High-Energy Approximations in Potential Scattering Theory*

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The relationship between the eikonal approximation of Glauber and the impact-parameter approximation of Blankenbecler and Goldberger is clarified. This is done by comparing the approximate forms for the scattering amplitude for nonrelativistic scattering by a potential with the Born expansion, which at high energy assumes the natural form of a series of terms in inverse powers of the center-of-mass momentum p. It is found that both approximations reproduce the first Born term and, to order $1/p$, the on-energy-shell part of the second Born term, for *all* momentum transfers. Furthermore, the Glauber approximation correctly gives the leading contribution (in powers of $1/p$) to the on-energy-shell part of *every* term in the Born series, whereas the Blankenbecler-Goldberger approximation fails to do this for Born terms beyond the second. However, since the comparison to the Born series shows that both approximations neglect contributions of order $1/p^2$, neither approximation, in the simple forms commonly used, is correct beyond terms of order $1/p$.

I. INTRODUCTION

Two familiar high-energy approximations for scattering amplitudes in nonrelativistic potential theory are the eikonal approximation of Glauber[1] and the impact-parameter approximation of Blankenbecler and Goldberger.[2] Although both have been taken over and used a great deal in phenomenological analyses of scattering data (after a suitable redefinition of what one means by a potential), the relationship between the two does not seem to be fully appreciated.[3] The main purpose of this paper is to clarify this relationship in the context of nonrelativistic potential scattering theory. This will be done by comparing the approximate forms to the Born expansion of the scattering amplitude, which at high energy assumes the form of a series of terms in inverse powers of the center-of-mass (c.m.) momentum p. This shows that both approximations reproduce the first Born term and, to order $1/p$, the on-energy-shell part (energy-conserving intermediate state) of the second Born term, for *all* momentum transfers. Furthermore, the Glauber approximation correctly gives the leading contribution (in powers of $1/p$) to the on-energy-shell part of *every* Born term, while the Blankenbecler-Goldberger approximation fails to do this for Born terms beyond the second. This latter point is demonstrated by explicitly comparing the approximate amplitudes to the exact amplitude for the special case of scattering by a Yukawa potential in the limit of "exchange mass" going to zero (range becoming infinite). However, for general potentials (of finite range) both approximations neglect contributions of order $1/p^2$, and consequently neither, in the simple form commonly used, is correct beyond order $1/p$.[4]

II. DIFFERENT EXPANSIONS

In the eikonal approximation of Glauber, the scattering amplitude for the nonrelativistic scattering of two spinless particles of mass m is given by[5]

$$T_G(p,\Delta)=-ip\int_0^\infty db\, bJ_0(b\Delta)[e^{i\chi(p,b)}-1], \qquad (1)$$

where $J_0(x)$ is the zeroth-order Bessel function of the first kind. The momentum transfer Δ is related to the c.m. momentum p and angle of scattering θ by

$$\Delta=2p\sin\tfrac{1}{2}\theta, \qquad (2)$$

and the eikonal function $\chi(p,b)$ is related to the potential $V(r)$ that appears in the Schrödinger equation by

$$\chi(p,b)=-\left(\frac{m}{2p}\right)\int_{-\infty}^\infty dz\, V((b^2+z^2)^{1/2}). \qquad (3)$$

Defining the potential function $V(b)$ by

$$V(b)=-\tfrac{1}{2}m\int_{-\infty}^\infty dz\, V((b^2+z^2)^{1/2}) \qquad (4)$$

gives

$$\chi(p,b)=(1/p)V(b), \qquad (5)$$

and consequently the Glauber amplitude (1) takes the form

$$T_G(p,\Delta)=-ip\int_0^\infty db\, bJ_0(b\Delta)[e^{iV(b)/p}-1]. \qquad (6)$$

Expanding the exponential generates a series of

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[1] R. J. Glauber, *Lectures in Theoretical Physics* (Interscience, New York, 1959), Vol. I, pp. 315–414.

[2] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962); M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), pp. 617–621.

[3] See, for example, H. D. I. Arbarbanel, S. D. Drell, and F. J. Gilman, Phys. Rev. **177**, 2458 (1969).

[4] This conclusion is not new. As Blankenbecler and Goldberger point out, the wave function that leads to their approximation satisfies the Schrödinger equation to the same order as the wave function that leads to the Glauber approximation, namely, to terms of order $1/p^2$. In fact, there are an infinite number of such wave functions. Cf. R. Blankenbecler and M. L. Goldberger, Ref. 2.

[5] For simplicity, a spherically symmetric potential is assumed. In Eq. (82) on p. 342 of Glauber's article (Ref. 1), $pb\theta$ has been replaced by $2pb\sin\frac{1}{2}\theta$, and consequently the small-angle restriction no longer holds. Also, $p=m_{\text{red}}\,v\equiv\frac{1}{2}mv$ and $\hbar=1$.

terms involving powers of the potential[6]:

$$T_G(p,\Delta)=\sum_{n=1}^{\infty}\left(\frac{i}{p}\right)^{n-1}\left(\frac{1}{n!}\right)\int_0^{\infty}db\, bJ_0(b\Delta)[V(b)]^n. \quad (7)$$

The nth term in this series corresponds, in a way to be discussed below, to the nth term in the expansion of the exact scattering amplitude in the (infinite) Born series.

In the impact-parameter approximation of Blankenbecler and Goldberger, the scattering amplitude for the nonrelativistic scattering of spinless particles of mass m is given by[7]

$$T_{BG}(p,\Delta)=\int_0^{\infty}db\, bJ_0(b\Delta)\left[\frac{V(b)}{1-(i/2p)V(b)}\right], \quad (8)$$

where $V(b)$ is defined by Eq. (4).

Expanding this into a series of terms in powers of the potential gives[8]

$$T_{BG}(p,\Delta)=\sum_{n=1}^{\infty}\left(\frac{i}{2p}\right)^{n-1}\int_0^{\infty}db\, bJ_0(b\Delta)[V(b)]^n. \quad (9)$$

Each term in this series also corresponds, in a way to be discussed below, to a term in the Born series. However, the Glauber series (7) and the Blankenbecler-Goldberger series (9) differ for terms of $n\geqslant 3$ (corresponding to Born terms of third or higher order).

The exact scattering amplitude for the nonrelativistic scattering of spinless particles of mass m by a potential satisfies the Lippmann-Schwinger equation[9]

$$T(\mathbf{p},\mathbf{p}')=V(\mathbf{p},\mathbf{p}')+\left(\frac{1}{2\pi^2}\right)\int d^3k\frac{V(\mathbf{p},\mathbf{k})T(\mathbf{k},\mathbf{p}')}{k^2-p^2-i\epsilon}, \quad (10)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final c.m. momenta, respectively, and

$$V(\mathbf{p},\mathbf{p}')=-\left(\frac{m}{4\pi}\right)\int d^3r\; V(r)e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}} \quad (11)$$

is the first Born term (first Born approximation to the scattering amplitude).

The infinite Born series is obtained from the Lippmann-Schwinger equation by iteration:

$$T_B(\mathbf{p},\mathbf{p}')=V(\mathbf{p},\mathbf{p}')+\sum_{n=1}^{\infty}\left(\frac{1}{2\pi^2}\right)^n\int d^3k_1\cdots d^3k_n \times\frac{V(\mathbf{p},\mathbf{k}_1)V(\mathbf{k}_1,\mathbf{k}_2)\cdots V(\mathbf{k}_n,\mathbf{p}')}{(k_1^2-p^2-i\epsilon)(k_2^2-k_1^2-i\epsilon)\cdots(k_n^2-k_{n-1}^2-i\epsilon)}. \quad (12)$$

The mth term in this series is the mth Born term (m equals the power to which the potential appears), and the sum of the first m terms gives the scattering amplitude through the mth Born approximation.

To show the relationship between the Born expansion (12) and the Glauber series (7) and the Blankenbecler-Goldberger series (9), we proceed as follows. First, we note that the first Born term can be written in the *exact* form[10]

$$V(\mathbf{p},\mathbf{p}')=\int_0^{\infty}db\, bJ_0(b\Delta)V(b)\equiv V(p,\Delta), \quad (13)$$

where $V(b)$ is defined by Eq. (4). By the replacement[11]

$$\frac{1}{k_1^2-p^2-i\epsilon}\rightarrow P\frac{1}{k_1^2-p^2}+i\pi\delta(k_1^2-p^2), \quad (14)$$

the second Born term can be broken up into an off-energy-shell contribution (virtual intermediate state) and an on-energy-shell contribution (real intermediate state). To order $1/p$, the on-energy-shell contribution to this term can be calculated by using the exact representation (13) for the first Born term and then performing the angular integration to leading order in inverse powers of the momentum, with the result[12]

$$\left(\frac{1}{2\pi^2}\right)\int d^3k_1 V(\mathbf{p},\mathbf{k}_1)V(\mathbf{k}_1,\mathbf{p}')[i\pi\delta(k_1^2-p^2)] =\left(\frac{i}{2p}\right)\int_0^{\infty}db\, bJ_0(b\Delta)[V(b)]^2+\cdots, \quad (15)$$

where the remaining terms are of order V^2/p^3. The off-energy-shell contribution to the second Born term, which comes from the principal-value integration, is

[6] Lest the reader be concerned about the legitimacy of expanding the eikonal form into a power series, we remind him that Schiff showed that the Born series sums up to the Glauber form if each Born term is evaluated (in configuration space) in the stationary phase approximation. Cf. L. I. Schiff, Phys. Rev. **103**, 443 (1956).

[7] Cf. M. L. Goldberger and K. M. Watson, Ref. 2, p. 620.

[8] Again, lest the reader be concerned about expanding the impact-parameter form into a power series, we note that the manner in which this approximate form is obtained by summing the Born series will be explained below.

[9] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[10] This follows from (11) by performing the integration in the coordinate system with its z axis in the direction $\hat{z}=(\mathbf{p}+\mathbf{p}')/|\mathbf{p}+\mathbf{p}'|$.

[11] In this respect "on-energy-shell" means "replace the propagator by its δ-function part" *only* in the second Born term. In the higher Born terms certain combinations of principal-value integrations and δ-function integrations which come from cross terms in the product of the propagators are also included in the "on-energy-shell" contribution. For an example of this in the relativistic case, where "on-energy-shell" becomes "on-mass-shell," see M. E. Ebel and R. J. Moore, Phys. Rev. **177**, 2470 (1969).

[12] To perform the angular integration it is helpful to use the integral representation of the Bessel function: $J_0(b\Delta)=(1/2\pi)\times\int d\phi\, e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{b}}$.

of order V^2/p^2. In general, the nth Born term has a term of order V^n/p^{n-1} which is an on-energy-shell contribution, and a term of order V^n/p^n which is an off-energy-shell contribution.

So, to order $1/p$, the scattering amplitude is

$$T_{\mathrm{B}}(p,\Delta)=\int_0^\infty db\, bJ_0(b\Delta)V(b)+\left(\frac{i}{2p}\right)\int_0^\infty db\, bJ_0(b\Delta)[V(b)]^2+\cdots \quad (16)$$

for *all* momentum transfers. For the sake of completeness, we note that the corrections of order $1/p^2$ to Eq. (16) arise from two sources: (1) the off-energy-shell part of the second Born term and (2) the on-energy-shell part of the third Born term.[13]

Comparing the Born expansion (16) to the Glauber expansion (7) and the Blankenbecler-Goldberger expansion (9) shows that to order $1/p$ both the eikonal approximation and the impact-parameter approximation agree with the exact amplitude for *all* momentum transfers. In particular, both give the correct on-energy-shell contribution to order $1/p$, but both neglect the off-energy-shell contribution which is of leading order $1/p^2$.[14] As will be explicitly demonstrated below, it happens that the eikonal approximation of Glauber also correctly reproduces the on-energy-shell contribution to leading order for *every* Born term (to order $1/p^{n-1}$ for the nth Born term), whereas the impact-parameter approximation of Blankenbecler-Goldberger fails to correctly give the on-energy-shell contribution (to leading order) to the third- or higher-order Born terms.

III. YUKAWA-POTENTIAL INFINITE-RANGE LIMIT

We now consider the scattering by the attractive Yukawa potential of strength g^2 and range μ^{-1}:

$$V(r)=-g^2e^{-\mu r}/r\,, \quad (17)$$

in the limit of "exchange mass" μ going to zero (range going to infinity). Each term in the series (7) and (9) will be evaluated in this limit and the series summed, and the results compared to the exact scattering amplitude for this problem.

For the Yukawa potential (17) the potential function is given by[15]

$$V(b)=mg^2K_0(\mu b)\,, \quad (18)$$

where $K_0(x)$ is the zeroth-order modified Bessel function of the second kind. Substitution of this potential function in the series (7) and (9) gives

$$T_{\mathrm{G}}(p,\Delta)=mg^2\sum_{N=1}^{\infty}\left(\frac{img^2}{p}\right)^{N-1}\left(\frac{1}{N!}\right)\times\int_0^\infty db\, bJ_0(b\Delta)[K_0(\mu b)]^N \quad (19)$$

and

$$T_{\mathrm{BG}}(p,\Delta)=mg^2\sum_{N=1}^{\infty}\left(\frac{img^2}{2p}\right)^{N-1}\times\int_0^\infty db\, bJ_0(b\Delta)[K_0(\mu b)]^N. \quad (20)$$

To proceed further, the integral

$$I_N(\Delta,\mu)=\int_0^\infty db\, bJ_0(b\Delta)[K_0(\mu b)]^N,\quad N\geq 1 \quad (21)$$

must be evaluated. The integration can be performed exactly for $N=1$ and $N=2$ only. However, in the limit $\mu\to 0$, it can be shown that[16] ($\Delta\neq 0$)

$$I_N(\Delta,\mu)\underset{\mu\to 0}{=}\left(\frac{N}{\Delta^2}\right)\left[\ln\left(\frac{\Delta}{\mu}\right)\right]^{N-1},\quad N\geq 1. \quad (22)$$

Using this result in the expansions (19) and (20) gives

$$T_{\mathrm{G}}(p,\Delta)\underset{\mu\to 0}{=}\left(\frac{mg^2}{\Delta^2}\right)\sum_{N=1}^{\infty}\left[\frac{img^2\ln(\Delta/\mu)}{p}\right]^{N-1}\Big/(N-1)!\,, \quad (23)$$

which can be summed to give

$$T_{\mathrm{G}}(p,\Delta)\underset{\mu\to 0}{=}\left(\frac{mg^2}{\Delta^2}\right)e^{-i(mg^2/p)\ln(\mu/\Delta)} \quad (24)$$

and

$$T_{\mathrm{BG}}(p,\Delta)\underset{\mu\to 0}{=}\left(\frac{mg^2}{\Delta^2}\right)\sum_{N=1}^{\infty}\left[\frac{img^2\ln(\Delta/\mu)}{2p}\right]^{N-1}N\,, \quad (25)$$

which can be summed to give

$$T_{\mathrm{BG}}(p,\Delta)\underset{\mu\to 0}{=}\left(\frac{mg^2}{\Delta^2}\right)\left[1+i\left(\frac{mg^2}{2p}\right)\ln\left(\frac{\mu}{\Delta}\right)\right]^{-2}. \quad (26)$$

The exact solution to this problem is[17] ($\Delta\neq 0$)

$$T_{\mathrm{B}}(p,\Delta)\underset{\mu\to 0}{=}\left(\frac{mg^2}{\Delta^2}\right)e^{-i(mg^2/p)\ln(\mu/\Delta)}. \quad (27)$$

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[13] Recently, Sugar and Blankenbecler have developed a scheme for calculating the corrections of type (1) to the Glauber approximation. Cf. R. L. Sugar and R. Blankenbecler, Phys. Rev. **183**, 1387 (1969). As will be made clear below, the Glauber approximation already includes the correction (2).

[14] See Ref. 13.

[15] After substituting (17) in (4), make the variable change $z=b\sinh\eta$ and then use integral (7) on p. 182 of G. N. Watson, *Theory of Bessel Functions* (Cambridge U. P., New York, 1922).

[16] This result is found indirectly as follows. The integral $\int_0^\infty db\, bJ_0(b\Delta)e^{\lambda K_0(\mu b)}$ is considered. For $\Delta\gg\mu$ the dominant contribution comes from the region of small b, so the replacement $K_0(z)\to -\ln(z)$ is made. This converts the integral to the form $\int_0^\infty db\, b^{1+x}J_0(b)$, which can be integrated using integral (1) on p. 391 of Watson, Ref. 15. Expanding both the original integral and this result in a power series in λ and comparing terms gives result (22).

[17] C. Kacser, Nuovo Cimento **13**, 303 (1959).

It should be noted that every term which does not vanish as μ tends to zero is accounted for in this amplitude. The form of the exact solution is not surprising since in this limit the Yukawa potential goes into a $1/r$ potential, and it is well known that the complete Born series for scattering by a $1/r$ potential sums up to give just the first Born term times a complex factor of modulus 1.[18]

Comparison of (24), (26), and (27) shows that the Glauber approximation agrees identically with the exact result in this case, while the Blankenbecler-Goldberger approximation agrees with the exact result only through terms of order g^4/p. To understand why the eikonal approximation of Glauber gives the exact amplitude in this case, recall that Schiff was able to derive the Glauber form by actually summing the infinite Born series, after evaluating each individual Born term in the stationary phase approximation.[19] In hindsight, what he actually did was to obtain the *correct* on-energy-shell contribution to leading order in $1/p$ for each Born term, and then show that the series summed to the eikonal form.[20] However, in the small-μ limit (long range) of Yukawa scattering, all the other corrections, both on-energy-shell and off-energy-shell, go to zero, in fact, as fast as μ, and consequently the Glauber approximation gives the exact result in this limit.[21] On the other hand, the impact-parameter approximation of Blankenbecler-Goldberger only gives the correct on-energy-shell contribution through the second Born term. The reason for this is that the Blankenbecler-Goldberger approximation really amounts to evaluating to leading order in $1/p$ each Born term in the *approximation of replacing every propagator by its δ-function part*, and this does not include all of the on-energy-shell contribution for Born terms higher than the second.[22,23]

IV. CONCLUSION

Comparison of the approximate amplitudes with the Born-series expansion has shown that both reproduce the first Born term, and to order $1/p$ the on-energy-shell part of the second Born term, for *all* momentum transfers. Also, the Glauber approximation correctly gives the on-energy-shell contribution to leading order (in powers of $1/p$) of *every* Born term, whereas the Blankenbecler-Goldberger approximation fails to do this for Born terms beyond the second. However, since both approximations neglect contributions of order $1/p^2$, neither approximation, in the simple form commonly used, is correct beyond terms of order $1/p$.

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[18] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951); R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), Chap. 14.

[19] L. I. Schiff, Ref. 6.

[20] Recent work in high-energy perturbation theory in quantum electrodynamics has shown that, apparently, inclusion of diagrams required by gauge invariance (which means both crossed and uncrossed multiphoton ladder-type diagrams) leads to a cancellation of off-mass-shell contributions to leading order in $1/s$ in each order of the perturbation theory, with the result that the infinite sets of diagrams considered sum up to an eikonal form. Cf. H. Cheng and T. T. Wu, Phys. Rev. Letters **22**, 666 (1969); Phys. Rev. **186**, 1611 (1969).

[21] This is in agreement with the observation that the corrections to the Glauber approximation go to zero as the range of the potential goes to infinity. Cf. R. L. Sugar and R. Blankenbecler, Ref. 13.

[22] See Ref. 11.

[23] This can be shown explicitly by replacing every propagator in the nth Born term by its δ-function part, and performing the angular integrations to leading order in $1/p$. This gives $(i/2p)^{n-1} \times \int_0^\infty db\; bJ_0(b\Delta)V^n(b)$ for the nth Born term, and so the Born series (12) sums up to the Blankenbecler-Goldberger approximation (8).