IV. CONCLUSION

A method of general applicability was developed to evaluate, in terms of single dispersion integrals over closed-form analytic functions, the contribution of box diagrams for interactions involving particles with spin. In particular, it was applied to a case of frequent interest: the interaction of spin-$\frac{1}{2}$ fermions of mass M with pseudoscalar bosons of mass μ. The technique for evaluating the absorptive parts with relative ease was a type of partial-fraction expansion yielding numerators free of angular dependence in the phase-space integrals. The correctness of the result was ensured in two different ways. (a) The terms in the absorptive parts which themselves have an imaginary part were checked by explicit calculation of the double-spectral functions via the s and t channels independently. (b) The other terms in the absorptive parts were checked by ensuring their cancellation, upon passing to the double-spectral integral form, by the residues of the absorptive parts at the spurious kinematical singularities.

Several surprising factorizations occurred, bringing out natural positive subtraction points. We underlined the care needed to use them in the presence of kinematical singularities. Another unexpected result derived by explicit calculation lies in the improved convergence of the absorptive parts under the usual substitution made for the convenient evaluation of partial waves. It was shown that this allows one to do away completely with the subtraction in this useful representation.

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Investigation of Some Dual Amplitudes with Cuts

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We study two simple ways of adding an infinite number of direct-channel two-particle thresholds to the Veneziano picture, which usually contains only resonances in the direct channel. We investigate the high-energy behavior, positivity, and l-plane structure of the resulting amplitudes, which turn out to have Regge cuts in addition to Regge poles in the crossed channel.

I. INTRODUCTION

A LONG time had to pass, during which numerous data fits were made, and a deeper understanding of the nature of Regge cuts achieved, before it was realized that cuts in the angular momentum plane could not be neglected in Regge theory. The situation was different with Veneziano theory, which was known from the outset to violate unitarity by giving rise to zero-width resonances.[1,2] In fact, it was introduced as a first step of a bootstrap procedure in which complete crossing symmetry was given priority over unitarity, which was to be treated as an approximation in a further step. Many attempts at unitarization have been made, and it seems that any reasonable unitarization procedure would either result in cuts or have to use cuts as an input.[3,4] The cuts in the models we are planning to investigate are introduced in simple ways, and it will be obvious that unitarization is not the immediate objective. Nevertheless, it will be seen that to study such cuts is not totally unreasonable, since they lead to interesting new ideas.

We will explore the possibility of generating Regge cuts in the crossed channel through the introduction of an infinite number of direct-channel two-particle thresholds with real square-root branch points.[5] We will find that, just as we can imagine a world composed of an infinite number of zero-width direct-channel resonances, which Reggeizes as the direct-channel energy increases giving rise to Regge poles, we can equally well imagine a world composed of an infinite number of direct-channel two-particle thresholds with real branch points, that gives rise to Regge cuts in the crossed channel. It will be seen that the latter picture possesses novel properties similar to those possessed by the former: crossing symmetry, Regge behavior, likelihood of positivity, absence of fixed poles, etc. It also suffers from similar diseases: lack of Regge behavior on the real axis (except on the average), branch points and trajectories can only be real, etc. In fact, the two pictures, instead of being introduced separately, may be interconnected in a "cross-duality" scheme wherein direct-channel resonances give rise to Regge cuts, and direct-channel thresholds give rise to Regge poles in the crossed channel. Furthermore, it appears that the "cross-duality" picture may have certain advantages over the "direct-duality" one, in

[1] For a comprehensive review of narrow-resonance models, see D. Sivers and J. Yellin, Rev. Mod. Phys. (to be published).
[2] For a short review of the Veneziano model, see J. D. Jackson, Rev. Mod. Phys. 42, 12 (1970).
[3] A. Martin, Phys. Letters 29B, 431 (1970).
[4] For a comprehensive list of references to works on unitarization, see Ref. 2.
[5] M. O. Taha, Nuovo Cimento Letters 3, 861 (1970); Phys. Rev. D (to be published).

which resonances give rise to Regge poles and thresholds give rise to Regge cuts.

We stress again that unitarity is not sought in the models we are about to discuss. The poles and cuts in one channel are introduced separately and not allowed to interact, in order that the cuts may push the poles into the second sheet, as is usually the case in a smearing procedure.[3] However, these models may be looked upon as a first step of an approximation scheme, in which cuts are introduced on the same footing with poles, simply to provide freedom from meromorphism. An obvious next step would be to allow the poles and cuts of one channel to interact with each other, in an effort to rid the models from some of their diseases.

II. INTRODUCTION OF AMPLITUDES. DISCUSSION OF HIGH-ENERGY BEHAVIOR AND POSITIVITY

The scattering amplitude for two spinless equal-mass particles may be written in the form

$$M(s,t,u)=A(s,t)+A(t,u)+A(s,u)\,, \tag{1}$$

where

$$A(s,t)=A(t,s)\,.$$

If we restrict ourselves to zero-width resonances in the direct channel dual to Regge poles with linear Regge trajectories in the crossed channel, the simplest form of $A(s,t)$ will be given by the well-known Veneziano term $B(-\alpha(s),\,-\alpha(t))$, where the α's are the Regge trajectories in the energy plane. Now that we want to enlarge the picture by allowing for an infinite number of two-particle thresholds, we give ourselves the liberty to split $A(s,t)$ into two parts[6]:

$$A(s,t)=A_1(s,t)+A_2(s,t)\,. \tag{2}$$

Obviously there is more than one possible way of writing A_1 and A_2. Nevertheless, we can distinguish two extreme possibilities. (1) Resonances and thresholds are contained in separate parts; more precisely, A_1 contains the resonances and A_2 the thresholds. (2) A_1 contains the s-channel resonances and t-channel thresholds, while the t-channel resonances and s-channel thresholds are contained in A_2. These two possibilities will be referred to henceforth as "direct duality" and "cross duality," respectively,[5] and will be discussed separately below. To simplify writing we will adopt the following notations for the leading cut and pole trajectories:

$$\begin{aligned} X&=\alpha^c(s)=a(s-s_0)\,,\quad Y=\alpha^c(t)=a(t-t_0)\,,\\ x&=\alpha^p(s)=b(s-s_0)\,,\quad y=\alpha^p(t)=b(t-t_0)\,,\\ W&=\alpha^c(u)=a(u-u_0)\,,\\ w&=\alpha^p(u)=b(u-u_0)\,,\end{aligned} \tag{3}$$

where $s_0=4m^2$ and s_R are the first threshold and resonance in the s channel.

A. Direct-Duality Amplitudes

The resonance part A_1 in Eq. (2) is given by the Veneziano term

$$\begin{aligned} A_1(s,t)=T(x,y)&=C_1B(-x,\,-y)\\ &=C_1\Gamma(-x)\Gamma(-y)/\Gamma(-x-y)\,, \end{aligned} \tag{4}$$

while the part that contains the thresholds, A_2, is given by

$$A_2(s,t)=T(X,Y)$$

$$=C_2\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}B(\lambda-X,\lambda-Y) \tag{5a}$$

$$=C_2\Gamma(\tfrac{3}{2})\int_0^\infty \frac{dz\,z^{-X-1}(1-z)^{-Y-1}}{[q-\ln z(1-z)]^{3/2}}\,, \tag{5b}$$

where q is a parameter used to adjust the decrease of the exponential. This amplitude was proposed by Taha,[5] and also by Matveev, Stoyanov, and Tavkhelidze[7] with a slight difference in the integrand. The transformation between (5a) and (5b) is achieved through the identity

$$\int_0^\infty d\lambda\,\lambda^{z-1}e^{-q\lambda}=\Gamma(z)q^{-z}.$$

The Veneziano term (4) contains an infinite number of resonances in the direct channel and Reggeizes as the direct-channel energy approaches infinity, except on the real axis, where Regge behavior holds only for the average of the amplitude. An analogous picture emerges when we consider the amplitude $T(X,Y)$, which can be written in the form

$$T(X,Y)=\pi C_2\int_0^\infty d\lambda$$

$$\times\lambda^{1/2}e^{-q\lambda}[\cot\pi(\lambda-X)+\cot\pi(\lambda-Y)]$$

$$\times\frac{\Gamma(X+Y+1-2\lambda)}{\Gamma(X-\lambda+1)\Gamma(Y-\lambda+1)}\,. \tag{6}$$

On expansion, the part of $T(X,Y)$ singular in $X-\lambda$ yields an infinite number of equally spaced s-channel thresholds at $X=n$ $(s=s_0+n/a,\ n=0,\,1,\,\ldots)$:

$$T(X,Y)=C_2\sum_{n=0}^\infty\int_0^\infty d\lambda$$

$$\times\lambda^{1/2}e^{-q\lambda}\frac{\Gamma(Y-\lambda+1+n)}{\Gamma(n+1)\Gamma(Y-\lambda+1)}\frac{1}{\lambda-(X-n)}\,.$$

[6] The splitting of $A(s,t)$ needs to be justified. In a work on a generalized interference model R. Jengo [Phys. Letters **28B**, 606 (1969)] stated some sufficient conditions which, if satisfied by A_1 and A_2, would justify the splitting. The amplitudes A_1 and A_2 introduced below do not satisfy Jengo's conditions. Thus they have no relation to Jengo's interference model, as Taha (Ref. 5) suggested the case might be.

[7] V. A. Matveev, D. T. Stoyanov, and A. N. Tavkhelidze, JINR Report No. E2-4978, 1970 (unpublished).

As $X\to\infty$ $(s\to\infty)$,[8] $T(X,Y)$ Reggeizes in the crossed channel (we exclude the real axis), giving rise to an infinite number of equally spaced Regge cuts at $\alpha^c(t)=Y=n=0,\,1,\,\ldots$:

$$T(X,Y)\to\pi C_2\int_0^\infty\frac{d\lambda\,\lambda^{1/2}e^{-q\lambda}\cot\pi(\lambda-Y)X^{Y-\lambda}}{\Gamma(Y-\lambda+1)}$$

$$\to C_2X^Y\sum_{n=0}^{\infty}\int_0^\infty\frac{d\lambda\,\lambda^{1/2}e^{-q\lambda}X^{-\lambda}}{[\lambda-(Y-n)]}.$$

The discontinuities across these cuts are proportional to

$$\frac{1}{\Gamma(n+1)}\frac{X^Y}{(\ln qX)^{3/2}},$$

and have the expected Regge behavior. X and Y are defined in (3).

Thus, we conclude that the above choice of A_1 and A_2 results in a duality between direct-channel resonances and crossed-channel Regge poles on one hand, and a duality between direct-channel thresholds and crossed-channel Regge cuts on the other hand.

B. Cross-Duality Amplitudes

The second possibility is achieved by pictorially breaking each of $T(x,y)$ and $T(X,Y)$ into two halves and attaching the resonances in x with the thresholds in Y and the thresholds in X with the resonances in y. Thus we obtain

$$A_1(s,t)=T(x,Y)=C\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}B(-x,\lambda-Y)\qquad(7\text{a})$$

$$=C\Gamma(\tfrac{3}{2})\int_0^\infty\frac{dz\,z^{-x-1}(1-z)^{-Y-1}}{[q-\ln z]^{3/2}},\qquad(7\text{b})$$

$$A_2(s,t)=A_1(t,s)=T(X,y).\qquad(8)$$

These amplitudes were proposed by Taha (with $q=1$) and called by him "cross-duality" amplitudes.[5] Upon expansion in direct-channel (s-channel) singularities, we find $T(x,Y)$ to be composed of an infinite number of equally spaced zero-width resonances, while $T(X,y)$ is made up of an infinite number of equally spaced thresholds with real square-root branch points:

$$T(x,Y)=C\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\frac{1}{x-n}\int_0^\infty d\lambda\times\lambda^{1/2}e^{-q\lambda}\frac{\Gamma(Y-\lambda+n+1)}{\Gamma(Y-\lambda+1)},\qquad(9)$$

$$T(X,y)=C\sum_{n=0}^{\infty}\frac{\Gamma(y+n+1)}{\Gamma(n+1)\Gamma(y+1)}\int_0^\infty d\lambda\times\lambda^{1/2}e^{-q\lambda}\frac{1}{\lambda-(X-n)}.\qquad(10)$$

On the other hand, as x, $X\to\infty$ $(s\to\infty)$, $T(x,Y)$ and $T(X,y)$ Reggeize in the crossed channel, giving Regge-cut and Regge-pole behavior, respectively:

$$T(x,Y)\to C\sum_{n=0}^{\infty}\frac{x^Y}{\Gamma(n+1)}\int_0^\infty d\lambda\times\lambda^{1/2}e^{-q\lambda}\frac{x^{-\lambda}}{\lambda-(Y-n)},\qquad(11)$$

with discontinuities across the cuts proportional to

$$\frac{1}{\Gamma(n+1)}\frac{x^Y}{(\ln qx)^{3/2}},\qquad(12)$$

$$T(X,y)\to\Gamma(\tfrac{3}{2})q^{-3/2}X^Y/\Gamma(y+1).\qquad(13)$$

Thus we obtain from Eqs. (9), (11), (12), and (10) and (13) the following picture: Direct-channel resonances are dual to Regge cuts, while direct-channel thresholds are dual to Regge poles in the crossed channel.

We conclude this section with a remark concerning positivity. The various amplitudes $T(X,Y)$, $T(x,Y)$, etc., are not smearings of the Veneziano amplitudes $T(x,y)$, as one might think, and thus we are not free to conclude, as was proved by Martin, that positivity should hold for the former wherever it holds for the latter.[3] On the other hand, the integral forms of these amplitudes, (5a) and (7a), may be looked upon as infinite sums over the Veneziano-type terms

$$\Gamma(\lambda_j-X)\Gamma(\lambda_j-Y)/\Gamma(2\lambda_j-X-Y),\quad \lambda_j=j\Delta\lambda,\quad j=0,\,1,\,\ldots\qquad(14)$$

with the exponentially decreasing factors $\lambda_je^{-q\lambda_j}$ that enhance the contribution from small values of λ. If the tails of these infinite sums beyond $\lambda_j\sim1$ are made negligible by choosing the parameter q to be large enough, the necessary condition for positivity,[9] in this case

$$\lambda<1-\alpha^c(0),$$

will be satisfied. Obviously the detailed discussion of positivity for terms of the type (14) follows the line intended for an ordinary Veneziano term.[1] It is hoped, however, that a summation (or, alternatively, an integration) over such terms with appropriate weight factors is more likely to satisfy positivity than a single Veneziano term.

[8] We note that taking the limit $X\to\infty$ inside the integral, where λ can also be very large, is justified since only small values of λ contribute appreciably to the integral. The latter fact will be used repeatedly throughout our work.

[9] R. Oehme, Nuovo Cimento Letters 1, 420 (1969).

III. l-PLANE STRUCTURE OF DIRECT-DUALITY AMPLITUDES

From Eqs. (1), (2), (4), and (5), we obtain for the full amplitude in the direct-duality case

$$A_D(s,t,u)=[T(x,y)+T(X,Y)]+[T(w,y)+T(W,Y)]+[T(x,w)+T(X,W)]. \quad (15)$$

Our aim is to project the partial-wave amplitudes in the t channel. We will ignore the last square bracket in (15), which has neither poles nor cuts in the physical t channel. Furthermore, it can be demonstrated that this bracket goes to zero faster than any power as s, $u\to\infty$ for fixed t, if we exclude the real axis: If we define

$$\nu=\tfrac{1}{2}(x-w)=x-f=-w+f,$$
$$\nu'=\tfrac{1}{2}(X-W)=X-g=-W+g,$$

where f and g are linear functions of t, the desired result follows immediately:

$$T(x,w)=C\Gamma(-\nu-f)\Gamma(\nu-f)/\Gamma(-2f) \underset{\nu\to\infty}{\longrightarrow} -\frac{\pi C}{\Gamma(-2f)}\frac{\nu^{-2f-1}}{\sin\pi(\nu+f)}, \quad \mathrm{Im}|\nu|>0$$

$$T(X,W)=C\Gamma(\lambda-\nu'-g)\Gamma(\lambda+\nu'-g)/\Gamma(2\lambda-2g) \underset{\nu'\to\infty}{\longrightarrow} -\frac{\pi C}{\Gamma(2\lambda-2g)}\frac{\nu'^{2\lambda-2f-1}}{\sin\pi(\nu'+f-\lambda)}, \quad \mathrm{Im}|\nu'|>0.$$

If we ignore the last square bracket in (15), the expansion of A_D in terms of its right- and left-hand singularities in s will have the following forms:

$$A_D{}^R(s,t,u)=C_1\sum_{n=0}^{\infty}\frac{R_n(y)}{n-x}+C_2\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}\frac{R_n(Y-\lambda)}{\lambda+n-X}, \quad (16)$$

$$A_D{}^L(s,t,u)=C_1\sum_{n=0}^{\infty}\frac{R_n(y)}{n-w}+C_2\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}\frac{R_n(Y-\lambda)}{\lambda+n-W}, \quad (17)$$

where

$$R_n(\zeta)=\Gamma(\zeta+n+1)/\Gamma(n+1)\Gamma(\zeta+1). \quad (18)$$

The partial-wave amplitudes are given by the formula[10]

$$\mathcal{A}_D{}^{\pm}(l,t)=\frac{1}{2}\int_{-1}^{1}P_l(z_t)A_D{}^{\pm}(s,t)\,dz_t, \quad (19)$$

where

$$z_t=1+2s/(t-t_0)=1+2[X-\alpha^c(0)]/Y=1+2a[x-\alpha^p(0)]/bY, \quad (20)$$

and $A_D{}^{\pm}$ give the even and odd parts of A_D, respectively:

$$A_D{}^{\pm}(s,t)=A_D{}^R[t,s(z_t,t)]\pm A_D{}^L[t,s(-z_t,t)]. \quad (21)$$

We finally obtain the following result for the partial-wave amplitudes:

$$\mathcal{A}_D(l,t)=\frac{C_1}{bY/2a}\sum_{n=0}^{\infty}R_n(y)Q_l\left(1+\frac{n-\alpha^p(0)}{bY/2a}\right)+\frac{C_2}{\frac{1}{2}Y}\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}R_n(Y-\lambda)\times Q_l\left(1+\frac{n+\lambda-\alpha^c(0)}{\frac{1}{2}Y}\right), \quad (22)$$

where $\mathcal{A}_D$ is identical with $\mathcal{A}_D{}^+$ and $\mathcal{A}_D{}^-$ at even and odd l, respectively.

Let us investigate the behavior of $\mathcal{A}_D$ in l and t.

(i) $\mathcal{A}_D$ possesses fixed poles in l at $l=-1, -2, \ldots$. The residues of these poles are proportional to

$$\beta_N\sim\frac{C_2}{bY/2a}\sum_{n=0}^{\infty}R_n(y)P_{N-1}\left(1+\frac{n-\alpha^p(0)}{bY/2a}\right)+\frac{C_2}{\frac{1}{2}Y}\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}R_n(Y-\lambda)\times P_{N-1}\left(1+\frac{n+\lambda-\alpha^c(0)}{\frac{1}{2}Y}\right). \quad (23)$$

This is a consequence of the fact that the Legendre function Q_l has fixed poles at $l=-N$ with residues proportional to P_{N-1}. The first part of (23), which is the contribution of the Veneziano term, was shown to vanish by Fivel and Mitter.[11] The second part may be shown to vanish in a similar manner. We only have to notice that (23) cannot be continued as it stands to negative l irrespective of the values of y, Y, and λ. Consider, for example, the case $l=-1$ ($N=1$). The integrand in (22), in this case, contains the following infinite sum:

$$\sum_{n=0}^{\infty}\Gamma(Y-\lambda+1)/\Gamma(Y-\lambda+1)\Gamma(n+1), \quad (24)$$

which converges only in the region $\mathrm{Re}(Y-\lambda)<-1$. It is easy to show that the infinite sum (24), and therefore the residue β_1, vanishes in this region. If we now introduce an analytic continuation of (22) to $l=-1$ valid in the region $\mathrm{Re}(Y-\lambda)>-1$, then by analyticity the corresponding β_1 will have to vanish

[10] P. D. B. Collins and E. Squires, *Springer Tracts in Modern Physics: Vol. 45, Regge Poles in Particle Physics* (Springer-Verlag, New York, 1968).

[11] D. I. Fivel and P. K. Mitter, Phys. Rev. 183, 1240 (1969).

in this region. Thus it follows that β_1 vanishes for all values of λ and Y. A similar argument may be used for $N>1$.

(ii) $\mathcal{A}_D$ contains moving singularities that appear as divergences of the infinite sums in (22):

$$\mathcal{A}_D(l,t)\sim\sum_{n=0}^{\infty} n^{y-l-1}[A^p(l,t)+B^p(l,t)n^{-1}+O(n^{-2})] +\int_0^\infty d\lambda \sum_{n=0}^{\infty} n^{Y-\lambda-l-1} \times[A^c(l,\lambda,t)+B^c(l,\lambda,t)n^{-1}+O(n^{-2})] \sim \frac{A^p(l,t)}{l-y}+\frac{B^p(l,t)}{l-(y+1)}+\cdots +\int_0^\infty d\lambda\left[\frac{A^c(l,\lambda,t)}{\lambda-(l-Y)}+\frac{B^c(l,\lambda,t)}{\lambda-(l-Y-1)}+\cdots\right]. \quad (25)$$

In deriving (25) we used the following formulas:

$$Q_l(z)\sim z^{-l-1}\quad (z\to\infty),\qquad \sum_{n=0}^{\infty} n^{-x}=1/(x-1)+(\text{terms regular in } x). \quad (26)$$

We also made use of the fact that only small values of λ contribute appreciably to the integral.[8]

Thus, just as an infinite number of direct-channel resonances diverge, giving rise to (moving) Regge poles in the crossed channel, an infinite number of direct-channel thresholds diverge in order to give rise to (moving) Regge cuts in the crossed channel. The leading Regge pole and Regge cut are found from (25) to be

$$\alpha^p=y=b(t-t_R),\quad \alpha^c=Y=a(t-t_0),$$

while the nonleading (satellite) poles and cuts are

$$l_n^p=y+n,\quad l_n^c=Y+n,\quad n=1,2,\ldots.$$

A^p and B^p in (25) are the residues of the leading and first satellite poles, while A^c and B^c are the discontinuities across the leading and first satellite cuts.

(iii) As we investigate the threshold behavior of $\mathcal{A}_D$, we should keep in mind that the first part of (22) is contributed by the amplitude $T(x,y)$, which has only poles in t. Even though this part may be shown to contain the correct threshold factor $Y^l\sim(t-t_0)^l$, it is not relevant to speak of its threshold behavior since $T(x,y)$ has no cuts in t. On the other hand, the second part is contributed by $T(X,Y)$, which possesses cuts in t, as we have already seen. If we restrict ourselves to the latter part and make use of formula (26) near the elastic threshold, i.e., near $Y=a(t-t_0)=0$, we obtain

$$\mathcal{A}_D\sim C_2(\tfrac{1}{2}Y)^l\int_0^\infty d\lambda \times\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}\frac{\Gamma(-\lambda+n+1)}{\Gamma(-\lambda+1)\Gamma(n+1)} \times[n+\lambda-\alpha^c(0)]^{-l-1}. \quad (27)$$

Thus we conclude that $\mathcal{A}_D$ has the correct threshold behavior, $Y^l\sim(t-t_0)^l$. However, we can safely make this conclusion only in the regions of l and λ where the above sum converges. In regions where the sum diverges a suitable analytic continuation must be introduced before we can draw a conclusion about the threshold behavior of $\mathcal{A}_D$.

(iv) From (25) we find that the discontinuities across the Regge cuts are proportional to

$$(l-Y-n)^{1/2}\exp[-(l-Y-n)]$$

and thus have the right behavior dictated by unitarity: They vanish at the branch points as square roots and fall off to zero away from the branch points. This behavior, however, is a result of deliberate construction and should not be taken to reflect some degree of unitarity existing in the model.

By construction, the Regge cuts in the crossed channel arise, as we have already seen, from summing an infinite number of direct-channel two-particle thresholds. In reality we are not certain that such thresholds do give rise to Regge cuts. In fact, we know for sure that the first (elastic) threshold ($s=s_0=4m^2$) is never responsible for Regge cuts in the t channel.[12]

IV. *l*-PLANE STRUCTURE OF CROSS-DUALITY AMPLITUDES

Equations (1), (2), (7), and (8) give the following form for the full amplitude in the cross-duality case:

$$A_{\rm CR}(s,t,u)=[T(x,Y)+T(X,y)]+[T(w,Y)+T(W,y)] +[T(x,W)+T(X,w)]. \quad (28)$$

For the same reasons stated in Sec. III, we drop the last square bracket of (28), as we intend to study the partial-wave amplitudes in the t channel. The cross-duality analogs of Eqs. (16) and (17) of Sec. III are

$$A_{\rm CR}^R=C\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}\frac{R_n(Y-\lambda)}{n-x} +C\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}\frac{R_n(y)}{\lambda+n-X},$$

$$A_{\rm CR}^L=C\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}\frac{R_n(Y-\lambda)}{n-w} +C\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}\frac{R_n(y)}{\lambda+n-W},$$

where $R_n(\zeta)$ is given by Eq. (18). Substituting these results into Eqs. (21) and (19), we obtain the following

[12] R. Oehme, in *Scottish Universities' Summer School in Physics* (Oliver and Boyd, London, 1964).

result for the partial-wave amplitude:

$$\mathcal{A}_{\mathrm{CR}}(l,t)=\frac{C}{bY/2a}\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}R_n(Y-\lambda)\times Q_l\left(1+\frac{n-\alpha^p(0)}{bY/2a}\right)+\frac{C}{\frac{1}{2}Y}\int_0^\infty d\lambda\,\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}R_n(y)\times Q_l\left(1+\frac{n+\lambda-\alpha^p(0)}{\frac{1}{2}Y}\right). \quad (29)$$

In a manner similar to that of Sec. III we can prove the vanishing of the residues of the fixed poles and discuss the threshold behavior of $\mathcal{A}_{\mathrm{CR}}(l,t)$. To investigate $\mathcal{A}_{\mathrm{CR}}$ for moving singularities in the l plane, we again make use of Eqs. (26) and (27) and obtain the following form for $\mathcal{A}_{\mathrm{CR}}$:

$$\mathcal{A}_{\mathrm{CR}}(l,t)\sim\frac{C}{bY/2a}\int_0^\infty d\lambda\frac{\lambda^{1/2}e^{-q\lambda}}{\Gamma(Y-\lambda+1)}\times\sum_{n=0}^{\infty}n^{Y-\lambda}\left(\frac{n}{bY/2a}\right)^{-l-1}[1+O(1/n)]+\frac{C}{\frac{1}{2}Y\Gamma(y+1)}\int_0^\infty d\lambda\times\lambda^{1/2}e^{-q\lambda}\sum_{n=0}^{\infty}n^{y}\left(\frac{n}{\frac{1}{2}Y}\right)^{-l-1}[1+O(1/n)],$$

$$\sim\int_0^\infty d\lambda\left[\frac{F^c(l,\lambda,t)}{\lambda-(l-Y)}+\frac{G^c(l,\lambda,t)}{\lambda-(l-Y-1)}+\cdots\right]+\frac{F^p(l,t)}{l-y}+\frac{G^p(l,t)}{l-(y-1)}+\cdots.$$

Thus, in contrast with the case in Sec. III [cf. Eq. (25)], we are dealing here with (moving) Regge cuts which appear as divergences of an infinite sum over direct-channel resonances, and (moving) Regge poles appearing as divergences of an infinite sum over direct-channel thresholds. The positions of the poles and the cuts are the same as those found in Sec. III.

So far we have discussed the direct-duality and cross-duality pictures in the realm of properties where they appear equally valid or invalid. Let us now comment on the possible advantages of the cross-duality picture. The direct-duality picture was realized by the simple addition of a threshold Regge-cut duality scheme to the already existing scheme of resonance Regge-pole duality. The sole achievement of this simple addition was freedom from the meromorphism dictated by the narrow-resonance model. None of the novel properties of this model was altered, and no cures were provided for its diseases. The pole trajectories are still strictly real. Furthermore, we had to make the assumption that direct-channel two-particle thresholds give rise to Regge cuts of the crossed channel. Obviously the cross-duality picture does not make use of this dubious assumption. Moreover, as can be seen from (9), the residues at the resonances in $x=\alpha^p(s)$ are polynomials in $Y=\alpha^c(t)$ instead of $y=\alpha^p(t)$. This gives us the liberty to make the pole trajectory α^p complex without risking the introduction of ancestors, as would inevitably be the case in the direct-duality picture where the residues at the resonances in $\alpha^p(s)$ are to be polynomials in $\alpha^p(t)$.[5]

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