}_q$, we are motivated to define

$$R_j{}^m=\sum_{N=0}^{\infty} z^{N+j}[a_j(N)\partial_0\nabla_m\nabla^{-2}A^0+b_j(N)A^T{}_m+c_j(N)\epsilon_{mpq}\partial_0\nabla_p\nabla^{-2}A^T{}_q]\,,\quad j=0,1,2,\ldots\,. \tag{35}$$

By demanding that $R_j{}^m$ belong to an indecomposable representation without a spin-0 component, we are guaranteed to find representations different from the Q-type representations.

In this subsection we look for irreducible and operator-irreducible representations, that is, for representations having just one spin-1 component. (In Sec. V C we will look for representations having several spin-1 components.) Our procedure is to commute $R_j{}^m$ once with J^{0k} and demand that the resulting expression contain no spin-0 component (be traceless), and that it contain $R_j{}^m$ as the spin-1 component. In other words, multiplying the resulting expression by ϵ_{imk} must give something proportional to $R_j{}^i$ itself. If the representation to which $R_j{}^i$ belongs is labeled by $[1,l_1(j)]$, then the proportionality factor is exactly $il_1(j)$.[2]

To do this calculation, we must solve three coupled difference equations for $a_j(N)$, $b_j(N)$, and $c_j(N)$. After summing over N, we find that $R_j{}^m$ takes the form

$$R_j{}^m=D(j)z^j[A_j(z)\partial_0\nabla_m\nabla^{-2}A^0+B_j(z)A^T{}_m+C_j(z)\epsilon_{mpq}\partial_0\nabla_p\nabla^{-2}A^T{}_q]. \quad (36)$$

$D(j)$ is an over-all multiplicative constant which will be determined in Sec. V D and A, B, and C are explicitly

$$A_j(z)=-[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{-2j}\times[1+(1-z)^{1/2}+(j-\tfrac{1}{2})j^{-1}z(1-z)^{-1/2}], \quad (37a)$$

$$B_j(z)=\tfrac{1}{2}[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{-2j}\times[2j-1-z+(2j-1-\tfrac{1}{2}j^{-1}z)(1-z)^{1/2}], \quad (37b)$$

$$C_j(z)=-\tfrac{1}{2}il_1(j)A_j(z), \quad (37c)$$

where[12]

$$l_1(j)=\pm(2j-1). \quad (38)$$

In Eqs. (37) and (38), $j=1, 2, 3, \ldots$. We have excluded $j=0$ to avoid repetition. When $j=0$, we find that $A_0(z)=-2z(1-z)^{-1/2}$, $B_0(z)=z(1-z)^{1/2}$, and $C_0(z)=il_1z(1-z)^{-1/2}$. This implies that $R_0{}^m=D(0)[D(1)]^{-1}\times(R_1{}^m)^*$. Hence, we take j to run from 1 to ∞ without loss of generality.

We have thus discovered that the representations $(1, \pm2j\mp1)$, $j=1, 2, 3, \ldots$, contribute to the sum in Eq. (1b).

C. S-Type Terms

Although a large number of representations have already been found which contribute to the transformation law for $A^T{}_i$, it is still not possible to choose $D(j)$ and $E(j)$ in Eqs. (36) and (34) such that the sum of the spin-1 components of all these representations gives $A^T{}_i$. Since we have identified all spin-multiplicity-1 representations which could contribute to $A^T{}_i$, we look for spin-multiplicity-2 representations.

We have discovered one of these representations [it is labeled $(1,1)_{n=2}$ and its conjugate is $(1,-1)_{n=2}$] and we describe it below.

The subspace part of the representation is R_0 which transforms as $(1,\pm1)$. [$R_0{}^i$, the spin-1 component of R, is given in Eqs. (36)–(38).] We call the nonsubspace part S. S^i, the spin-1 component of S, is given by

$$S^i=C\{2[(\beta\lambda-1)z(1-z)^{-1/2}+(1-z)^{1/2}]\partial_0\nabla_i\nabla^{-2}A^0+[1+\lambda\beta^{-1}z+2z](1-z)^{1/2}A^T{}_i-[\beta(1-z)^{1/2}+\lambda\beta^2z(1-z)^{-1/2}]\epsilon_{imn}\partial_0\nabla_m\nabla^{-2}A^T{}_n\}, \quad (39)$$

where C is an over-all multiplicative constant, $\beta=il_1=\mp i$, and λ is an arbitrary constant which gives the relative normalization of R_0 to S. For simplicity we choose $\lambda=\beta^{-1}$. Then

$$S^i=C\{2(1-z)\partial_0\nabla_i\nabla^{-2}A^0+(1-z)^{1/2}(1+z)A^T{}_i-\beta(1-z)^{-1/2}\epsilon_{imn}\partial_0\nabla_m\nabla^{-2}A^T{}_n\}. \quad (40)$$

The result in Eq. (39) follows from the commutation relation

$$-i[S^i,J^{0k}]=(x^k\partial^0-x^0\nabla_k)S^i+\Phi^{ik} \quad (41)$$

and from the equation

$$\epsilon_{ijk}\Phi^{jk}=R_0{}^i+\beta S^i. \quad (42)$$

The infinitesimal algebra that R satisfies is given in Sec. V B. Figure 1 gives a schematic picture of the infinitesimal algebra of the $(1,\pm1)_{n=2}$ representations.

D. Addition of Q-, R-, and S-Type Terms

It remains only to choose the coefficients C in Eq. (40), $D(j)$ in Eq. (36), and $E(j)$ in Eq. (34) in such a way that

$$\sum_{j=0}^{\infty} Q_j{}^i+\sum_{j=1}^{\infty} R_j{}^i+S^i+\text{H.c.}=A^T{}_i. \quad (43)$$

[Without loss of generality, we choose not to sum over R_0 (see Sec. V C) in Eq. (43) because we showed in Sec. V B that R_0 is proportional to $R_1{}^*$.] This calculation involves much tedious algebra. We list the *unique* result below:

$$C=\tfrac{1}{2}, \quad (44a)$$

$$D(1)=\tfrac{3}{4}, \quad (44b)$$

$$D(j)=[(j-1)(2j-1)^2]^{-1}(-4)^{-j}, \quad j\geq2 \quad (44c)$$

$$E(0)=-2, \quad (44d)$$

$$E(j)=-4[(2j+1)(2j-1)^2]^{-1}(-4)^{-j}, \quad j\geq1. \quad (44e)$$

Equation (44) implies Eq. (43), which implies that $A^T{}_i$ transforms as the spin-1 component of the direct sum of representations

$$\sum_{j=0}^{\infty}\oplus(0,2j)\oplus\sum_{j=1}^{\infty}\oplus(1,\pm2j\mp1)\oplus(1,\pm1)_{n=2}. \quad (45)$$

We have thus verified the assertion made in Eq. (1b).

VI. OBSERVATIONS

A. Reduction to Free-Field Case

In free-field radiation-gauge electrodynamics A^0 vanishes and $A^T{}_i$ decomposes into left- and right-handed

[12] The $\pm$ sign identifies a conjugate pair of representations. See Ref. 2 for details.

helicity states

$$A^T{}_i = C_i + C_i^*, \tag{46a}$$

where

$$C_i, C_i^* = \tfrac{1}{2} A^T{}_i \pm \tfrac{1}{2} i \epsilon_{ijk} \partial_0 \nabla_j \nabla^{-2} A^T{}_k \tag{46b}$$

and C_i and C_i^* transform as the spin-1 components of $(1, \pm 1)$.[5]

We check that the transformation laws in Eq. (1) reduce in the noninteracting limit to Eq. (46). As the current $j^\mu \to 0$, $A^0 \to 0$ and $zA^T{}_i \to 0$ [see Eq. (3)]. Hence, $Q_j{}^0$, $Q_j{}^m$, and $R_j{}^m$ vanish for all j. The *only* nonvanishing contribution to $A^T{}_i$ comes from the noninteracting limit of S^i in Eq. (40) which is precisely the expression in Eq. (46b). Thus, the noninteracting limit of electrodynamics displays a remarkable phenomenon: As $j^\mu \to 0$, the irreducible subrepresentation $R_0{}^i$ of the multiplicity-2 representation dissolves away, leaving S^i, which in this limit becomes irreducible. In short, the "glue" which connects $R_0{}^i$ and S^i when $j^\mu \neq 0$ becomes "unstuck" when $j^\mu = 0$.

B. Other Areas of Research

We list below some possible continuations of this research.

(i) It is remarkable that l_1 for the representations in Eqs. (24b), (31), and (45) always increases by steps of two. Of course, this is not sufficient to indicate Regge behavior because one would also have to find a sequence of increasing masses which corresponds to the sequence of l_1's. This would be a difficult task since the fields associated with these values of l_1 are dependent fields. Nevertheless, our results are suggestive and merit further investigation.[13]

(ii) Because we have discovered a set of fields which systematically display in covariant form the physical content of electrodynamics and which do not suffer any sort of gauge transformation, we have completely eliminated gauge transformations and gauge invariance from quantum electrodynamics. Hence there is now no structural difference between electrodynamics and massive field theories. Low-energy theorems and Ward identities which are ordinarily proved using gauge invariance and covariance should be directly provable in this new framework.

(iii) It would be interesting to extend the present formalism to include magnetic charge.

(iv) Using this work on interacting massless spin-1 particles as a model, one might try to formulate a theory of interacting massless spin-2 particles (gravitons).

ACKNOWLEDGMENT

One of us (C. M. B.) wishes to thank Dr. Carl Kaysen for the hospitality extended to him at the Institute for Advanced Study.

[13] One possible continuation might be found in the study of light-cone commutators. For example, R. A. Brandt [Phys. Rev. Letters **23**, 1260 (1969)] has drawn physical consequences from the connection between Regge theory and light-cone commutators. The assumption that commutators on the light cone behave like $(x^0)^{-\alpha}$ corresponds neatly with the example in Sec. III.