The intermediate-coupling models do not have analogs of the second superconvergence conditions [Eqs. (26)]. The only solution for the mass matrix obtained from these equations for the representations ($V=0$, **56**) and ($V=2$, **56**) is the trivial one of mass degeneracy within each of these multiplets. In the strong-coupling model,[15] such a result would have implied the vanishing of the scattering amplitude itself. Here the mass degeneracy is an acceptable solution, while not a satisfactory one.

In conclusion, we believe that this work gives a better perspective to the results of Cronstrom and Noga.[2]

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Critique of a Proposed Dynamical Group for Relativistic Quantum Mechanics*

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The dynamical group $\tilde{G}_5$ for relativistic quantum mechanics phenomenologically suggested by Aghassi, Roman, and Santilli is derived from the analysis of symmetry properties of Lagrangians and corresponding equations of motion for a free relativistic particle. All physical observables such as position, momentum, angular momentum, and mass squared are represented by well-defined operators which close the algebra of the dynamical group $\tilde{G}_5$. The unitary irreducible representations of this group, which are possible states of the physical system, are found. The particles accommodated in the single unitary irreducible representations have various spins starting from the lowest spin value and going up to infinity in integral steps. The mass-squared operator $P_\mu P^\mu$ lies in the enveloping algebra of $\tilde{G}_5$, and its eigenvalues are not necessarily quantized and can have any positive or negative values. It is pointed out that this group has several failures and thus it cannot be accepted as the reliable dynamical group for particles within relativistic quantum mechanics.

I. INTRODUCTION

THE hypothesis that the dynamics of the quantal interacting system can be completely described by some dynamical group has been verified for almost all interesting quantum-mechanical problems.[1] In the approach using dynamical groups, instead of postulating the Hamiltonian for the quantum-mechanical system we postulate a dynamical group. Then the quantum-mechanical wave functions are supposed to form the basis for the unitary irreducible representation of the group in question which is generated by the operators of the physical observables. The same idea was consequently used in strong-interaction physics with the great hope of predicting hadron states with their masses and mutual coupling constants.[2,3] This approach to particle physics became rather popular recently because various bootstrap schemes[4] and superconvergence relations following from the proper Regge behavior of the scattering amplitude are formulated in the group-theoretical language.[5]

One essential shortcoming of the models mentioned above is connected with the mass spectrum. The relation for the mass spectrum is solvable only in models which are not fully relativistic invariant, such as those in strong coupling[6] and in bootstrap theory,[7] while in the models with relativistic invariance the condition imposed on the hadron masses[5] can be solved only if rough approximations are made.[8]

Other attempts which were made to obtain the mass spectrum in relativistic-invariant theories tried to combine the Poincaré group with some semisimple

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internal symmetry group in a nontrivial way. These attempts were, of course, disappointing because O'Raifeartaigh's theorem[9] prevents the emergence of a discrete mass spectrum. The explanation for this failure can be given by the Flato-Sternheimer theorem[10] according to which any extension of the Poincaré groups by some semisimple Lie group is trivial in the sense that the mass-squared operator $P_\mu P^\mu$ will always be a Casimir operator of the extended algebra. This simply implies that if we want to have the various particles with different masses in the same unitary irreducible representation of some dynamical group G, then G must be the extension of the Poincaré group by some non-semisimple group in order to have a relativistic invariant theory where the $P_\mu P^\mu$ operator will not be the Casimir operator of the group G.

However, the most interesting approach to the mass spectrum of particles is treated in theories with infinite multiplets and infinite-component wave equations. In those theories, $P_\mu P^\mu$ is not a Casimir operator and O'Raifeartaigh's theorem cannot be applied since the dynamical group is confirmed to the rest frame only. The results obtained in this approach are more than encouraging.[11]

The first dynamical group for relativistic quantum mechanics of elementary particles which is not restricted to the rest frame only was recently suggested by Aghassi, Roman, and Santilli.[12] These authors proposed phenomenologically the so-called $\tilde{G}_5$ group generated by the set of operators which can be identified with the physical observables in the framework of relativistic quantum mechanics such as position, momentum, angular momentum, and particle mass squared. A similar group was also considered by Castell,[13] but not in such an elegant form as in Ref. 12.

The group $\tilde{G}_5$ is the nontrivial extension of the Poincaré group by some non-semisimple group so that O'Raifeartaigh's theorem is not applicable, and therefore it is possible to have particles with different masses in single unitary irreducible representations. Indeed the mass-squared operator $P_\mu P^\mu$ lies in the enveloping algebra of $\tilde{G}_5$ and enables us to calculate the mass spectrum of particles accommodated in the single representation. To the best of our knowledge, the model suggested by these authors[12] is the only one so far which is not restricted to the rest frame and is fully relativistic invariant with a clear physical interpretation of the generators of the group in question.

The aim of this paper is to show that the aforementioned group $\tilde{G}_5$ is a direct consequence of symmetry properties of the Lagrangian and corresponding equations of motion for a free relativistic particle provided that a *naive* transition from classical to quantum mechanics is performed. Furthermore, we shall find the unitary irreducible representations of this group which are assumed to represent the physical states. From the knowledge of the representations, we are able to find the physical properties of the particles furnishing the single unitary irreducible representations. To conclude this section, it should be stressed that even though the structure of this dynamical group is enormously attractive, the results obtained in this model are unphysical, indicating that this particular dynamical group is not suitable for describing particles in relativistic quantum mechanics.

II. KINEMATICAL AND DYNAMICAL SYMMETRIES

We shall study first the free-particle states in classical nonrelativistic mechanics from the group-theoretical point of view. The Lagrangian for a free particle in classical mechanics has the simple form

$$\mathcal{L}=\sum_k \left(\frac{dx_k}{dt}\right)^2, \quad k=1,2,3 \tag{2.1}$$

and is clearly invariant under rotations and translations in three-dimensional Euclidean space. Thus the symmetry group of this Lagrangian is the inhomogeneous rotation group $ISO(3)$, which is the semidirect product of the rotation group $SO(3)$ with the three-dimensional Abelian group $T_3{}^a$:

$$ISO(3)=SO(3)\times T_3{}^a. \tag{2.2}$$

Here $SO(3)$ is generated by the three angular momentum operators $\mathcal{T}_k$, and $T_3{}^a$ contains three translation operators P_k. However, the equations of motion

$$d^2x_k/dt^2=0, \tag{2.3}$$

derived from the Lagrangian (2.1), are covariant also with respect to the Galilei transformations

$$x_k \to x_k+v_kt, \tag{2.4a}$$

$$t\to t+\tau, \tag{2.4b}$$

where v_k and τ are unrestricted parameters independent of time t, in addition to the group transformations (2.2). This implies immediately that the symmetry group of the equations of motion (2.3) becomes larger than the symmetry group of the corresponding Lagrangian (2.1). By detailed analysis,[12] it can be shown that the symmetry group of the equations of motion (2.3) is G_4, which has the following Lie group structure:

$$G_4=\{T_3{}^a\otimes T_1{}^\tau\}\times\{T_3{}^v\times SO(3)\}. \tag{2.5}$$

Here $T_3{}^v$ denotes the three-dimensional Abelian group generated by the generators Q_k connected with the

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[12] J. J. Aghassi, P. Roman, and R. M. Santilli, Phys. Rev. D **1**, 2753 (1970); J. Math. Phys. (to be published).
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velocity transformations (2.4a), while T_1^τ is the one-dimensional time translation group generated by the operator H corresponding to the transformations (2.4b), and the symbols $\otimes$ and $\times$ stand for the direct and semidirect products, respectively. The operator H has important physical meaning because it plays the role of the evolution operator with respect to the time t. We can see that the symmetry group $ISO(3)$ of the Lagrangian is only the subgroup of the symmetry group G_4 of the equations of motion which represent the dynamics. Therefore, we shall refer to the symmetry of the Lagrangian as a kinematical symmetry while the symmetry of the equations of motion will be referred to as a dynamical symmetry. From this point of view, the group G_4 plays the role of the dynamical group for the free particle in nonrelativistic classical mechanics. It is worthwhile to note that one of the Casimir operators of G_4 is the operator P_kP_k, the eigenvalue of which represents (up to a multiplicative constant) the energy of the particle, which can be used as the label for the unitary irreducible representations of the group in question. It can be easily verified that the physical observables such as the position x_k and the momentum p_k of our particle are the matrix elements of the generators of G_4, namely,

$$x_k=\langle X_k\rangle\equiv M^{-1}\langle Q_k\rangle \tag{2.6a}$$

and

$$p_k=\langle P_k\rangle\,, \tag{2.6b}$$

where M is the mass of the particle under consideration.

Having identified the position X_k and momentum P_k operators in the dynamical group G_4, we can start to study the transition from classical to quantum nonrelativistic mechanics on the basis of group theory. This transition consists in requiring that the commutator between the position X_k and momentum P_k operators be nonvanishing, namely, the relation

$$[X_l,P_r]=i\delta_{lr}\,, \tag{2.7a}$$

which in terms of Q_k operators can be rewritten in the form

$$[Q_k,P_r]=i\delta_{kr}M\,. \tag{2.7b}$$

The last commutator implies that in quantum mechanics we are not dealing any more with the dynamical group G_4 but with a larger group which we denote by $\tilde{G}_4$. Its Lie group structure can be found easily by taking into account relation (2.7b) along with the remaining commutators defining the group G_4. One finds that the dynamical group characterizing the free particle in nonrelativistic quantum mechanics has the following Lie group structure:

$$\tilde{G}_4=\{T_1^M\otimes T_1^\tau\otimes T_3^a\}\times\{T_3^v\times SU(2)\}\,. \tag{2.8}$$

Here the additional one-dimensional Abelian group T_1^M is responsible for the emergence of the constant M in the commutator (2.7b). It is to be mentioned that one of the Casimir operators of the group $\tilde{G}_4$ has the form

$$B=P_kP_k-2MH\,, \tag{2.9}$$

where H plays the role of the Hamiltonian, the eigenvalue of which is the energy of the quantum-mechanical system. However, the last equation tells us that the unitary irreducible representation of $\tilde{G}_4$ labeled by B contains states with different energies, which was not the case in the classical dynamical group G_4. It should also be mentioned that particles with different masses M belong to different unitary irreducible representations because the constant M commutes with everything and can be used as the label for the representations.

All results presented so far were well known long ago but we have presented them here to have a good analogy for deriving the dynamical group for relativistic quantum mechanics. Before we attack this problem, we shall study the symmetry properties of the free particle in classical relativistic mechanics. To begin with, we write the Lagrangian for a free particle in relativistic classical mechanics as

$$\mathfrak{L}=g^{\mu\nu}U_\mu U_\nu\equiv g^{\mu\nu}\,dx_\mu/d\tau\cdot dx_\nu/d\tau\,, \tag{2.10}$$

where U_ν is the four-velocity, τ is the proper time, and $g^{\mu\nu}$ is the metric tensor defined as

$$g^{00}=-g^{kk}=1\,,\quad g_{\mu\nu}=0\quad\text{for}\quad \mu\neq\nu\,,$$

and $\mu,\nu=0,1,2,3$, $k=1,2,3$.

The Lagrangian $\mathfrak{L}$ in (2.10) is obviously invariant under homogeneous Lorentz transformations,

$$x_\mu\longrightarrow\Lambda_\mu{}^\nu x_\nu\quad(\Lambda_\mu{}^\nu\Lambda_\nu{}^\rho=g_\mu{}^\rho)\,, \tag{2.11a}$$

as well as under translations,

$$x_\mu\longrightarrow x_\mu+a_\mu\,. \tag{2.11b}$$

As is well known, the transformations (2.11) are generated by the Poincaré group P which is the semidirect product of the homogeneous Lorentz group $SO(3,1)$ with the four-dimensional Abelian group T_4^a and satisfies the following Lie algebra:

$$[\mathcal{T}_{\mu\nu},\mathcal{T}_{\rho\sigma}]=i(g_{\nu\rho}\mathcal{T}_{\mu\sigma}+g_{\mu\sigma}\mathcal{T}_{\nu\rho}-g_{\mu\rho}\mathcal{T}_{\nu\sigma}-g_{\nu\sigma}\mathcal{T}_{\mu\rho})\,, \tag{2.12a}$$

$$[\mathcal{T}_{\mu\nu},P_\rho]=i(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu)\,, \tag{2.12b}$$

and

$$[P_\mu,P_\nu]=0\,. \tag{2.12c}$$

Here $\mathcal{T}_\mu$ and P_μ are the generators of $SO(3,1)$ and T_4^a, respectively. From our point of view, the Poincaré group is considered as the kinematical symmetry group of the system under considerations.

The equations of motion for a free particle in classical relativistic mechanics take the following simple form:

$$d^2x_\mu/d\tau^2=0\,, \tag{2.13}$$

and they possess a larger symmetry than the Lagrangian from which they are derived. Clearly, Eq. (2.13) is covariant under the following group of transformations:

$$x_\mu \to x_\mu + b_\mu' \tau, \tag{2.14a}$$

$$\tau \to \tau + \sigma', \tag{2.14b}$$

where b_μ' and σ' are unrestricted parameters independent of the proper time τ. The form of the transformations (2.14) reminds us very strongly of the group of Galilei transformations (2.4). For the sake of convenience, we shall use, instead of the proper time τ, the dimensionless parameter u defined as

$$u = \tau / l m_0, \tag{2.15}$$

where m_0 is the mass of the particle under consideration, and l is a constant having the dimension of $(\text{length})^2$. It is clear that the equations of motion (2.13) are invariant under the transformations

$$x_\mu \to x_\mu + b_\mu u \tag{2.16a}$$

and

$$u \to u + \sigma, \tag{2.16b}$$

where b_μ and σ are again some unrestricted parameters. The transformations given by Eqs. (2.16) are generated by two Abelian groups $T_4{}^b$ and $T_1{}^\sigma$, the generators of which we denote by Q_μ and S, respectively, and fulfil the set of commutation relations

$$[Q_\mu, Q_\nu] = [S, P_\mu] = [S, \mathcal{T}_{\mu\nu}] = 0, \tag{2.17a}$$

$$[Q_\mu, P_\nu] = 0, \tag{2.17b}$$

$$[\mathcal{T}_{\mu\nu}, Q_\rho] = i(g_{\nu\rho} Q_\mu - g_{\mu\rho} Q_\nu), \tag{2.17c}$$

$$[Q_\mu, S] = i P_\mu. \tag{2.17d}$$

It should be noted that the operator S plays the role of the evolution operator with respect to the parameter u. The Lie algebra defined by the commutators (2.12) and (2.17) generates the Lie group G_5 with the structure

$$G_5 = \{T_1{}^\sigma \otimes T_4{}^a\} \times \{T_4{}^b \times SO(3,1)\}. \tag{2.18}$$

This group can be considered as the dynamical group describing the motion of the free particle in classical relativistic mechanics. It is possible to verify that the mass-squared operator $P_\mu P^\mu$ is the Casimir operator of the group G_5, which implies that particles with different masses belong to different unitary irreducible representations of form incoherent states.

Before proceeding further we investigate the u development of the operator X_μ, defined as

$$X_\mu \equiv l Q_\mu, \tag{2.19}$$

from its initial value $X(0)$ at $u=0$ to an arbitrary u, $X_\mu(u)$. The X_μ, however, obeys the following transformation properties:

$$X_\mu \to e^{ius} X_\mu e^{-ius} = X_\mu(u) = X_\mu(0) + ulP_\mu = X_\mu(0) + \tau P_\mu / m_0. \tag{2.20}$$

Upon taking the expectation value of the last equation, we see that this becomes the classical equation of motion, where $\langle P_\mu / m_0 \rangle$ is the expectation value of the four-velocity, and X_μ can be considered as the position operator of the particle in classical relativistic mechanics.

Now we are prepared to make the crucial step in our discussion. This crucial step deals with the transition from classical to relativistic quantum mechanics. As was shown by Johnson in his detailed paper[14] and later by Aghassi, Roman, and Santilli,[12] the operator X_μ defined by Eq. (2.19) has all the necessary properties to be the well-defined position operator in relativistic quantum mechanics. Therefore, the transition from classical to relativistic quantum mechanics consists in requiring the commutator $[P_\mu, Q_\nu]$ to be nonvanishing; namely, we postulate, in accordance with the previous authors,[12,14]

$$[P_\mu, Q_\nu] = i g_{\mu\nu} l^{-1}, \tag{2.21}$$

where we have taken into account the relation between the operators X_μ and Q_μ given by Eq. (2.19). The last commutator is the simplest covariant generalization of the nonrelativistic commutator (2.7a). Postulating this commutator, we have obtained a new dynamical group describing the free particle in relativistic quantum mechanics which was first suggested by Castell[13] and later by Aghassi, Roman, and Santilli.[12] Our algebra differs from theirs only in irrelevant changes in notation and physical interpretations. This dynamical group is denoted by $\tilde{G}_5$ and its Lie algebra is defined by the relations (2.12), (2.17a), (2.17c), (2.17d), and by (2.21). It is worthwhile to mention that the meaning of the generators $\mathcal{T}_{\mu\nu}$ introduced in Eqs. (2.12) is to be changed; they are to be considered as the operators of the total (relativistic) angular momentum so that they become the generators of the $SL(2,c)$ group. The Lie group structure of $\tilde{G}_5$ can be written as

$$\tilde{G}_5 = \{T_1{}^\theta \otimes T_4{}^a \otimes T_1{}^\sigma\} \times \{T_4{}^b \times SL(2,c)\}, \tag{2.22}$$

where $T_1{}^\theta$ represents the one-parametric Abelian group which is connected with the emergence of the constant l in Eq. (2.21). The most interesting and important feature of this dynamical group is the fact that the mass-squared operator $P_\mu P^\mu$ is no longer the Casimir operator of $\tilde{G}_5$. This implies immediately that the mass spectrum of the particles involved in relativistic quantum mechanics can be quantized, and that particles with different masses belong to the same unitary irreducible representation of the group $\tilde{G}_5$.

It should be stressed that we know the physical meaning of all operators forming the Lie algebra of $\tilde{G}_5$ except for the meaning of the constant l. Further we can observe that the operator S, besides its role as the evolution operator with respect to u, is connected with the mass of particles accommodated in the single unitary irreducible representations of $\tilde{G}_5$. By direct

[14] J. E. Johnson, Phys. Rev. **181**, 1755 (1969).

computation one can verify that the operator D, defined as

$$D = P_\mu P^\mu + 2l^{-1}S, \quad (2.23)$$

is the Casimir operator of $\tilde{G}_5$, the eigenvalue of which can be used to label representations. From the last equation, it is evident that S is the mass-squared operator up to multiplicative and additive constants. This implies that the mass-squared operator lies in the algebra of this dynamical group and enables us to determine the mass spectrum of particles belonging to the single unitary irreducible representation of $\tilde{G}_5$.

At the end of this section we would like to mention that the unitary irreducible representations labeled by the various D are equivalent because each two of them are connected by a simple unitary transformation. The reason for this is that the operator S occurs only inside the commutators of the $\tilde{G}_5$ algebra. This property of the algebra allows us to redefine the operators S as follows:

$$S \to S + \tfrac{1}{2}Dl, \quad (2.24)$$

and the last relation, combined with Eq. (2.23), tells us that the representations corresponding to any D are equivalent to the representation labeled by $D=0$.[12]

We can conclude this section with the statement that free-particle states in relativistic quantum mechanics furnish the unitary irreducible representations of the dynamical group $\tilde{G}_5$. Therefore, the problem of characterizing the physical states has been reduced to the study of the unitary irreducible representations of the group in question. These representations are given in the next section.

III. UNITARY IRREDUCIBLE REPRESENTATIONS OF $\tilde{G}_5$

Let us assume from the beginning that there exist representations of the Lie algebra of $\tilde{G}_5$ defined by the commutators (2.12), (2.17a), (2.17c), (2.17d), and (2.21) as linear operators on some vector space. Since successive operations upon a vector with a series of operators are well defined, we can use these linear operators to define the product of two or more of them. All such products with their linear combinations define the universal enveloping algebra of $\tilde{G}_5$. We have mentioned in Sec. II that the generators $\mathcal{T}_{\mu\nu}$ [Eq. (2.12)] in our dynamical group $\tilde{G}_5$ play the role of the total (relativistic) angular momenta. Therefore, these operators can be split into two parts, namely, into the external $M_{\mu\nu}$ and into the intrinsic $T_{\mu\nu}$ angular momenta defined as

$$M_{\mu\nu} = l(Q_\mu P_\nu - Q_\nu P_\mu) \quad (3.1)$$

and

$$T_{\mu\nu} = \mathcal{T}_{\mu\nu} - M_{\mu\nu}, \quad (3.2)$$

respectively.

Having defined $M_{\mu\nu}$ and $T_{\mu\nu}$, we can easily verify the following set of commutation relations:

$$[T_{\mu\nu}, T_{\rho\sigma}] = i(g_{\nu\rho}T_{\mu\sigma} + g_{\mu\sigma}T_{\nu\rho} - g_{\mu\rho}T_{\nu\sigma} - g_{\nu\sigma}T_{\mu\rho}), \quad (3.3)$$

$$[T_{\mu\nu}, Q_\rho] = [T_{\mu\nu}, P_\rho] = [T_{\mu\nu}, S] = 0, \quad (3.4)$$

and

$$[P_\mu, P_\nu] = [Q_\mu, Q_\nu] = [P_\mu, S] = 0, \quad (3.5a)$$

$$[P_\mu, Q_\nu] = ig_{\mu\nu}l^{-1}, \quad (3.5b)$$

$$[Q_\mu, S] = iP_\mu. \quad (3.5c)$$

This algebra is the direct consequence of the enveloping algebra of $\tilde{G}_5$ and both of them are completely equivalent if the relations (3.1) and (3.2) are taken into account. Since $\mathcal{T}_{\mu\nu}$ is uniquely related to $T_{\mu\nu}$ and to $M_{\mu\nu}$ by

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + M_{\mu\nu}, \quad (3.6)$$

we may use the algebra (3.3)–(3.5) as the basis for the algebra of $\tilde{G}_5$. Thus we can find all representations for the algebra of $\tilde{G}_5$ by constructing the representations for the algebra (3.3)–(3.5).

The commutators (3.3)–(3.5) tell us that we are dealing with the direct product of two subalgebras. The algebra (3.3), involving only the operators $T_{\mu\nu}$, defines the homogeneous Lorentz group $SL(2,c)_T$, the representations of which are completely known.[15] The algebra (3.5) involving the operators P_μ, Q_μ, and S and the constant l is denoted by $\mathcal{A}$, the representations of which we are going to find. In such a way the enveloping algebra of the dynamical group $\tilde{G}_5$ is decomposed in the form

$$\tilde{G}_5 = SL(2,C)_T \otimes \mathcal{A} \quad (3.7)$$

and its unitary irreducible representations are obtained by taking the direct product of the separate unitary irreducible representations of $SL(2,C)$ and $\mathcal{A}$. The last equation expresses the fact that the enveloping algebras of these two groups are the same.

The unitary irreducible representations of $SL(2,C)$ are, as is well known, specified by two numbers j_0 and ν which are related with the eigenvalues of two Casimir operators C_1 and C_2, defined as

$$C_1 = \tfrac{1}{2}T_{\mu\nu}T^{\mu\nu} \to j_0^2 - 1 - \nu^2 \quad (3.8a)$$

and

$$C_2 = \tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma} \to 2j_0\nu. \quad (3.8b)$$

Here j_0 can be any integer or half-integer number, while ν is any real number (the principal series) or $j_0=0$ and ν is a purely imaginary number fulfilling the restriction $|\nu| \leq 1$ (the supplementary series).

The state within the unitary irreducible representation of $SL(2,C)$ is denoted by $|j_0,\nu; j,m\rangle$, where j and m denote the spin and its third component, respectively, of the particle belonging to the representation characterized by j_0 and ν. The spin j and its third component

[15] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (MacMillan, New York, 1963).

take the following discrete values:

$$j=j_0,\ j_0+1,\ \ldots,\ \infty$$
$$m=-j,\ -j+1,\ \ldots,\ j. \tag{3.9}$$

This simply implies that the particle states furnishing the unitary irreducible representation of the dynamical group $\tilde{G}_5$ form an infinite tower of spin states starting with the lowest spin value $j=j_0$ and going up in integral steps to infinity.

Now we turn our attention to the construction of unitary irreducible representations of the algebra $\mathcal{A}$ defined by the commutators (3.5). These representations will be specified by two numbers l and D corresponding to the constant l and to the eigenvalue of one Casimir operator D defined by Eq. (2.23). The state within the single unitary irreducible representation labeled by a given l and D must be specified by five numbers corresponding to the total number of mutually commuting operators of the Cartan subalgebra of $\mathcal{A}$. We select these five numbers to be the eigenvalues of the four-momentum p_1, p_2, p_3, p_0 and mass squared $m^2=p_\mu p^\mu$ so that a state is denoted as $|l,D; p_1,p_2,p_3,p_0,m^2\rangle$. The realization of the algebra $\mathcal{A}$ [Eq. (3.5)] can be easily constructed if we define the operators Q_μ and S as follows:

$$Q_\mu=-il^{-1}g_{\mu\nu}\partial/\partial p_\nu, \tag{3.10a}$$

$$S=-\tfrac{1}{2}lP_\mu P^\mu+\tfrac{1}{2}Dl. \tag{3.10b}$$

The last relations represent the unitary irreducible representations of the algebra $\mathcal{A}$, and these are the solutions to the problem we wanted to solve. Furthermore, it is clear that for every unitary irreducible representation the states $|l,D;\ p_1,p_2,p_3,p_0,m^2\rangle$ are expressed as square integrable functions $\Psi_{lD}(p_1,p_2,p_3,p_0,m^2)$ of the four-momentum and mass squared $m^2=p_\mu p^\mu$. The inner product $(\Psi_{lD},\Psi_{l'D'})=\delta_{ll'}\delta_{DD'}$ is obtained by an integration over the manifold $p_\mu p^\mu=m^2$ and by a summation or integration over the variable m^2.

The mass squared of the particles belonging to the same unitary irreducible representation is not necessarily quantized and can have any positive or negative value. The reason for this is that the representation functions $\Psi_{lD}(p_1,p_2,p_3,p_0,m^2)$ can be any class of square-integrable functions of p_μ and $m^2=p_\mu p^\mu$. The possibility of having the positive and negative values for the mass squared of particles in the same unitary irreducible representation is a great drawback of this dynamical group. This fact tells us that the generators representing physical observables can transform physical states into unphysical ones. This failure of this model is sufficiently strong to throw doubt on the reliability of this dynamical group to describe properly the free particles in relativistic quantum mechanics.

An additional reason why this group is unacceptable follows from the relation (3.7). This equation tells us that either each mass state occurs infinitely many times with spins $j=j_0,\ j_0+1,\ \ldots,\ \infty$ (the principal series); or there are only spin-zero particles (the supplementary series). This is because $P_\mu P^\mu$ lies entirely in the algebra $\mathcal{A}$ and there is no connection between $SL(2,C)_T$ and $\mathcal{A}$.

The unphysical results we encountered here are due to the postulate (2.21), which is a very naive covariant generalization of the nonrelativistic quantum-mechanical commutator. Therefore, this simple form of the corresponding commutator cannot be accepted, since it leads to the completely unphysical results obtained.

IV. CONCLUSIONS

We have studied the symmetry properties of Lagrangians for free particles in nonrelativistic and relativistic classical mechanics. The symmetry group which leaves the aforementioned Lagrangians invariant under the group transformations is referred to as the kinematical group. Furthermore, we have extended our symmetry investigation to the equations of motion which were derived from previous Lagrangians. The symmetry group of the equations of motion is always larger than the kinematical group, and we refer to this group as the dynamical group of the system. From this point of view, the Poincaré group is only the kinematical group while its extension to the group G_5 plays the role of the dynamical group characterizing the motion of a free particle in classical relativistic mechanics.

The transition from classical to relativistic quantum mechanics was performed by requiring that the commutator between the position and momentum operators be nonvanishing. These operators were previously defined in the classical dynamical group G_5. In such a way we have obtained from the classical dynamical group G_5 a new group $\tilde{G}_5$ which is supposed to be the dynamical group describing the elementary particles in relativistic quantum mechanics. All particle observables such as position, momentum, angular momentum, and mass are well-defined operators and form the closed algebra for the dynamical group $\tilde{G}_5$. Since observables operate on the space of physical states, the physical states must form a representation space of the dynamical algebra of observables, namely, the algebra of $\tilde{G}_5$. From this, it follows that the unitary irreducible representations of $\tilde{G}_5$ are possible physical states of the particles in relativistic quantum mechanics.

We have found the unitary irreducible representations of this dynamical group $\tilde{G}_5$. Each of these contains infinitely many particle states with different masses. The mass spectrum of the particles in the same representation is not necessarily quantized and the mass-squared value can be positive or negative. These properties of the group $\tilde{G}_5$ are very disappointing and tell us that generators representing physical observables can transform physical states with positive mass-squared into unphysical states having negative mass-squared. Therefore, this group cannot be accepted as the reliable dynamical group for particles in relativistic quantum mechanics. A further unpleasant feature of this model is concerned with the fact that to each mass state we have infinitely many particles with different spins varying from some lowest spin value j_0 and going up to infinity or that there are only spin-zero particles in the world.

All these unphysical features of the presented theory have their origin in postulating the naive and simplest covariant generalization of the commutator (2.21) between position and momentum operators. To obtain some reasonable physical results within the framework of this kind of theory, we must give up the aforementioned commutator and replace it by some other relativistically covariant form.[13] This replacement will induce, however, a radical change in the structure of the dynamical group here investigated.

To conclude this discussion, it should be stressed that relativistic quantum mechanics using the naive commutator (2.21) leads to completely unphysical results, and thus the dynamical group $\tilde{G}_5$, which is mainly based on this commutator, has no chance of giving any reasonable physical predictions.

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Compositeness Criterion for Unstable Particles

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The problem of making unstable elementary particles equivalent to resonances is investigated in a soluble model field theory. It is demonstrated that the irreducible part of the scattering amplitude develops a pair of complex conjugate poles on the second sheet of the energy plane, one of them corresponding to an S-wave resonance. The poles in the vertex and inverse propagator induced by these complex poles of the irreducible part of the scattering amplitude are such that the so-called Jin-MacDowell cancellation holds good. We show that under the conditions $Z_1=0$ and $Z_3=0$, the S-wave resonance completely replaces the unstable particle.

I. INTRODUCTION

IT has been demonstrated by several authors[1] that the stable elementary particles become composite under vanishing of the wave function and the vertex-function renormalization constants. In the models employed by these authors, the poles of the reducible part of the scattering amplitude are made to cancel each other, and the pole of the irreducible part which corresponds to the dynamical state is made to move to the elementary-particle pole position. In this paper, we examine this type of compositeness mechanism for an unstable elementary particle.

We define unstable particles according to the suggestion by Peierls[2] that they correspond to the complex poles of the propagator analytically continued to unphysical sheets. For our purpose, we consider a model field theory which consists of V, N, and θ particles, where N and θ are stable, but V is unstable. We show that the irreducible part of the $N\theta$ scattering amplitude analytically continued onto the second sheet develops a complex conjugate pair of poles corresponding to resonance and antiresonance of N and θ. We demonstrate that these poles of the irreducible part get canceled by the corresponding induced poles in the reducible part, i.e., the Jin-MacDowell[3] cancellation holds good for resonant states also. Conditions are found under which the pair of poles due to the unstable V particle gets canceled with the induced poles of the reducible part and the $N\theta$ resonance pole replaces the unstable V-particle pole.

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